ON REDUCTION OF A SMOOTH SYSTEM LINEAR IN THE CONTROL

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ON REDUCTION OF A SMOOTH SYSTEM LINEAR IN THE CONTROL

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ABSTRACT. A method is presented for reducing a smooth system linear in the control on an $n$-dimensional manifold $M$ to a nonlinear system on an $(n-1)$-dimensional manifold. This reduction is used to obtain sufficient conditions for a high order of local controllability of the system, and the problem of a time-optimal control of the angular momentum of a rotating rigid body is investigated.

Bibliography: 7 titles.

§1. Introduction

In this article a method is presented for investigating a controllable system of the form

$$\dot{x} = f(x) + g(x)u$$

(1.1)
on a smooth $n$-dimensional manifold $M$. Here $x \in M$, $u \in \mathbb{R}$, $f(x)$ and $g(x)$ are complete smooth vector fields on $M$, and the admissible controls $u(t)$ are bounded measurable functions of $t$.

It is shown that (1.1) can be reduced to a system nonlinear in the control with an $(n-1)$-dimensional phase space. This reduction is used here to obtain sufficient conditions for local controllability of high order for system (1.1), and also in the problem of time-optimal control of the rotation of an asymmetric rigid body by means of a moment applied along an axis fixed in the body.

§2. Preliminary material

We introduce some notation which mainly follows [1]. Denote by $C^\infty(M)$ the algebra of infinitely differentiable functions on $M$. We must deal below with operators $B$ and families of operators $B_t$ ($t \in \mathbb{R}$) mapping $C^\infty(M)$ into itself. Following [1], we define the properties of continuity, differentiability, integrability, etc., of a family of operators $B$ with respect to $t$ in the weak sense: $B_t$ has property (*) with respect to $t$ if the function $B_t \varphi$ has property (*) with respect to $t$ for all $\varphi \in C^\infty(M)$.

A vector field on $M$ is defined to be an arbitrary derivation of the algebra $C^\infty(M)$, i.e., a linear mapping $X$ of $C^\infty(M)$ into itself such that $X(\varphi_1 \varphi_2) = (X\varphi_1)\varphi_2 + \varphi_1(X\varphi_2)$. If we introduce local coordinates on $M$, then the field $X$ can be written in the form $X = \sum_i X_i \partial/\partial x_i$, where $X_i \in C^\infty(M)$. The value of a vector field $X$ at a point $x \in M$ is a vector, denoted by $x \circ X$, in the tangent space $T_xM$.
The Lie bracket (commutator) \([X, Y]\) of vector fields \(X\) and \(Y\) is defined by the formula \([X, Y]f = X(Yf) - Y(Xf)\). In local coordinates, \([X, Y] = \frac{\partial Y}{\partial x}X - \frac{\partial X}{\partial x}Y\). As we know, the commutator \([X, Y]\) is also a vector field; the Lie bracket introduces the structure of a Lie algebra in the space of vector fields. For a vector field \(X\) the linear operator \(\text{ad} X\) is defined on the space of vector fields by the formula \((\text{ad} X)Y = [X, Y]\).

Finally, a nonautonomous vector field \(X_t (t \in \mathbb{R})\) is defined to be a family of vector fields integrable with respect to \(t\).

Consider a diffeomorphism \(P\) of \(M\) onto itself. It determines an automorphism of the algebra \(C^\infty(M)\) by the formula \((Pf)(x) = f(P(x))\) for \(f \in C^\infty(M)\). This automorphism of \(C^\infty(M)\) is also called a diffeomorphism and is denoted by the same symbol \(P\). So that there will be no confusion, we denote the image of a point \(x\) under a diffeomorphism \(P\) by \(x \circ P\), and the value of a function \(f\) at \(x\) by \(x \circ f\).

Following [1], we define a flow \(P_t\) to be an absolutely continuous family of diffeomorphisms. It is easy to show that the composition \(P_t^{-1} \circ (d/dt)P_t\) is a family of derivations of \(C^\infty(M)\) that is integrable with respect to \(t\), i.e., a nonautonomous vector field \(X_t\). It follows from the equality \(P_t^{-1} \circ (d/dt)P_t = X_t\) that
\[
\frac{d}{dt} P_t = P_t \circ X_t. \tag{2.1}
\]
Thus, any flow \(P_t\) is generated by some nonautonomous vector field \(X_t\) in view of the differential equation (2.1). A solution of (2.1) will be denoted by \(\exp\int_0^t X_t \, dt\) and called [1] a chronological exponential. If the vector field \(X_t\) is autonomous, i.e., \(X_t = X\), then the flow generated by this field is denoted by \(e^{tU}\).

According to [1], the chronological exponential can be expanded in a series
\[
\exp\int_0^t X_t \, dt = I + \int_0^t X_t \, dt + \int_0^t \int_0^t X_{t_1} \circ X_{t_2} \, dt_1 \, dt_2 + \cdots. \tag{2.2}
\]

We also give a variational formula for the chronological exponential [1]:
\[
\exp\int_0^t (X_t + Y_t) \, dt = \exp\int_0^t \exp\int_0^t \text{ad} X_s \, ds Y_t \, dt \circ \exp\int_0^t X_t \, dt. \tag{2.3}
\]

In (2.3) the operator exponential \(Q_t = \exp\int_0^t \text{ad} X_s \, ds\) is an absolutely continuous family of operators on the space of vector fields that satisfies the equation
\[
\frac{d}{dt} Q_t Z = Q_t \circ (\text{ad} X_t) Z
\]
for any vector field \(Z\). The flow \(\exp\int_0^t \exp\int_0^s \text{ad} X_s \, ds Y_t \, dt\) (see (2.3)) was called a perturbation flow in [1].

We consider a controllable system on \(M\) of the form
\[
\dot{x} = X(x, u), \quad u \in U. \tag{2.4}
\]
The right-hand side of (2.4) can be regarded as a family \(\mathcal{X} = \{X(x, u): u \in U\}\) of vector fields depending on the parameter \(u \in U\). It will be assumed that the vector field \(X(x, u)\) is complete for any \(u\).

The orbit \(\mathcal{O}_x\) of system (2.4) at the point \(x \in M\) is defined to be the set of points of the form
\[
\mathcal{O}_x = \{ x \circ (e^{t_1 X_{t_1}} \circ e^{t_2 X_{t_2}} \circ \cdots \circ e^{t_k X_{t_k}}): t_i \in \mathbb{R}, X_i \in \mathcal{X} \}.
\]
Obviously, if \(x' \in \mathcal{O}_x\), then \(\mathcal{O}_{x'} = \mathcal{O}_x\).
THEOREM 2.1 (SUSSMANN [2]). For any point \( x \in M \) the orbit \( \mathcal{O}_x \) is a smooth submanifold of \( M \) that is invariant for system (2.4).

The positive orbit \( \mathcal{O}_x^+ \) of system (2.4) at a point \( x \in M \) is defined to be the set of points of the form
\[
\mathcal{O}_x^+ = \{ x \circ (e^{t_1 X_1} \circ \cdots \circ e^{t_n X_n}) : t_i \in \mathbb{R}, t_i \geq 0, X_i \in \mathcal{F} \}.
\]
Obviously, \( \mathcal{O}_x^+ \subseteq \mathcal{O}_x \).

Denote by \( \mathcal{L}[\mathcal{F}] \) the smallest Lie algebra of vector fields such that \( \mathcal{L}[\mathcal{F}] \supseteq \mathcal{F} \). The rank of the controllable system (2.4) at a point \( x \in M \) is defined to be
\[
\dim \text{span}\{ x \circ X : X \in \mathcal{L}[\mathcal{F}] \}.
\]

THEOREM 2.2 [2]. The value of any vector field \( X \in \mathcal{L}[\mathcal{F}] \) at a point \( x' \in \mathcal{O}_x \) is a tangent vector to \( \mathcal{O}_x \). In particular, the rank of system (2.4) at a point \( x' \in \mathcal{O}_x \) does not exceed \( \dim \mathcal{O}_x \).

The following condition is assumed in what follows (it is true, in particular, for all real analytic systems): the rank of (2.4) at any point \( x' \in \mathcal{O}_x \) coincides with the dimension \( \dim \mathcal{O}_x \). What is more, for our purposes it suffices to consider the restriction of system (2.4) to the orbit \( \mathcal{O}_x \), which enables us to assume without loss of generality that the orbit \( \mathcal{O}_x \) coincides with the manifold \( M \), and that the rank of (2.4) is equal to \( \dim M \) at each point. In this case we have

THEOREM 2.3 (KRENER [2]). If the rank of system (2.4) at each point \( x \in M \) is equal to \( \dim M \) and \( \mathcal{O}_x = M \), then the set of interior points of the positive orbit \( \mathcal{O}_x^+ \) is dense in \( \mathcal{O}_x^+ \).

THEOREM 2.4 (SUSSMANN AND JURDJEVIC [2]). If the rank of system (2.4) at a point \( x \) is equal to \( \dim M \), then for any \( T > 0 \) the set of attainability \( A_{\leq T,x} \) of (2.4) from the point \( x \) in a time \( \leq T \) has nonempty interior, and \( A_{t} \) is dense in \( A_{\leq T,x} \).

Along with the Lie algebra \( \mathcal{L}[\mathcal{F}] \) we consider the smallest Lie subalgebra \( \mathcal{L}^0[\mathcal{F}] \) of it containing all the fields of the form \( X^1 - X^2 \) (\( X^1, X^2 \in \mathcal{F} \)) and \([Y^1, Y^2] \) (\( Y^1, Y^2 \in \mathcal{L}[\mathcal{F}] \)). We call \( \dim \text{span}\{ x \circ X : X \in \mathcal{L}^0[\mathcal{F}] \} \) the exact rank of the system. Obviously, the exact rank of the system does not exceed its rank.

THEOREM 2.5 [2]. If the exact rank of system (2.4) at a point \( x \) is equal to \( \dim M \), then for any \( T > 0 \) the set of attainability \( A_{T,x} \) of (2.4) from \( x \) in the time \( T \) has nonempty interior, and \( \text{int} A_{T,x} \) is dense in \( A_{T,x} \).

§3. Reduction of the controllable system (1.1)

We consider a controllable system (1.1) and an admissible control \( u(t) \). The flow \( \Phi_t \) generated by the differential equation
\[
\dot{x} = f(x) + g(x)u(t)
\]
(3.1) can be represented in the form of the chronological exponential
\[
\Phi_t = \exp \left( \int_0^t (f(x) + g(x)u(\tau)) \, d\tau \right).
\]
According to the variational formula (2.3),
\[
\exp \int_0^T \left( f + gu(\tau) \right) d\tau = \exp \int_0^T e^{\int_0^\tau u(s) \, ds} \, df \, d\tau \circ e^{\int_0^\tau \tau \, df}.
\]
(3.2)
or, with the notation \( v(\tau) = \int_0^\tau u(s) \, ds \),
\[
\exp \int_0^T \left( f + gu(\tau) \right) d\tau = \exp \int_0^T e^{v(\tau)} \, df \, d\tau \circ e^{v(\tau)}.
\]
(3.3)
The right-hand side of (3.3) is the composition of the flows generated by the respective nonautonomous vector fields \( e^{v(\tau)} \, df \) and \( v(t) \, g \).

Consider a neighborhood \( V \) of a point \( \tilde{x} \in M \) such that \( g \mid_V \neq 0 \). We define on \( V \) an equivalence relation that puts in a single class all the points lying on a single trajectory of the vector field \( g \mid_V \), and we denote by \( V^g \) the quotient set by this equivalence relation. We can regard \( V^g \) as a set of segments of trajectories of the vector field \( g \). Obviously, \( V^g \) can be parametrized by the points of the set \( N \cap V \), where \( M \supset N \) is an \((n - 1)\)-dimensional submanifold of \( M \) transversal to the trajectories of the field \( g \) in the neighborhood of \( \tilde{x} \).

Suppose that \( g \neq 0 \) on the whole manifold \( M \) and, moreover, satisfies the "nonrecurrence" conditions: for each point \( x \in M \) there exist a neighborhood \( V_x \ni x \) and an \((n - 1)\)-dimensional manifold \( N_x \subset M \) transversal to \( g \) such that any trajectory of \( g \) intersects the set \( V_x \cap N_x \) in a unique point. In particular, the "nonrecurrence" condition holds when \( M = \mathbb{R}^n \) and \( g \) is a constant vector field. Under these conditions the equivalence relation can be defined globally on the manifold \( M \). The corresponding quotient manifold (the manifold of trajectories of \( g \)) is denoted by \( M^g \).

We consider the family of vector fields \( F_v = e^{v \, df} \) \((v \in \mathcal{R})\), and prove that it is well defined on \( M^g \), i.e., under the action of the diffeomorphism \((e^g)_* \), a vector field in the family \( F_v \) passes into a vector field in the same family. Indeed, under the action of \((e^g)_* \) the field \( F_v = e^{v \, df} \) passes \([1]\) into the field \( e^{v \, df} F_v = e^{v \, df} e^{v \, df} = e^{(v + v) \, df} = F_{v + v} \), i.e., the group of diffeomorphisms \((e^g)_* \) carries the family \( F_v \) into itself. We prove

**Proposition 1.** Let \( M^g \) be the quotient manifold described above, \( \pi \) the canonical projection of \( M \) onto \( M^g \), and \( D_{T, \tau} (D_{\leq T, \tau}) \) the set of attainability in the time \( T \) \((\leq T)\) from a point \( \tilde{y} \) for the controllable system
\[
y = y \circ F_v = y \circ (e^{v \, df})
\]
(3.4)
on the manifold \( M^g \), where essentially bounded measurable scalar functions \( v(t) \) are taken as the controls. The set of attainability \( A_{T, \tau} (A_{\leq T, \tau}) \) of (1.1) in the time \( T \) \((\leq T)\) from a point \( \tilde{x} \) is contained in the inverse image \( \pi^{-1}(D_{\tau, \tau}(\pi)) \) \((\pi^{-1}(D_{\leq T, \tau}(\pi)))\), and if the exact rank (the rank) of system (1.1) at \( \tilde{x} \) is equal to \( \dim M \), then the interior of \( A_{T, \tau} (A_{\leq T, \tau}) \) is dense in
\[
\pi^{-1}(D_{\tau, \tau}(\pi)) \) \((\pi^{-1}(D_{\leq T, \tau}(\pi)))\).

**Remark.** In other words, Proposition 1 means that the sets \( A_{T, \tau} (A_{\leq T, \tau}) \) and \( \operatorname{int} A_{T, \tau} \) \((\operatorname{int} A_{\leq T, \tau})\) are contained and everywhere dense in the \"cylinder\"; that is, \"swept out\" in the motion of the set \( D_{\tau, \tau}(\pi) \) \((D_{\leq T, \tau}(\pi))\) along trajectories of \( g \).

**Proof of Proposition 1.** Let \( \tilde{u}(t) \) be a fixed admissible control of (1.1), and \( T \) a fixed time. Setting \( \tilde{v}(t) = \int_0^t \tilde{u}(\tau) \, d\tau \), we get by (3.3) that
\[
\exp \int_0^T \left( f + g\tilde{v}(\tau) \right) d\tau = \exp \int_0^T F_{\tilde{v}(t)} \circ x \circ \tilde{u}(T) \circ g.
\]
(3.5)
Obviously, the point $\tilde{\chi} \circ (\exp \int_0^T F_v(t) \, dt \circ e^{b(T)}g)$ is contained in $\pi^{-1}(D_{T, \pi(\tilde{x})})$, since $\tilde{x} \circ \exp \int_0^T F_v(t) \, dt \in D_{T, \pi(\tilde{x})}$; and this proves the inclusion $A_{T, \tilde{x}} \subseteq \pi^{-1}(D_{T, \pi(\tilde{x})})$.

To prove the second part of Proposition 1 we use the auxiliary

**Lemma 2** [1]. The point

$$\tilde{y} \circ \left( \exp \int_0^T F_v(t) \, dt \right) = \tilde{y} \circ \left( \exp \int_0^T e^{v(t)adg} \, dt \right)$$

depends continuously on $v(\cdot)$ in the metric of $L_2[0, T]$.

Suppose that $\tilde{x} \in \pi^{-1}(D_{T, \pi(\tilde{x})})$ and $b(\cdot)$ is a corresponding control carrying system (3.4) from the point $\pi(\tilde{x})$ to $\pi(\tilde{x})$ in time $T$. We consider on $\mathbb{M}$ the differential equation $\dot{x} = x \circ e^{b(t)adg}$ and the trajectory $\tilde{x}(t)$ of it satisfying $\tilde{x}(0) = \tilde{x}$. Let $\tilde{x}(T) = \tilde{z}$. Since $b(\cdot)$ carries system (3.4) from $\pi(\tilde{x})$ to $\pi(\tilde{x})$ in time $T$, the points $\tilde{x}$ and $\tilde{z}$ lie on a single trajectory of $g$ in view of (3.5), i.e., $\dot{x} = \tilde{x} \circ e^{s_T}$.

Choose an absolutely continuous function $v(\cdot)$ in the $\delta$-neighborhood of $v(\cdot)$ in the $L_2[0, T]$-metric and satisfying the conditions $v^s(0) = 0$ and $v^s(T) = s$. We let $u^s(t) = \delta^s(t)$, and consider the Cauchy problem $\dot{x}^s(t) = f(x) + g(x)u^s(t)$, $x(0) = \tilde{x}$. According to (3.3), a solution $x^s(t)$ of this Cauchy problem is defined by

$$x^s(t) = \tilde{x} \circ \left( \exp \int_0^t e^{v^s(\tau)adg} \, d\tau \circ e^s \right).$$

By choosing $\delta$ sufficiently small it is possible by Lemma 2 to make the point $\tilde{x} \circ (\exp \int_0^T e^{v^s(\tau)adg} \, d\tau) \circ e^{s_T}$ arbitrarily close to $\tilde{z} = \tilde{x} \circ \exp \int_0^T e^{v^s(\tau)adg} \, d\tau$, and thereby to make $x^s(t)$ arbitrarily close to $\tilde{x} = \tilde{x} \circ e^{s_T}$.

Thus, it is proved that the set of attainability $A_{T, \tilde{x}}$ is dense in $\pi^{-1}(D_{T, \pi(\tilde{x})})$. According to Theorem 2.5, int $A_{T, \tilde{x}}$ is dense in $A_{T, \tilde{x}}$. Consequently, int $A_{T, \tilde{x}}$ is dense in $\pi^{-1}(D_{T, \pi(\tilde{x})})$.

Analogous arguments can be carried out for the set $A_{T, \tilde{x}}$. Proposition 1 is proved.

It follows from Proposition 1 that the investigation of the set of attainability of the controllable system (1.1) of order $n$ can be reduced to an investigation of a system (3.4) of order $n - 1$ which, contrary to (1.1), is nonlinear (and often nondegenerate) in the control.

Proposition 1 admits a natural generalization to the case of a system linear in the vector-valued control $u = (u_1, \ldots, u_l)$ and of the form

$$\dot{x} = f(x) + \sum_{j=1}^l g_j(x)u_j. \quad (3.6)$$

Suppose that the fields $\{g_i(x), i = 1, \ldots, l\}$ are linearly independent at each point $x \in M$ and generate an involutory $l$-dimensional distribution $G$ on $M$. By the Frobenius theorem, there exist functions $b_{ij}(x), i, j = 1, \ldots, l$, such that the vector fields $\tilde{g}_j(x) = \sum_{i=1}^l b_{ij}(x)g_i(x)$ form a basis for the distribution $G$ and have commutator $[\tilde{g}_i, \tilde{g}_j] = 0$ for any $i$ and $j$. Obviously, the determinant of the matrix $B = \|b_{ij}(x)\|$ is nonzero on $M$. Let $B^{-1}(x) = C(x) = \|c_{ij}(x)\|$; then

$$g_i(x) = \sum_{j=1}^l c_{ij}(x)\tilde{g}_j(x), \quad i = 1, \ldots, l. \quad (3.7)$$

Substituting (3.7) into (3.6) and introducing the new controls $v_j = \sum_{i=1}^l c_{ij}(x)u_i, j = 1, \ldots, l$, we get that (3.6) can be reduced to the system

$$\dot{x} = f(x) + \sum_{i=1}^l \tilde{g}_i(x)v_j \quad (3.8)$$
with pairwise commuting fields \( \dot{g}_i(x) \), \( i = 1, \ldots, l \), which generate the same distribution \( G \) on \( M \).

According to the Frobenius theorem, \( M \) is stratified into the integral manifolds of the distribution \( G \). A literal repetition of the arguments given above with the integral curves of \( g \) replaced by the integral manifolds of \( G \) enables us to define from \( G \) an equivalence relation on \( M \) and an \((n - l)\)-dimensional quotient manifold \( M^G \) by this equivalence relation.

**Proposition 1'.** Let \( M^G \) be the indicated quotient manifold, \( \pi \) the canonical projection of \( M \) onto \( M^G \), and \( D_{\pi}(x) \) the set of attainability in the time \( T \) (\( \leq T \)) from a point \( \dot{y} \in M^G \) for the controllable system

\[
\dot{y} = y \circ \left( e^{\sum_{i=1}^l w_i \text{ad} \dot{g}_i} \right) \tag{3.9}
\]
on \( M^G \), where the essentially bounded scalar functions \( w_i(t) \) are taken as controls. The set of attainability \( A_{\pi}(x) \) of system (3.6) (or (3.8)) in time \( T \) (\( \leq T \)) from a point \( \hat{x} \in M \) is contained in the inverse image \( \pi^{-1}(D_{\pi}(x)) \) (\( \pi^{-1}(D_{\pi}(\hat{x})) \)), and if the exact rank (the rank) of (3.6) at \( \hat{x} \) is equal to \( \dim M \), then the interior of \( A_{\pi}(x) \) is dense in \( \pi^{-1}(D_{\pi}(x)) \).

Thus, by Proposition 1', the investigation of the system (3.6) of order \( n \) with \( l \)-dimensional control reduces to the investigation of the system (3.9) of order \( n - l \).

**§4. Sufficient conditions for local controllability**

Let us consider a controllable system (1.1) and a trajectory \( \dot{x}(t) \) of this system with the initial condition \( \dot{x}(0) = \hat{x} \) generated by an admissible control \( \bar{u}(t) \). We introduce a special norm in the space of controls \( u(\cdot) \); namely, we let

\[
\|u(\cdot)\|_{[0,T]} = \sup_{t_1, t_2 \in [0,T]} \left| \int_{t_1}^{t_2} u(\tau) d\tau \right| .
\]

This kind of norm is used in investigating sliding regimes [3]; therefore, the metric generated by it is called the sliding regime metric.

For what follows it is convenient to introduce the notation \( A_{\pi}(x) \) for the set of attainability of system (1.1) in time \( T \) from the point \( \hat{x} \) by means of a control \( u(\cdot) \) with \( \|u(\cdot)\|_{[0,T]} < \varepsilon \).

**Definition.** Let \( \hat{x}(\cdot) \) be the trajectory of (1.1) generated by the zero control, \( \hat{x}(0) = \hat{x} \). Then system (1.1) is weakly locally controllable from the point \( \hat{x} \) in time \( T \) if \( \hat{x}(T) \in \text{int} A_{\pi}(\hat{x}) \) for all \( \varepsilon > 0 \).

**Proposition 3.** Consider on \( M \) the two-parameter family of vector fields \( Z_{t,v} = e^{t \text{ad} f} e^{v \text{ad} \dot{g} - f} \), and let

\[
\Theta_{\pi}(\hat{x}) = \text{con} \left\{ \hat{x} \circ Z_{t,v} : 0 \leq t \leq T, \|v\| \leq \varepsilon \right\}, \tag{4.1}
\]

\[
\Xi_{\pi}(\hat{x}) = \text{con} \left\{ \Theta_{\pi}(\hat{x}) \cup \left\{ \hat{x} \circ g, \hat{x} \circ (-g) \right\} \right\}. \tag{4.2}
\]

(here \( \text{con} B \) denotes the convex cone generated by a set \( B \); \( \Theta_{\pi}(\hat{x}) \) and \( \Xi_{\pi}(\hat{x}) \) are thus convex cones lying in the tangent plane \( T_{\hat{x}} M \)).

Suppose that \( \gamma(t) \) is the trajectory of system (1.1) generated by the zero control, \( \gamma(0) = \hat{x} \), and \( \gamma(s) (s \geq 0) \) is a curve on \( M \) with \( \gamma(0) = \hat{x} \). If \( \gamma'(0) \in \text{int} \Xi_{\pi}(\hat{x}) \) for all \( \varepsilon > 0 \), then for any \( \varepsilon > 0 \) the point \( \gamma(s) \circ e^{Tt} \) lies in \( \text{int} A_{\pi}(\hat{x}) \) for all sufficiently small \( s \geq 0 \).
PROOF OF PROPOSITION 3. For an arbitrary control $u(\cdot)$ we represent the trajectory of (1.1) generated by it in the form of the chronological exponential $\exp \int_0^T (f + gu(\tau)) \, d\tau$. According to (3.5),

$$\exp \int_0^T (f + gu(\tau)) \, d\tau = \exp \int_0^T e^{(r)(ad)g} \, d\tau \circ e^{(r)g},$$

where $v(t) = \int_0^t u(\tau) \, d\tau$. We represent the vector field $e^{(r)(ad)g}$ in the form

$$e^{(r)(ad)g} = f + e^{(r)(ad)g} - f.$$  

By the variational formula (2.3),

$$\exp \int_0^T (f + gu(\tau)) \, d\tau = \exp \int_0^T e^{(r)(ad)g} \, d\tau \circ e^{(r)g},$$

or [1]

$$\exp \int_0^T (f + gu(\tau)) \, d\tau = \exp \int_0^T e^{(r)(ad)g} \, d\tau \circ e^{(r)g}.$$  

Combination of (4.3) and (4.4) gives us that

$$\exp \int_0^T (f + gu(\tau)) \, d\tau = \exp \int_0^T e^{(r)(ad)g} \, d\tau \circ e^{(r)g},$$

or [1]

$$\exp \int_0^T (f + gu(\tau)) \, d\tau = \exp \int_0^T e^{(r)(ad)g} \, d\tau \circ e^{(r)g}.$$  

We prove that

$$\text{con}\{(\Theta_{T,e}(\tilde{x})) \cup \{\tilde{x} \circ (\pm e^{(r)ad}g)\}\} = \Xi_{T,e}(\tilde{x})$$

(cf. (4.2)). To do this we first show that $(\tilde{x} \circ (\pm e^{(r)ad}g)) \in \Theta_{T,e}(\tilde{x})$. Since $(\tilde{x} \circ Z_{t,\pm e}) \in \Theta_{T,e}(\tilde{x})$ for $t \in [0, T]$ and $|v| \leq \varepsilon$, and $\Theta_{T,e}(\tilde{x})$ is a convex cone, it follows that

$$\frac{d}{dv} \bigg|_{v=0} (\tilde{x} \circ Z_{t,\pm e}) \in \Theta_{T,e}(\tilde{x}).$$

A direct computation yields

$$\frac{d}{dv} \bigg|_{v=0} (\tilde{x} \circ Z_{t,\pm e}) = \tilde{x} \circ (\pm e^{(r)ad}g) = \tilde{x} \circ (\mp e^{(r)ad}g) \in \Theta_{T,e}(\tilde{x}).$$

We now prove that

$$\tilde{x}(\pm e^{(r)ad}g) \in \Xi_{T,e}(\tilde{x}).$$

(4.7)

Obviously,

$$\frac{d}{dt} (\tilde{x} \circ (\pm e^{(r)ad}g)) = \tilde{x} \circ (\pm e^{(r)ad}g) \in \Theta_{T,e}(\tilde{x}) \subseteq \Xi_{T,e}(\tilde{x}).$$

On the other hand, $(\tilde{x} \circ (\pm e^{(r)ad}g)) = (\tilde{x} \circ (\pm g)) \in \Xi_{T,e}(\tilde{x})$ for $t = 0$. Since $\Xi_{T,e}(\tilde{x})$ is a convex cone, we get (4.7) and, in particular, $(\tilde{x} \circ (\pm e^{(r)ad}g)) \in \Xi_{T,e}(\tilde{x})$, which implies that

$$\text{con}\{(\Theta_{T,e}(\tilde{x})) \cup \{\tilde{x} \circ (\pm e^{(r)ad}g)\}\} \subseteq \Xi_{T,e}(\tilde{x}).$$

To prove the reverse inclusion we show that $(\tilde{x} \circ (\pm (g - e^{(r)ad}g)))$ lies in $\Theta_{T,e}(\tilde{x})$. Indeed,

$$\frac{d}{dt} (\tilde{x} \circ (\pm (g - e^{(r)ad}g))) = (\tilde{x} \circ (\mp e^{(r)ad}g)) \in \Theta_{T,e}(\tilde{x}).$$
Since $g - e^{t\text{ad}^f/g} = 0$ for $t = 0$, we get that $g - e^{t\text{ad}^f/g} \in \Theta_{T,e}(\tilde{x})$ for all $t \in [0, T]$. The equality (4.6) is proved.

We consider the set
\[ C_{T,\tilde{x}} = \left\{ \tilde{x} \circ \left( \exp \int_0^T Z_{t,v(t)} dt \circ e^{V(T)e^{T\text{ad}^f/g}} \right) \right\}, \]
where the $v(\cdot)$ are absolutely continuous functions with $|v| \leqslant \varepsilon$. By (4.5), it suffices to show that for small $s > 0$ the points of the curve $\gamma(s)$ with $\gamma(0) \in \text{int} \Xi_{T,e}(\tilde{x})$ lie in $C_{T,\tilde{x}}$.

Let $Y_{t,v(t)} = v(t)e^{t\text{ad}^f/g}$. Then
\[ C_{T,\tilde{x}} = \left\{ \tilde{x} \circ \left( \exp \int_0^T Z_{t,v(t)} dt \circ e^{Y_{T,v(t)}} \right) \right\}. \quad (4.8) \]

Note that $Z_{t,v(t)} = Y_{t,v(\cdot)} = 0$ for $v(\cdot) = 0$. Using formula (2.2) for the exponentials on the right-hand side of (4.8), we get that
\[ \exp \int_0^T Z_{t,v(t)} dt \circ e^{Y_{T,v(t)}} = I + \int_0^T Z_{t,v(t)} dt + Y_{T,v(\cdot)} + \cdots, \quad (4.9) \]
where the dots stand for terms of higher than first order of smallness in $Z$ and $Y$. Let
\[ W(v(\cdot)) = \int_0^T Z_{t,v(t)} dt + Y_{T,v(\cdot)}. \]
The range of the mapping $W$ when $v(\cdot)$ is replaced by the set of absolutely continuous functions with $|v| \leqslant \varepsilon$ is a convex subset of $T_{\tilde{x}}M$. The interior of the cone spanned by it coincides with $\text{int} \Xi_{T,e}(\tilde{x})$ by the definition of $\Xi_{T,e}(\tilde{x})$.

Therefore the points of any curve $\gamma(s)$ ($s > 0$) lying in $\text{int} \Xi_{T,e}(\tilde{x})$ for small $s > 0$ also lie in the interior of the range of $W$ for $|s| < \delta$ if $\delta$ is small.

Arguments analogous to those used in proving the maximum principle (see, for example, [3], Theorem VII.1) imply the existence of a $\delta', 0 < \delta' < \delta$, such that for $|s| < \delta'$ the points of the curve $\gamma(s)$ lie in $C_{T,\tilde{x}}$, and this proves Proposition 3.

The next result follows directly from the proof of Proposition 3.

**Proposition 4.** If
\[ 0 \in \text{int} \Xi_{T,e}(\tilde{x}) = \text{int \, con} \{ \{ \tilde{x} \circ (e^{t\text{ad}^f/g} - f) : 0 \leqslant t \leqslant T, |v| \leqslant \varepsilon \} \cup \{ \tilde{x} \circ (\pm g) \} \}, \quad (4.10) \]
then system (1.1) is weakly locally controllable from the point $\tilde{x}$ in time $T$.

### §5. Algebraic conditions for weak local controllability

Everywhere in this section we consider a controllable system (1.1) with the extra condition $\tilde{x} \circ f = 0$. Denote by $\Phi$ the Jacobi matrix $\Phi = \tilde{x} \circ (d\tilde{f}/dx)$. Then $\tilde{x} \circ (e^{t\text{ad}^f/X}) = \tilde{x} \circ (e^{t\Phi X})$ for any vector field $X$ on $M$. In this case condition (4.10) for system (1.1) takes the form
\[ 0 \in \text{int} \Xi_{T,e}(\tilde{x}) = \text{int \, con} \{ \{ e^{t\Phi}(\tilde{x} \circ (e^{t\text{ad}^g/f}) : 0 \leqslant t \leqslant T, |v| \leqslant \varepsilon \} \cup \{ \tilde{x} \circ (\pm g) \} \}. \quad (5.1) \]

Since the matrix $e^{t\Phi}$ determines a linear transformation of the tangent space $T_{\tilde{x}}M$, it follows from (4.6) that
\[ \Xi_{T,e}(\tilde{x}) = \left\{ e^{t\Phi}(\text{con} \{ \tilde{x} \circ (e^{t\text{ad}^g/g}) : |v| \leqslant \varepsilon \} \cup \{ \tilde{x} \circ (\pm g) \}) : 0 \leqslant t \leqslant T \right\}. \]

Let us investigate the set $\text{con} \{ \tilde{x} \circ (e^{i\text{ad}^g/g}) : |v| \leqslant \varepsilon \} \cup \{ \tilde{x} \circ (\pm g) \}$. To do this we consider the smallest even $j \geqslant 0$ such that
\[ \{ \tilde{x} \circ ((\text{ad} g)^j f) \} \not\subset \text{span} \{ \tilde{x} \circ ((\text{ad} g)^i f) : 1 \leqslant i < j \} \cup \{ \tilde{x} \circ g \}. \quad (5.2) \]
If condition (5.2) does not hold for any even \( j \), then we set \( j = +\infty \). Let

\[
\mathcal{L}_x = \begin{cases} 
\text{span} \left\{ \tilde{x} \circ ((\text{ad} g)^i f) : 1 \leq i < j \right\} \cup \left\{ \tilde{x} \circ (g) \right\} & \text{if } j < +\infty, \\
\text{span} \left\{ \tilde{x} \circ ((\text{ad} g)^i f) : 1 \leq i < +\infty \right\} \cup \left\{ \tilde{x} \circ (g) \right\} & \text{if } j = +\infty.
\end{cases}
\]

**Proposition 5.** The linear space \( \mathcal{L}_x \) and the vector \( \tilde{x} \circ ((\text{ad} g)^i f) \) (if \( j < +\infty \)) are contained in the cone

\[
\mathcal{F}_x = \text{con} \left\{ \left\{ \tilde{x} \circ (e^{u \text{ad} g f}) : |v| \leq \varepsilon \right\} \cup \left\{ \tilde{x} \circ \left( \pm g \right) \right\} \right\} \subseteq T_x M.
\]

The proof is by contradiction. If this assertion is false, then, since \( \mathcal{F}_x \) is convex, there exist a vector \( q \in \mathcal{L}_x \) and a covector \( \psi \in T_{T_x M}^* \) (\( \psi \neq 0 \)) such that with the notation \( \varphi(v) = (\psi, (\tilde{x} \circ (e^{u \text{ad} g f}))) \) we have

\[
\left( \left( \langle \psi, (\tilde{x} \circ ((\text{ad} g)^k f)) \right) > 0 \right) \lor \left( \langle \psi, q \right) > 0 \right) \]

\[
\land \left( \forall v: |v| \leq \varepsilon, \varphi(v) \leq 0 \right) \land \left( \langle \psi, \tilde{x} \circ (g) = 0 \right).
\]

(5.3)

Obviously, \( \varphi(0) = 0 \). It follows from (5.3) that the first nonzero derivative \( \varphi^{(k)}(0) \) must be even, and \( \varphi^{(k)}(0) < 0 \). We prove that \( k \geq j \). Indeed, if \( k < j \) is even and \( \varphi^{(l)}(0) = (\langle \psi, (\tilde{x} \circ ((\text{ad} g)^l f)) \rangle = 0 \) for all \( l < k \), then by the definition of \( j \)

\[
\left\{ \tilde{x} \circ ((\text{ad} g)^k f) \right\} \in \text{span} \left\{ \left\{ \tilde{x} \circ ((\text{ad} g)^l f) : l < k \right\} \cup \left\{ \tilde{x} \circ (g) \right\},
\]

and, consequently, \( \varphi^{(k)}(0) = (\psi, (\tilde{x} \circ ((\text{ad} g)^k f))) = 0 \). Thus, \( k \geq j \), and hence \( \varphi^{(j)}(0) = (\psi, (\tilde{x} \circ ((\text{ad} g)^j f))) \leq 0 \), which contradicts (5.3). Proposition 5 is proved.

Proposition 6 and 5 is

**Proposition 6.** Let \( \mathcal{X} \) be the cone generated by the space \( \mathcal{L}_x \) and the vector \( (\tilde{x} \circ ((\text{ad} g)^i f)) \) (if \( j < +\infty \)). If \( \Phi = \tilde{x} \circ \frac{df}{dx} \) and \( \text{con} \left\{ e^{t\Phi} \mathcal{X} : 0 \leq t \leq T \right\} = T_x M \), then system (1.1) is weakly locally controllable from \( \tilde{x} \) in time \( T \).

**Proof.** By Proposition 5, \( \mathcal{F}_x \supseteq \mathcal{X} \), and hence

\[
0 \in \text{int} T_x M = \text{int} \text{con} \left\{ e^{t\Phi} \mathcal{X} : 0 \leq t \leq T \right\}
\]

\[
\subseteq \text{int} \text{con} \left\{ e^{t\Phi} \mathcal{F}_x : 0 \leq t \leq T \right\} = \text{int} \Xi_{T_x}(\tilde{x}),
\]

i.e., condition (4.10) of Proposition 4 holds. Proposition 6 is proved.

We deduce from Proposition 6 that system (1.1) is weakly locally controllable from the point \( \tilde{x} \) in some sufficiently large time \( T \).

Let \( \mathcal{L}_x \) be the subspace of \( T_x M \) defined above, and let \( \mathcal{L}_x^0 \) be the smallest \( \Phi \)-invariant subspace of \( T_x M \) containing \( \mathcal{L}_x \). In this case \( \Phi \) is well defined on the quotient space \( T_x M/\mathcal{L}_x^0 \). If \( \mathcal{L}_x \) coincides with \( T_x M \), then by Proposition 6 the system is weakly locally controllable from \( \tilde{x} \) in any time \( T > 0 \). In the opposite case we have

**Proposition 7.** If the vector \( (\tilde{x} \circ ((\text{ad} g)^i f)) \) does not belong to any nontrivial \( \Phi \)-invariant subspace of \( T_x M/\mathcal{L}_x^0 \) and all the eigenvalues of \( \Phi \) on \( T_x M/\mathcal{L}_x^0 \) are nonreal, then system (1.1) is weakly locally controllable from \( \tilde{x} \) in a sufficiently large time \( T > 0 \).

**Proof.** We consider an arbitrary covector \( \psi \in (T_x M/\mathcal{L}_x^0)^* \), \( \psi \neq 0 \). If

\[
(\psi, (\tilde{x} \circ \left( e^{i\Phi} \left( (\text{ad} g)^j f \right) \right)) = 0,
\]
then this means that the $\Phi$-invariant subspace $\text{span}\{ \dot{x} \circ (e^{t\Phi}((\text{ad}g) f)f)), t \in R \}$ contains the vector $(\dot{x} \circ ((\text{ad}g) f)f))$ and is orthogonal to $\psi$, i.e., does not coincide with $T_xM/L^0_x$, which contradicts the condition.

Suppose that $\omega(t) = \langle \psi, (\dot{x} \circ (e^{t\Phi}((\text{ad}g) f)f)) \rangle \neq 0$. We prove that $\omega(t)$ changes sign on some interval $[0, T]$. Indeed, let $R(\lambda)$ be the characteristic polynomial of $\Phi$ on $T_xM/L^0_x$, $R(\Phi) = 0$, and consider the differential operator $R(\frac{d}{dt})$. Obviously, $R(\frac{d}{dt})\omega = 0$, i.e., $\omega(t)$ is a nonzero solution of a linear homogeneous equation with constant coefficients. Since all the eigenvalues of $\Phi$ are nonreal, $\omega(t)$ has the form

$$
\omega(t) = \sum_{k=1}^{m} e^{\alpha_k t} \left( P_{\alpha_k}(t) \cos \beta_k t + Q_{\beta_k}(t) \sin \beta_k t \right) = \sum_{s=1}^{N} a_s e^{\alpha_s t} \begin{pmatrix} \cos \beta_s t \\ \sin \beta_s t \end{pmatrix}.
$$

(5.4)

On the right-hand side of (5.4) we single out all the monomials corresponding to the largest of the $\alpha_s$, and then we single out those of them for which the power $r_s$ of $t$ is maximal. Obviously, for large $t$ the sign of $\omega(t)$ is determined by the sum of these monomials, i.e., by an expression of the form

$$
e^{\alpha_t} \left( \sum_{i=1}^{m} \left( a_i \cos \beta_i t + b_i \sin \beta_i t \right) \right), \quad \beta_i \neq 0.
$$

As is known, any nonzero trigonometric polynomial of the form

$$
P(t) = \sum_{i=1}^{m} \left( a_i \cos \beta_i t + b_i \sin \beta_i t \right)
$$

is a function of variable sign on any interval of the form $(t, +\infty)$, which proves that $\omega(t)$ is of variable sign.

Since the choice of $\psi$ was arbitrary, what has been proved implies that the cone $\mathcal{H}_T = \text{con}\{ e^{t\Phi}(\dot{x} \circ ((\text{ad}g) f)f)) : 0 \leq t \leq T \}$ is a complement of $L^0_x$ for all sufficiently large $T$, i.e., $\mathcal{H}_T + L^0_x = T_xM$, and hence, by the inclusion $\mathcal{H}_T + L^0_x \subseteq \text{con}\{ e^{t\Phi}x : 0 \leq t \leq T \}$, we find ourselves under the conditions of Proposition 6, i.e., system (1.1) is weakly locally controllable from $\dot{x}$ in a sufficiently large time $T$. Proposition 7 is proved.

We now investigate weak local controllability of system (1.1) in an arbitrarily small time $T > 0$. Obviously, if there is a number $m$ such that $\text{span}\{ \Phi^k \dot{x} : 0 \leq k \leq m \} = T_xM$, then for any arbitrarily small $T > 0$ the conditions of Proposition 6 hold for the system (1.1); hence we have

**PROPOSITION 8.** Let $j$ be the index defined in Proposition 5. If there exists a number $m$ such that

$$
\text{span}\left\{ \dot{x} \circ ((\text{ad}f)^i(\text{ad}g)^j) : 0 \leq k \leq m, 1 \leq i < j \right\} \cup \{ \dot{x} \circ g \} = T_xM,
$$

(5.5)

then system (1.1) is weakly locally controllable from $\dot{x}$ in any (arbitrarily small) time $T > 0$.

**REMARK.** The following condition for local controllability of system (1.1) in an arbitrarily small time $T > 0$ was presented in [4].

**THEOREM [4].** Suppose that $S^k(f, g)$ is the linear hull of the values at a point $\dot{x}$ of all possible commutators of the vector fields $f$ and $g$, with $g$ appearing at most $k$ times. If $\dot{x} \circ f = 0$ and

1) $S^k(f, g)$ coincides with $T_xM$ for some $k$,

2) $S_{i+1}(f, g) = S_i(f, g)$ for any odd $i$,

then system (1.1) is locally controllable from $\dot{x}$ in an arbitrarily small time $T > 0$. 

$$
\text{span}\left\{ \dot{x} \circ ((\text{ad}f)^i(\text{ad}g)^j) : 0 \leq k \leq m, 1 \leq i < j \right\} \cup \{ \dot{x} \circ g \} = T_xM,
$$

(5.5)

then system (1.1) is weakly locally controllable from $\dot{x}$ in any (arbitrarily small) time $T > 0$. 

**PROPOSITION 8.** Let $j$ be the index defined in Proposition 5. If there exists a number $m$ such that
A comparison of condition (5.5) in Proposition 8 with conditions 1) and 2) in the theorem shows that these two assertions do not reduce to each other.

§6. Time-optimality in the problem of controlling the angular momentum of a rotating rigid body

The free rotation of a rigid body is described by the Euler equation (see [5]):

\[ \dot{K} = K \times BK, \]

where \( K \in \mathbb{R}^3 \) is the angular momentum vector in a coordinate system connected with the body, \( B \) is the symmetric \( 3 \times 3 \) matrix inverse to the inertia tensor of the body \( A \), and the sign "\( \times \)" denotes the vector product in \( \mathbb{R}^3 \). Denote by \( I_1 < I_2 < I_3 \) the principal central moments of inertia of the body (the body is dynamically asymmetric), and by \( J_1 > J_2 > J_3 \) the quantities inverse to them (\( J_1, J_2, \) and \( J_3 \) are the eigenvalues of the matrix \( B \)).

If a controlling moment is applied to the body along an axis \( \overline{L} \) passing through the center of mass, then the controlled motion of the angular momentum vector \( K \) is described by

\[ \dot{K} = K \times BK + Lu, \]

where \( L \) is the unit vector on the axis \( \overline{L} \).

We assume that the axis \( \overline{L} \) is in general position: \( \overline{L} \) does not coincide with any of the principal axes of inertia of the body and does not lie in one of the planes of the separatrices \( \Pi_1 \) and \( \Pi_2 \) given in the principal axes by the equations

\[ \sqrt{J_1 - J_2} \kappa_1 + \sqrt{J_2 - J_3} \kappa_3 = 0. \]

It follows from results in [6] that the exact rank (and thus also the rank) of system (6.1) is equal to 3 when \( \overline{L} \) is in general position. The same is obviously true for the time-reversed system (6.1), denoted by \( (6.1 - ) \). Hence, the conditions of Theorems 2.4 and 2.5 (see §2) and Proposition 1 in §3 are satisfied for systems (6.1) and \( (6.1 - ) \).

For a controllable system (6.1) we consider the time-optimal problem

\[ T \to \min \]

with boundary conditions

\[ K(0) = \tilde{K}, \quad K(T) = \dot{\tilde{K}}. \]

To investigate problem (6.1)–(6.3) we apply the reduction described in §3 to system (6.1), setting \( f = K \times BK \) and \( g = L \). As a result we get the planar system

\[ \dot{K} = K \circ (e^{\alpha f}f) = K \circ (e^{v g} \circ f \circ e^{-v g}), \]

which, since \( g = L \) is a constant field, is equivalent to the system \( \dot{K} = K \circ (e^{v g}f) \) or

\[ \dot{K} = (K + vL) \times B(K + vL). \]

We remark that in the case of a constant field \( g = L \) the quotient manifold \( (\mathbb{R}^3)^g \) can be identified with the plane \( P \) passing through the origin \( O \) and perpendicular to the axis \( \overline{L} \). Under this identification system (6.4) on \( (\mathbb{R}^3)^g \) is carried into the system

\[ \dot{K} = (K + vL) \times B(K + vL) - \langle (K + vL) \times B(K + vL), L \rangle L \]

\[ = (K + vL) \times B(K + vL) - \langle K \times B(K + vL), L \rangle L, \]

whose right-hand side is the projection of the right-hand side of (6.4) on \( P \). Any trajectory of system (6.5) generated by an absolutely continuous control \( v(t) \) is the projection on \( P \) of some (nonunique!) trajectory of (6.4).
We introduce the Cartesian coordinate system $Oy_1y_2$ on $P$ by directing the $Oy_1$ axis along the vector $L \times BL$ and the axis $Oy_2$ along the vector $L \times (L \times BL)$. In this coordinate system (6.5) takes the form
\[
\dot{y}_1 = b_{13}y_2^2 + (-b_{23}y_1 + (b_{11} - b_{33})y_2)v + v^2;
\]
\[
\dot{y}_2 = -b_{13}y_1y_2 + ((b_{22} - b_{11})y_1 + b_{23}y_2)v,
\]
(6.6)
where the $b_{ij}$ are the components of the tensor $B$ in the basis $L, L \times BL, L \times (L \times BL)$. Obviously, $b_{ij} = b_{ji}$, and a direct computation gives us also that $b_{13} < 0$ and $b_{22} - b_{11} \neq 0$.

For the controllable system (6.6) let us consider the time-optimal problem with the conditions
\[
y(0) = y, \quad y(T) = \bar{y}, \quad T \rightarrow \min (y = (y_1, y_2)).
\]
(6.7)
We establish a connection between the optimal trajectories of problems (6.1)–(6.3) and those of problem (6.6)–(6.7).

**Definition.** A control $u(t)$ and the trajectory $K(t)$ generated by it for system (6.1) are said to be strongly locally optimal if for any points $K_1 = K(t_1)$ and $K_2 = K(t_2)$ there exists a $\delta$-neighborhood of $u(t)$ in the sliding regime metric ($\delta$ is the same for all the pairs of points $K_1, K_2$ of the trajectory $K(t)$) such that $T > t_2 - t_1$ for any control $u(\cdot)$ in this $\delta$-neighborhood that carries system (6.1) from $K_1$ to $K_2$ in the time $T$.

**Definition.** A control $v(t)$ and the trajectory $y(t)$ generated by it for system (6.6) are said to be locally optimal if there exists a $\delta$-neighborhood of $v(t)$ in the $L_{\infty}[0, T]$-metric such that $T > t_2 - t_1$ for any points $y_1 = y(t_1)$ and $y_2 = y(t_2)$ of the trajectory $y(t)$ and any control $v(\cdot)$ in this $\delta$-neighborhood that carries system (6.6) from $y_1$ to $y_2$ in the time $T$.

Let $\bar{v}(t)$ and $\bar{y}(t)$ be locally optimal for system (6.6), with $\bar{v}(\cdot)$ absolutely continuous, and let $\bar{u}(t)$ and $\bar{K}(t)$ be a control and the corresponding trajectory of (6.1) that pass under the reduction of (6.1) to (6.6) into $\bar{v}(t)$ and $\bar{y}(t)$, respectively. By the definition of the reduction (see §3), the $\delta$-neighborhood of $\bar{u}(\cdot)$ in the sliding regime metric is mapped under the reduction inside the $\delta$-neighborhood of $\bar{v}(\cdot)$ in the $L_{\infty}[0, T]$-metric. This implies immediately that the local optimality of $\bar{v}(t)$ and $\bar{y}(t)$ for system (6.6) yields the strong local optimality of the corresponding pair $\bar{u}(t), \bar{K}(t)$ for (6.1).

It turns out that the time-optimal problem (6.1)–(6.3) under consideration has many strongly locally optimal trajectories, but does not have any globally optimal ones. Namely, we have the following assertion.

**Proposition 9.** For any point $\bar{K} \in R^3$ there exists a one-parameter family of strongly locally time-optimal trajectories $K^u(t)$ of system (6.1) emanating from $\bar{K}$ and generated by the corresponding controls $u^u(t)$.

**Proof.** For the reduced controllable system (6.6) we form the Hamiltonian
\[
H = \psi_1(b_{13}y_2^2 + (-b_{23}y_1 + (b_{11} - b_{33})y_2)v + v^2) + \psi_2(-b_{13}y_1y_2 + ((b_{22} - b_{11})y_1 + b_{23}y_2)v),
\]
(6.8)
and write out the conjugate system
\[
\dot{\psi}_1 = -\partial H/\partial y_1 = b_{23}v\psi_1 + (b_{13}y_2 - (b_{22} - b_{11})v)\psi_2,
\]
\[
\dot{\psi}_2 = -\partial H/\partial y_2 = -(2b_{13}y_2 + (b_{11} - b_{33})v)\psi_1 + (b_{13}y_1 - b_{23}v)\psi_2.
\]
(6.9)
Obviously, if $\psi_1 < 0$, then the Hamiltonian $H$, which is quadratic in $v$, attains for $\lambda = -|(-b_{23}^2 + b_{11})y_1 + b_{13}y_2| - b_{23}^2$ a maximum equal to

$$ H_{\text{max}} = b_{13}(\psi_1 y_2^2 - \psi_2 y_1 y_2) - \beta^2/4\psi_1, $$

where, for brevity, $\beta$ denotes the coefficient of $v$ in (6.8). Obviously, the strengthened Legendre condition $\partial^2 H/\partial v^2 = \psi_1 < 0$ holds for $\psi_1 < 0$, and $H_{\text{max}} > 0$ under the additional condition $\text{sgn} \psi_2 = \text{sgn} y_1 y_2$ (with the inequality $b_{13} < 0$ taken into account), i.e., the corresponding transversality condition holds in problem (6.6)-(6.7).

Substituting (6.10) into (6.6) and (6.9), we get a system of fourth-order differential equations. Specifying the initial conditions $y_1(0) = \bar{y}_1, y_2(0) = \bar{y}_2, \psi_1(0) = -1, \psi_2(0) = \alpha$ ($\alpha$ is a parameter, and $\text{sgn} \alpha = \text{sgn} y_1 y_2$), we get the family of trajectories $y^\alpha(\cdot), \psi^\alpha(\cdot)$ of this system, and from (6.10) the corresponding family of controls $v^\alpha(\cdot)$. The maximum principle in combination with the strengthened Legendre condition and the transversality condition ensures the local time-optimality of some part of any of the trajectories $y^\alpha(\cdot)$.

By the foregoing, any pair $u^\alpha(\cdot), K^\alpha(\cdot)$ passing under reduction into the pair $v^\alpha(\cdot), y^\alpha(\cdot)$ is strongly locally time-optimal for system (6.1). Proposition 9 is proved.

**PROPOSITION 10.** In problem (6.1)-(6.3) there exists a minimizing sequence of controls $\{u_n(\cdot)\}$ carrying system (6.1) from $\bar{K}$ to $\hat{K}$ in time $T_n$, where $\lim_{n \to \infty} T_n = 0$. In other words, system (6.1) can be carried from $\bar{K}$ to $\hat{K}$ in an arbitrarily small time $T > 0$.

**REMARK.** Generally speaking, an assertion stronger than Propositions 9 and 10 is valid. It can be shown that for any fixed compact set $C \subset \mathbb{R}^3$ (for example, a compact ball) containing $\bar{K}$ and $\hat{K}$ and for the set of trajectories $\gamma$ of (6.1) going from $\bar{K}$ to $\hat{K}$ in a time $T_\gamma$ while remaining in $C$ we have that $\inf_{\gamma} T_\gamma = T_{C, \bar{K}, \hat{K}} > 0$. If $C_n$ is a collection of compact balls such that $C_1 \subset C_2 \subset \cdots$ and $\bigcup C_i = \mathbb{R}^3$, then $\lim_{n \to \infty} T_{C_n, \bar{K}, \hat{K}} = 0$.

**PROOF OF PROPOSITION 10.** We first formulate and prove an auxiliary lemma.

**LEMMA 11.** The statement of Proposition 10 is true for the reduced system (6.6).

**PROOF OF LEMMA 11.** In polar coordinates $(r, \varphi)$ ($y_1 = r \cos \varphi, y_2 = r \sin \varphi$) system (6.6) takes the form

$$\dot{r} = r \cdot F(\cos \varphi, \sin \varphi) v + \cos \varphi v^2, \quad (6.11)$$

$$\dot{\varphi} = -b_{13} r \sin \varphi - (1/r) \sin \varphi v^2 + G(\cos \varphi, \sin \varphi) v, \quad (6.12)$$

where $F$ and $G$ are homogeneous polynomials of degree 2, and $G(\pm 1, 0) = b_{22} - b_{11} \neq 0$.

We prove that (6.6) has trajectories $\gamma$ beginning and ending on the positive semi-axis $Oy_1$ and encircling the origin $O$. We remark that the first and second terms on the right-hand side of (6.12) have (since $b_{13} < 0$) different signs. Setting

$$\dot{\varphi}_1(r, \varphi) = \dot{\varphi}_2(\varphi) = \begin{cases} 0, & \sin \varphi > \epsilon, \\ \pm k, & \sin \varphi < -\epsilon, \\ (b_{22} - b_{11}), & |\sin \varphi| \leq \epsilon, \end{cases} \quad (6.13)$$

we get that for all $\rho_0 > 0$ there exist a sufficiently large $k$ and a sufficiently small $\epsilon > 0$ such that for $|r| \geq \rho_0$ we have (by (6.12) and (6.13))

$$\dot{\varphi} \geq a > 0, \quad (6.14)$$
i.e., $\varphi$ is monotonically increasing along any trajectory $\gamma$ of system (6.11), (6.12) generated by the control (6.13) and contained in the region $r > \rho_0$.

We prove the existence of such a trajectory. Since $\rho_0 > 0$ is arbitrary, it suffices to prove the existence of a trajectory of system (6.11) generated by the control (6.13) and not passing through $O$. It follows from (6.6), (6.12), and (6.13) that any trajectory (6.6) passing through $O$ at the time $t$ is tangent to the axis $Oy_1$, and $\lim_{t \to 0} \varphi(t) = \pi - 0$.

Let us fix $\rho_0$ and take the initial point $\bar{y}$ on the axis $Oy_1$ with polar coordinates $r = \rho_1, \varphi = 0$ ($\rho_1 > \rho_0$). Since $\dot{\delta}(\varphi)$ is a bounded function, the right-hand side of (6.11) admits the estimate

$$|\dot{r}| \leq \mu r + \nu. \quad (6.15)$$

It follows [7] from the differential inequality (6.15) that as $\varphi$ varies along the trajectory from $\varphi(0) = 0$ to $\varphi(t_*) = \arcsin \epsilon$ we have that $r(t) \geq \rho_1 e^{-\mu t_*} - \nu t_*$, or, by (6.14),

$$r(t) \geq \rho_1 e^{-\mu \arcsin \epsilon / a} - \nu \arcsin \epsilon / a.$$ 

As $\varphi$ varies along the trajectory from $\arcsin \epsilon$ to $\pi - \arcsin \epsilon$ the control $\dot{\delta}(\varphi)$ is equal to 0 in view of (6.13), and (6.11) implies that $r(t) = \text{const}$. As $\varphi$ varies along the trajectory from $\pi - \arcsin \epsilon$ to $\pi$ we get from (6.14) and (6.15) that

$$r(t) \geq r(t_*) e^{-\mu \arcsin \epsilon / a} - \nu \arcsin \epsilon / a,$$

or

$$r(t) \geq \rho_1 \cdot e^{-2\mu \arcsin \epsilon / a} - 2\nu \arcsin \epsilon / a.$$ 

Obviously, by choosing $\epsilon$ sufficiently small we can get that $r(t) > \rho_0$, which is what was required.

Thus, the trajectory $\gamma$ of system (6.6) generated by the control (6.13) and beginning on the positive semi-axis $Oy_1$ does not pass through the origin and, in view of the monotone variation of $\varphi$ along $\gamma$, returns to the positive semi-axis $Oy_1$ in a finite amount of time $t_0$. Similarly, it is possible to construct a trajectory $\Gamma$ of (6.6) that completes two circuits about $O$ in a finite amount of time $T_\Gamma$ (see the figure) and is generated by the control $\dot{\delta}(\cdot)$.
We remark that system (6.6) (as well as (6.1)) has an obvious self-similarity—it is invariant with respect to the change of variables \( y_1 \to ay_1, \ y_2 \to ay_2, \ v \to av, \ t \to a^{-1}t \) \((a > 0)\). Consequently, the curve \( \Gamma^a = a\Gamma \) also is an admissible trajectory of system (6.6) generated by the control \( \delta^a(\varphi) = a\delta(\varphi) \), and its circuit time is \( T_{\Gamma^a} = a^{-1}T_\Gamma \).

We prove that if \( \hat{y} \) and \( \hat{y} \) are arbitrary points of the plane \( P \) and \( \varepsilon > 0 \), then \( \hat{y} \) can be reached from \( \hat{y} \) by means of (6.6) with the help of some control \( w(\cdot) \) in a time \( T \leq \varepsilon \).

Choose \( a > 0 \) such that 1) the points \( \hat{y} \) and \( \hat{y} \) are covered by the trajectory \( \Gamma^a \), and 2) \( a^{-1}T_\Gamma \leq \varepsilon/3 \). It follows from the form of the right-hand side of (6.6) that by choosing \( v \) large in absolute value we can ensure an arbitrarily rapid motion of system (6.6) in the positive direction of the axis \( Oy_1 \) along a trajectory close to the horizontal. Similarly, for the reversed-time system (6.6) a control \( v \) large in absolute value ensures an arbitrarily rapid displacement in the negative direction of the \( Oy_1 \)-axis. Consequently, there exists a control \( v^1(t) \) carrying system (6.6) from \( \hat{y} \) to a point \( y^1 \) on the trajectory \( \Gamma^a \) in a time \( \tau_1 \leq \varepsilon/3 \), as well as a control \( v^2(t) \) carrying the reversed-time system (6.6) from \( \hat{y} \) to a point \( y^2 \in \Gamma^a \) in a time \( \tau_2 \leq \varepsilon/3 \). The latter means that system (6.6) goes from \( y^2 \) to \( \hat{y} \) with the help of the control \( v^2(t) \) in the same time \( \tau_2 \leq \varepsilon/3 \). Passage of (6.6) from \( y^1 \) to \( y^2 \) by means of the control \( \delta^a(t) = a\delta(\alpha^{-1}t) \) along the trajectory \( \Gamma^a \) takes place in the time \( \tau_0 \leq T_{\Gamma^a} \leq \varepsilon/3 \) (see the figure).

The desired control \( w(\cdot) \) is determined by

\[
 w(t) = \begin{cases} 
 v^1(t), & 0 \leq t \leq \tau_1, \\
 \delta^a(t), & \tau_1 < t \leq \tau_1 + \tau_0, \\
 v^2(t), & \tau_1 + \tau_0 \leq t \leq \tau_1 + \tau_0 + \tau_2.
\end{cases}
\]

Obviously, \( w(t) \) carries the system (6.6) from \( \hat{y} \) to \( \hat{y} \) in time \( \tau_1 + \tau_0 + \tau_2 \leq \varepsilon \). Lemma 11 is proved.

Let us consider again the time-optimal problem (6.1)–(6.3). We project the points \( \hat{K} \) and \( \hat{K} \) onto the plane \( P \) into the respective points \( \hat{y} = \pi(\hat{K}) \) and \( \hat{y} = \pi(\hat{K}) \), and consider the \( \delta \)-neighborhood \( U_\delta(\hat{y}) \) of \( \hat{y} \). By Lemma 11, for any \( \varepsilon > 0 \) and any \( y \in U_\delta(\hat{y}) \) there exists a control \( w(t) \) carrying system (6.6) from \( \hat{y} \) to \( y \) in a time \( \varepsilon/2 \). Let \( D_{\leq \varepsilon/2,\hat{y}} \) be the set of attainability of system (6.6) from \( \hat{y} \) in a time \( \leq \varepsilon/2 \); then \( U_\delta(\hat{y}) \subseteq D_{\leq \varepsilon/2,\hat{y}} \).

By Proposition 1, the interior of the set of attainability \( A_{\leq \varepsilon/2,\hat{K}} \) of system (6.1) from \( \hat{K} \) in a time \( \leq \varepsilon/2 \) is dense in \( \pi^{-1}(U_\delta(\hat{y})) \subseteq \pi^{-1}(D_{\leq \varepsilon/2,\hat{y}}) \), i.e., in the cylinder \( C_\delta \) with base \( U_\delta(\hat{y}) \subseteq P \) and generator parallel to \( \overline{L} \). Obviously,

\[
 \hat{K} \in \pi^{-1}(\hat{y}) \subseteq \text{int} \ C_\delta, \quad \hat{K} \subseteq \text{clos int} \ A_{\leq \varepsilon/2,\hat{K}}.
\]

As mentioned above, Theorem 2.5 in §2 is applicable to system (6.1–) (system (6.1) with reversed time). In particular, the set of attainability \( A_{\leq \varepsilon/2,\hat{K}} \) of this system in a time \( \leq \varepsilon/2 \) from the point \( \hat{K} \) has a nonempty interior that is dense in \( A_{\leq \varepsilon/2,\hat{K}} \). It follows from the inclusions

\[
 \hat{K} \in A_{\leq \varepsilon/2,\hat{K}}, \quad \hat{K} \subseteq \text{int} \ C_\delta, \quad \hat{K} \subseteq \text{clos int} \ A_{\leq \varepsilon/2,\hat{K}} \supseteq C_\delta
\]

that \( \text{int} A_{\leq \varepsilon/2,\hat{K}} \cap C_\delta \neq \emptyset \), and, consequently,

\[
 \text{int} A_{\leq \varepsilon/2,\hat{K}} \cap \text{int} A_{\leq \varepsilon/2,\hat{K}} \neq \emptyset.
\]
If \( K^1 \in \text{int } A_{\leq \varepsilon/2, \bar{K}} \cap \text{int } A_{\leq \varepsilon/2, \bar{K}} \), then (6.1) can be brought from \( \bar{K} \) to \( K^1 \) in a time \( \leq \varepsilon/2 \) and from \( K^1 \) to \( \bar{K} \) in a time \( \leq \varepsilon/2 \), hence from \( \bar{K} \) to \( \bar{K} \) in a time \( \leq \varepsilon \). Proposition 10 is proved.

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