Geometric Control and Geometry of Vector Distributions

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Rank $k$ vector distribution $\Delta$ on the $n$-dimensional smooth manifold $M$ is a vector subbundle of the tangent bundle $TM$:

$$\Delta = \{\Delta_q\}_{q \in M}, \quad \Delta_q \subset T_q M, \quad \dim \Delta_q = k.$$ 

Distributions $\Delta$ and $\Delta'$ are called locally equivalent at $q_0$ if $\exists$ a neighborhood $O_{q_0}$ and a local diffeomorphism $\Phi : O_{q_0} \to O_{q_0}$ such that $\Phi_*\Delta_{q} = \Delta'_{\Phi(q)}$, $\forall q \in O_{q_0}$.

Horizontal paths: $t \mapsto q(t), \dot{q}(t) \in \Delta_q(t)$.

Local bases: $f_1, \ldots, f_k \in \text{Vec}M$,

$$\Delta_q = \text{span}\{f_1(q), \ldots, f_k(q)\}, \quad q \in O_{q_0}.$$ 

Horizontal paths are admissible trajectories of the control system: $\dot{q} = \sum_{i=1}^{k} u_i f_i(q)$. 
Let $\Delta'_q = \text{span}\{f'_1(q), \ldots, f'_k(q)\}$. We have $\Phi_*\Delta_q = \Delta'_{\Phi(q)}$ iff $\Phi_*f_i = \sum_{j=1}^k a_{ij}f'_j$, where $a_{ij} \in C^\infty(O_{q_0})$, 

$$\det \begin{pmatrix} a_{11}(q) & \ldots & a_{1n}(q) \\ & \cdots & \\ a_{n1}(q) & \ldots & a_{n1}(q) \end{pmatrix} \neq 0.$$ 

In other words, the distributions are equivalent iff the control systems are equivalent by the feedback and state transformations.

Flag of the distribution:

$$\Delta^l_q = \text{span}\{(\text{ad} f_{i_j} \cdots \text{ad} f_{i_1} f_{i_0})(q) : 0 \leq j < l\},$$

where $\text{ad} f g \overset{\text{def}}{=} [f, g]$ is the Lie bracket.

Subspaces $\Delta^l_q$ do not depend on the basis of $\Delta$ since $\text{ad} f (ag) = a\text{ad} f g + (fa)g$ but the structure of the generated by $f_1, \ldots, f_k$ Lie subalgebra of $\text{Vec}M$ essentially depends on the basis.
Local parameterization of the space of distributions: $M \approx \mathbb{R}^n$, $TM \approx \mathbb{R}^n \times \mathbb{R}^n$,

$$\Delta : \mathbb{R}^n \rightarrow G_k(\mathbb{R}^n),$$

where $G_k(\mathbb{R}^n)$ is the Grassmann manifold of $k$-dim. subspaces of $\mathbb{R}^n$. Recall that $G_k(\mathbb{R}^n)$ is a smooth $k(n-k)$-dim. manifold. Indeed, all $k$-dim. subspaces that are transversal to a fixed $(n-k)$-dim. subspace can be identified with graphs of linear maps from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$ (i.e. with $k \times (n-k)$ -matrices) and form a coordinate chart of $G_k(\mathbb{R}^n)$:

The space of rank $k$ distributions is thus locally parameterized by $C^\infty(\mathbb{R}^n; \mathbb{R}^k(n-k))$. 

![Diagram](image-url)
On the other hand, local diffeomorphisms of \( \mathbb{R}^n \) form an open subset of \( C^\infty(\mathbb{R}^n; \mathbb{R}^n) \). A smooth change of coordinates allows to normalize no more than \( n \) of \( k(n - k) \) functions. The space of equivalence classes should be at least as “massive” as \( C^\infty(\mathbb{R}^n; \mathbb{R}^{k(n-k)-n}) \).

1. \( k(n - k) \leq n \), i.e. \( k = 1 \) or \( k = n - 1 \) or \( n = 4, k = 2 \). Generic distributions can be completely normalized:

   \[ k = 1 \] – rectification of vector fields;

   \[ k = n - 1 \] – Darboux normal forms for differential 1-forms;

   \[ n = 4, k = 2 \] – Engel structure.

2. \( k(n - k) > n \). Any classification of generic distributions must contain “functional parameters”.


First nontrivial case: \( n = 5, \ k = 2 \lor 3 \).

**Theorem.** Let \( \mathcal{D}_k(\mathbb{R}^n) \) be the space of germs of \( k \)-distributions in \( \mathbb{R}^n \). If \( k(n-k) > n \), then \( \exists \) a residual subset \( \mathcal{U} \subset \mathcal{D}_k(\mathbb{R}^n) \) s.t. no one distribution from \( \mathcal{U} \) possesses a basis generating a finite dimensional Lie algebra.

Main steps of the proof:

1. If two distributions possess bases which generate finite dimensional Lie algebras and have equal bracket relations, then the distributions are locally equivalent.

2. Take a Hall basis of Lie polynomials in \( k \) indeterminates and consider the set of all multiplication tables of Lie algebras additively generated by first \( m \) elements of this basis. The set of pairs:

\[ \langle \text{multipl. table}, \ \text{codim. n Lie subalgebra} \rangle \]
forms a semi-algebraic subset of the appropriate vector space. Each pair generates a germ of a $k$-tuple of vector fields in $\mathbb{R}^n$. Moreover, $\forall N > 0$ the set of $N$-jets of these germs is a semi-algebraic subset of the space of $N$-jets and dimension of this subset does not depend on $N$.

3. The group of $N$-jets of diffeomorphisms acts on the space of jets of distributions and codimension of the orbits of this action tends to $\infty$ as $N \rightarrow \infty$.

**Looking for invariants**

The growth vector:

$$(\dim \Delta_q, \dim \Delta^2_q, \dim \Delta^3_q, \ldots).$$

We mainly study distributions with maximal growth vector (generic case). If $k = 2$, then
maximal growth is: \((2, 3, 5, 8, \ldots)\); in general:
\((k, k(k + 1)/2, k(k + 1)(2k + 1)/6, \ldots)\).

If \(k(n - k) \leq n, k > 1\), then any maximal growth vector distribution possesses a basis generating the nilpotent \(n\)-dimensional Lie algebra. This is not true, if \(k(n - k) > n\).

Natural questions:

- Equivalence problem for the maximal growth vector distributions: Given two distributions, how to check are they locally equivalent or not?

- How to characterize the distributions which possess bases generating the \(n\)-dim. nilpotent Lie algebra?
• Is there a chance to make effective the above theorem?

Cartan equivalence method, in principle, provides the answer to first two questions for the following values of \((k, n)\): (2, 5) (E. Cartan), (3, 6) (R. Bryant), and (4, 7) (R. Montgomery).

“Optimal control” approach

The space of horizontal paths:
\[
\Omega_\Delta = \{ \gamma : [0, 1] \to M : \dot{\gamma}(t) \in \Delta_{\gamma(t)}, \ 0 \leq t \leq 1 \},
\]
\[
\Omega_\Delta \subset H^1([0, 1]; M). \quad \text{Boundary mappings:}
\]
\[
\partial_t : \gamma \mapsto (\gamma(0), \gamma(t)) \in M \times M.
\]
Critical points of \(\partial_1|_{\Omega_\Delta}\) are singular curves of \(\Delta\). Any singular curve is a critical point of \(\partial_t \ \forall t \in [0, 1]\).
Moreover, any singular curve possesses a singular extremal, i.e. a curve \( \lambda : [0, 1] \to T^*M \) in the cotangent bundle to \( M \) s.t. \( \lambda(t) \in T^*_{\gamma(t)}M \),
\[
(\lambda(t), -\lambda(0))D_{\gamma} \partial_t = 0, \quad \forall t \in [0, t].
\]

We set:
\[
\Delta^\perp_q = \{ \nu \in T^*_q M : \langle \nu, \Delta_q \rangle = 0, \nu \neq 0 \},
\]
\[
\Delta^\perp = \bigcup_{q \in M} \Delta^\perp_q.
\]

Let \( \sigma \) be the canonical symplectic structure on \( T^*M \). The PMP implies: A curve \( \lambda \) in \( T^*M \) is a singular extremal iff it is a characteristics of the form \( \sigma|_{\Delta^\perp} \); in other words,
\[
\dot{\lambda}(t) \in \ker \left( \sigma|_{\Delta^\perp} \right), \quad 0 \leq t \leq 1.
\]

Characteristic variety:
\[
C_\Delta = \{ z \in \Delta^\perp : \ker \sigma_z|_{\Delta^\perp} \neq 0 \}.
\]
We have: $C_{\Delta} = \Delta^{2\perp}$ if $k = 2$; $C_{\Delta} = \Delta^{\perp}$ if $k$ is odd; typically, $C_{\Delta}$ is a codim 1 submanifold of $\Delta$ if $k$ is even.

Regular part of the characteristic variety:

$$C^0_{\Delta} = \left\{ z \in C_{\Delta} : \dim \ker \sigma_z \big|_{\Delta^{\perp}} \leq 2, \dim \ker \sigma_z \big|_{\Delta^{\perp} \cap T_zC_{\Delta}} = 1 \right\}.$$ 

If $k = 2$, then $C^0_{\Delta} = \Delta^{2\perp} \setminus \Delta^{3\perp}$.

Submanifold $C^0_{\Delta}$ is foliated by singular extremals and by the fibers $T^*_qM \cap C^0_{\Delta}$. 
The movement along singular extremals is not fiber-wise!

Canonical projection:

\[ F : C^0 \rightarrow C_\Delta^0 / \{ \text{sing. ext. foliation} \} \].

Let \( \lambda \) be a sing. extremal associated to a sing. curve \( \gamma \). Consider a family of subspaces

\[ J^0_\lambda(t) = T_\lambda F(T^*_\gamma(t)M \cap C^0_\Delta) \]
of the space

$$T_\lambda C^0_\Delta / \{ \text{sing. ext. foliation} \} \cong T_{\lambda(0)} C^0_\Delta / T_{\lambda(0)} \Delta.$$  

Then $$t \mapsto J^0_\lambda(t)$$ is a curve in the Grassmannian. Geometry of these curves reflects the dynamics of the fibers along sing. extremals and contains the fundamental information about distribution $$\Delta$$.

Let $$n = 5$$; two interesting cases $$k = 2$$ and $$k = 3$$ are essentially equivalent:

$$\dim \Delta_q = 2 \Rightarrow \dim \Delta^2_q = 3;$$

$$C^0_\Delta = C^0_{\Delta^2} = \Delta^2 \perp, \, \dim(C^0_{\Delta^2} \cap T^*_q M) = 2.$$  

Reconstruction of the 2-distribution from the 3-distribution:

$$\Delta = \{ \dot{\gamma}(t) \in TM : \gamma \text{ is a sing. curve of } \Delta^2 \}.$$
Let $\pi : T_p(T^*M) \rightarrow T_qM$ be the differential of the projection $T^*M \rightarrow M$; then $\pi(J_0^0(t)) \subset p^\perp \subset T_qM$ and $t \mapsto \pi(J_0^0(t))$ is a curve in the projective plane $\mathbb{P}(p^\perp/\dot{\gamma})$.

**Proposition:** Distribution $\Delta$ has a basis generating the 5-dim. nilpotent Lie algebra iff this curve is a quadric $\forall p, q$.

In general, let $K_p(q) \subset p^\perp$ be the osculating quadric to this curve: $K_p(q)$ is zero locus of a signature $(2, 1)$ quadratic form on $p^\perp/\dot{\gamma}$. Finally, $\mathcal{K}(q) = \bigcup_{p \in \Delta^2_q} K_p(q)$ is zero locus of a $(3, 2)$ quadratic form on $T_qM$. 
$K(q)$, $q \in M$ is and intrinsically “raised” from $\Delta$ conformal structure on $M$; $\Delta_q \subset K(q)$.

Assume that $k = 2$, $n \geq 5$. Let $p \in C^0_\Delta$, $\lambda$ the sing. extremal through $p$ and $\gamma$ the corresponding singular curve. We set:

$$J_\lambda(t) = D_\lambda F \left( \pi^{-1} \Delta_{\gamma(t)} \right) \subset T_p C^0_\Delta / T_p \lambda.$$  

Then $J_\lambda(t) \supset J^0_\lambda(t)$ and $J_\lambda(t)$ is a Lagrangian subspace of the symplectic space $T_p C^0_\Delta / T_p \lambda$. In other words, $J_\lambda(t)^\bot = J_\lambda(t)$, where

$$S^\bot \overset{\text{def}}{=} \{ \zeta \in T_p C^0_\Delta : \sigma(\zeta, S) = 0 \}, \ S \subset T_p.$$  

Given $s \in \mathbb{R} \setminus \{0\}$, $s\lambda$ is the singular extremal through $sp \in C^0_\Delta$. Hence $T_p(\mathbb{R}p) \subset J_\lambda(t)$, $\forall t$ and $J_\lambda(t) \subset T_p(\mathbb{R}p)^\bot$.

Final reduction: $\Sigma_p = T_p(\mathbb{R}p)^\bot / T_p \mathbb{R}p$ is a symplectic space, $\dim \Sigma_p = 2(n - 3)$. Then $J_\lambda(t)$ is a Lagrangian subspace of $\Sigma_p$. 
Important property: $J_\lambda(t) \cap J_\lambda(\tau) = 0$ for sufficiently small $|t - \tau| \neq 0$.

Take projectors: $\pi_{t\tau} : \Sigma_p \to \Sigma_p,$

$$\pi_{t\tau}\big|_{J_\lambda(t)} = 0, \quad \pi_{t\tau}\big|_{J_\lambda(\tau)} = 1.$$ 

**Lemma:**

$$\text{tr} \left( \frac{\partial^2 \pi_{t\tau}}{\partial t \partial \tau} \right) = \left( n - 3 \right)^2 \frac{t - \tau}{(t - \tau)^2} + g_\lambda(t, \tau),$$

where $g(t, \tau)$ is a smooth symmetric function of $(t, \tau)$.

“Ricci curvature” on $\lambda$: $\rho_\lambda(\lambda(t)) \overset{\text{def}}{=} g_\lambda(t, t)$.

Chain rule: let $\varphi : \mathbb{R} \to \mathbb{R}$ be a change of the parameter; then $\rho_{\lambda \circ \varphi}(\lambda(\varphi(t))) =$

$$\rho_\lambda((\varphi(t))) \dot{\varphi}^2(t) + (n - 3)^2 S(\varphi),$$

where $S(\varphi) = \frac{\dddot{\varphi}(t)}{e\varphi(t)} - \frac{3}{4} \left( \frac{\dot{\varphi}(t)}{\varphi(t)} \right)^2.$
“Ricci curvature” \( \rho \) can be killed by a local reparametrization. A parametrization which kills \( \rho \) is called *projective*; it is defined up to a Möbius transformation.

Let \( t \) be a projective parameter, then the quantity:

\[
A(\lambda(t)) = \left. \frac{\partial^2 g}{\partial \tau^2}(t, \tau) \right|_{\tau=t} (dt)^4
\]

is called the *fundamental form* on \( \lambda \).

For arbitrary parameter: \( A(\lambda(t)) = \\
\left( \left. \frac{\partial^2 g}{\partial \tau^2} \right|_{\tau=t} - \frac{3}{5(n-3)^2}\rho_\lambda(t)^2 - \frac{3}{2}\ddot{\rho}_\lambda(t) \right) (dt)^4. \)

Assume that \( A(\lambda(t)) \neq 0 \), then the identity \( |A(\lambda(s)) (\frac{d}{ds})| = 1 \) defines a unique (up to a translation) *normal parameter* \( s \).

Let \( z \in C^0_\Delta \) and \( \lambda_s \) is the normally parameterized singular extremal through \( z \). We set

\[
\bar{\rho}(z) = \rho_{\lambda_s}(z),
\]
the projective Ricci curvature. Then $z \mapsto \bar{\rho}(z)$ is a function on $C^0_\Delta$ which depends only on $\Delta$.

Back to the $(2, 5)$ distributions. Such a distribution admits a basis generating the 5-dim. nilpotent Lie algebra iff $A \equiv 0$.

**Example.** A radius 1 ball is rolling over the radius $r$ ball without slipping or twisting, $1 < r \leq \infty$. Admissible velocities form a $(2, 5)$ distribution. Then (I. Zelenko): $\text{sgn}(A) = \text{sgn}(r - 3)$;

$$\bar{\rho} = \frac{4\sqrt{35}(r^2 + 1)}{3\sqrt{(r^2 - 9)(9r^2 - 1)}}.$$  

In particular, the distributions corresponding to different $r$ are mutually non equivalent and the distribution corresponding to $r = 3$ admits a basis generating the 5-dim. nilpotent Lie algebra.