SWITCHING IN TIME-OPTIMAL PROBLEM WITH CO-DIMENSION 1 CONTROL.

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Abstract. In this paper we analyse local regularity of time-optimal controls and trajectories for an \( n \)-dimensional affine control system with a control parameter, taking values in a \( k \)-dimensional closed ball. In the case of \( k = n - 1 \), we give sufficient conditions in terms of Lie bracket relations for all optimal controls to be smooth or to have only isolated jump discontinuities.

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1. Introduction

In this paper, we continue to study singularities of the extremals of the time-optimal problem for the control system of the form:

\[ \dot{q} = f_0(q) + \sum_{i=1}^{k} u_i f_i(q), \quad q \in M, \ (u_1, \ldots, u_k) \in U, \]  

(1.1)

where \( M \) is a smooth \( n \)-dimensional manifold, \( U = \{ u \in \mathbb{R}^k : ||u|| \leq 1 \} \) is the \( k \)-dimensional ball, and \( f_0, f_1, \ldots, f_k \) are smooth vector fields. We also assume that \( f_1(q), \ldots, f_k(q) \) are linearly independent in the domain under consideration.

If \( k = n \), then all extremals are smooth; otherwise they may be nonsmooth and there exists a vast literature dedicated to the case \( k = 1 \). Some references can be found in paper [2], where we studied the simplest intermediate case \( k = 2, n = 3 \). It appears that the developed in [2] techniques work in much more general setting than we expected and can be efficiently applied to any pair \( k < n \) giving a clear explicit description of less degenerate singularities (see Theorem 3.4 of the current paper).

Moreover, if \( k = n - 1 \), \( q \in M \), and \( f_0, f_1, \ldots, f_{n-1} \) is a generic germ of \( n \)-tuple of vector fields at \( q \), then the germs of extremal at \( q \) may have only these less degenerate singularities. More precisely, let us define a vector \( a \in \mathbb{R}^{n-1} \) and a matrix \( A \in \text{so}(n-1) \) by the formulas:

\[ a(q) = \{ \det (f_1(q), \ldots, f_{n-1}(q), [f_0, f_i](q)) \}_{i=1}^{n-1}, \]
\[ A(q) = \{ \det (f_1(q), \ldots, f_{n-1}(q), [f_i, f_j](q)) \}_{i,j=1}^{n-1}, \]
where $[\cdot,\cdot]$ is a Lie bracket. We have the following:

**Theorem 1.1.** If

$$(1.1) \quad a(q) \notin A(q)S^{n-2},$$

then there exists a neighbourhood $O_q$ of $q$ in $M$ such that any time-optimal trajectory contained in $O_q$ is piecewise smooth with no more than $1$ non smoothness point.

Here $S^{n-2} = \{ u \in \mathbb{R}^{n-1} : ||u|| = 1 \}$ is the unit sphere.

If $n = 3$, $k = 2$, then inequality (1.1) reads:

$$(1.2) \quad \det^2 (f_1(q), f_2(q), [f_0, f_1](q)) + \det^2 (f_1(q), f_2(q), [f_0, f_2](q)) \neq \det^2 (f_1(q), f_2(q), [f_1, f_2](q)).$$

In this case, the result of Theorem 1.1 follows from [2, Th. 3.1], but the cited result of [2] is a bit stronger than this. Indeed, assumption (1.2) is more restrictive than the used in [2, Th. 3.1] assumption

$$\text{rank}\{f_1(q), f_2(q), [f_0, f_1](q), [f_0, f_2](q), [f_1, f_2](q)\} = 3.$$

In the next section we recall necessary background from the optimal control theory: the Pontryagin maximum principle and the Goh condition. Theorem 1.1 is a corollary of the main result stated in Section 3 and proved in Section 4. The proof is based on the blow-up techniques and the structure of partially hyperbolic equilibria.

2. Preliminaries

In this section we recall some basic definitions in Geometric Control Theory. For a more detailed introduction, see [3].

**Definition 2.1.** Given a $n$-dimensional manifold $M$, we call $\text{Vec}(M)$ the set of $C^2$ vector fields on $M$: $f \in \text{Vec}(M)$ if and only if $f$ is a smooth map with respect to $q \in M$ taking value in the tangent bundle,

$$f : M \rightarrow TM,$$

such that if $q \in M$ then $f(q) \in T_q M$.

Each vector field defines a dynamical system

$$\dot{q} = f(q),$$

i. e. for each initial point $q_0 \in M$ it admits a solution $q(t, q_0)$ on an opportune time interval $I$, such that $q(0, q_0) = q_0$ and

$$\frac{d}{dt}q(t) = f(q(t)), \quad \text{a. e.} \ t \in I.$$

**Definition 2.2.** $f \in \text{Vec}(M)$ is a complete vector field if, for each initial point $q_0 \in M$, the solution $q(t, q_0)$ of the dynamical system $\dot{q} = f(q)$ is defined for every $t \in \mathbb{R}$. If $f \in \text{Vec}(M)$ has a compact support, it is a complete vector field.

In our local study, we may assume without lack of generality that all vector fields under consideration are complete.

**Definition 2.3.** A control system in $M$ is a family of dynamical systems

$$\dot{q} = f_u(q), \quad \text{with } q \in M, \{f_u\}_{u \in U} \subseteq \text{Vec}(M),$$

parametrized by $u \in U \subseteq \mathbb{R}^k$, called space of control parameters.

Instead of constant values $u \in U$, we are going to consider $L^\infty$ time depending functions taking values in $U$. Thus, we call $\mathcal{U} = \{u : I \rightarrow U, u \in L^\infty\}$ the set of admissible controls and study the following control system

$$(2.1) \quad \dot{q} = f_u(q), \quad \text{with } q \in M, u \in \mathcal{U}.$$
With the following theorem we want to show that, choosing an admissible control, it is guaranteed the locally existence and uniqueness of the solution of a control system for every initial point.

**Theorem 2.4.** Fixed an admissible control \( u \in U \), \((2.1)\) is a non-autonomous ordinary differential equation, where the right-hand side is smooth with respect to \( q \), and measurable essentially bounded with respect to \( t \), then, for each \( q_0 \in M \), there exists a local unique solution \( q_u(t, q_0) \) such that \( q_u(0, q_0) = q_0 \) and it is lipschitzian with respect to \( t \).

**Definition 2.5.** We denote \( A_{q_0} = \{ q_u(t, q_0) : t \geq 0, u \in U \} \), the attainable set from \( q_0 \).

We will write \( q_u(t) = q_u(t, q_0) \) if we do not need to stress that the initial position is \( q_0 \).

**Definition 2.6.** An affine control system is a control system of the following form

\[
\dot{q} = f_0(q) + \sum_{i=1}^{k} u_i f_i(q), \quad q \in M
\]

where \( f_0, \ldots, f_k \in \text{Vec}(M) \) and \( (u_1, \ldots, u_k) \in U \), taking values in the set \( U \subseteq \mathbb{R}^k \). The uncontrollable term \( f_0 \) is called drift.

**2.1. Time-optimal problem.**

**Definition 2.7.** Given the control system \((2.1)\), \( q_0 \in M \) and \( q_1 \in A_{q_0} \), the time-optimal problem consists in minimizing the time of motion from \( q_0 \) to \( q_1 \) via admissible trajectories:

\[
\begin{align*}
\dot{q} &= f_u(q) \quad u \in U \\
q_u(0, q_0) &= q_0 \\
q_u(t_1, q_0) &= q_1 \\
t_1 &\rightarrow \min
\end{align*}
\]

We call these minimizer trajectories time-optimal trajectories, and time-optimal controls the correspondent controls.

**2.1.1. Existence of time-optimal trajectories.** Classical Filippov’s Theorem (See [3]) guarantees the existence of a time-optimal control for the affine control system if \( U \) is a convex compact and \( q_0 \) is sufficiently close to \( q_1 \).

**2.2. First and second order necessary optimality condition.** Now we are going to introduce basic notions about Lie brackets, Hamiltonian systems and Poisson brackets, so that we present the first and second order necessary conditions of optimality: Pontryagin Maximum Principle, and Goh condition.

**Definition 2.8.** Let \( f, g \in \text{Vec}(M) \), we define their Lie brackets the following vector field

\[
[f, g](q) = \frac{1}{2} \frac{\partial^2}{\partial t^2} \bigg|_{t=0} e^{-tg} \circ e^{-tf} \circ e^{tg} \circ e^{tf}(q), \quad \forall q \in M.
\]

where \( e^{-tf} \) is the flow defined by \( f \).
Given the canonical projection $\pi_\lambda$ Then there exists $\lambda \in T^* M$, such that for almost all \((Pontryagin Maximum Principle - time-optimal problem)\) Theorem 2.13

2.2.1. Pontryagin Maximum Principle.

Definition 2.9. An Hamiltonian is a smooth function on the cotangent bundle $h \in C^\infty(T^* M)$. The Hamiltonian vector field is the vector field associated to $h$ via the canonical symplectic form $\sigma$

$$\sigma_\lambda(\cdot, \overline{h}) = d\lambda h.$$ We denote

$$\dot{\lambda} = \overline{h}(\lambda), \quad \lambda \in T^* M,$$

the Hamiltonian system, which corresponds to $h$. Let \((x_1, \ldots, x_n)\) be local coordinates in $M$ and \((\xi_1, \ldots, x_1, \ldots, x_n)\) induced coordinates in $T^* M$, $\lambda = \sum_{i=1}^{n} \xi_i dx_i$. The symplectic form has expression $\sigma = \sum_{i=1}^{n} d\xi_i \wedge dx_i$. Thus, in canonical coordinates, the Hamiltonian vector field has the following form

$$\overline{h} = \sum_{i=1}^{n} \left( \frac{\partial h}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i} \frac{\partial}{\partial \xi_i} \right).$$ Therefore, in canonical coordinates, it is

$$\begin{cases}
\dot{x}_i = \frac{\partial h}{\partial \xi_i} \\
\dot{\xi}_i = -\frac{\partial h}{\partial x_i}
\end{cases}$$

for $i = 1, \ldots, n$.

Definition 2.10. The Poisson brackets \(\{a, b\} \in C^\infty(T^* M)\) of two Hamiltonians $a, b \in C^\infty(T^* M)$ are defined as follows: \(\{a, b\} = \sigma(\overline{a}, \overline{b})\); the coordinate expression is:

$$\{a, b\} = \sum_{k=1}^{n} \left( \frac{\partial a}{\partial \xi_k} \frac{\partial b}{\partial x_k} - \frac{\partial a}{\partial x_k} \frac{\partial b}{\partial \xi_k} \right).$$

Remark 2.11. Let us recall that, given $g_1$ and $g_2$ vector fields in $M$, considering the Hamiltonians $a_1(\xi, x) = \langle \xi, g_1(x) \rangle$ and $a_2(\xi, x) = \langle \xi, g_2(x) \rangle$, it holds

$$\{a_1, a_2\}(\xi, x) = \langle \xi, [g_1, g_2](x) \rangle.$$ Remark 2.12. Given a smooth function $\Phi$ in $C^\infty(T^* M)$, and $\lambda(t)$ solution of the Hamiltonian system $\dot{\lambda} = \overline{h}(\lambda)$, the derivative of $\Phi(\lambda(t))$ with respect to $t$ is the following

$$\frac{d}{dt} \Phi(\lambda(t)) = \{h, \Phi\}(\lambda(t)).$$

2.2.1. Pontryagin Maximum Principle.

Theorem 2.13 (Pontryagin Maximum Principle - time-optimal problem). Let an admissible control $\overline{a}$, defined in the interval $t \in [0, \tau_1]$, be time-optimal for the system \([2,7]\), and let the Hamiltonian associated to this control system be the action on $f_a(q) \in T_q^* M$ of a covector $\lambda \in T_q^* M$: \(H_a(\lambda) = \langle \lambda, f_a(q) \rangle\).

Then there exists $\lambda(t) \in T_q^* M$, for $t \in [0, \tau_1]$, called extremal never null and lipschitzian, such that for almost all $t \in [0, \tau_1]$ the following conditions hold:

(1) $\dot{\lambda}(t) = \overline{H}_a(\lambda(t))$

(2) $\overline{H}_a(\lambda(t)) = \max_{u \in U} H_a(\lambda(t))$ (Maximality condition)

(3) $H_a(\lambda(t)) \geq 0$.

Given the canonical projection $\pi : TM \to M$, we denote $q(t) = \pi(\lambda(t))$ the extremal trajectory.
2.2.2. Goh condition. Finally, we present the Goh condition, on the singular arcs of the extremal trajectory, in which we do not have information from the maximality condition of the Pontryagin Maximum Principle. We state the Goh condition only for affine control systems\(\text{[2.2]}\).

Theorem 2.14 (Goh condition). Let \(\bar{q}(t), t \in [0,t_1]\) be a time-optimal trajectory corresponding to a control \(\bar{u}\). If \(\bar{u}(t) \in \text{int}U\) for any \(t \in (\tau_1, \tau_2)\), then there exist an extremal \(\lambda(t) \in T^*_qM\) such that
\[
\langle \lambda(t), [f_i, f_j](q(t)) \rangle = 0, \quad t \in (\tau_1, \tau_2), \ i, j = 1, \ldots, m.
\] (2.4)

2.3. Consequence of the optimality conditions. In this paper we are going to investigate the local regularity of time-optimal trajectories for the \(n\)-dimensional affine control system with a \(k\)-dimensional control:
\[
\hat{q} = f_0(q) + \sum_{i=1}^{k} u_i f_i(q), \quad q \in M, u \in U
\] (2.5)
where the space of control parameters is the \(k\)-dimensional closed unitary ball: \(U = \{u \in \mathbb{R}^k : ||u|| \leq 1\}\).

By the Pontryagin Maximum Principle, every time-optimal trajectory of our system has an extremal in the cotangent bundle \(T^*M\) that satisfies a Hamiltonian system, given by the maximized Hamiltonian.

Notation 2.15. Let us call \(h_{ij}(\lambda) = \langle \lambda, f_i(\lambda) \rangle\), \(f_{ij}(q) = [f_i, f_j](q)\), \(f_{ijk}(q) = [f_i, [f_j, f_k]](q)\), \(h_{ij}(\lambda) = \langle \lambda, f_i(\lambda) \rangle\), and \(h_{ijk}(\lambda) = \langle \lambda, f_{ijk}(\lambda) \rangle\), with \(\lambda \in T^*_qM\) and \(i, j, k \in \{0, 1, \ldots, k\}\).

Moreover, we denote the following vector \(H_{ij}(\lambda) = \{h_{ij}(\lambda)\}_i \in \mathbb{R}^k\) and \(k \times k\) matrix \(H_{ij}(\lambda) = \{h_{ij}(\lambda)\}_ij\) with respect to \(\lambda \in T^*M\).

Definition 2.16. The singular locus \(\Lambda \subseteq T^*M\), is defined as follows:
\[
\Lambda = \{\lambda \in T^*M : h_1(\lambda) = \ldots = h_k(\lambda) = 0\}.
\]

The following proposition is an immediate corollary of the Pontryagin Maximum Principle.

Proposition 2.17. If an extremal \(\lambda(t), t \in [0,t_1]\), does not intersect the singular locus \(\Lambda\), then \(\forall t \in [0,t_1]\)
\[
\hat{u}(t) = \left(\frac{h_1(\lambda(t))}{(h_1^2(\lambda(t)) + \ldots + h_k^2(\lambda(t)))^{1/2}}, \ldots, \frac{h_k(\lambda(t))}{(h_1^2(\lambda(t)) + \ldots + h_k^2(\lambda(t)))^{1/2}}\right).
\] (2.6)

Moreover, this extremal is a solutions of the Hamiltonian system defined by the Hamiltonian \(\mathcal{H}(\lambda) = h_0(\lambda) + h_1^2(\lambda) + \ldots + h_k^2(\lambda)\). Thus, it is smooth.

Definition 2.18. We will call bang arc any smooth arc of a time-optimal trajectory \(q(t)\), whose correspondent time-optimal control \(\hat{u}\) lies in the boundary of the space of control parameters: \(\hat{u}(t) \in \partial U\).

Corollary 2.19. An arc of a time-optimal trajectory, whose extremal is out of the singular locus, is a bang arc.

From Corollary 2.19 we already have an answer about the regularity of time-optimal trajectories: every time-optimal trajectory, whose extremal lies out of the singular locus, is smooth.

However, we do not know what happen if an extremal touches the singular locus, optimal controls may be not always smooth.
**Definition 2.20.** A switching is a discontinuity of an optimal control. Given \( u(t) \) an optimal control, \( t \) is a switching time if \( u(t) \) is discontinuous at \( t \). Moreover given \( q_u(t) \) the admissible trajectory, \( \dot{q} = q_u(t) \) is a switching point if \( \dot{t} \) is a switching time for \( u(t) \).

A concatenation of bang arcs is called bang-bang trajectory. An arc of an optimal trajectory that admits an extremal totally contained in the singular locus \( \Lambda \), is called singular arc.

### 3. Statement of the result

Let us assume that \( \dim M = n \) and study the time-optimal problem for the following system

\[
\dot{q} = f_0(q) + \sum_{i=1}^{k} u_i f_i(q), \quad q \in M, u \in \mathcal{U},
\]

where \( k < n \), \( f_0, f_1, \ldots, f_k \) are smooth vector fields, and \( \mathcal{U} = \{ u \in \mathbb{R}^k : ||u|| \leq 1 \} \); we also assume that \( f_1, \ldots, f_k \) are linearly independent in the domain under consideration, and \( f_{ij} = [f_i, f_j] \) with \( i, j \in \{0, 1, \ldots, k\} \).

**Notation 3.1.** Recalling Notation 2.15, let us introduce the following abbreviated notation:

\[ \mathcal{H}_{0i} := \mathcal{H}_{0i}(\lambda), \mathcal{H}_{1j} := \mathcal{H}_{1j}(\lambda), \]

chosen an opportune \( \lambda \in \Lambda_{\bar{q}} \).

In order to prove Theorem 1.1 we are going to study extremals for any control system of the form (3.1) with \( k < n \) in a neighbourhood of \( \lambda \in \Lambda_{\bar{q}} \subseteq T^*_\mu M \) such that

\[
\mathcal{H}_{0i} \notin \mathcal{H}_{1j} S^{k-1},
\]

where \( S^{k-1} = \{ u \in \mathbb{R}^k : ||u|| = 1 \} \) is the unit sphere.

**Remark 3.2.** If \( k = n - 1 \), we should choose \( \lambda = f_1(\bar{q}) \wedge \ldots \wedge f_{n-1}(\bar{q}) \). One can notice that conditions (1.1) and (3.2) are equivalent.

From Corollary 2.19 we already know that every arc of a time-optimal trajectory, whose extremal lies out of \( \Lambda \), is bang, and so smooth. Thus, we are interested to study arcs of a time-optimal trajectories, whose extremals passes through \( \Lambda \) or lies in \( \Lambda \).

The fist step is to investigate if our system admits singular arcs.

**Proposition 3.3.** Assuming (3.2), there are no optimal extremals in \( O_{\bar{q}} \) that lie in the singular locus \( \Lambda \) for a time interval.

Thanks to Proposition 3.3 if it holds (3.2), the description of optimal extremals in a neighbourhood of \( \lambda \) is essentially reduced to the study of the solutions of the Hamiltonian system with a discontinuous right-hand side, defined by the Hamiltonian \( \mathcal{H}(\lambda) = h_0(\lambda) + \sqrt{h_1^2(\lambda) + \ldots + h_k^2(\lambda)} \).

**Theorem 3.4.** Assume that condition (3.2) is satisfied.

If it holds

\[
\mathcal{H}_{0i} \notin \mathcal{H}_{1j} B^k,
\]

where \( B^k = \{ u \in \mathbb{R}^k : ||u|| < 1 \} \), then there exists a neighborhood \( O_{\bar{q}} \subseteq T^* M \) such that for any \( z \in O_{\bar{q}} \) and \( \lambda > 0 \) there exists a unique contained in \( O_{\bar{q}} \) extremal \( t \mapsto \lambda(t, z) \) with the condition \( \lambda(t, z) = z \), defined on an interval \( t \in (\alpha, \beta) \), where \( \alpha < t < \beta \). Moreover, \( \lambda(t, z) \) continuously depends on \( (t, z) \) and every extremal in \( O_{\bar{q}} \) that passes through the singular locus is piece-wise smooth with only one switching.
Besides that, if \( u \) is the control correspondent to the extremal that passes through \( \bar{\lambda} \), and \( \bar{t} \) is its switching time, we have:

\[
\begin{equation}
(3.4) \quad u(\bar{t} \pm 0) = [\pm d \text{Id} - H_{IJ}]^{-1} H_{0I} ,
\end{equation}
\]

with \( d > 0 \) such that

\[
(3.5) \quad \langle [d^2 \text{Id} - H_{IJ}^2]^{-1} H_{0I}, H_{0I} \rangle = 1.
\]

If it holds

\[
(3.6) \quad H_{0I} \in H_{IJ}B^k,
\]

then there exists a neighborhood \( O_{\bar{\lambda}} \subset T^*M \) such that no one optimal extremal intersects singular locus in \( O_{\bar{\lambda}} \); all close to \( \bar{q} \) optimal trajectories are smooth bang arcs.

Note that \( H_{IJ}B^k = H_{IJ}S^{k-1} \) if the matrix \( H_{IJ} \) is degenerate, and that this matrix is always degenerate for odd \( k \). Hence, assuming (3.2), we have the following possibilities:

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>( k ) is odd</td>
</tr>
<tr>
<td>(B)</td>
<td>( k ) is even and ( H_{IJ} ) is degenerate</td>
</tr>
<tr>
<td>(C')</td>
<td>( k ) is even, ( H_{IJ} ) is non-degenerate and ( H_{0I} \notin H_{IJ}B^k )</td>
</tr>
</tbody>
</table>

It holds (3.3) if it is verified one of the following scenarios:

<table>
<thead>
<tr>
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</tr>
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</tr>
<tr>
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</tr>
<tr>
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<td>( k ) is even, ( H_{IJ} ) is non-degenerate and ( H_{0I} \notin H_{IJ}B^k )</td>
</tr>
</tbody>
</table>

It holds (3.6) if it is verified the following scenario:

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C'')</td>
<td>( k ) is even, ( H_{IJ} ) is non-degenerate and ( H_{0I} \in H_{IJ}B^k )</td>
</tr>
</tbody>
</table>

Remark 3.5. In general, the flow of switching extremals from Theorem 3.4 is not locally Lipschitz with respect to the initial value. In [2] was found a simple counterexample that can be easily generalized to any \( k < n \).

Since the Pontryagin Maximum Principle is a necessary but not sufficient condition of optimality, even if we have found extremals that passes through the singular locus, we cannot guaranty that they are all optimal, namely that their projections in \( M \) are time-optimal trajectory. In some cases they are certainly optimal, in particular, for linear system with an equilibrium target, where to be an extremal is sufficient for optimality. We plan to study general case in a forthcoming paper.

4. Proof

In this Section we are going to present at first the proof of Theorem 3.4, secondly we are going to prove Proposition 3.3. All together, these statements contain Theorem 1.1.

4.1. Proof of Theorem 3.4. Let us present the Blow-up technique, in order to analyse the discontinuous right-hand side Hamiltonian system, defined by

\[
(4.1) \quad \mathcal{H}(\lambda) = h_0(\lambda) + \sqrt{h_2^2(\lambda)} + \cdots + h_k^2(\lambda),
\]

in a neighbourhood \( O_{\bar{\lambda}} \) of \( \bar{\lambda} \).
\[ \mathcal{H}(\xi, x) = h_0(\xi, x) + \sqrt{h_1^2(\xi, x) + \ldots + h_k^2(\xi, x)}. \]

Since \( f_1, \ldots, f_k \) are linearly independent everywhere, we can define \( n - k \) never null vector fields \( f_{k+1}, \ldots, f_n \), such that \( \{f_1, \ldots, f_n\} \) form a basis at any \( q \in M \), then we will have the correspondent \( h_j(\xi, x) = \langle \xi, f_j(x) \rangle \), with \( j = k + 1, \ldots, n \). Therefore, we are allowed to consider the following smooth change of variables

\[ \Phi : (\xi, x) \rightarrow ((h_1, \ldots, h_n), x), \]

so the singular locus becomes the subspace

\[ \Lambda = \{(h_1, \ldots, h_n), x) : h_1 = \ldots = h_k = 0 \}. \]

**Notation 4.1.** In order not to do notations even more complicated, we call \( \lambda \) any point defined with respect to the new coordinates \(((h_1, \ldots, h_n), x), \) and \( \bar{\lambda} \) what corresponds to the singular point.

Thus, let us define the blow-up technique.

**Definition 4.2.** The blow-up technique is defined in the following way:

We make a change of variables: \((h_1, \ldots, h_k) = (\rho u_1, \ldots, \rho u_k) \) with \( \rho \in \mathbb{R}^+ \) and \((u_1, \ldots, u_k) \in S^{k-1} \). Instead of considering the components \( h_1, \ldots, h_k \) of the singular point \( \bar{\lambda} \) in \( \Lambda \), as the point \((0, \ldots, 0) \) in the \( k \)-dimensional euclidean space, we will consider it as a sphere \( S^{k-1} \), where \( \{\rho = 0\} \).

\[ ((h_1, \ldots, h_k, h_{k+1}, \ldots, h_n), x) \rightarrow ((\rho, u_1, \ldots, u_k, h_{k+1}, \ldots, h_n), x) \]

**Figure 2. Blow-up technique**

Let us notice that it is good to denote \( u := (u_1, \ldots, u_k) \) the \( S^{k-1} \)-coordinates. As it is already know from Proposition 2.17 every optimal control \( \bar{u} \), that corresponds to an extremal \( \lambda(t) \) out of \( \Lambda \), satisfies formula 2.6; therefore \( \bar{u} \) lies on \( \partial U = S^{k-1} \), and it is the normalization of the vector \((h_1(\lambda(t)), \ldots, h_k(\lambda(t)))\).

It is useful denote \( f_u(x) = u_1 f_1(x) + \ldots + u_k f_k(x) \)

and \( h_u(\lambda) = \langle \xi, f_u(x) \rangle ; \) and finally we can see that

\[ h_u(\lambda) = \sqrt{h_1^2 + \ldots + h_k^2}, \]

namely \( h_u(\lambda) = \rho \), because \( h_u(\lambda) = u_1 h_1 + \ldots + u_k h_k \), and \( u_i = \frac{h_i}{\sqrt{h_1^2 + \ldots + h_k^2}} \) for all \( i \in \{1, \ldots, k\} \).

Hence, with this new formulation the maximized Hamiltonian becomes

\[ \mathcal{H}(\lambda) = h_0(\lambda) + h_u(\lambda). \]
Thanks to Notation 2.15, Remarks 2.12 and 2.11, the Hamiltonian system has the following form:

\[
\begin{align*}
\dot{x} &= f_0(x) + f_u(x) \\
\dot{p} &= \langle H_{0I}(\lambda), u \rangle \\
\dot{u} &= \frac{1}{\rho} (H_{0I}(\lambda) - \langle H_{0I}(\lambda), u \rangle) u - H_{IJ}(\lambda) u \\
\dot{h}_j &= h_{0j}(\lambda) + h_{u_j}(\lambda), \quad j \in \{k + 1, \ldots, n\}.
\end{align*}
\]  

(4.4)

Claim 4.3. If assumption (3.3) is satisfied at the singular point \( \lambda \), then in \( S^{k-1} \)

\[
(4.5) \\
\langle d \, \text{Id} - H_{IJ}^2 \rangle^{-1} H_{0I} = 1.
\]

The function (4.3) has no zero if it holds assumption (3.7).

Proof. Denoting \( Z := \langle H_{0I}, u \rangle \), we are looking for \( u \in S^{k-1} \) and \( Z \in \mathbb{R} \) such that

\[
H_{0I} = (Z \, \text{Id} + H_{IJ}) u.
\]

We already know that, if \( Z = 0 \), then there is no \( u \in S^{k-1} \) such that \( H_{0I} = H_{IJ} u \), by assumption (3.2). Moreover, since \( H_{IJ} \) is a skew-symmetric matrix, if \( Z \neq 0 \) then \( (Z \, \text{Id} + H_{IJ}) \) is invertible, and

\[
u = (Z \, \text{Id} + H_{IJ})^{-1} H_{0I}.
\]

Let us consider the function

\[
(4.8) \\
Z \mapsto ||(Z \, \text{Id} + H_{IJ})^{-1} H_{0I}||
\]

that will be continuous even and monotone in the domains \((-\infty, 0) \) and \((0, +\infty) \).

We are going to verify if and in which cases the function (4.8) takes value \( 1 \) two or zero times. Let us observe that

\[
\lim_{Z \to \pm \infty} ||(Z \, \text{Id} + H_{IJ})^{-1} H_{0I}|| = 0^+, \quad \text{for } d > 0
\]

moreover, if \( H_{IJ} \) is a degenerate matrix, then

\[
\lim_{Z \to 0^\pm} ||(Z \, \text{Id} + H_{IJ})^{-1} H_{0I}|| = +\infty,
\]

thus, by monotonicity and continuity of (4.3), there will be a value \( Z = d > 0 \) such that

\[
||d \, \text{Id} + H_{IJ}^{-1} H_{0I}|| = 1.
\]

It means that there exist \( u_+ \) and \( u_- \) zeros of the function (4.5) such that \( ||H_{0I}, u_\pm || = d \).

We will assume \( \langle H_{0I}, u_+ \rangle > 0 \) and \( \langle H_{0I}, u_- \rangle < 0 \). These facts happen in scenarios (A) and (B) of condition (3.3).

If \( H_{IJ} \) is non-degenerate, then (4.8) is a continuous function for all \( Z \in \mathbb{R} \), hence

\[
\lim_{Z \to 0} ||(Z \, \text{Id} + H_{IJ})^{-1} H_{0I}|| = ||H_{IJ}^{-1} H_{0I}||
\]

and the function will have two or no zeros if and only if \( ||H_{IJ}^{-1} H_{0I}|| > 1 \) or \( ||H_{IJ}^{-1} H_{0I}|| < 1 \), namely \( H_{0I} \notin H_{IJ} B^k \) or \( H_{0I} \in H_{IJ} B^k \).

These are, indeed scenarios (C') and (C'').
Lemma 4.4. Given the singular point \( \tilde{\lambda} \) and any \( \tilde{\lambda}_{uv} \) such that it holds
\[
H_{\partial I} - \langle H_{\partial I}, u \rangle u - H_{I^1}u = 0.
\]
We consider \( O_{\lambda_{uv}} \) neighborhoods of \( \tilde{\lambda}_{uv} \), and a neighbourhood \( O_{\lambda} \) small enough such that
\[\forall \tilde{\lambda} \in O_{\lambda} \setminus \bigcup_{u} O_{\lambda_{uv}} \text{ it holds} \]
\[
H_{\partial I}(\tilde{\lambda}) - \langle H_{\partial I}(\tilde{\lambda}), \dot{u} \rangle \dot{u} - H_{I^1}(\tilde{\lambda})\dot{u} \neq 0.
\]
For each connected component \( O \) of \( O_{\lambda} \setminus \bigcup_{u} O_{\lambda_{uv}} \) there exist constants \( c > 0 \) and \( \alpha > 0 \) such that if an extremal \( \lambda(t) \) lies in \( O \) for a time interval \( I = (\tau_1, \tau_2) \), with \( \lambda(\tau_1) \notin \Lambda \), then it holds the following inequality:
\[\rho(t) \geq ce^{-\alpha t}\rho(\tau_1), \text{ for } t \in I.\]

Proof. Let us call
\[v(\lambda) = H_{\partial I}(\lambda) - \langle H_{\partial I}(\lambda), u \rangle u - H_{I^1}(\lambda)u,
\]
by construction, for all \( \lambda \in O \) it holds
\[||v(\lambda)|| > 0.
\]
Since in the compact set \( O \) the map \( \lambda \to v(\lambda) \) is continuous and not null, then there exist constants \( c_1 > 0 \) and \( c_2 > 0 \) such that, for all \( \lambda \in O \),
\[c_1 \geq ||v(\lambda)|| \geq c_2 > 0.
\]
Given the extremal \( \lambda(t) \) in \( O \), we can observe that
\[
\frac{d}{dt}\rho(t)||v(\lambda(t))|| = \rho(t)\frac{\langle v(\lambda(t)), A(\lambda(t)) \rangle}{||v(\lambda(t))||} = \rho(t)\tilde{A}(\lambda(t))
\]
where
\[A(\lambda(t)) = H_{\partial I}(\lambda(t)) - \langle H_{\partial I}(\lambda(t)), u(t) \rangle u(t) - H_{I^1}(\lambda(t))u(t).
\]
Let us notice that for any Hamiltonian \( h(\lambda) \) its time-derivative along \( \lambda(t) \) is
\[
\dot{h}(\lambda(t)) = \{h_0 + \rho, h\}(\lambda(t)) = \{h_0, h\}(\lambda(t)) + \{\rho, h\}(\lambda(t)) = \{h_0, h\}(\lambda(t)) + 1 \sum_{i=1}^k h_i(\lambda(t))\{h_i, h\}(\lambda(t)) = \{h_0, h\}(\lambda(t)) + \sum_{i=1}^k u_i(t)\{h_i, h\}(\lambda(t))
\]
and it is bounded.
As consequence each component of \( A(\lambda(t)) \) is bounded too, and \( \tilde{A}_{I^1} \) is bounded from below by a negative constant \( C \)
\[\tilde{A}_{I^1} \geq C.
\]
Finally, we can see that
\[
\frac{d}{dt} \left[ \frac{\rho(t)||v(\lambda(t))||}{\exp \left( \int_0^t C ||v(\lambda(s))||^{-1} ds \right)} \right] \geq 0,
\]

hence, for each \( t \geq \tau_1 \), by the monotonicity:
\[
\rho(t) \geq \rho(\tau_1) \frac{||v(\lambda(\tau_1))||}{||v(\lambda(t))||} \exp \left( \int_{\tau_1}^t C ||v(\lambda(s))||^{-1} ds \right) \geq \rho(\tau_1) \frac{c_1}{c_2} \exp \left( \frac{c_2 - c_1}{c_2} \right).
\]
Denoting \( c := \frac{c_1}{c_2} \) and \( \alpha := -\frac{c_2}{c_2} \), the thesis follows. \( \square \)
4.1.2. Case \( H_{01} \in H_{11}B_k \).
Lemma 4.4 and Claim 4.3 immediately imply the following Corollary:

**Corollary 4.5.** If we give condition (3.2) and assumption (3.3), there exists a unique extremal that passes through \( \bar{\lambda} \) in \( O_{\lambda} \).

Moreover, as we saw from Claim 4.3, at \( \bar{\lambda} \), we rescale the time considering the time \( t(s) \) such that \( \frac{dt}{ds}(s) = \rho(s) \) and we obtain the following system

\[
\begin{align*}
\dot{x'} &= \rho \left( f_0(x') + f_u(x') \right) \\
\dot{\rho'} &= \rho \langle H_{01}(\lambda), u \rangle \\
\dot{u'} &= H_{01}(\lambda) - \langle H_{01}(\lambda), u \rangle u - H_{11}(\lambda)u \\
\dot{h_j'} &= \rho \left( h_{0j}(\lambda) + h_{uj}(\lambda) \right), \quad j \in \{k + 1, \ldots, n\}
\end{align*}
\]

(4.9)

with a smooth right-hand side.

This system has an invariant subset \( \{ \rho = 0 \} \) in which only the \( u \)-component is moving. Moreover, as we saw from Claim 4.3, if \( \lambda \in \{ \rho = 0 \} \) there are two equilibria \( \lambda_{u-} \) and \( \lambda_{u+} \), such that \( \langle H_{01}, u+ \rangle > 0 \) and \( \langle H_{01}, u- \rangle < 0 \).

Let us present the Shoshihataishvili’s Theorem [12] that explain how is the behaviour of the solutions in \( O_{\lambda_{u-}} \) and \( O_{\lambda_{u+}} \) neighbourhoods of the equilibria \( \lambda_{u-} \) and \( \lambda_{u+} \) in \( T^*M \).

**Theorem 4.7** (Shoshihataishvili’s Theorem). In \( \mathbb{R}^n \), let the \( C^k \)-germ, \( 2 \leq k < \infty \), of the family

\[
\begin{align*}
\dot{z} &= Bz + r(z, \varepsilon), \\
\dot{\varepsilon} &= 0, \quad z \in \mathbb{R}^n, \varepsilon \in \mathbb{R}^l,
\end{align*}
\]

be given, where \( r \in C^k(\mathbb{R}^n \times \mathbb{R}^l) \), \( r(0,0) = 0 \), \( \partial_r r(0,0) = 0 \), and \( B : \mathbb{R}^n \to \mathbb{R}^n \) is a linear operator whose eigenvalues are divided into three groups:

\[
\begin{align*}
\text{I} &= \{ \lambda_i, 1 \leq i \leq k^0 \mid \text{Re} \lambda_i = 0 \} \\
\text{II} &= \{ \lambda_i, k^0 + 1 \leq i \leq k^0 + k^- \mid \text{Re} \lambda_i < 0 \} \\
\text{III} &= \{ \lambda_i, k^0 + k^- + 1 \leq i \leq k^0 + k^- + k^+ \mid \text{Re} \lambda_i > 0 \}
\end{align*}
\]

\( k^0 + k^- + k^+ = n \).

Let the subspaces of \( \mathbb{R}^n \), which are invariant with respect \( B \) and which correspond to these groups be denoted by \( X \), \( Y^- \) and \( Y^+ \) respectively, and let \( Y^- \times Y^+ \) be denoted by \( Y \).

Then the following assertions are true:

1. There exists a \( C^{k-1} \) manifold \( \gamma^0 \) that is invariant with respect to the germ \( \gamma^0 \), may be given by the graph of mapping \( \gamma^0 : X \times \mathbb{R}^l \to Y \), \( y = \gamma^0(x, \varepsilon) \), and satisfies \( \gamma^0(0,0) = 0 \) and \( \partial_x \gamma^0(0,0) = 0 \).
(2) The germ of the family (4.10) is homeomorphic to the product of the multidimensional saddle \( \dot{y}^+ = y^+, \dot{y}^- = -y^- \), and the germ of the family
\[
\begin{aligned}
\dot{x} &= Bx + r_1(x, \varepsilon), \\
\dot{\varepsilon} &= 0,
\end{aligned}
\]
where \( r_1(x, \varepsilon) \) is the \( x \)-component of the vector \( r(z, \varepsilon), z = (x, \gamma(0)(x, \varepsilon)) \), i.e. the germ of (4.10) is homeomorphic to the germ of the family
\[
\begin{aligned}
\dot{y}^+ &= y^+, \dot{y}^- = -y^- \\
\dot{x} &= Bx + r_1(x, \varepsilon), \dot{\varepsilon} = 0.
\end{aligned}
\]

Due to the fact that \( \tilde{\lambda}_{u_-} \) and \( \tilde{\lambda}_{u_+} \) belong to the invariant subset \( \{ \rho = 0 \} \), where the components \( \rho, h_j \) with \( j \in \{ k + 1, \ldots, n \} \) and \( x \) are fixed, we can observe that Jacobian matrix of (4.9) have the following eigenvalues: \( \partial u'_{\tilde{\lambda}_{u_\pm}}(H_{0I}, u_{\pm}) \) that corresponds to the \( \rho \)-coordinate, the eigenvalues of the matrix \( \partial u'_{\tilde{\lambda}_{u_\pm}} \) that correspond to the \( u \)-coordinate, and \( 2n - k \) 0-eigenvalues corresponding to the other coordinates.

Let us observe that
\[
\partial u'_{\tilde{\lambda}_{u_\pm}} = -[(H_{0I}, u_{\pm}) I + H_{IJ} + u_{\pm} H_{0I}^T]
\]
where \( H_{0I}^T \) is the row vector. Thus, we can calculate
\[
\det \left[ \partial u'_{\tilde{\lambda}_{u_\pm}} - \mu I \right] = - \det \left[ (H_{0I}, u_{\pm}) I + H_{IJ} + u_{\pm} H_{0I}^T \right] = - \det \left[ (H_{0I}, u_{\pm}) I + H_{IJ} \right] \det \left[ I + u_{\pm} H_{0I}^T \right] = - \det \left[ (H_{0I}, u_{\pm}) I + H_{IJ} \right] (1 + (H_{0I}, u_{\pm})).
\]

Since \( H_{IJ} \) is skew symmetric and reduced in normal form, from this equality we can claim that the real part of all those eigenvalues is \( - (H_{0I}, u_{\pm}) \).

By Claim 4.3 we know that \( (H_{0I}, u_-) \) and \( (H_{0I}, u_+) \) are not null with opposite sign. Hence, assuming \( (H_{0I}, u_-) < 0 \), we can conclude that in a neighbourhood of \( \tilde{\lambda}_{u_-} \) there is a stable 1-dimensional submanifold with respect to \( \rho \) and an unstable submanifold with respect to \( u \). Analogously in a neighbourhood of \( \tilde{\lambda}_{u_+} \), we can notice the unstable 1-dimensional submanifold with respect to \( \rho \) and the stable one with respect to \( u \).

Central manifolds \( \gamma_0 \) of Theorem 4.7 applied to the equilibria \( \tilde{\lambda}_{u_\pm} \) are \( (2n-k) \)-dimensional submanifolds defined by the equations \( \rho = 0, u = u_{\pm} \). The dynamics on the central manifold is trivial: all points are equilibria.

Hence, according to the Shoshitaishvili Theorem, there is a trajectory from the one-dimensional asymptotically stable invariant submanifold that tends to the equilibrium point \( \tilde{\lambda}_{u_-} \) as \( s \to +\infty \), and analogously there is a trajectory from the one-dimensional asymptotically unstable invariant submanifold that escapes from the equilibrium point \( \tilde{\lambda}_{u_+} \) as \( s \to -\infty \).

Thanks to Lemma 4.4 and the consequences of Shoshitaishvili’s Theorem, we obtain that exactly one extremal enters submanifold \( \rho = 0 \) at \( \tilde{\lambda}_{u_-} \) and exactly one extremal goes out of this submanifold at \( \tilde{\lambda}_{u_+} \). Thus, there exists only one extremal, solution of (4.4), that passes through \( \lambda \).

Moreover, the same result in the same neighbourhood is valid for any \( \tilde{\lambda} \in \Lambda \) sufficiently close to \( \bar{\lambda} \) with \( \bar{\lambda} \) playing the role of parameter \( \varepsilon \) in the Shoshitaishvili Theorem.
Finally, we are going to show that the extremal that we found passes through $\bar{\lambda}$ in finite time. At the moment it is known that there exists $\lambda(t(s))$ that satisfies (4.9) and it reaches $\bar{\lambda}$ at equilibrium, so $\lambda(t(s))$ attains and escapes from $\bar{\lambda}$ in infinite time $s$.

Thus, let us estimate the time $\Delta t$ that this extremal needs to reach $\bar{\lambda}$.

Due to the facts that $\langle H_{01}, u_- \rangle < 0$ and $\langle H_{01}(\lambda), u \rangle$ is continuous in $\bar{\lambda}$, there exist a neighbourhood $O_{\bar{\lambda}}$ of $\bar{\lambda}$, in which $\langle H_{01}(\lambda), u \rangle$ is bounded from above by a negative constant $c_1 < 0$, namely $\langle H_{01}(\lambda), u \rangle|_{O_{\bar{\lambda}}} < c_1 < 0$.

Hence, in $O_{\bar{\lambda}}$ we have the following estimate of the derivative $\rho'$

$$\rho' = \rho \langle H_{01}(\lambda), u \rangle < \rho c_1,$$

consequently until $\rho(s) > 0$, it holds

$$\int_{s_0}^{s} \frac{\rho'}{\rho} ds < \int_{s_0}^{s} c_1 ds,$$

then this inequality implies $\log(\rho(s)) < c_1(s - s_0) + \log(\rho(s_0))$, and so

$$\rho(s) < \rho(s_0)e^{c_1(s - s_0)}.$$

Since $\frac{d}{ds}t(s) = \rho(s)$, the amount of time that we want to estimate is the following

$$\Delta t = \lim_{s \to \infty} t(s) - t(s_0) = \int_{s_0}^{\infty} \rho(s) ds,$$

therefore,

$$\Delta t = \int_{s_0}^{\infty} \rho(s) ds < \rho(s_0) \int_{s_0}^{\infty} e^{c_1(s - s_0)} ds = \frac{\rho(s_0)}{-c_1} < \infty.$$

The amount of time in which this extremal goes out from $\bar{\lambda}$ may be estimate in an analogous way. $\square$

By the previous Proposition and the fact that every extremal out of $\Lambda$ is smooth, it is proven that there exist a neighbourhood $O_{\lambda} \subset T^*M$ such that for any $z \in O_{\lambda}$ and $\hat{t} > 0$ there exists a unique extremal $t \mapsto \lambda(t, z)$ contained in $O_{\lambda} \subset T^*M$ with condition $\lambda(\hat{t}, z) = z$ defined on an interval $t \in (\alpha, \beta)$, where $\alpha < \hat{t} < \beta$.

Let us conclude the proof with the following Proposition.

**Proposition 4.8.** The map $(t, z) \mapsto \lambda(t, z)$, $(t, z) \in (\alpha, \beta) \times O_{\lambda}$ is continuous.
Proof. It remains to prove the continuity of the flow \((t, z) \to \lambda(t, z)\) with respect to \(z\), thus we prove that for each \(\varepsilon > 0\) there exists a neighbourhood \(O^{\lambda}_{\bar{\lambda}}\) such that the maximum time interval of the extremals in this neighbourhood \(\Delta_{O^{\lambda}_{\bar{\lambda}}} t\) is less than \(\varepsilon\).

As we saw previously, the extremal through \(\lambda\) will arrive and go out at \(u_-\) and \(u_+\), then we can distinguish three parts of the extremals close to \(\lambda\): the parts in \(O^{\lambda}_{\bar{\lambda}_-}\) and in \(O^{\lambda}_{\bar{\lambda}_+}\), and that part in the middle that is close to \(\rho = 0\).

In this last region, since each \(\rho\)-component is close to 0 and the correspondent time interval with time \(s\) is bounded, then \(\Delta t\) is arbitrarily small with respect to \(O^{\lambda}_{\bar{\lambda}}\).

Hence, in \(O^{\lambda}_{\bar{\lambda}_-}\) we are going to show that there exists a sequence of neighbourhoods of \(\bar{\lambda}_{u_-}\)

\[
\left( O^{\rho}_{u_-} \right)_R,
\]

such that

\[
\lim_{R \to 0^+} \Delta_{O^{\rho}_{u_-}} t = 0.
\]

For simplicity, we are going to prove this fact in \(O^{\lambda}_{\bar{\lambda}_-}\), because the situations in \(O^{\lambda}_{\bar{\lambda}_+}\) is equivalent.

Let us denote \(O^{\rho}_{u_-}\) a neighbourhood of \(\bar{\lambda}_{u_-}\) such that \(O^{\rho}_{u_-} \subseteq O^{\lambda}_{\bar{\lambda}_-}\), for each \(\lambda \in O^{\rho}_{u_-}\) \(\rho < R\) and \(|u - u_-| < R\). Therefore, we can define

\[
M_R = \sup_{\lambda \in O^{\rho}_{u_-}} \langle H_{0I}(\lambda), u \rangle,
\]

and assume that it is strictly negative and finite, due to the fact that we can choose \(O^{\lambda}_{\bar{\lambda}_-}\) in which \(\langle H_{0I}(\lambda), u \rangle\) is strictly negative and finite.

Hence, for every extremal \(\lambda(t(s))\) in \(O^{\rho}_{u_-}\), until its \(\rho\)-component is different that zero, it holds

\[
\frac{\dot{\rho}(s)}{\rho(s)} < M_R,
\]

then

\[
\rho(s) < \rho(s_0)e^{M_R(s - s_0)},
\]

for every \(s > s_0\).

Consequently, \(\Delta_{O^{\rho}_{u_-}} t\) can be estimated in the following way:

\[
\Delta_{O^{\rho}_{u_-}} t < \int_{s_0}^{\infty} \rho(s_0)e^{M_R(s - s_0)} ds = \frac{\rho(s_0)}{-M_R} < \frac{R}{-M_R}.
\]

Due to the fact that \(\lim_{R \to 0^+} \frac{R}{-M_R} = 0\), we have proved that for each \(\varepsilon > 0\) there exists \(O^{\rho}_{u_-}\) such that \(\Delta_{O^{\rho}_{u_-}} t < \varepsilon\). 

\(\square\)

4.2. Proof of Proposition 3.3 Let us assume that there exist a time-optimal control \(\bar{u}\), and an interval \((\tau_1, \tau_2)\) such that \(\bar{u}\) corresponds to an extremal \(\lambda(t)\) in \(O_{\bar{\lambda}}\), and \(\lambda(t) \in \Lambda\), \(\forall t \in (\tau_1, \tau_2)\). By construction, for \(t \in (\tau_1, \tau_2)\) it holds

\[
\left\{ \begin{array}{c}
\frac{d}{dt}h_1(\lambda(t)) = 0 \\
\quad \vdots \\
\frac{d}{dt}h_k(\lambda(t)) = 0.
\end{array} \right.
\]

(4.13)

Since the maximized Hamiltonian associated to \(\bar{u}\) is

\[
\mathcal{H}_\bar{u}(\lambda) = h_0(\lambda) + \bar{u}_1h_1(\lambda) + \ldots + \bar{u}_kh_k(\lambda),
\]

by Remark 2.12 (4.13) implies

\[
H_{0I}(\lambda(t)) - H_{IJ}(\lambda(t))\bar{u} = 0.
\]

Moreover, due to condition (3.2), we can claim that, choosing \(O_{\bar{\lambda}}\) small enough, \(H_{0I}(\lambda(t)) \notin H_{IJ}(\lambda(t))\overline{B^k}\) or \(H_{0I}(\lambda(t)) \in H_{IJ}(\lambda(t))B^k\), for all \(t \in (\tau_1, \tau_2)\).
If $H_0(\lambda(t)) \notin H_{IJ}(\lambda(t))B_k$, we arrive to a contradiction, because in this case $||\tilde{u}|| > 1$ but the norm of admissible controls is less equal than 1. On the other hand, if $H_0(\lambda(t)) \in H_{IJ}(\lambda(t))B_k$, such extremals might exist, but they are not optimal by the Goh Condition, presented at Subsection 2.2.2.

References


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