2nd workshop
Mathematical Challenges of Zero-Range Physics: rigorous results and open problems
SISSA, Trieste, 7-10 November 2016

Programme

Monday 7 November 2016

11:11 ***opening***
11:20 Sergio Albeverio • *Point interactions – Some souvenirs and some new developments*

lunch

14:50 Yvan Castin • *Unitary gases – I*
15:50 ***coffee break***
16:20 Ivan Castin • *Unitary gases – II*
17:20 Alessandro Teta • *A Class of Hamiltonians for a three-particle fermionic system at unitarity*

Tuesday 8 November 2016

9:20 Giulia Basti • *Efimov effect for a system of two identical fermions and a different particle*
10:20 ***coffee break***
10:50 Andrea Ottolini • *On point interactions realised as Ter-Martirosyan–Skornyakov Hamiltonians*
11:50 Claudio Cacciapuoti • *Existence of ground state for the non-linear Schrödinger equation on star-like graphs*

lunch

14:50 Andrea Posilicano • *On the Calderon uniqueness problem with non-regular conductivity*
15:50 ***coffee break***
16:20 Konstantin Pankrashkin • *Eigenvalue inequalities and absence of threshold resonances for waveguide junctions*
17:20 Markus Holzmann • *An extension theoretic approach for the study of Dirac operators with δ-shell interactions supported on surfaces*

Wednesday 9 November 2016

9:20 Vladimir Lotoreichik • *Optimization of the lowest eigenvalue for surface δ-interactions and for Robin Laplacians*
10:20 ***coffee break***
10:50 Raffaele Carlone • *The 1D Dirac equation with concentated non-linearity*
11:50 Lorenzo Tentarelli • *Well-posedness of the 2D non-linear Schrödinger equation with concentrated non-linearity*

lunch

14:50 Davide Fermi • *Casimir energy for singular potentials concentrated on a plane*
15:50 ***coffee break***
16:20 Raffaele Scandone • *On fractional powers Schrödinger operators with point interaction*
17:20 Rodolfo Figari • *Quantum beating in a non-linear point interaction double-well potential*

Thursday 10 November 2016

9:20 *discussion session*
10:20 ***coffee break***
10:50 *discussion session*

lunch

Point interactions – some souvenirs and some new developments

Sergio Albeverio
University of Bonn and HCM Bonn

Efimov effect for a system of two identical fermions and a different particle

Giulia Basti
University of Rome "La Sapienza"

In the 1970's it was first pointed out by the physicist Vitaly Efimov that a three-body system may present an infinite number of bound states even if there are no bound states in any of the two-body sub-systems. This remarkable phenomenon known as the Efimov effect has been deeply studied both in the mathematical and in the physical literature and it has been shown to be due to the presence of a zero-energy resonance in at least two of the two-body sub-systems. We study a system of two identical fermions with unit mass and a third different particle and prove that the Efimov effect could occur only if the mass of the third particle is smaller than a certain threshold. Remarkably, the mass threshold we find is the same obtained in the study of the occurrence of the Thomas effect in the corresponding system with point interactions.

Existence of ground state for the non-linear Schrödinger equation on star-like graphs

Claudio Cacciapuoti
University of Insubria

We consider a non-linear Schrödinger equation (NLS) on a star-like graph (a graph composed by a compact core to which a finite number of half-lines are attached). At the vertices of the graph interactions of delta-type can be present and an overall external potential is admitted. Our goal is to show that the NLS dynamics on a star-like graph admits a ground state of prescribed mass m under mild and natural hypotheses. By ground state of mass m we mean a minimizer of the NLS energy functional constrained to the manifold of mass (L^2-norm) equal to m. When existing, the ground state is an orbitally stable standing wave for the NLS evolution. We prove that a ground state exists whenever the quadratic part of the energy admits a simple isolated eigenvalue at the bottom of the spectrum (the linear ground state) and m is sufficiently small. This is a major generalization of a result previously obtained for a graph with a single vertex (a star graph) with a delta interaction in the vertex and without potential terms.

The 1D Dirac equation with concentrated non-linearity

Raffaele Carlone
University of Naples Federico II

We define and study the Cauchy problem for a 1D non-linear Dirac equation with non-linearities concentrated at one point. Global well-posedness is provided and conservation laws for mass and energy are shown. Several examples, including non-linear Gesztesy-Seba models and the concentrated versions of the Bragg Resonance, Gross-Neveu, and Soler type models are given. In the last part of the talk we discuss also the linear stability for Gesztesy-Seba and Soler type models.

Unitary gases

Yvan Castin
LKB-ENS Paris

A gas is at the unitary limit when its constitutive particles interact via a potential of infinite scattering length and negligible range. It then exhibits interesting scale invariance properties, whose physical consequences we shall discuss. In particular, we shall speak about the 3-body and the 4-body Efimov effect, that break this scale invariance, and we shall present the progress and the challenges in the analytical calculation of virial coefficients of a unitary gas. We shall make the link with the ongoing cold atom experiments at ENS in the group of Christophe Salomon.

Casimir energy for singular potentials concentrated on a plane

Davide Fermi
University of Insubria

Within the framework of Euclidean quantum field theory, the partition function of a scalar field on a compact domain at finite temperature is heuristically related to the determinant of the elliptic operator associated to the free theory. Müller's relative zeta function technique allows to give a rigorous formulation of these arguments, even when the domain underlying the field theory is non-compact. This technique is used to investigate a model involving a delta-like potential concentrated on a plane; the potential consists of a singular perturbation of the free operator, which depends non-trivially on the energy of the modes propagating parallel to the plane. Finally, an explicit expression for the renormalized Casimir energy is derived by means of analytic continuation, in the spirit of zeta regularization. (Joint work with Claudio Cacciapuoti and Andrea Posilicano.)

Quantum beating in a point interaction non-linear double-well potential

Rodolfo Figari
University of Naples Federico II

In recent works we examined the suppression of quantum beating in a one dimensional non-linear double-well potential, made up of two focusing non-linear point interactions. The reduction of the evolution problem to the solution of a system of coupled non-linear Volterra integral equations allows to obtain analytic and numerical results confirming that, already for a nonlinearity exponent well below the critical value, there is complete suppression of the typical beating behavior characterizing the linear quantum case.

An extension theoretic approach for the study of Dirac operators with δ-shell interactions supported on surfaces

Markus Holzmann
Graz University of Technology

In this talk I will discuss how the abstract concept of quasi boundary triples can be used to study the spectral properties of Dirac operators with electrostatic δ-shell interactions supported on surfaces in $\mathbb{R}^3$. For this purpose, I will recall the definition of quasi boundary triples and their associated Weyl functions and some properties that allow the systematic investigation of the spectral properties of some associated operators. To illustrate these ideas I will apply them to some well-known types of Schrödinger operators. Then, the quasi boundary triple technique will be used to introduce the Dirac operator with an electrostatic δ-shell interaction. This operator is formally given by $A_\eta := A_0 + \eta\delta_\Sigma I_4$, where A_0 is the free Dirac operator, I_4 is the 4×4 identity matrix, $\eta \in \mathbb{R}$ and Σ is the boundary of a smooth domain. After establishing the self-adjointness of A_η, it turns out that some of the spectral properties of A_η are of a different nature, if $\eta = 2c$, where c denotes the speed of light, or $\eta \neq 2c$. In particular, I address the questions of the smoothness of functions in $\operatorname{dom} A_\eta$, the structure of the discrete and essential spectrum, existence and completeness of the wave operators for the pair $\{A_\eta, A_0\}$ and the non-relativistic limit of A_η, as $c \to \infty$.

Optimization of the lowest eigenvalue for surface δ-interactions and for Robin Laplacians

Vladimir Lotoreichik
UJF-CAS Prague

The celebrated Faber-Krahn inequality implies that, among all domains of a fixed volume, the ball minimizes the lowest eigenvalue of the Dirichlet Laplacian. This result can be viewed as a spectral counterpart of the well-known geometric isoperimetric inequality. The aim of this talk is to discuss generalizations of the Faber-Krahn inequality for optimization of the lowest eigenvalues for:

- Schrödinger operators with δ-interactions supported on conical surfaces and open arcs [1, 3];
- Robin Laplacians on exterior domains and planes with slits [2, 3].

Beyond clear physical relevance of δ-interactions and Robin Laplacians, a purely mathematical motivation to consider these optimization problems stems from the fact that standard methods, going back to the papers of Faber and Krahn, are not directly applicable. Moreover, in some cases the shape of the optimizer bifurcates as the boundary parameter varies. Another interesting novel aspect is related to the fact that the optimizer does not always exist. The results in the talk are obtained in collaboration with P. Exner and D. Krejčiřík.

References

[1] P. Exner and V. Lotoreichik, A spectral isoperimetric inequality for cones, *to appear in Lett. Math. Phys., arXiv:1512.01970.*

[2] D. Krejčiřík and V. Lotoreichik, Optimisation of the lowest Robin eigenvalue in the exterior of a compact set, *submitted, arXiv:1608.04896.*

[3] V. Lotoreichik, Spectral isoperimetric inequalities for δ-interactions on open arcs and for the Robin Laplacian on planes with slits, *submitted, arXiv:1609.07598.*

On point interactions realised as Ter-Martirosyan–Skornyakov Hamiltonians

Andrea Ottolini
Stanford University

For quantum systems of zero-range interaction we discuss the mathematical scheme within which modelling the two-body interaction by means of the physically relevant ultra-violet asymptotics known as the "Ter-Martirosyan–Skornyakov condition" gives rise to a self-adjoint realisation of the corresponding Hamiltonian. This is done within the self-adjoint extension scheme of Kreĭn, Višik, and Birman. We show that the Ter-Martirosyan–Skornyakov asymptotics is a condition of self-adjointness only when is imposed in suitable functional spaces, and not just as a point-wise asymptotics, and we discuss the consequences of this fact on a model of two identical fermions and a third particle of different nature.

Eigenvalue inequalities and absence of threshold resonances for waveguide junctions

Konstantin Pankrashkin
Paris-Sud University

We consider Dirichlet Laplacians in domains consisting of several half-finite cylinders attached to a bounded center, and we derive a sufficient condition for the absence of resonances and embedded eigenvalues at the bottom of the essential spectrum. Some applications to the limits of domains shrinking to graphs will be explained in the talk.

On the Calderon uniqueness problem with non-regular conductivity

Andrea Posilicano
University of Insubria

We sketch a proof of uniqueness in Calderon's problem for a class of conductivities having singularities of single-layer type along compact hypersurfaces. We use scattering theory at fixed energy for Schrödinger operators with δ-type distributional potentials supported on hypersurfaces (joint work with Andrea Mantile and Mourad Sini).

On fractional powers of Schrödinger operators with point interactions

Raffaele Scandone
SISSA Trieste

Singular perturbations of the d-dimensional Laplacian arise naturally in the context of quantum systems of particles subject to interactions of zero range, thus supported on manifolds with positive co-dimension. For $d = 3$ it is well-known that an interaction supported at $x = 0$ is realised by one element of a one-parameter family $\{-\Delta_\alpha \,|\, \alpha \in (-\infty, +\infty]\}$ of self-adjoint operators on $L^2(\mathbb{R}^3)$, each of which acts as the free negative Laplacian on functions supported away from the origin. In this talk, after a quick overview of the definition and the main properties of $-\Delta_\alpha$, we present some new results concerning the characterization of fractional domains $\widetilde{H}^s_\alpha := \mathcal{D}((-\Delta_\alpha + 1)^{s/2})$, the analog of classical Sobolev spaces in the free case. One main motivation for studying such spaces is the investigation of the non-linear Schrödinger equation with singular perturbations: in this direction we provide some new partial results.

Well-posedness of the two-dimensional non-linear Schrödinger equation with concentrated non-linearity

Lorenzo Tentarelli
University of Naples Federico II

I will talk about Schrödinger equations with non-linear point interactions in dimension two. More precisely, I will focus on local and global well-posedness, thus completing the investigation initiated in [AT] and [ADFT] in the one and the three-dimensional cases. In particular, I will show that in both the attractive and the repulsive case there is local well-posedness (as well as mass and energy conservation), while global existence can be guaranteed only in the repulsive case (whereas in the attractive case blowing-up solutions may arise). This is a joint work with R. Carlone and M. Correggi.

References

[AT] Adami R., Teta A., A Class of Nonlinear Schrödinger Equations with Concentrated Nonlinearity, *J. Funct. Anal.* **180** (2001), 148–175.

[ADFT] Adami R., Dell'Antonio G., Figari R., Teta A., The Cauchy problem for the Schrödinger equation in dimension three with concentrated nonlinearity, *Ann. I. H. Poincaré – AN* **20** (2003), 477–500.

A class of Hamiltonians for a three-particle Fermionic system at unitarity

Alessandro Teta
University of Rome "La Sapienza"

We consider a quantum mechanical three-particle system made of two identical fermions of mass one and a different particle of mass m, where each fermion interacts via a zero-range force with the different particle. In particular we study the unitary regime, i.e., the case of infinite two-body scattering length. The Hamiltonians describing the system are, by definition, self-adjoint extensions of the free Hamiltonian restricted on smooth functions vanishing at the two-body coincidence planes, i.e., where the positions of two interacting particles coincide.