Dispersion and dissipation: a model problem

alberto.maspero

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1 Introduction

Structural stability and the persistence of coherent structures under perturbations of a dynamical system are fundamental issues in dynamical systems theory with implications in many fields of application. In the context of discrete, finite dimensional Hamiltonian systems, this issue is addressed by the celebrated KAM theorem, which guarantees the persistence of most invariant tori of the unperturbed dynamics under small Hamiltonian perturbations. For infinite dimensional systems, defined by Hamiltonian partial differential equations (PDEs), KAM type methods have recently been used to obtain results on the persistence of periodic and quasi-periodic solutions, in the case where solutions are defined on compact spatial domains with appropriate boundary conditions. The compactness of the spatial domain ensures discreteness of the spectrum associated with the unperturbed dynamics. Therefore this situation is the generalization of the finite dimensional case to systems with an infinite number of discrete oscillators and frequencies.
In this notes we consider these questions in the context of Hamiltonian systems for which the unperturbed dynamics has associated with it discrete and continuous spectrum. This situation arises in the study of Hamiltonian PDEs governing functions defined on unbounded spatial domains or, more generally, extended systems. The physical picture is that of a system which can be viewed as an interaction between one or more discrete oscillators and a field or continuous medium. In contrast to the KAM theory, where nonresonance implies persistence, we find here that resonant nonlinear interaction between discrete (bound state) modes and continuum (dispersive radiation) modes leads to energy transfer from the discrete to continuum modes. This mechanism is responsible for the eventual time-decay and nonpersistence of trapped states. The rate of timedecay, however, is very slow and hence such a trapped state can be thought of as a metastable state.

This paper is devoted to the study of the following questions:

(1) Do small amplitude spatially localized and time-periodic solutions persist for typical non-linear and Hamiltonian perturbations?

(2) What is the character of general small amplitude solutions to the perturbed dynamics?

(3) How are the structures of the unperturbed dynamics manifested in the perturbed dynamics?

We start by giving three examples of problems.

Particle interacting with an external field. Consider a mechanical particle which interacts with an external dispersive field. The Hamiltonian of the system is

$$H(p,q,u,v)=\frac{|p|^2}{2m}+V(q)+\frac{1}{2}\int_{\mathbb{R}^3}(v^2+|\nabla u|^2+m^2u^2)\,dx+\epsilon\int_{\mathbb{R}^3}V(x+q)u(x)\,dx$$

Assume that V has a minimum at 0, and that its Taylor expansion at 0 is $V(q)=\sum_j \omega_j^2 q_j^2$.

Asymptotic stability of the zero solution of Klein- Gorgon equation on $\mathbb{R}^3$ Consider a NLKG in $\mathbb{R}^3$

$$u_{tt}-\Delta u+Vu+m^2u+\beta'(u)=0\ ,\qquad x\in\mathbb{R}^3$$

with $-\Delta+V(x)+m^2$ a positive short range Schrödinger operator, and β' a smooth function having a zero of order 3 at the origin and growing at most like u^3 at infinity.

The system is Hamiltonian in $H^1(\mathbb{R}^3,\mathbb{R})\times L^2(\mathbb{R}^3,\mathbb{R})$ endowed with the standard symplectic form

$$\Omega\left((u_1,v_1);\ (u_2,v_2)\right):=\langle u_1,v_2\rangle_{L^2}-\langle u_2,v_1\rangle_{L^2}$$

and Hamiltonian

$$\begin{aligned}
H&=H_L+H_P\\
H_L&:=\int_{\mathbb{R}^3}\frac{1}{2}(v^2+|\nabla u|^2+Vu^2+m^2u^2)\,dx\\
H_P&:=\int_{\mathbb{R}^3}\beta(u)\,dx.
\end{aligned}$$

If $V(x)$ is real valued and of Schwartz class, then one has that the set of discrete eigenvalues $\sigma_d(-\Delta+V)=\{-\lambda_j^2\}_{j=1}^n$ is finite, contained in $(-\infty,0)$, with each eigenvalue of finite multiplicity. We take a mass m^2 such that $-\Delta+V+m^2>0$ and order the eigenvalues as $-\lambda_1^2\le\cdots\le-\lambda_n^2$. Set $\omega_j:=\sqrt{m^2-\lambda_j^2}$. Furthermore $-\Delta+V+m^2$ has continuous spectrum in $[m^2,\infty)$.

Associate to any ω_j an L^2 eigenvector $\psi_j(x)$, real valued and normalized. Furthermore one has $\phi_j\in\mathcal{S}(\mathbb{R}^3,\mathbb{R})$. Set $P_du:=\sum\langle u,\phi_j\rangle\phi_j$ the projector on the discrete spectrum, and $P_c:=\mathbb{1}-P_d$ the projector on the continuous spectrum. Denote

$$u=\sum_j q_j\phi_j+P_cu\ ,\qquad v=\sum_j p_j\phi_j+P_cv\ .$$

Introduce the operator

$$B := P_c(-\Delta + V + m^2)^{1/2} P_c \ ,$$

and the complex variables

$$\xi_j := \frac{q_j\sqrt{\omega_j} + \mathrm{i}\frac{p_j}{\sqrt{\omega_j}}}{\sqrt{2}} \ , \quad f := \frac{B^{1/2}P_c u + \mathrm{i}B^{-1/2}P_c v}{\sqrt{2}}$$

The transformation is symplectic, the new phase space is $\mathbb{C}^n \times P_c H^1$ and the equation of motion take the form

$$\dot{\xi}_j = -\mathrm{i}\frac{\partial H}{\partial \bar{\xi}_j} \ , \quad \dot{f} = -\mathrm{i}\nabla_{\overline{f}} H \ .$$

The form of H_L and H_P are respectively

$$H_L = \sum_{j=1}^{n} \omega_j |\xi_j|^2 + \left(Bf, \overline{f}\right) \ ,$$

$$H_P(\xi, f) = \int_{\mathbb{R}^3} \beta\left(\sum \frac{\xi_j + \overline{\xi}_j}{\sqrt{2\omega_j}}\phi_j(x) + B^{-1/2}\frac{f + \overline{f}}{\sqrt{2}}\right) dx$$

One sees clearly that the linearized system is given by

$$\mathrm{i}\dot{\xi}_j = \omega_j \xi_j \ , \quad \mathrm{i}\dot{f} = Bf \ ,$$

which are discrete harmonic oscillators coupled with a continuous field which exhibits dispersive behaviour. Thus the unperturbed system is given by quasi-periodic motions and a field which scatters to zero, in the sense that for every pair (r, s) which is admissable, i.e.

$$\frac{2}{r} + \frac{3}{s} = \frac{3}{2}, \quad 6 \geq s \geq 2, \ r \geq 2 \ ,$$

we have

$$\|f\|_{L^r_t W_x^{\frac{1}{s} - \frac{1}{r} + \frac{1}{2}, s}} \leq C\|f\|_{H^1} \ .$$

The nonlinearity H_P couples the discrete and the continuous mode.

Question: do the quasi-periodic orbit persist under perturbation?

Dynamics of soliton of NLS on $\mathbb{R}^3$ interacting with external potential. Consider the defocusing NLS in $\mathbb{R}^3$

$$\mathrm{i}\dot{\psi} = -\Delta\psi - \beta'(|\psi|^2)\psi + \epsilon V\psi$$

where V is a potential of Schwarz class, β is a smooth focusing nonlinearity. The phase space is $H^1(\mathbb{R}^3)$ endowed with the real scalar product

$$\langle \psi_1, \psi_2 \rangle = 2\mathrm{Re} \int_{\mathbb{R}^3} \psi_1 \overline{\psi}_2 \ ,$$

and the symplectic form is given by

$$\omega(\psi_1, \psi_2) = \langle E\psi_1, \psi_2 \rangle \ , \qquad E := \mathrm{i}$$

Assume that for $\epsilon = 0$, the NLS admits moving solitary waves solutions, namely solutions of the form

$$\psi(x,t) = e^{-\mathrm{i}(\gamma + \frac{p}{2m} \cdot (x-q))} \eta_m(x-q) \ ,$$

where the parameter p, q, γ, m fulfill

$$p = p_0 \ , \quad q = \frac{p}{m} t + q_0 \ , \quad \gamma = mt + \frac{p^2}{4m} t + \gamma_0 \ , \quad m = m_0$$

while η_m is the ground state, namely it solves the variational problem

$$\inf_{\|\psi\|_{L^2} = m} \int_{\mathbb{R}^3} |\nabla \psi(x)|^2 - \beta(|\psi(x)|^2) \, dx \ .$$

We denote by $\mathcal{T}$ the *soliton manifold*

$$\mathcal{T} := \bigcup_{p,q,m,\gamma} e^{-\mathrm{i}(\gamma + \frac{p}{2m} \cdot (x-q))} \eta_m(x-q)$$

which is invariant 8-dimensional manifold. Moreover the coordinates p, q, m, γ are symplectic on it.

Then the following theorem holds

Theorem 1.1 (Bambusi, M.)**.** *There exists a sufficiently small neighbourhood* $\mathcal{U}$ *of the soliton manifold and* canonical coordinates (p, q, ϕ) *such that the Hamiltonian is these coordinates is given by*

$$H(p, q, \phi) = \epsilon^{1/2} H_{mech}(p, q) + H_{L_0}(\phi) + H_P(p, q, \phi) \ ,$$

where

$$H_{mech}(p, q) := \frac{|p|^2}{2m} + V^{eff}(q) \ , \quad V^{eff}(q) = V * \eta_m^2$$

$$H_{L_0}(\phi) = \frac{1}{2} \langle E L_0 \phi, \phi \rangle$$

where L_0 *is the operator which, when expressed as acting on the real and imaginary part of* $\phi = u + \mathrm{i}v$*, has the form*

$$L_0 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \ .$$

The operator L_0 has continuous spectrum $\sigma_c(L_0) = \bigcup_{\pm} \pm \mathrm{i}[\mathcal{E}, \infty)$ and discrete spectrum $\sigma_d(L_0) = \{\pm \mathrm{i}\omega_j\}_{j=1}^n$. Once again one can reduce to the structure before.

2 The model hamiltonian

In each of the three cases above one reduce to the study of the following model hamiltonian system

$$H(\xi, \phi) = \omega|\xi|^2 + \left(L_0\phi_c, \overline{\phi}_c\right) + H_P(\xi, \phi)$$

The phase space is $\mathbb{C} \times H^1(\mathbb{R}^3)$, and the hamiltonian vector field takes the form

$$\dot{\xi} = -\mathrm{i}\frac{\partial H}{\partial \overline{\xi}} \ , \quad \dot{\phi}_c = -\mathrm{i}\nabla_{\overline{\phi}_c} H \ ,$$

the poisson bracket are

$$\{H, K\} := \mathrm{i}\left(\frac{\partial H}{\partial \xi}\ \frac{\partial K}{\partial \overline{\xi}} - \frac{\partial H}{\partial \overline{\xi}}\frac{\partial K}{\partial \xi}\right) + \mathrm{i}(\nabla_\phi H, \nabla_{\overline{\phi}} K) - \mathrm{i}(\nabla_{\overline{\phi}} H, \nabla_\phi K)$$

This is the hamiltonian of a discrete oscillator coupled with a continuous field. The following are typical assumptions on the spectrum of the operator L_0:

(H1) $\sigma_c(L_0) = [\mathcal{E}, \infty)$

(H2) $\mathcal{E}$ is neither an eigenvalue nor a resonance for L_0, i.e. there are no nonzero solution of $L_0\psi = \mathcal{E}\psi$ with $|\psi(x)| \sim \langle x\rangle^{-1}$.

(H3) there is no $n \in \mathbb{Z}$ such that $\omega n = \mathcal{E}$.

(H4) the linear propagator $e^{-\mathrm{i}L_0 t}$ fulfills Strichartz estimates

Clearly, Strichartz estimates change according to the operator L_0. Here I will assume that the following Strichartz estimates hold:

Theorem 2.1. *Fix $d \geq 1$, and call a pair (r, s)* admissable *if $2 \leq r, s \leq \infty$ and*

$$\frac{2}{r} + \frac{d}{s} = \frac{d}{2} \ , \quad (r, s, d) \neq (2, \infty, 2) \ .$$

Then for every admissable exponents (r, s), $(\widetilde{r}, \widetilde{s})$ we have the homogeneous Strichartz estimate

$$\|e^{-\mathrm{i}L_0 t}\phi\|_{L^r_t W^{1,s}_x} \leq \|\phi\|_{H^1_x} \ , \tag{2.1}$$

the dual homogeneous Strichartz estimate

$$\|\int_{\mathbb{R}} e^{-\mathrm{i}L_0\tau}F(\tau)\ d\tau\|_{H^1_x} \leq \|F\|_{L^{\widetilde{r}'}_t W^{1,\widetilde{s}'}_x} \ , \tag{2.2}$$

and the inhomogeneous Strichartz estimate

$$\|\int_{\tau<t} e^{-\mathrm{i}L_0(t-\tau)}F(\tau)\ d\tau\|_{L^r_t W^{1,s}_x} \leq \|F\|_{L^{\widetilde{r}'}_t W^{1,\widetilde{s}'}_x} \ , \tag{2.3}$$

where $(\widetilde{r}', \widetilde{s}')$ is the dual conjugate of $(\widetilde{r}, \widetilde{s})$.

Remark 2.1. *We will always use the inhomogeneous Strichartz estimate in the localized version*

$$\| \int_0^t e^{-\mathrm{i}L_0(t-\tau)} F(\tau)\ d\tau \|_{L_t^r[0,t]W_x^{1,s}} \leq \|F\|_{L_t^{\tilde{r}'}[0,t]W_x^{1,\tilde{s}'}} \ ,$$

Remark 2.2. *In this exposition we will use the case $d = 3$. Furthermore we want to use $r = 2$. The reason is that the dual Strichartz esponent is $r' = 2$, thus we are able to close the estimate. More precisely, we apply the inhomogeneous Strichartz estimate with admissable exponents* $(2, 6)$ *(and the dual is* $(2, 6/5)$*):*

$$\| \int_0^t e^{-\mathrm{i}L_0(t-\tau)} F(\tau)\ d\tau \|_{L_t^2[0,t]W_x^{1,6}} \leq \|F\|_{L_t^2[0,t]W_x^{1,6/5}}$$

Concerning the nonlinearity we assume that it is smooth as a function on $\mathbb{C} \times H^1(\mathbb{R}^3)$.

We want to study the dynamics of such system. We consider first the case $H_P = 0$, the continuum field and the particle are decoupled and the dynamics is given simply by

$$\xi(t) = e^{-\mathrm{i}t\omega}\xi_0, \quad \phi(t) = e^{-\mathrm{i}L_0 t}\phi_0 \ ,$$

where $\xi_0 \in \mathbb{C}$ and $\phi_0 \in H^1$ are the initial data. Here we see that the particle oscillates with periodic motions of frequency ω, while the continuum field disperse to zero due to Strichartz estimates. Thus the unperturbed motion carries periodic motions for the particles. This means that the linear problem, when $H_P = 0$, has periodic motions.

It is a very natural question what happens to such periodic motions when we turn the perturbation: do such periodic motions persist?
In order to do this, one tries to decouple the oscillator and the field to higher order. Here comes a big difference with the finite volume setting. Indeed, in this last setting (typically $\phi \in H^1(\mathbb{T}^d)$), the operator L_0 displays a pure discrete spectrum (eventually with multiplicity) and with eigenvalues which are typically separating more and more at infinity. Then one can impose the condition that the spectrum of the discrete oscillator does not enter in resonance with the spectrum of the operator L_0. Typically one needs to impose the so called first order Melnikov condition, which reads something as follows: there exists $\gamma, \tau > 0$ such that

$$|\omega n \pm \lambda| \geq \frac{\gamma}{\langle n \rangle^\tau} \ , \quad \forall n \in \mathbb{Z} \ , \lambda \in \sigma(L_0) \ .$$

The typical result is that, for not too bad operators L_0, one can find a Cantor set of large measure such that, if ω belongs to such set, than the condition above is fulfilled.

We come back to the case of infinite volume setting. Now the spectrum of L_0 is purely continuous: in this case it is impossible to impose the first Melnikov condition, which indeed it is always violated: we will always find an integer n_0 such that $\omega n_0 > \mathcal{E}$, i.e.

$$\omega n_0 \in \sigma(L_0) \ .$$

Thus a resonance between the discrete mode and the continuous field is created. The question now is: which is the effect of such resonance? Our aim is to show that such resonance creates a coupling between the oscillator and the continuum field which produce a dissipative behaviour on the oscillator, leading to the instability of the periodic unperturbed motion: the periodic orbit is destroyed. The reason behind this behaviour is that a coupling between the oscillator and the field leads to an exchange of energy between the two: when the oscillator transfer energy to the field, the field (being dispersive) transfer energy to infinity. This results in a loss of energy in the oscillator.

For the sake of clarity, we choose the simplest possible nonlinearity, which is linear in the variable ϕ_c

$$H_P = \xi^\mu \bar{\xi}^\nu \int \Phi_{\mu\nu}\, \phi_c + \xi^\nu \bar{\xi}^\mu \int \overline{\Phi}_{\mu\nu}\, \overline{\phi}\ .$$

Here $\Phi_{\mu\nu}$ are Schwartz functions. The total hamiltonian is thus

$$H(\xi,\phi) = \omega|\xi|^2 + (L_0\phi,\phi) + \xi^\mu\bar{\xi}^\nu\,(\Phi_{\mu\nu},\phi) + \xi^\nu\bar{\xi}^\mu\,\left(\overline{\Phi}_{\mu\nu},\overline{\phi}\right)$$

where we defined $(u,v) := \int uv$ the L^2 real scalar product.

We proceed trying to decouple the discrete and the continuous spectrum order by order. In particular we try to delete the terms which are of order $\mu+\nu$ in ξ and order 1 in ϕ. To to this, we look for a generating function of the form

$$\chi(\xi,\phi) = \xi^\mu\bar{\xi}^\nu \int \Psi_{\mu\nu}\,\phi_c + \xi^\nu\bar{\xi}^\mu \int \overline{\Psi}_{\mu\nu}\,\overline{\phi}\ .$$

Then, denoting by $\mathcal{T}$ the time 1-flow of χ, we have that

$$H\circ\mathcal{T} = H_L + \{H_L,\chi\} + H_P + h.o.t.$$

and we want to choose χ is such a way that $\{H_L,\chi\}+H_P$ is as simplest as possible. One computes that

$$\{H_L,\xi^\mu\bar{\xi}^\nu\} = -\mathrm{i}\omega(\mu-\nu)\xi^\mu\bar{\xi}^\nu \tag{2.4}$$

$$\{H_L,\xi^\mu\bar{\xi}^\nu\int\Psi\,\phi\} = -\mathrm{i}\xi^\mu\bar{\xi}^\nu\int\phi\,(L_0-\omega(\nu-\mu))\Psi \tag{2.5}$$

$$\{H_L,\xi^\nu\bar{\xi}^\mu\int\Psi\,\overline{\phi}\} = \mathrm{i}\xi^\nu\bar{\xi}^\mu\int\overline{\phi}\,(L_0-\omega(\nu-\mu))\Psi \tag{2.6}$$

Thus by choosing $\Psi_{\mu\nu}$ such that

$$-\mathrm{i}(L_0-\omega(\nu-\mu))\Psi_{\mu\nu} = \Phi_{\mu\nu}$$

we can eliminate the monomials in H_P. Clearly everything work in case $\omega(\nu-\mu)<\mathcal{E}$, which allows us to set

$$\Psi_{\mu\nu} = \mathrm{i}(L_0-\omega(\nu-\mu))^{-1}\Phi_{\mu\nu}\ .$$

Thus the terms which we cannot delete so easily are the ones in which the discrete spectrum is in resonance with the continuous spectrum, namely one has

$$\omega(\nu-\mu) > \mathcal{E} \ .$$

It is interesting to notice that it is enough to expand H_P in a Taylor series of sufficiently high order to have monomials such that $\omega(\nu-\mu) > \mathcal{E}$. In other words, such resonances are *unavoidable*! This is a consequence of having continuous spectrum. We can (and we will) elimate such terms with the help of the regularized resolvent. Anyhow, we have to proceed at the level of vector field.

Thus it is necessary to understand the effect of the resonant interaction of the continuous spectrum and the discrete spectrum. For the sake of clarity, I will simplify even more our model, and consider a toy model which contains just 1 monomial which is resonant.

2.1 A toy model

More precisely, we will consider the following "Toy Model":

$$H(\xi,\phi) = \omega|\xi|^2 + \left(L_0\phi, \overline{\phi}\right) + (G,\phi) + \left(\overline{G}, \overline{\phi}\right) \tag{2.7}$$

where

$$G = \overline{\xi}^{\nu}\Phi \ , \quad \overline{G} = \xi^{\nu}\overline{\Phi} \ ,$$

and such that

$$\omega\nu > \mathcal{E} \ .$$

These are exaclty two of the monomials which we were not able to delete before. Since we want to study the dynamics close to 0, it is convenient to rescale the variables and define

$$\xi := \varepsilon\xi', \ \ \phi = \varepsilon\phi'$$

and we rescale also the time by $t' = \varepsilon t$. The new Hamiltonian (dropping the prime) and defining $\epsilon := \varepsilon^{\mu+\nu-1}$ reads

$$H(\xi,\phi) = \omega|\xi|^2 + \left(L_0\phi, \overline{\phi}\right) + \epsilon\,(G,\phi) + \epsilon\left(\overline{G}, \overline{\phi}\right) \tag{2.8}$$

Then the equation of motions are

$$\begin{aligned} \mathrm{i}\dot{\xi} &= \omega\xi + \epsilon\left(\frac{\partial G}{\partial \overline{\xi}}, \phi\right) \\ \mathrm{i}\dot{\phi} &= L_0\phi + \epsilon\overline{G} \end{aligned} \tag{2.9}$$

We will prove the following result

Theorem 2.2. *Let $\phi_0 \in H^1(\mathbb{R}^3)$, $\xi_0 \in \mathbb{C}$. Assume that the*

$$\omega\nu > \mathcal{E} \tag{2.10}$$

(that is, $\omega\nu \in \sigma(L_0)$). Furthermore, assume that the Nonlinear Fermi Golden Rule (NFGR) holds:

$$\big(\Phi, \delta(L_0 - \omega\nu)\overline{\Phi}\big) > 0 \tag{2.11}$$

Consider the dynamics given by the Hamiltonian above. Then one has that

$$\lim_{t\to\infty} |\xi(t)| = 0$$

and furthermore there exists $\widetilde{\phi} \in H^1(\mathbb{R}^3)$ such that

$$\|\phi(t) - e^{-\mathrm{i}\Delta t}\widetilde{\phi}\|_{H^1} \to 0 \ , \quad t \to \infty \ .$$

One can also say that 0 is asymptotically stable.

It is interesting to contrast our results with those known for Hamiltonian partial differential equations for a function $u(x,t)$, where x varies over a compact spatial domain, e.g. periodic or Dirichelt boundary conditions. For nonlinear wave equations with periodic boundary conditions in x, KAM type results have been proved; invariant tori, associated with a nonresonance condition persist under small perturbations. The nonresonance hypotheses of such results fail in the current context, a consequence of the continuous spectrum associated with unbounded spatial domains. In our situation, non-vanishing resonant coupling (condition (2.11)) provides the mechanism for the radiative decay and therefore nonpersistence of localized periodic solutions.

Remark 2.3. *In the more general situation in which the nonlinearity is more complicated, or there are several discrete modes, one performs Birkhoff normal form in order to reduce to the toy model.*

3 Proof

We prove the theorem by a bootstrap assumption. More precisely, suppose that the following estimates hold [1]

$$\|\phi\|_{L^r_t[0,T]W^{1,s}_x} \le C_1 \ , \qquad \text{for all admissible pairs } (r,s) \tag{3.1}$$

$$\epsilon\|\xi^\nu\|_{L^2_t[0,T]} \le C_2 \ , \qquad \text{for all integers } \nu \text{ such that } \omega\nu > \mathcal{E} \tag{3.2}$$

for fixed constant $C_1, C_2 > 0$. Then we will prove that (3.1) and (3.2) imply the same estimate but with C_1, C_2 replaced by $C_1/2, C_2/2$. Then (3.1) and (3.2) hold with $[0,T)$ replaced by $[0,\infty)$.

We begin with the following remark. By conservation of energy, it follows that

$$\omega|\xi|^2 + \|\phi\|^2_{H^1} \le H(\xi_0, \phi_0) + \epsilon|\xi|^\nu\|\phi\|_{H^1}$$

[1] the scaling of the ξ variable is given by the fact that the original $\xi \sim \varepsilon$, hence we expect in the scaled variable $\varepsilon^\nu\|\xi^\mu\|_{L^r_t[0,T]} \le C_2\varepsilon$, which givev the condition (3.2)

which implies the L^∞ controll

$$\sup_{t\in\mathbb{R}} (|\xi(t)| + \|\phi(t)\|_{H^1}) \le C_3 \ .$$

We start by studying the equation for ϕ, which is given by the first of (2.9).

Lemma 3.1. *Under the assumption* (3.1) *and* (3.2)*, one has that*

$$\|\phi\|_{L^r_t[0,T]W^{1,s}_x} \le \frac{C_1}{2} \ , \qquad \textit{for all admissible pairs } (r,s)$$

Proof. By Duhamel one has

$$\phi(t) = e^{-\mathrm{i}L_0 t}\phi_0 + \epsilon \int_0^t e^{-\mathrm{i}L_0(t-\tau)} \left(\overline{\xi}^\nu(\tau)\Phi\right) d\tau \ .$$

Using Strichartz, the assumptions (3.1) and (3.2), and assumption (2.10), one gets immediately

$$\begin{aligned}\|\phi\|_{L^r_t[0,T]W^{1,s}_x} \le & \|\phi_0\|_{H^1} + 2\epsilon \|\xi^\nu \Phi\|_{L^2_t[0,T]W^{1,6/5}_x} \\ \le & \|\phi_0\|_{H^1} + 2C\,\epsilon \|\xi^\nu\|_{L^2_t[0,T]} \\ \le & \|\phi_0\|_{H^1} + 2CC_2\end{aligned}$$

Thus the thesis follows provided that $C_1 > 2\|\phi_0\|_{H^1}$ and $C_1 > 4CC_2$. □

We turn to the equation for ξ. To study such equation, we need to decouple (2.9) by defining the new variable

$$g := \phi + \epsilon \overline{Y}$$

where

$$\overline{Y} = \xi^\nu \overline{\Psi} \ ,$$

with a function $\overline{\Psi}$ to be determined. The equation for $\dot{g}$ is given by

$$\begin{aligned}\dot{g} = & -\mathrm{i}L_0 g \\ & + \epsilon\mathrm{i}\left[L_0\overline{Y} - \omega\xi\frac{\partial\overline{Y}}{\partial\xi} - \overline{G}\right] \\ & - \mathrm{i}\epsilon^2\,\frac{\partial\overline{Y}}{\partial\xi}\left(\frac{\partial G}{\partial\overline{\xi}}, \phi\right) \ .\end{aligned}$$

We want to kill the term of order ϵ. Thus we impose that the second line above equal zero. Explicitly, this means that

$$(L_0 - \omega\nu)\,\overline{Y} = \overline{G} \ ,$$

and thus we need to solve the equations

$$(L_0 - \omega\nu)\overline{\Psi} = \overline{\Phi} \ .$$

By our assumption $\nu\omega > \mathcal{E}$, this is exacly the resonance between the discrete and the continuous spectrum. We will analyze such resonance. In order to invert the operator above, we have to regularize the resolvent. For any $\lambda \in (\mathcal{E}, \infty)$ one defines the regularized resolvent as

$$R_{L_0}^{\pm}(\lambda) := \lim_{\varepsilon \to 0^+} (L_0 - \lambda \mp \mathrm{i}\varepsilon)^{-1}$$

The following result is well known and goes under the name of Limiting absorption principle (or Agmon theorem):

Theorem 3.1 (Limiting absorption principle)**.** *Let f be a test function on $\mathbb{R}^3$, let $\lambda > \mathcal{E}$, and let $\sigma > 1/2$. Then one has*

$$\|R_{L_0}^{\pm}(\lambda) f\|_{L^{2,-\sigma}(\mathbb{R}^3)} \leq C_\sigma \frac{1}{\lambda^{1/2}} \|f\|_{L^{2,\sigma}(\mathbb{R}^3)}$$

where $C_\sigma > 0$ depends only on σ, and $L^{2,s}(\mathbb{R}^3)$ is the weighted norm

$$\|f\|_{L^{2,s}(\mathbb{R}^3)} := \|\langle x\rangle^s f\|_{L^2_x(\mathbb{R}^3)} \ .$$

For a proof of such result, see Appendix A. In the applications, one has to prove such estimate for the operator L_0.

In particular, the limiting absorption principle states that for $\lambda \in (\mathcal{E}, \infty)$, the operator $R_{L_0}^{\pm}(\lambda) \in B(L^{2,\sigma}, L^{2,-\sigma})$ for $\sigma > 1/2$. Now we can solve the equations above and define

$$\overline{\Psi} := R_{L_0}^{+}(\nu\omega)\overline{\Phi} \ , \qquad \Psi := \overline{R_{L_0}^{+}(\nu\omega)\Phi} = R_{L_0}^{-}(\nu\omega)\overline{\Phi} \ .$$

By the limiting absorption principle and the fact that $\Phi \in L^{2,\sigma}$ for $\sigma > 1/2$, one has

Lemma 3.2. *We have $Y \in L^{2,-\sigma}$ for all $\sigma > 1/2$.*

It is clear now that also $g \in L^{s,-\sigma}$. We want to prove that we can control its norm.

In order to estimate g, we need the following Strichartz estimates:

Lemma 3.3. *For any Φ in the Schwartz class, any complex valued function $h \in L^2_t$, any $\lambda \in (\mathcal{E}, \infty)$ one has*

$$\|\langle x\rangle^{-\sigma} \mathcal{U}(t,0) R_{L_0}^{\pm}(\lambda)\Phi\|_{L^2_x} \preceq \frac{1}{\langle t\rangle^{3/2}} \|\langle x\rangle^{\sigma}\Phi\|_{L^2_x} \ , \tag{3.3}$$

$$\|\int_0^t \mathcal{U}(t,s)\, h(s)\, R_{L_0}^{\pm}(\lambda)\Phi\, ds\|_{L^2_t L^{2,-\sigma}_x} \preceq \|h\|_{L^2_t} \|\Phi\|_{L^{2,\sigma}_x} \ . \tag{3.4}$$

We are now ready to study the dynamics of g.

Lemma 3.4. *Under the assumption* (3.1) *and* (3.2)*, one has that for any $a >$* 1/2

$$\|g\|_{L^2_t[0,T] L^{2,-a}} \leq C(\|\phi_0\|_{L^2}, |\xi_0|) + \epsilon^2 C_2 C_3$$

Proof. Once again using Duhamel one has

$$g(t) = e^{-\mathrm{i}L_0 t} g_0 - \mathrm{i}\epsilon^2 \int_0^T e^{-\mathrm{i}L_0(t-\tau)} \left[\frac{\partial \overline{Y}}{\partial \xi}(\tau) \left(\frac{\partial G}{\partial \bar{\xi}}(\tau), \phi(\tau) \right) \right] d\tau \ .$$

Then by (3.3), using that $g_0 = \phi_0 + \overline{Y}(0)$, it follows that

$$\begin{aligned} \|e^{-\mathrm{i}tL_0} g_0\|_{L^2_t[0,T]L^{2,-a}_x} \leq & \|e^{-\mathrm{i}tL_0}\phi_0\|_{L^2_t[0,T]L^{2,-a}_x} + 2\|e^{-\mathrm{i}tL_0} Y(0)\|_{L^2_t[0,T]L^{2,-a}_x} \\ \leq & \|e^{-\mathrm{i}tL_0}\phi_0\|_{L^2_t[0,T]L^6_x} + 2\|e^{-\mathrm{i}tL_0}\xi_0^\nu R^{\pm}_{L_0}(\omega\nu)\Phi\|_{L^2_t[0,T]L^{2,-a}_x} \\ \leq & \|\phi_0\|_{L^2} + \|\Phi\|_{L^{2,a}_x}|\xi_0^\nu| \preceq C(\|\phi_0\|_{L^2}, |\xi_0|) \end{aligned}$$

We consider the integral term, and more precisely we estimate

$$\| \int_0^T e^{-\mathrm{i}L_0(T-\tau)} \left[\frac{\partial \overline{Y}}{\partial \xi}(\tau) \left(\frac{\partial G}{\partial \bar{\xi}}(\tau), \phi(\tau) \right) \right] d\tau \|_{L^2_t[0,T]L^{2,-a}_x} \tag{3.5}$$

Using estimate (3.4) we get

$$\begin{aligned} \|(3.5)\|_{L^2_t[0,T]L^{2,-a}_x} \preceq & \|\xi^{\nu-1}(\tau) \left(\frac{\partial G}{\partial \bar{\xi}}(\tau), \phi(\tau) \right) \|_{L^2_t[0,T]} \ \|\Phi\|_{L^{2,a}_x} \\ \preceq & \| |\xi|^{2(\nu-1)} \ \|\Phi\|_{L^{6/5}_x} \ \|\phi\|_{L^6_x} \|_{L^2_t[0,T]} \\ \preceq & \sup_{t\in[0,T]} \|\xi(t)\|^{2(\nu-1)} \ \|\phi\|_{L^2_t[0,T]L^6_x} \preceq C_3^{2(\nu-1)} \frac{C_1}{2} \end{aligned}$$

□

Finally we are able to estimate ξ. In order to do this, first we substitute $\phi = g - \epsilon\overline{Y}$ into the first of (2.9), getting

$$\mathrm{i}\dot{\xi} = \omega\xi + \epsilon \left(\frac{\partial G}{\partial \bar{\xi}}, g \right) - \epsilon^2 \left(\frac{\partial G}{\partial \bar{\xi}}, \overline{Y} \right)$$

We want to show now that the last term above produces a dissipative behaviour in the equation. To understand why, we compute the last term explicitly. One has that

$$\left(\frac{\partial G}{\partial \bar{\xi}}, \overline{Y} \right) = \nu \frac{\xi^\nu \bar{\xi}^\nu}{\bar{\xi}} c$$

where we defined the complex coefficient

$$c := (\Phi, \overline{\Psi}) = (\Phi, R^-_{L_0}(\omega\nu)\overline{\Phi})$$

Thus the equation for $\dot{\xi}$ is

$$\mathrm{i}\dot{\xi} = \omega\xi + \epsilon \left(\frac{\partial G}{\partial \bar{\xi}}, g \right) - \epsilon^2 \nu \frac{\xi^\nu \bar{\xi}^\nu}{\bar{\xi}} c$$

which we rewrite as

$$\dot{\xi} = \Xi(\xi, \overline{\xi}) + \mathcal{R}(t)$$

where

$$\begin{aligned}\Xi(\xi, \overline{\xi}) &= -\mathrm{i}\omega\xi + \epsilon^2 \mathcal{N}(\xi)\\ \mathcal{N}(\xi) &= \mathrm{i}\nu \frac{\xi^\nu \overline{\xi}^\nu}{\overline{\xi}} c\\ \mathcal{R}(t) &= -\mathrm{i}\epsilon \left(\frac{\partial G}{\partial \overline{\xi}}, g \right)\end{aligned}$$

We are ready to analyze the dynamics of the function

$$H_d(\xi) := \omega |\xi|^2$$

In particular we consider its time derivative which is given by

$$\begin{aligned}\frac{d}{dt} H_d(\xi) &= 2Re(\omega \overline{\xi} \dot{\xi})\\ &= 2Re(\omega \overline{\xi} \Xi) + 2Re(\omega \overline{\xi} \mathcal{E}) = 2Re(\omega \overline{\xi} \mathcal{N}) + 2Re(\omega \overline{\xi} \mathcal{R})\\ &= -\epsilon^2 \omega \nu |\xi|^{2\nu} 2Im(c) + 2Re(\omega \overline{\xi} \mathcal{R})\end{aligned}$$

Now we analyze the term $Im(c)$, which is the source of dissipation. We have that

$$\left(\Phi, R^-_{L_0}(\omega\nu)\overline{\Phi}\right)$$

Lemma 3.5. *We have that* $Im\left(\left(\Phi, R^-_{L_0}(\omega\nu)\overline{\Phi}\right)\right) \geq 0$.

Proof. Let $W^+ := \lim_{t\to\infty} e^{\mathrm{i}tH} e^{-\mathrm{i}tH_0}$ be the wave operator. It is well known that W^+ exists, it is invertible, and it is an isometry w.r.t. the scalar product $\langle u, v\rangle := \int u\overline{v}$. Furthermore, one has the *interwiening relation*

$$f(L_0)W^+ = W^+ f(-\Delta + \mathcal{E})$$

for any analytic function f. for a proof of these properties see Appendix B. We write

$$\left(\Phi, \overline{R^+_{L_0}(\nu\omega)\Phi}\right) = \langle \Phi, R^+_{L_0}(\nu\omega)\Phi\rangle = \lim_{\epsilon\to 0^+} \langle \Phi, (L_0 - \nu\omega - \mathrm{i}\epsilon)^{-1}\Phi\rangle$$

Define F s.t. $\Phi = W^+ F$. Then

$$\begin{aligned}\langle \Phi, (L_0 - \nu\omega - \mathrm{i}\epsilon)^{-1}\Phi\rangle &= \langle W^+F, (L_0 - \nu\omega - \mathrm{i}\epsilon)^{-1} W^+ F\rangle\\ &= \langle W^+F, W^+(-\Delta + \mathcal{E} - \nu\omega - \mathrm{i}\epsilon)^{-1} F\rangle\\ &= \langle F, (-\Delta + \mathcal{E} - \nu\omega - \mathrm{i}\epsilon)^{-1} F\rangle\end{aligned}$$

It follows that

$$\begin{aligned}\lim_{\epsilon\to 0^+} Im\langle \Phi, (L_0 - \nu\omega - \mathrm{i}\epsilon)^{-1}\Phi\rangle &= \lim_{\epsilon\to 0^+} Im\langle F, (-\Delta + \mathcal{E} - \nu\omega - \mathrm{i}\epsilon)^{-1} F\rangle\\ &= \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^3} \frac{\epsilon}{(|\xi|^2 - (\omega\nu - \mathcal{E}))^2 + \epsilon^2} |\widehat{F}(\xi)|^2 \, d\xi \geq 0\end{aligned}$$

Let us elaborate more on the last condition: passing to spherical coordinates

$$\int_0^\infty \frac{\epsilon\ \rho^2}{(\rho^2-(\omega\nu-\mathcal{E}))^2+\epsilon^2} f(\rho)\ d\rho\ , \qquad f(\rho)=\int_0^{2\pi}\int_0^{\pi} |\widehat{F}(\rho,\theta,\phi)|^2\ \sin\theta\ d\theta d\phi d\rho$$

and with the substitution $\rho^2-\lambda=r$ we get

$$\frac{1}{2}\int_{-(\omega\nu-\mathcal{E})}^{\infty} \frac{\epsilon}{r^2+\epsilon^2}\widetilde{f}\left(\sqrt{r+(\omega\nu-\mathcal{E})}\right)\ dr\ , \qquad \widetilde{f}(x)=xf(x)$$

At this point it is clear that

$$\frac{1}{2}\int_{-(\omega\nu-\mathcal{E})}^{\infty} \frac{\epsilon}{r^2+\epsilon^2}\widetilde{f}\left(\sqrt{r+(\omega\nu-\mathcal{E})}\right)\ dr \to \pi\widetilde{f}\left(\sqrt{\omega\nu-\mathcal{E}}\right)\ , \quad \epsilon\to 0\ .$$

Namely we get

$$\langle\Phi,(L_0-\nu\omega-\mathrm{i}\epsilon)^{-1}\Phi\rangle \to \pi\sqrt{\omega\nu-\mathcal{E}}\int_{|\xi|^2=\omega\nu-\mathcal{E}} |\widehat{F}(\xi)|^2\ d\xi$$

□

Remark 3.1. *One can also invoke Plemelji formula from funcional calculus:*

$$R^-_{L_0}(\lambda)=PV\,(L_0-\lambda)^{-1}+\mathrm{i}\pi\delta\,(L_0-\lambda)$$

Now we are ready to ask the *Nonlinear Fermi Golden Rule* , which essentially tells that this quantity is strictly positive:

$$\Gamma\equiv(\Phi,\delta\,(L_0-\lambda)\,\overline{\Phi})>0\ . \qquad \text{(NFGR)}$$

`aggiungere commenti su quanto questa condizione e generica`

We are ready to prove the bootstrap assumption: integrating the derivative of the energy we get

$$H_d(\xi(t))-H_d(\xi(0))=-\epsilon^2\omega\nu\Gamma\int_0^t|\xi|^{2\nu}+2\int_0^t Re(\omega\bar{\xi}\mathcal{R})$$

and using that $H_d(\xi(t))$ is positive, $\Gamma>0$, we obtain that

$$\begin{aligned}\epsilon^2\omega\nu\Gamma\int_0^t|\xi(\tau)|^{2\nu}\ d\tau &\le H_d(\xi(0))+2\epsilon\nu\int_0^t|\xi(\tau)|^{\nu}\|g(\tau)\|_{L^{2,-\sigma}_x}\ d\tau\\ &\le C(|\xi_0|)+(C(\|\phi_0\|,|\xi_0|)+\epsilon^2C_3^{2(\nu-1)}C_1)\epsilon\left(\int_0^t|\xi(\tau)|^{2\nu}\ d\tau\right)^{1/2}\end{aligned}$$

from which one deduces the inequality

$$\epsilon\|\xi^\nu\|_{L^2_t[0,T]}\le(C(\|\phi_0\|,|\xi_0|)+\epsilon^2C_3C_1)$$

By choosing C_2 large enough, it follows that

$$\epsilon \|\xi^\nu\|_{L^2_t[0,T]} \leq \frac{C_2}{2} \ ,$$

thus proving the bootstrap assumption.

It is also possible to estimate the time-decay of ξ. Indeed $y := |\xi|^2$ fulfills the approximate equation

$$\frac{d}{dt} y = -c y^{nu}$$

which decays at $t \to \infty$ as

$$y(t) \sim \frac{1}{t^{\frac{1}{\nu - 1}}}$$

A Limiting absorption principle

We sketch the proof of the limiting absorption principle. We prove first the case of the free resolvent:

Case $L_0 = -\Delta$. Let $\lambda > 0$ and $f, g \in C_c^\infty(\mathbb{R}^n)$. By Parseval formula

$$\langle R_0(\lambda) f, g \rangle = \int \frac{\widehat{f}(\xi)\, \overline{\widehat{g}(\xi)}}{|\xi|^2 - \lambda} \, d\xi$$

Then one verifies that

$$\lim_{\substack{\lambda \to k \in \mathbb{R}^+ \\ \pm Im \lambda > 0}} \langle R_0(\lambda) f, g \rangle = \pm \frac{\pi \mathrm{i}}{2\sqrt{k}} \int_{|\xi| = \sqrt{k}} \widehat{f}(\xi)\, \overline{\widehat{g}(\xi)} \, dS^{n-1} + \text{p.v.} \int \frac{\widehat{f}(\xi)\, \overline{\widehat{g}(\xi)}}{|\xi|^2 - k} \, d\xi \ .$$

Now we need two bound the two integrals above. It is clear at this point that in order to make sense of the integrals above, we need $\widehat{f}$ and $\widehat{g}$ to have some regularity. To bound the first integral, we use the trace lemma which refers to the statement that for every $f \in L^{2, \frac{1}{2}+}$, there is a restriction of $\widehat{f}$ to any (compact) hypersurface, and this restriction belongs to L^2 relative to surface measure:

Lemma A.1 (Trace Lemma). *Let Γ be a C^∞ compact $n-1$ dimensional manifold imbedded in R^n. Let $d\zeta$ be the measure induced on Γ by the Lebesgue measure dx, and denote by $L^2(\Gamma)$ the class of L^2 functions on Γ with respect to the measure $d\zeta$. For any given $m > 1/2$ the restriction map $\Pi : H^m(\mathbb{R}^n) \to L^2(\Gamma)$, $u \mapsto u|_\Gamma$ is bounded.*

Such lemma implies immediately that for $\widehat{f}, \widehat{g}$ in $H^{\frac{1}{2}+}$, then

$$\left| \int_{|\xi| = \sqrt{k}} \widehat{f}(\xi)\, \overline{\widehat{g}(\xi)} \, dS^{n-1} \right| \leq \|\widehat{f}|_{S^{n-1}}\|_{L^2(S^{n-1})} \, \|\widehat{g}|_{S^{n-1}}\|_{L^2(S^{n-1})} \preceq \|\widehat{f}\|_{H^{\frac{1}{2}+}} \, \|\widehat{g}\|_{H^{\frac{1}{2}+}}$$

and it is enough to remark that $\|\widehat{f}\|_{H^{\frac{1}{2}+}} \equiv \|f\|_{L^{2,\frac{1}{2}+}}$.

Now we bound the second integral. Clearly is enough localize around the sphere $|\xi|^2 = k$, the rest being obvious. Thus let $\chi(\xi)$ a radial cut off function which equals 1 in $||\xi|^2 - k| > 1$. then it is enough to bound

$$\int \chi(\xi) \frac{\widehat{f}(\xi)\,\overline{\widehat{g}(\xi)}}{|\xi|^2 - k}\, d\xi = \int \frac{\chi(\xi)}{(|\xi| + \sqrt{k})(|\xi| - \sqrt{k})} \widehat{f}(\xi)\,\overline{\widehat{g}(\xi)}\, d\xi \tag{A.1}$$

Using the Sobolev embedding $H^{\frac{1}{2}+} \hookrightarrow L^{\frac{2n}{n-1}+}$ and Hölder inequality, we get

$$\big|(\mathrm{A.1})\big| \preceq \frac{1}{\sqrt{k}} \|\frac{\chi(\xi)}{|\xi| - \sqrt{k}}\|_{L^{n-}} \|\widehat{f}\|_{L^{\frac{2n}{n-1}+}} \|\widehat{g}\|_{L^{\frac{2n}{n-1}+}} \preceq \|\widehat{f}\|_{H^{\frac{1}{2}+}} \|\widehat{g}\|_{H^{\frac{1}{2}+}}$$

It follows that $R_0(\lambda \pm \mathrm{i}0)$ exists as a bounded map from $L^{2,\sigma}$ to $L^{2,-\sigma}$ for any $\sigma > 0$ (the norm exploding when $\sigma \to 1/2$.

Remark A.1. *An easy modification of this argument show that $R_0^{\pm}(\lambda)$ is bounded from $L^{2,\sigma}$ to $\mathcal{H}^{2,-\sigma}$. More precisely, one has that $\|\Delta R_0^{\pm}(\lambda) f\|_{L^{2,-\sigma}} \leq \|f\|_{L^{2,\sigma}}$.*

Case $L_0 = -\Delta + V$. Now consider the case with V a Schwartz potential. We need a preliminary definition and result due to Agmon.

Definition A.1. *A function $u \in H^2_{\mathrm{loc}}(\mathbb{R}^n)$ will be said to be k-*outgoing *if there exists $k > 0$, $v \in L^{2,\sigma}$, $\sigma > 1/2$, such that*

$$u = R^+_{-\Delta}(k^2) f$$

Then the following theorem holds

Theorem A.1 (Agmon theorem on decay of eigenfunction)**.** *Let $u \in H^2_{\mathrm{loc}}(\mathbb{R}^n)$ be k-*outgoing *and such that it fulfills*

$$-\Delta u + V u = k^2 u$$

where V is a real potential of class Schwartz. Then $u \in \mathcal{H}^{2,\sigma}$ for every $\sigma > 0$. In particular $u \in D(-\Delta + V)$ and it is an eigenvector of $-\Delta + V$.

One transfer the free resolvent estimate to $R^{\pm}_{L_0}(\lambda)$ by means of the resolvent estimate

$$R^{\pm}_{L_0}(\lambda) = R^{\pm}_{-\Delta}(\lambda) - R^{\pm}_{-\Delta}(\lambda) V R^{\pm}_{L_0}(\lambda)$$
$$R^{\pm}_{L_0}(\lambda) = \left(\mathbb{1} + R^{\pm}_{-\Delta}(\lambda) V\right)^{-1} R^{\pm}_{-\Delta}(\lambda)$$

Now $S = S(\lambda) := \mathbb{1} + R_{-\Delta}(\lambda) V$ is a perturbation of the identity by the compact operator $R_{-\Delta}(\lambda) V : L^{2,-\sigma} \to L^{2,-\sigma}$ with $\sigma > 1/2$ provided $|V(x)| \leq \frac{1}{(1+|x|)^{1+}}$.

The compactness here follows from the fact that the resolvent gains two derivatives in the weighted L^2 space. Thus by Fredholm alternatives, one has that S^{-1} exists iff $Sf = 0$ implies $f = 0$ for any $f \in L^{2,-\sigma}$. But $Sf = 0$ means that $f = -R^{+}_{-\Delta}(\lambda)(Vf)$, thus f is λ-outgoing. Furthermore $Sf = 0$ is formally equivalent to $(-\Delta + V)f = \lambda f$. Since $\lambda > 0$, it follows by Agmon lemma A.1 that in fact f is an eigenvector of $-\Delta + V$. But positive embedded eigenvalues do not exist by Kato's theorem. Hence $S(\lambda)^{-1} : L^{2,-\sigma} \to L^{2,-\sigma}$ exists for all $\lambda > 0$ provided $\sigma > 1/2$.

Furthermore, $S(\lambda)$ converges to the identity operator as $\lambda \to \infty$ which then implies that $S(\lambda)^{-1}$ is uniformly bounded for all $\lambda > \lambda_0$.

B Wave operator

Consider the following linear Schrödinger equation:

$$\mathrm{i}\psi = (-\Delta + V)\psi \ , \quad \psi_0 = P_c H^1 \tag{B.1}$$

where P_c is the projection on the continuous spectrum of $H := -\Delta + V$. Here we assume that V is a potential which decays together with its derivatives:

$$|\partial_x^\alpha V(x)| \leq C\langle x\rangle^{-\mu-|\alpha|} \tag{B.2}$$

for $|\alpha| \leq 2$ and $\mu > 1$ (short range interaction).

The question that we want to answer is if $\psi(t)$ scatters to a solution of the free Schrödinger equation

$$\mathrm{i}\psi = -\Delta\psi$$

In other words, we ask if there exists $\widetilde{\psi} \in H^1$ such that

$$\|e^{-\mathrm{i}Ht} P_c\psi - e^{-\mathrm{i}H_0 t}\widetilde{\psi}\|_{L^2} \to 0 \ , \quad t \to \infty$$

We can reformulate the asymptotic completeness property by defining the *wave operator*

$$W^{\pm}\phi := \lim_{t\to\pm\infty} e^{\mathrm{i}Ht}\, e^{-\mathrm{i}H_0 t}\phi \ . \tag{B.3}$$

We will show that under the considtion $\mu > 1$, then such limit exists always in L^2. Furthemore, the operator is an isometry, since both $e^{\mathrm{i}Ht}$ and $e^{-\mathrm{i}H_0 t}$ are isometry.

Furthermore, the most important property is the following:

$$HW^{\pm} = W^{\pm}H_0 \tag{B.4}$$

Indeed, by changing variables, we can obtain the following intertwining relations

$$e^{-\mathrm{i}Ht}W^{\pm} = W^{\pm}e^{-\mathrm{i}H_0 t} \ .$$

Differentiating these relations at t = 0, we obtain (B.4).

Let $L_{sc} := RanW^+$ the set of scattering states. We have shown that $L_{sc} \subset P_c L^2$. The property of asymptotic completeness states that

$$L_{sc} = P_c L^2 \qquad \text{or} \qquad P_d L^2 \oplus L_{sc} = L^2 \ ,$$

i.e., that the scattering states and bound states span the entire state space L^2.

Theorem B.1. *Let $V \in L^2$. Then the wave operator exists.*

Proof. Denote $W^t := e^{\mathrm{i}Ht} e^{-\mathrm{i}H_0 t}$. since $\|W^t\| = 1$, it suffices to prove the existence of the limit (B.3) on functions $\phi \in L^2 \cap L^1$ (the existence of the limit for $\phi \in L^2$ will then follow by approximating ϕ by elements of the dense subspace $L^2 \cap L^1$). For $t \geq t'$, we write

$$W^t\phi - W^{t'}\phi = \int_{t'}^{t} \frac{d}{d\tau} W^\tau \phi \; d\tau$$

Using that $\frac{d}{d\tau} e^{\mathrm{i}H\tau} = \mathrm{i}He^{\mathrm{i}H\tau}$ (and similarly for $e^{-\mathrm{i}H_o\tau}$, we get

$$\frac{d}{d\tau} W^\tau = \mathrm{i}He^{\mathrm{i}H\tau}e^{-\mathrm{i}H_0\tau} + e^{\mathrm{i}H\tau}(-\mathrm{i}H_0)e^{-\mathrm{i}H_0\tau} = e^{\mathrm{i}H\tau}\mathrm{i}Ve^{-\mathrm{i}H_0\tau} \ .$$

Thus, using the Minkowski inequality $\| \int f(\tau) \; d\tau\| \leq \int \|f(\tau)\| \; d\tau$ and the unitarity of $e^{\mathrm{i}H\tau}$, it follows that

$$\|W^t\phi - W^{t'}\phi\| \leq \int_{t'}^{t} \|Ve^{-\mathrm{i}H_0\tau}\phi\| \; d\tau \ .$$

But if $V \in L^2$, it follows that

$$\|Ve^{-\mathrm{i}H_0\tau}\phi\| \leq \|V\| \|e^{-\mathrm{i}H_0\tau}\phi\|_{L^\infty} \leq \|V\| \; \frac{1}{\langle\tau\rangle^{3/2}} \|\phi\|_{L^1}$$

where we used the time-decaying estimate for the free hamiltonian H_0. thus we obtain that

$$\int_{t'}^{t} \|Ve^{-\mathrm{i}H_0\tau}\phi\| \; d\tau \leq \|V\| \|\phi\|_{L^1} \int_{t'}^{t} \frac{1}{\langle\tau\rangle^{3/2}} \; d\tau \to 0, \quad t, t' \to \infty \ .$$

In other words, for any sequence $t_j \to \infty$, $\{W^{t_j}\phi\}$ is a Cauchy sequence, and so $\{W^t\phi\}$ converges as $t \to \infty$. Convergence for $t \to -\infty$ is proved in the same way. $\square$

We construct now the inverse of $W^\pm$. Define

$$Z^\pm\phi := \lim_{t\to\pm\infty} e^{\mathrm{i}H_0 t} e^{-\mathrm{i}Ht}\phi \ . \tag{B.5}$$

Then, if the limit above exists, we have that

$$Z^\pm W^\pm = \mathbb{1} \ , \quad W^\pm Z^\pm = \mathbb{1} \ .$$

Indeed define $Z^t = e^{\mathrm{i}H_0 t} e^{-\mathrm{i}Ht}$. Then $Z^t W^t = W^t Z^t = \mathbb{1}$ for every time t. Thus also $\lim_{t\to\infty} Z^t W^t = \lim_{t\to\infty} W^t Z^t = \mathbb{1}$.

We prove now that such limit exists. Assume now that we know the Strichartz estimate

$$\|e^{\mathrm{i}\tau H}\phi\|_{L_t^2 L_x^6} < \infty$$

holds.