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THE NONLINEAR SCHRÖDINGER EQUATION AS A RESONANT NORMAL FORM

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Abstract. Averaging theory is used to study the dynamics of dispersive equations taking the nonlinear Klein Gordon equation on the line as a model problem: For approximatively monochromatic initial data of amplitude ϵ, we show that the corresponding solution consists of two non interacting wave packets, each one being described by a nonlinear Schrödinger equation. Such solutions are also proved to be stable over times of order $1/\epsilon^2$. We think that this approach puts into a new light the problem of obtaining modulations equations for general dispersive equations. The proof of our results requires a new use of normal forms as a tool for constructing approximate solutions.

1. Introduction.

The aim of this paper is to show that the nonlinear Schrödinger equation (NLS) appears as a resonant normal form of some dispersive non linear equations. To be definite we will concentrate on the nonlinear Klein Gordon equation (in the appendix we will briefly discuss also the Fermi Pasta Ulam system), namely

$$u_{tt} - u_{xx} + mu + \sigma u^3 = 0 \ , \quad t \in \Re \ , \quad x \in \Re \tag{1}$$

with m a positive parameter, and σ a real parameter. We recall that multiple time expansions provides the standard way to derive NLS as a modulation equation for small amplitude almost monochromatic solutions of such an equation (see e.g. [1]); indeed NLS appears as an equation that must be satisfied in order to eliminate the so called secular terms that would cause a linear in time growth of the solution. We also recall that rigorous justifications of such a scheme can be found e.g. in [2, 3]. Kalyakin also gave an algorithm that allows to construct higher order approximations to the solution of the original problem.

In the present paper we will discuss the derivation of NLS from the point of view of averaging theory of infinite dimensional Hamiltonian systems. In particular we will show that there exists a canonical transformation that transforms the system into a new one, which, up to a small error, has an invariant manifold on which the dynamics is completely resonant. Then we perform a resonant normal form, obtaining that on this manifold the dynamics is given by two decoupled NLS, the first one controlling the modulation of wave packets traveling to the right, and the second one controlling the modulation of wave packets traveling to the left; the two packets do not interact. We will also give rigorous estimates on the errors (see theorem 4.1). Similar results have been obtained (from a different point of view) in [[4, 5]]. Finally we prove that solutions starting $\mathcal{O}(\epsilon^2)$ close to such approximatively monochromatic solutions remain $\mathcal{O}(\epsilon^2)$ close to them up to times of order $\mathcal{O}(\epsilon^{-2})$, here ϵ is the order of magnitude of the initial datum as well as the parameter controlling the width of the wave packet; as far as we know this result is new.

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We think that our approach has several advantages with respect to the usual one. In particular it makes clear that NLS is just the first term of a resonant normal form that in principle could be computed up to any order. Moreover, our method provides approximate equations. In principle this makes possible to give a complete description of the dynamics, at least for initial data in the domain of validity of the normal form, which is a considerable improvement with respect to the classical method that only allows to approximatively describe some solutions.

We point out that our approach might be relevant for the study of equipartition properties in FPU type systems (see [6]). We plan to study this problem in detail in a future paper.

From the technical point of view we develop a theory of averaging for systems with perturbations that contain the derivatives of the unknown function. As far as we know this is new. The main point is that we use canonical perturbation theory to construct approximate solutions of the original problem, and only later we prove that such approximate solutions are actually close to exact solutions. We now illustrate briefly this point: suppose one is given a partial differential equation of the form

$$\dot{u} = Au + \mu X(u) \ , \tag{2}$$

with A a linear operator generating a unitary group, X a nonlinear operator and μ a small parameter (that in the case of eq. (1) will be ϵ^2). Suppose one is able to find a coordinate transformation $u = \mathcal{T}(v)$ (usually provided by more or less standard averaging methods) with the property that the equation for the transformed variable v takes the form

$$\dot{v} = Av + \mu N(v) + \mu^2 R(v) \ , \tag{3}$$

where N is a suitable normal form, and R a remainder. If R is a smooth operator then it is clear that the solutions of the normal form equation

$$\dot{w} = Aw + \mu N(w) \tag{4}$$

approximate well the solutions of the complete problem (3). However, if R contains derivatives of the unknown function, then it could happen that solutions of the equation (4), and solutions of (3) are very different. In particular, in order to ensure that they are close to each other one should be able to prove some a priori estimate for higher derivatives of the solution of (3) and this is seldom possible.

So we proceed as follows: given a solution $w(t)$ of (4) we construct an approximate solution $\bar{u}(t) := \mathcal{T}(w(t))$ of the original system. One easily sees that the error $r(t) := u(t) - \bar{u}(t)$ with respect to a true solution $u(t)$, fulfills the equation

$$\dot{r} = Ar + \mu \left\{ X(\bar{u} + r) - X(\bar{u}) \right\} + \mu^2 dT(w(t))R(w(t)) \ , \tag{5}$$

the main point is that *the remainder is evaluated on the approximate solution* $w(t)$, so it gives a small contribution if the approximate solution is smooth, a property that is much easier to check than the smoothness of exact solutions. Assume that $R(w(t))$ is bounded, then in order to control the norm of r one has just to apply Gronwall lemma to estimate the solution of (5). This is what we will do, and this gives (for equation (2) a control of order μ of the solution up to times of order $1/\mu$. We point out that, in case one is interested in controlling the solution for a longer time scale then one has to study in more detail the linearization of (5) (see sect. 5 for more details).

Finally we point out that our paper contains one more novelty, namely a method that we use to extend averaging theory to the case where the fast variable is not an angle, but varies in $\Re$ (see theorem 3.3).

The paper is organized as follows: In sect. 4 we reduce the Klein Gordon equation to a form suitable for the application of our methods, in sect. 3 we state our normal form results showing the appearance of NLS as a resonant normal form; in sect. 4 we follow the procedure illustrated above to use the normal form in order to approximate the solutions of the original system. In sect. 5 we discuss some possible extensions of our approach. In sect. 6 we give the proof of the normal form results, and in sect. 7 we add some useful technical lemmas. In the appendix we show how to apply our approach to the FPU system.

2. **Preparation.** In this section we reduce the nonlinear Klein Gordon equation (1) to a form suitable for the application of our method, such a form is given by (14). As it will be clear the procedure of the present section applies directly also to the more general equation

$$u_{tt} + \mathcal{L}u + \sigma u^3 = 0 \ ,$$

where $\mathcal{L}$ is a positive polynomial in $i\partial_x$. In the appendix we will show that it applies also to the FPU system.

Consider now eq. (1); introduce new independent variables $x_0 = x$, $x_1 = \epsilon x$ and consider a new unknown function $u(x_0, x_1, t)$, fulfilling the equation

$$u_{tt} - \left(\frac{\partial}{\partial x_0} + \epsilon\frac{\partial}{\partial x_1}\right)^2 u + mu + \epsilon^2\sigma u^3 = 0 \ , \quad x_1 \in \Re \ , \quad x_0 \in \Re \ , \quad t \in \Re \ , \tag{6}$$

so that $\epsilon u(x, \epsilon x, t)$ fulfills equation (1); we will impose u to be periodic in x_0 with some period that, in order to be definite, we fix equal to 2π, namely we write

$$u(x_0 - \pi, x_1, t) = u(x_0 + \pi, x_1, t) \ , \quad u_{x_0}(x_0 - \pi, x_1, t) = u_{x_0}(x_0 + \pi, x_1, t) \ , \tag{7}$$

for this reason we will consider only $x_0 \in [-\pi, \pi]$.

REMARK 1. *System (6) is Hamiltonian with Hamiltonian function given by*

$$H = H_L + \epsilon^2 f \tag{8}$$

where

$$\begin{aligned} H_L(p,u) &= \int_{-\pi}^{\pi} dx_0 \int_{\Re} dx_1 \left\{ \frac{p^2}{2} + \frac{1}{2}\left[\left(\frac{\partial}{\partial x_0} + \epsilon\frac{\partial}{\partial x_1}\right) u\right]^2 + \frac{m}{2}u^2 \right\} \ , \\ f(u) &= \sigma \int_{-\pi}^{\pi} dx_0 \int_{\Re} dx_1 \frac{u^4}{4} \end{aligned}$$

$p = u_t$ *is the momentum conjugated to* u*; the symplectic form is given by*

$$\Omega\left((p^1,u^1),(p^2,u^2)\right) := \int_{\Re} dx_1 \int_{-\pi}^{\pi} dx_0 \left(p^1(x_0,x_1)u^2(x_0,x_1) - p^2(x_0,x_1)u^1(x_0,x_1)\right) \ .$$

It is useful to expand both p and u in Fourier series with respect to x_0 and to Fourier transform with respect to x_1:

$$\begin{aligned} u(x_0,x_1) &= \tfrac{1}{2\pi}\textstyle\sum_{k_0\in\mathbf{Z}} \int_{\Re} \hat{q}_{k_0,k_1} e^{ik_0x_0 + ik_1x_1} dx_1 \\ p(x_0,x_1) &= \tfrac{1}{2\pi}\textstyle\sum_{k_0\in\mathbf{Z}} \int_{\Re} \hat{p}_{k_0,k_1} e^{ik_0x_0 + ik_1x_1} dx_1 \ . \end{aligned} \tag{9}$$

then $\hat{q}_{k_0,k_1}$ turns out to be canonically conjugated to $\hat{p}_{-k_0,-k_1}$; it is useful to introduce also complex variables $\hat{\xi}, \hat{\eta}$ by

$$\begin{aligned} \hat{\xi}_{k_0,k_1} &:= \frac{1}{\sqrt{2}}\left(\hat{q}_{k_0,k_1}\sqrt{\omega_{k_0}(\epsilon k_1)} - i\frac{\hat{p}_{k_0,k_1}}{\sqrt{\omega_{k_0}(\epsilon k_1)}}\right) \ , \\ \hat{\eta}_{k_0,k_1} &:= \frac{1}{\sqrt{2}}\left(\hat{q}_{k_0,k_1}\sqrt{\omega_{k_0}(\epsilon k_1)} + i\frac{\hat{p}_{k_0,k_1}}{\sqrt{\omega_{k_0}(\epsilon k_1)}}\right) \ , \end{aligned} \tag{10}$$

where

$$\omega_{k_0}(\epsilon k_1) := \sqrt{(k_0 + \epsilon k_1)^2 + m} \ ;$$

so that the quadratic part H_L of the Hamiltonian takes the diagonal form

$$H_L(\hat{\xi}, \hat{\eta}) = \sum_{k_0\in\mathbf{Z}} \int_{\Re} \omega_{k_0}(\epsilon k_1) \left(\hat{\xi}_{k_0,k_1}\hat{\eta}_{-k_0,-k_1}\right) dk_1 \ . \tag{11}$$

REMARK 2. *Real* u *correspond to sequences of functions* $\hat{\xi}, \hat{\eta}$ *such that*

$$\hat{\xi}^*_{k_0,k_1} = \hat{\eta}_{-k_0,-k_1} \ . \tag{12}$$

For the Hamiltonian vector field of a Hamiltonian function H we will use the notation X_H, so that the Hamilton equations are written in the form $\dot{z} = X_H(z)$, where $z = (\hat{\xi}, \hat{\eta})$ denotes a phase point. Explicitly they are given by [1]

$$\frac{d}{dt}\hat{\xi}_{k_0,k_1} = i\frac{\partial H}{\partial \hat{\eta}_{-k_0,-k_1}} \ , \quad \frac{d}{dt}\hat{\eta}_{k_0,k_1} = -i\frac{\partial H}{\partial \hat{\xi}_{-k_0,-k_1}} \ .$$

We expand the Hamiltonian in series of ϵ up to order ϵ^2; concerning the linear part one has

$$H = H_{0,L} + \epsilon H_{1,L} + \epsilon^2 H_{2,L} + \epsilon^3 R_L \tag{13}$$

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[1] for functionals H that do not depend on the derivatives of $\hat{\eta}$ and $\hat{\xi}$,

where

$$\begin{aligned} H_{0,L}(\hat{\xi},\hat{\eta}) &= \sum_{k_0\in\mathbf{Z}} \int_{\Re} \omega_{k_0} \left(\hat{\xi}_{k_0,k_1} \hat{\eta}_{-k_0,-k_1} \right) dk_1 \ , \\ H_{1,L}(\hat{\xi},\hat{\eta}) &= \sum_{k_0\in\mathbf{Z}} \int_{\Re} v_{k_0} \left(k_1 \hat{\xi}_{k_0,k_1} \hat{\eta}_{-k_0,-k_1} \right) dk_1 \ , \\ H_{2,L}(\hat{\xi},\hat{\eta}) &= \sum_{k_0\in\mathbf{Z}} \int_{\Re} \hbar_{k_0} \left(k_1^2 \hat{\xi}_{k_0,k_1} \hat{\eta}_{-k_0,-k_1} \right) dk_1 \ , \end{aligned}$$

and R_L is the remainder of the expansion; we have introduced, for $k_0 \geq 0$, the following objects

$$\begin{aligned} \omega_{\pm k_0} &:= \omega_{k_0}(\epsilon k_1)|_{k_1=0} \\ v_{\pm k_0} &:= \left. \frac{\partial \omega_{k_0}(\epsilon k_1)}{\partial(\epsilon k_1)} \right|_{(\epsilon k_1)=0} \\ \hbar_{\pm k_0} &:= \left. \frac{1}{2} \frac{\partial^2 \omega_{k_0}(\epsilon k_1)}{\partial(\epsilon k_1)^2} \right|_{(\epsilon k_1)=0} \ . \end{aligned}$$

In order to write in a simple way also the expansion of the nonlinear part, it is useful to go back to the variables x_1, i.e. to define the functions

$$\begin{aligned} \xi_{k_0}(x_1) &:= \frac{1}{\sqrt{2\pi}} \int_{\Re} \hat{\xi}_{k_0,k_1} e^{ik_1x_1} dk_1 \\ \eta_{k_0}(x_1) &:= \frac{1}{\sqrt{2\pi}} \int_{\Re} \hat{\eta}_{k_0,k_1} e^{ik_1x_1} dk_1 \ , \end{aligned}$$

so that one has

$$\begin{aligned} H_{0,L}(\xi,\eta) &= \sum_{k_0\in\mathbf{Z}} \omega_{k_0} \int_{\Re} \left(\xi_{k_0}(x_1) \eta_{-k_0}(x_1) \right) dx_1 \\ H_{1,L}(\xi,\eta) &= i \sum_{k_0\in\mathbf{Z}} v_{k_0} \int_{\Re} \left(\xi_{k_0}(x_1) \frac{\partial \eta_{-k_0}}{\partial x_1}(x_1) \right) dx_1 \\ H_{2,L}(\xi,\eta) &= \sum_{k_0\in\mathbf{Z}} \hbar_{k_0} \int_{\Re} \left(\frac{\partial \xi_{k_0}}{\partial x_1}(x_1) \frac{\partial \eta_{-k_0}}{\partial x_1}(x_1) \right) dx_1 \end{aligned}$$

and the Hamiltonian of the system takes the form

$$H = H_{0,L} + \epsilon H_{1,L} + \epsilon^2 H_{2,L} + \epsilon^2 f_0 + \epsilon^3 R_0 \tag{14}$$

where

$$\begin{aligned} f_0(\xi,\eta) &:= \frac{\sigma}{4(2\pi)^2} \int_{-\pi}^{\pi} dx_0 \int_{\Re} dx_1 \left[\sum_{k_0\in\mathbf{Z}} \frac{1}{\sqrt{2\omega_{k_0}}} \left(\xi_{k_0}(x_1) + \eta_{k_0}(x_1) \right) e^{ik_0x_0} \right]^4 \\ &\equiv \frac{\sigma}{2\pi} \sum_{k_0+j_0+l_0+i_0=0} \frac{1}{16\sqrt{\omega_{k_0}\omega_{j_0}\omega_{l_0}\omega_{i_0}}} \\ &\times \int_{\Re} \left(\xi_{k_0} + \eta_{k_0} \right) \left(\xi_{j_0} + \eta_{j_0} \right) \left(\xi_{l_0} + \eta_{l_0} \right) \left(\xi_{i_0} + \eta_{i_0} \right) dx_1 \end{aligned}$$

while R_0 is the remainder of the expansion; In what follows we will denote with a prime the differentiation with respect to x_1 (i.e. $' \equiv \frac{d}{dx_1}$).

We ensure now that R_0 is small in a suitable sense. To this end we have to introduce a topology in the phase space. So, consider the Banach space S^α of the functions $g(k_1)$ such that

$$\|g\|_{S^\alpha} := \int_{\Re} |g(k_1)| \left(1 + |k_1|^\alpha\right) dk_1 < \infty$$

Then the phase space $\mathcal{P}_\alpha$ is the space of the sequences $\hat{\xi}_{k_0,k_1}$, $\hat{\eta}_{k_0,k_1}$ in S^α such that

$$\left\| (\hat{\xi},\hat{\eta}) \right\|_\alpha := \sum_{k_0} \left(1 + |k_0|^\alpha\right) \left(\left\| \hat{\xi}_{k_0} \right\|_{S^\alpha} + \left\| \hat{\eta}_{k_0} \right\|_{S^\alpha} \right) < \infty$$

endowed by the natural norm. We will denote by by $B_\alpha(\rho)$ the ball in S^α of radius ρ centered at zero.

REMARK 3. *The C^α norm of the functions*

$$\begin{aligned} \xi(x_0,x_1) &= \tfrac{1}{2\pi}\textstyle\sum_{k_0\in\mathbf{Z}}\int_{\Re}\hat{\xi}_{k_0,k_1}\mathrm{e}^{ik_0x_0+ik_1x_1}\,dx_1 \\ \eta(x_0,x_1) &= \tfrac{1}{2\pi}\textstyle\sum_{k_0\in\mathbf{Z}}\int_{\Re}\hat{\eta}_{k_0,k_1}\mathrm{e}^{ik_0x_0+ik_1x_1}\,dx_1 \end{aligned}$$

are controlled by the $\mathcal{P}_\alpha$ norm of $\hat{\xi}$ and $\hat{\eta}$ respectively.

REMARK 4. *The Hamiltonian and the symplectic form are defined only in a dense subset of $\mathcal{P}_\alpha$. Nevertheless in* [7] *a formalism suitable to deal with such a situation was developed. In particular it was ensured that coordinate transformations leaving invariant the domain of the symplectic form and the symplectic form itself have standard properties of canonical transformations. Finally the Hamiltonian flow is canonical provided it is generated by a bounded vector field.*

Having fixed a positive ρ we will denote by

$$\|X_H\|^{\rho}_{\alpha,\beta} := \sup_{\|z\|_\alpha\le\rho}\|X_H(z)\|_\beta$$

the sup norm of the Hamiltonian vector field of a function H, when considered as a function from $\mathcal{P}_\alpha$ to $\mathcal{P}_\beta$.

From now on we will use the notation

$$a \preceq b$$

to mean "there exists a constant C independent of ϵ such that $a\le Cb$".

PROPOSITION 2.1. *For any positive ρ, one has*

$$\|X_{R_0}\|^{\rho}_{\alpha+3,\alpha}\preceq 1\ .$$

The proof is an immediate consequence of lemmas 7.1, 7.2 of sect. 7 and is left to the reader.

3. **Normal form results.** To start up consider the dynamics of the unperturbed Hamiltonian (i.e. of $H_{0,L}$) whose Hamilton equations are

$$\dot{\xi}_{k_0} = i\omega_{k_0}\xi_{k_0}\ ,\quad \dot{\eta}_{k_0} = -i\omega_{k_0}\eta_{k_0}\ , \tag{15}$$

whose solutions is obviously given by

$$\xi_{k_0}(x_1,t) = e^{i\omega_{k_0}t}\xi_{k_0}(x_1,0)\ ,\quad \eta_{k_0}(x_1,t) = e^{-i\omega_{k_0}t}\eta_{k_0}(x_1,0)\ , \tag{16}$$

namely the variables with index $\pm k_0$ rotate with frequency ω_{k_0}.

Then we will take into account the corrections to the dynamics due to the presence of the higher order terms. From standard perturbation theory one can expect that the resonant oscillators interact strongly due to the nonlinearity, while nonresonant oscillators are expected to behave independently. So we could try to separate the dynamics of oscillators corresponding to different values of $|k_0|$. However, due to the presence of infinitely many values of k_0 this is a too hard task. So, to start with, we tackle a simpler problem: namely to understand whether there exists an invariant manifold continuing to the nonlinear case the manifold

$$\mathcal{M}_{\pm1} := \left\{(\xi,\eta)\ :\ \xi_{k_0} = \eta_{k_0} = 0\ \forall\, k_0\ne\pm1\right\} \tag{17}$$

(which is invariant for the linearized flow) or whether there is at least some approximatively invariant manifold close to $\mathcal{M}_{\pm1}$. Since the unperturbed dynamics is almost periodic there is a quite standard technique for doing that, namely to look for a canonical transformation eliminating from the nonlinear part all terms which are linear in the variables $\xi_{k_0}(x_1)$ and $\eta_{k_0}(x_1)$ with $k_0\ne\pm1$.

THEOREM 3.1. *Fix $\rho>0$ and $\alpha\ge0$; then there exists a positive $\epsilon_* = \epsilon_*(\alpha,\rho_0)$ such that, if $|\epsilon|<\epsilon_*$ there exists a canonical analytical transformation $\mathcal{T}_0 : B_\alpha(2\rho_0/3)\to B_\alpha(\rho_0)$ with analytic inverse defined on $B_\alpha(\rho_0/3)$ which transforms the Hamiltonian (14) into the system*

$$H\circ\mathcal{T}_0 = H^{(1)} + \epsilon^3 R_1 \tag{18}$$

where

1) The Hamiltonian system

$$H^{(1)} := H_{L,0} + \epsilon H_{L,1} + \epsilon^2 H_{L,2} + \epsilon^2 f^{(1)} \tag{19}$$

has the invariant manifold $\mathcal{M}_{\pm1}$ and

$$f^{(1)}|_{\mathcal{M}_{\pm1}}(\xi_{\pm1},\eta_{\pm1}) \equiv f_0|_{\mathcal{M}_{\pm1}}(\xi_{\pm1},\eta_{\pm1}) = \frac{3\sigma}{8\omega_1^2 2\pi}\int_{\Re}dx_1(\xi_1+\eta_1)^2(\xi_{-1}+\eta_{-1})^2\ ;$$

2) $\epsilon^3 R_1$ is a small remainder, namely one has

$$\left\|X_{R_1}\right\|_{\alpha+3,\alpha}^{2\rho_0/3} \preceq 1$$

3) the transformation is close to identity

$$\sup_{\|z\|_\alpha \leq 2\rho_0/3} \left\|z - \mathcal{T}_0(z)\right\|_\alpha \preceq \epsilon^2 \ .$$

Now we fix attention on the normalized Hamiltonian $H^{(1)}$ (cf. (19)), and we study its dynamics on the invariant manifold $\mathcal{M}_{\pm 1}$.

In $\mathcal{M}_{\pm 1}$ the unperturbed Hamiltonian $H_{0,L}$, reduces to

$$\omega_1 \int_{\Re} \left(\xi_1 \eta_{-1} + \xi_{-1}\eta_1\right) dx_1 \ , \tag{20}$$

which consists of infinitely many resonant oscillators labelled by $x_1 \in \Re$, so it is natural to construct a resonant normal form, i.e. to average with respect to the unperturbed flow, and this will give rise to the nonlinear Schrödinger equation.

THEOREM 3.2. *Fix $\rho_1 > 0$ with $2\rho_0/3 > \rho_1$ and $\alpha \geq 0$, then, provided ϵ is small enough, there exists an analytic canonical transformation $\mathcal{T}_1 : B_\alpha(2\rho_1/3) \to B_\alpha(\rho_1)$, which transforms $H^{(1)}$ into the system*

$$H^{(1)} \circ \mathcal{T}_1 = H^{(2)} + \epsilon^3 R_2$$

where $\mathcal{T}_1$ leaves invariant $\mathcal{M}_{\pm 1}$ and acts as the identity on the variables $(\xi_{k_0}, \eta_{k_0})_{k_0 \neq \pm 1}$, and
1) The Hamiltonian system

$$H^{(2)} := H_{0,L} + \epsilon H_{1,L} + \epsilon^2 H_{2,L} + \epsilon^2 N_1 \tag{21}$$

has the invariant manifold $\mathcal{M}_{\pm 1}$ and the restriction of N_1 to $\mathcal{M}_{\pm 1}$ is given by

$$N_1\big|_{\mathcal{M}_{\pm 1}}(\xi_{\pm 1}, \eta_{\pm 1}) = \frac{1}{2\pi}\frac{3}{8}\frac{\sigma}{\omega_1^2}\int_{\Re} \left(\xi_1^2\eta_{-1}^2 + 4\xi_1\eta_{-1}\xi_{-1}\eta_1 + \xi_{-1}^2\eta_1^2\right) dx_1$$

2) $\epsilon^3 R_2$ is a small remainder, namely one has

$$\left\|X_{R_2}\right\|_{\alpha+2,\alpha}^{2\rho_1/3} \preceq 1 \ .$$

3) $\mathcal{T}_1$ is close to identity, namely one has

$$\sup_{\|z\|_\alpha \leq 2\rho_1/3} \left\|z - \mathcal{T}_1(z)\right\|_\alpha \preceq \epsilon^2 \ .$$

We now study the dynamics of $H^{(2)}$ on $\mathcal{M}_{\pm 1}$.

REMARK 5. *On $\mathcal{M}_{\pm 1}$ one has*

$$\left\{H_{0,L}, N_1\right\} = 0 \ , \tag{22}$$

and

$$\left\{H_{0,L}, H_{1,L}\right\} = \left\{H_{0,L}, H_{2,L}\right\} = 0 \ . \tag{23}$$

Due to these properties $H_{0,L}$ is a constant of motion for the dynamics of $H^{(2)}$ in $\mathcal{M}_{\pm 1}$. It means that in order to study the dynamics of $H^{(2)}$ on $\mathcal{M}_{\pm 1}$ one can limit oneself to study the reduced system H_R defined on the intersection of $\mathcal{M}_{\pm 1}$ with the level surfaces of $H_{0,L}$ by

$$H_R := H_{1,L} + \epsilon H_{2,L} + \epsilon N_1 \ . \tag{24}$$

Then, given a solution $z_R(t) \in \mathcal{M}_{\pm 1}$ of the Hamilton equations of (24), a solution of the equations of motion of (21) is given by

$$z_t := \Phi_0^t(z_R(\epsilon t)) \ ,$$

where Φ_0^t is the flow generated by $H_{0,L}$ and described by 16.

We study now the dynamics of H_R. Remark that the manifolds

$$\mathcal{M}_1 := \{\xi_{-1} = \eta_1 = 0\} \ , \quad \mathcal{M}_{-1} := \{\xi_1 = \eta_{-1} = 0\} \ ,$$

are invariant for this dynamics[2]. Let's concentrate on $\mathcal{M}_1$ and begin to study the dynamics here. To this end remark that here one has $\{H_{1,L}, H_R\} = 0$, as it is easily seen by remarking that here the dynamics generated by $H_{1,L}$ is given by

$$\begin{array}{rl} \dot{\xi}_1 = & v_1 \xi_1' \\ \dot{\eta}_{-1} = & v_1 \eta_{-1}' \end{array} \ , \quad \Leftrightarrow \quad \begin{array}{rl} \xi_1(x_1,t) = & \xi_1^0(x_1+v_1 t) \\ \eta_{-1}(x_1,t) = & \eta_{-1}^0(x_1+v_1 t) \ , \end{array}$$

where (ξ_1^0, η_{-1}^0) are the initial data, and it is now clear that v_1 is the group velocity.

So, it is again possible to pass to the reduced system with Hamiltonian which on $\mathcal{M}_1$ coincides with $\epsilon(H_{N,2} + N_1)$, whose equations of motion (on $\mathcal{M}_1$) are given by

$$\dot{\xi}_1 = i\epsilon\left(-\hbar_1 \xi_1'' + \nu \eta_{-1} \xi_1^2\right) \ , \quad \dot{\eta}_{-1} = -i\epsilon\left(-\hbar_1 \eta_{-1}'' + \nu \xi_1 \eta_{-1}^2\right) \tag{25}$$

with $\nu := 3\sigma/32\pi\omega_1^2$. It is immediate to see that for initial data fulfilling $\xi_1^* = \eta_{-1}$, which correspond to real (p,u), the equation (25) is the nonlinear Schrödinger equation. So we have the following

PROPOSITION 3.1. *Let $A(x_1,t)$ be a solution of the nonlinear Schrödinger equation*

$$\dot{A} = -i\hbar_1 A'' + i\nu A|A|^2 \ , \tag{26}$$

then

$$\xi_1(x_1,t) := A(x_1 + \epsilon v_1 t, \epsilon^2 t) e^{i\omega_1 t} \ , \quad \eta_{-1}(x_1,t) := \xi_1(x_1,t)^* \tag{27}$$

$$\eta_{k_0} = \xi_{k_0} = 0 \ , \quad \forall k_0 \ : \ |k_0| \neq \pm 1 \tag{28}$$

is a solution of the Hamilton equations of (21). Conversely, if $\xi_1 = \xi_1(x_1,t), \eta_{-1}(x_1,t) = \xi_1(x_1,t)^*, \xi_{-1} = \eta_1 = 0$, with also (28) fulfilled, is a solution of the Hamilton equations of (21), then there exist a solution A of the nonlinear Schrödinger such that (27) holds.

REMARK 6. *The same is true, mutatis mutandis, and in particular substituting v_1 with $-v_1$, for solutions in $\mathcal{M}_{-1}$.*

It is also possible to study the dynamics corresponding to general initial data, which is expected to consist of a wave traveling to the right and a wave traveling to the left that interact when they meet. The result of the interaction process is in principle difficult to be predicted a priori. However, remarking that H_R appears as a perturbation of a linear Hamiltonian one can try to use again canonical perturbation theory to study the dynamics. This can be done in a space of functions that enjoy suitable integrability properties. This is given by the standard Sobolev space $W^{s,1}$ of the functions admitting s weak derivatives which are $L^1(\Re)$. In particular remark that by Sobolev embedding theorem one has $C^k \supset W^{s,1}$ provided $s > 1+k$; moreover for $s > 1$ the space $W^{s,1}$ is an algebra and one has $S^\alpha \supset W^{s,1}$ with a continuous embedding provided $s > \alpha + 1$.

The canonical transformation of our forthcoming normal form theorem (as well as the canonical transformation defined in theorem 3.2) acts non trivially only on the variables with index $k_0 = \pm 1$, so it is useful to introduce a new phase space $\mathcal{F}_{s,\alpha}$ as the subspace of $\mathcal{P}_\alpha$, such that the functions $\xi_{\pm 1}, \eta_{\pm 1} \in W^{s,1}$. We endow such a space by the norm

$$\|(\xi,\eta)\|_\alpha := \sum_{k_0 \neq \pm 1} (1 + |k_0|^\alpha) \left(\left\| \hat{\xi}_{k_0} \right\|_{S^\alpha} + \left\| \hat{\eta}_{k_0} \right\|_{S^\alpha} \right) + \sum_{k_0 = \pm 1} \left(\|\xi_{k_0}\|_{W^{s,1}} + \|\eta_{k_0}\|_{W^{s,1}} \right)$$

Clearly one has $\mathcal{F}_{s,\alpha} \subset \mathcal{P}_\alpha$ with a continuous embedding. Moreover we will denote by $\mathcal{B}_s(\rho)$ the ball of radius ρ in $\mathcal{F}_{s,\alpha}$.

THEOREM 3.3. *Fix a positive $2\rho_1/3 > \rho_2$; then, provided ϵ is small enough there exists an analytic canonical coordinate transformation $\mathcal{T}_2 : \mathcal{B}_s(2\rho_2/3) \to \mathcal{B}_s(\rho_2)$ which acts as the identity on the variables $(\xi_{k_0}, \eta_{k_0})_{k_0 \neq \pm 1}$, and such that*

$$H_R \circ \mathcal{T}_2 = H_R^{(1)} + \epsilon^2 R_3 \ , \tag{29}$$

where

$$H_R^{(1)} := H_{1,L} + \epsilon H_{2,L} + \epsilon N_2 \tag{30}$$

has the invariant manifold $\mathcal{M}_{\pm 1}$, and the restriction of N_2 to $\mathcal{M}_{\pm 1}$ is given by

$$N_2\big|_{\mathcal{M}_{\pm 1}}(\xi_{\pm 1}, \eta_{\pm 1}) = \frac{\sigma}{2\pi} \frac{3}{8} \frac{1}{\omega_1^2} \int_{\Re} \left(\xi_1^2 \eta_{-1}^2 + \xi_{-1}^2 \eta_1^2 \right) dx_1$$

---

[2] this is due to the fact that N_1 does not contain terms which are linear in either the variables (ξ_1, η_{-1}) or the variables (ξ_{-1}, η_1)

and $H_{0,L} \circ \mathcal{T}_2 \equiv H_{0,L}$*; moreover the following inequalities hold*

$$\sup_{\|z\|_{\mathcal{F}_{s+2}} \leq 2\rho_2/3} \left\| X_{R_3}(z) \right\|_{\mathcal{F}_s} \preceq 1 \ , \qquad \sup_{\|z\|_{\mathcal{F}_s} \leq 2\rho_2/3} \|z - \mathcal{T}_2(z)\|_{\mathcal{F}_s} \preceq \epsilon$$

REMARK 7. *For the normalized system with Hamiltonian* $H_R^{(1)}$*, the function* $H_{1,L}$ *is clearly an integral of motion, so one can pass again to the reduced system*

$$H_{2,L} + N_2 \ , \tag{31}$$

whose equations of motion (on $\mathcal{M}_{\pm 1}$*) are simply two decoupled NLS. It follows that the dynamics of (30) consists of two* **non–interacting** *modulated wave packets, one traveling to the right, and the other traveling to the left; the shape of each one of the packets is modulated by a nonlinear Schrödinger equation.*

4. **Use of the normal form.** We use now the above normal form results and in particular theorem 3.3 in order to construct some approximate solutions of the original equations of motion.

In order to explain the idea let us consider a smooth solution $\tilde{\zeta}(t)$ of the equations of motion of (31), or more precisely of the Hamilton equations of

$$H^{(3)} := H_{0,L} + \epsilon H_{1,L} + \epsilon^2 H_{2,L} + \epsilon^2 N_2 \tag{32}$$

and construct the function

$$\bar{z}(t) := (\mathcal{T}_0 \circ \mathcal{T}_1 \circ \mathcal{T}_2)(\tilde{\zeta}(t)) \ ; \tag{33}$$

we claim this is the wanted approximative solution of the original system. To show this we write down the equation it fulfills.

PROPOSITION 4.1. *Fix* $\alpha \geq 0$*, and let* $\tilde{\zeta}(t)$*,* $t \in [-T/\epsilon^2, T/\epsilon^2]$*, with a suitable* $T > 0$*, be a solution of the Hamilton equations of* $H^{(3)}$ *belonging to* $C^0([-T/\epsilon^2, T/\epsilon^2], \mathcal{F}_{s,\alpha})$ *with* $s > \alpha + 4$*; then* $\bar{z}(t)$ *defined by (33) belongs to*

$$C^0([-T/\epsilon^2, T/\epsilon^2], \mathcal{P}_{\alpha+3}) \cap C^2([-T/\epsilon^2, T/\epsilon^2], \mathcal{P}_{\alpha+1}) \ ,$$

fulfills the equation

$$\frac{d\bar{z}}{dt}(t) = X_H(\bar{z}(t)) - \epsilon^3 \Delta(t)$$

with $\Delta(t)$ *a suitable function. Moreover the following inequalities are fulfilled*

$$\sup_{|t| \leq T/\epsilon^2} \left\| \bar{z}(t) - \tilde{\zeta}(t) \right\|_{\alpha+3} \preceq \epsilon \ , \qquad \sup_{|t| \leq T/\epsilon^2} \|\Delta(t)\|_{\alpha} \preceq 1 \ .$$

Proof. Denote by

$$(\mathcal{T}_{0*} X)(z) := d\mathcal{T}_0(\mathcal{T}_0^{-1}(z)) X(\mathcal{T}_0^{-1}(z)) \tag{34}$$

the standard push forward of the vector field X by $\mathcal{T}_0$, and similarly for $\mathcal{T}_{1*}$ and $\mathcal{T}_{2*}$. One has

$$\frac{d\bar{z}}{dt} = \left[\mathcal{T}_{0*} \mathcal{T}_{1*} \mathcal{T}_{2*} X_{H^{(3)}} \right](\bar{z}) \ . \tag{35}$$

On the other hand, by theorems 3.3, 3.2 and 3.1 one has

$$\begin{aligned} X_{H^{(2)}} &= \mathcal{T}_{2*} X_{H^{(3)}} + \epsilon^3 \mathcal{T}_{2*} X_{R_3} \\ X_{H^{(1)}} &= \mathcal{T}_{1*} X_{H^{(2)}} + \epsilon^3 \mathcal{T}_{1*} X_{R_2} \\ X_H &= \mathcal{T}_{0*} X_{H^{(1)}} + \epsilon^3 \mathcal{T}_{0*} X_{R_1} \end{aligned}$$

from which one immediately gets

$$\mathcal{T}_{0*} \mathcal{T}_{1*} \mathcal{T}_{2*} X_{H^{(3)}} = X_H - \epsilon^3 \left[\mathcal{T}_{0*} \mathcal{T}_{1*} \mathcal{T}_{*2} X_{R_3} + \mathcal{T}_{0*} \mathcal{T}_{1*} X_{R_2} + \mathcal{T}_{0*} X_{R_1} \right]$$

Denoting

$$\Delta(t) := \left[\mathcal{T}_{0*} \mathcal{T}_{1*} \mathcal{T}_{*2} X_{R_3} \right](\bar{z}(t)) + \left[\mathcal{T}_{0*} \mathcal{T}_{1*} X_{R_2} \right](\bar{z}(t)) + \left[\mathcal{T}_{0*} X_{R_1} \right](\bar{z}(t))$$

we have the thesis. □

In order to look for a solution of the original equations of motion close to $\bar{z}$ we write $z = \bar{z} + r$ and remark that r fulfills the equation

$$\dot{r} = X_H(\bar{z}(t) + r) - X_H(\bar{z}(t)) + \epsilon^3 \Delta(t) \ .$$

So we should estimate the solution of this equation. This can be done directly, however it is easier to go back to the original nonlinear Klein–Gordon equation and to estimate the error in the original equation.

We will consider
1)the solution $u(x,t)$ of eq.(1) with initial data

$$u_0(x) = \epsilon \frac{a(\epsilon x)e^{ix} + a^*(\epsilon x)e^{-ix}}{\sqrt{2\pi}} \ , \quad p_0(x) = \epsilon \frac{b(\epsilon x)e^{ix} + b^*(\epsilon x)e^{-ix}}{\sqrt{2\pi}} \ , \tag{36}$$

2)the solution $A(x_1,t)$ of the NLS equation (c.f. (26)) with initial datum

$$A(x_1,0) = \frac{1}{\sqrt{2}}\left(a(x_1)\sqrt{\omega_1} - i\frac{b(x_1)}{\sqrt{\omega_1}}\right) \ ,$$

3) the solution $B(x_1,t)$ of NLS with initial datum

$$B(x_1,0) = \frac{1}{\sqrt{2}}\left(a(x_1)\sqrt{\omega_1} + i\frac{b(x_1)}{\sqrt{\omega_1}}\right) \ .$$

4)the approximate solution u_a defined by

$$u_a(x,t) := \frac{e^{i(x+\omega_1 t)}A(\epsilon(x+v_1t),\epsilon^2 t) + e^{i(x-\omega_1 t)}B(\epsilon(x-v_1t),\epsilon^2 t)}{\sqrt{2\omega_1}} + \text{c.c.} \ , \tag{37}$$

where c.c. means complex conjugated term.

Theorem 4.1. *Fix $\alpha \geq 1/2$, assume $a, b \in W^{s,1}$ with $s > \alpha + 4$, then there exists $T > 0$ such that one has*

$$\sup_{|t|\leq T/\epsilon^2} \|\epsilon u_a(.,t) - u(.,t)\|_{C^{\alpha+1/2}} \preceq \epsilon^2 \ , \qquad \sup_{|t|\leq T/\epsilon^2} \|\epsilon \dot{u}_a(.,t) - \dot{u}(.,t)\|_{C^{\alpha-1/2}} \preceq \epsilon^2$$

Proof. First remark that, by standard theory of semilinear equations (see for example [8]) one has that the Cauchy problem for NLS is locally well posed in $W^{s,1}$ (which is an algebra), and therefore there exists a positive T such that the solutions A and B exist up to this time. Use A and B to construct a solutions

$$\tilde{\zeta}(t) = (\tilde{\xi}_1(t), \tilde{\eta}_1(t), \tilde{\xi}_{-1}(t), \tilde{\eta}_{-1}(t))$$

of the Hamilton equations of $H^{(3)}$ (cf. eq.(32)), namely

$$\begin{aligned} \tilde{\xi}_1(x_1,t) &:= A(x_1+\epsilon v_1 t, \epsilon^2 t)e^{i\omega_1 t} \ , \quad \tilde{\eta}_{-1}(x_1,t) := \tilde{\xi}_1(x_1,t)^* \\ \tilde{\xi}_{-1}(x_1,t) &:= B(x_1-\epsilon v_1 t, \epsilon^2 t)e^{i\omega_1 t} \ , \quad \tilde{\eta}_1(x_1,t) := \tilde{\xi}_{-1}(x_1,t)^* \ . \end{aligned}$$

Define $\bar{\zeta} \equiv (\bar{\xi}_{k_0,k_1}, \bar{\eta}_{k_0,k_1}) := (\mathcal{T}_0 \circ \mathcal{T}_1 \circ \mathcal{T}_2)(\tilde{\zeta})$; using $\bar{\zeta}$ construct the approximate solution for the original system by

$$\begin{aligned} \bar{u}(x,t) &:= \frac{1}{2\pi}\sum_{k_0\in\mathbf{Z}}\int_{\Re} \frac{\bar{\xi}_{k_0,k_1}(t) + \bar{\eta}_{k_0,k_1}(t)}{\sqrt{2\omega_{k_0}(\epsilon k_1)}} e^{ik_0x+ik_1\epsilon x}dk_1 \\ \bar{p}(x,t) &:= \frac{1}{2\pi}\sum_{k_0\in\mathbf{Z}}\int_{\Re} \frac{\bar{\eta}_{k_0,k_1}(t) - \bar{\xi}_{k_0,k_1}(t)}{i\sqrt{2}} \sqrt{\omega_{k_0}(\epsilon k_1)} e^{ik_0x+ik_1\epsilon x}dk_1 \end{aligned}$$

by proposition 4.1 such functions fulfill the equations

$$\begin{aligned} \frac{\partial}{\partial t}\bar{u}(x,t) &= \bar{p}(x,t) + \epsilon^3\Delta_u(x,t) \\ \frac{\partial}{\partial t}\bar{p}(x,t) &= \frac{\partial^2}{\partial x^2}\bar{u}(x,t) - \epsilon^2\sigma\bar{u}(x,t)^3 + \epsilon^3\Delta_p(x,t) \end{aligned} \tag{38}$$

where Δ_u, Δ_p are the functions of proposition 4.1 represented in space x. By lemma 7.3 the estimates

$$\sup_t \|\Delta_u(.,t)\|_{S^{\alpha+1/2}} \preceq 1 \ , \quad \sup_t \|\Delta_p(.,t)\|_{S^{\alpha-1/2}} \preceq 1 \ .$$

hold. The norm S^α has to be computed in Fourier transform (with respect to x). Denote by $U := (u,p)/\epsilon$ the rescaled solution of the original Klein Gordon equation eq. (1), denote also $\bar{U} := (\bar{u},\bar{p})$, $\Delta_U := (\Delta_u, \Delta_p)$ and by A the linear operator describing the Klein Gordon equation; finally denote by $F(U) := (0, -\sigma u^3)$ the nonlinearity, we have that the equation for the (rescaled) remainder $r := U - \bar{U}$ takes the form:

$$\dot{r} = Ar + \epsilon^2[F(\bar{U}+r) - F(\bar{U})] + \epsilon^3\Delta_U \ , \tag{39}$$

which is the equation giving the error. In order to estimate its solution we have first to estimate the difference between the initial datum for $(\bar{u},\bar{p})$ and the original initial datum in order to estimate the initial value of r.

To this end denote

$$\tilde{u}(x,t) := \frac{1}{2\pi}\sum_{k_0=\pm1}\int_{\Re}\frac{\tilde{\xi}_{k_0,k_1}(t)+\tilde{\eta}_{k_0,k_1}(t)}{\sqrt{2\omega_{k_0}(\epsilon k_1)}}e^{ik_0x+ik_1\epsilon x}dk_1$$

$$\tilde{p}(x,t) := \frac{1}{2\pi}\sum_{k_0=\pm1}\int_{\Re}\frac{\tilde{\eta}_{k_0,k_1}(t)-\tilde{\xi}_{k_0,k_1}(t)}{i\sqrt{2}}\sqrt{\omega_{k_0}(\epsilon k_1)}e^{ik_0x+ik_1\epsilon x}dk_1$$

and remark that, by lemma 7.1 one has

$$\left\|\tilde{u}(.,0)-\frac{u_0(.)}{\epsilon}\right\|_{S^\alpha}\preceq\epsilon\left\|\tilde{\zeta}(0)\right\|_{\alpha+1}\ ,\quad \left\|\tilde{p}(.,0)-\frac{p_0(.)}{\epsilon}\right\|_{S^\alpha}\preceq\epsilon\left\|\tilde{\zeta}(0)\right\|_{\alpha+1}\ ,$$

and also

$$\|\tilde{u}(.,t)-u_a(.,t)\|_{S^\alpha}\preceq\epsilon\,\|\zeta(t)\|_{\alpha+1}\ , \tag{40}$$

On the other hand, by the estimates of the canonical transformations we have

$$\|\tilde{u}(.,t)-\bar{u}(.,t)\|_{S^{\alpha+1/2}}\preceq\epsilon\qquad\|\tilde{p}(.,t)-\bar{p}(.,t)\|_{S^{\alpha-1/2}}\preceq\epsilon \tag{41}$$

provided $\tilde{\xi},\tilde{\eta}\in W^{s,1}$ with $s>\alpha+1$, it follows that, under the regularity assumptions of the theorem one has that the initial datum r_0 to be inserted in (39) fulfills

$$\|r_0\|_{S^{\alpha+1/2}\times S^{\alpha-1/2}}\preceq\epsilon\ .$$

So a simple use of the Gronwall lemma and of the fact that the group generated by the Klein Gordon operator A is unitary gives

$$\sup_{|t|\leq T/\epsilon^2}\|r(t)\|_{S^{\alpha+1/2}\times S^{\alpha-1/2}}\preceq\epsilon$$

and using also (40), (41) and remark 3 one has the thesis. □

From the proof of the above theorem we also obtain the following result

THEOREM 4.2. *Fix $\alpha\geq1/2$, consider an initial datum (u_1,p_1) such that*

$$\|(u_1,p_1)-(u_0,p_0)\|_{S^{\alpha+1/2}\times S^{\alpha-1/2}}\preceq\epsilon^2\ ,$$

with u_0,p_0 given by (36), then for the corresponding solution $(u_1(x,t),\dot{u}_1(x,t))$ one has

$$\sup_{|t|\leq T/\epsilon^2}\|(\epsilon u_a(.,t),\epsilon\dot{u}_a(.,t))-(u_1(.,t),\dot{u}_1(.,t))\|_{S^{\alpha+1/2}\times S^{\alpha-1/2}}\preceq\epsilon^2$$

5. **Extensions.** In this paper we confined our attention on the submanifold $\mathcal{M}_{\pm1}$ of the phase space; it is possible to perform exactly the same study considering the manifold

$$\mathcal{M}_{\pm1,\dots,\pm n}:=\left\{(\xi,\eta)\ :\ \xi_{k_0}=\eta_{k_0}=0\ \forall k_0\ :\ k_0\notin\{\pm1,...,\pm n\}\right\}\ . \tag{42}$$

We explain this point more in detail. First one has to eliminate from the Hamiltonian the terms linear in the variables with index 0 or index larger than n obtaining that this manifold is approximatively invariant (in the same sense as $\mathcal{M}_{\pm1}$ is approximatively invariant for the Hamiltonian (18)). Then the computation of the normal form on this manifold (as in theorems 3.2 and 3.3) shows that the dynamics consists of $2n$ non interacting traveling waves each traveling with velocity $\pm v_j$ $(j=1,...,n)$ and modulated by an NLS equation with suitable parameters.

An other interesting extension pertains to the case of a quadratic nonlinearity, i.e. the case where the nonlinear term is u^2 instead of u^3. Here it happens that the first order average of the perturbation vanishes, and that NLS appears as the second order normal form. The procedure of sect. 3 can be performed also in this case leading to very similar results, however the procedure of sect. 4 has to be modified in order to obtain a meaningful result. Indeed, a straightforward application of such a procedure would show that NLS describes the original system only up to time scales of order $1/\epsilon$, a time scale over which the NLS dynamics is invisible. To obtain the time scale $1/\epsilon^2$ one has to study more in detail the equation (5), and in particular one has to remark that it has the form

$$\dot{r}=Ar+\epsilon dX(\bar{u}(t))r+\text{higher order terms}\ ; \tag{43}$$

if one just estimates the solution of this equation using only the fact that $dX(\bar{u}(t))$ is bounded than one is lead to the time scale $1/\epsilon$. To reach the time scale $1/\epsilon^2$ one has to study more carefully the linearized equation where the higher order terms in (43) are neglected. Such an equation is a small periodic perturbation of a time independent diagonal system, and so it can be studied using Floquet theory, or more precisely its infinite dimensional extension (see e.g. [9], which involves some subtleties due to regularity problems). In particular it can be shown that under suitable nonresonance conditions (fulfilled in our case) the solutions of such a linearized system are time periodic modulations of the solutions of the linear Klein Gordon equation. Exploiting such a property one gets the result. We point out that for the case of nonlinearity u^2 the validity of NLS has already been proved by different techniques in [2, 10].

Finally we discuss briefly the problem of computation of higher order terms of the normal forms. Althoght the formal computations are quite straightforward, their justification is far from trivial because of two different problems. The first one is the generation of the normalizing canonical transformation: the generating vector field turns out to be non smooth, therefore it is not obvious that it generates a flow. We have some ideas on how to overcome such a problem but the question is very technical and is left for future work. The second problem is the use of the normal form, indeed, in this case the same problem appears as for the case of quadratic perturbation of the Klein Gordon equation, namely the time scale over which the corrections are visible are longer than the time scale over which we are able to ensure the validity of the approximation. We have no ideas on how to solve this problem, we think that it deserves numerical investigation.

6. **Proof of the normal form results.** We start by proving theorem 3.1. We will generate the normalizing canonical transformations as the time ϵ^2 flow of the Hamiltonian vector field of a suitable Hamiltonian function χ_0. So, denote by $\mathcal{T}_0^{\epsilon^2}$ the time ϵ^2 flow of the vector field X_{χ_0}. One has

$$H\circ\mathcal{T}_0^{\epsilon^2}=H_{0,L}+\epsilon H_{1,L}+\epsilon^2H_{2,L}+\epsilon^2\left[\left\{\chi_0,H_{0,L}\right\}+f_0\right]+\text{h.o.t.} \tag{44}$$

where explicitly we have

$$\begin{aligned}\text{h.o.t.}\quad &=\epsilon(H_{1,L}\circ\mathcal{T}_0^{\epsilon^2}-H_{1,L})+\epsilon^2(H_{2,L}\circ\mathcal{T}_0^{\epsilon^2}-H_{2,L})\\ +\quad &\epsilon^2(f_0\circ\mathcal{T}_0^{\epsilon^2}-f_0)+(H_{0,L}\circ\mathcal{T}_0^{\epsilon^2}-H_{0,L}-\epsilon^2\left\{\chi_0,H_{0,L}\right\})\\ +\quad &\epsilon^3R_0\circ\mathcal{T}_0^{\epsilon^2}\ .\end{aligned} \tag{45}$$

As usual the idea is to determine χ_0 in such a way that the square bracket in (44) takes a suitable normal form. Explicitly the normal form must not contain terms linear in the variables ξ_{k_0},η_{k_0} with $k_0\neq\pm1$. It is easy to see that the wanted function χ_0 is given by

$$\chi_0(\xi,\eta):=\frac{\sigma}{2\pi}\frac{-i}{4\sqrt{\omega_1^3\omega_3}}\int_{\Re}G(\xi,\eta)dx_1$$

with

$$\begin{aligned}G(\xi,\eta)\quad:=\quad&\frac{\xi_1^3\xi_{-3}}{3\omega_1+\omega_3}+\frac{3\xi_1^2\eta_1\xi_{-3}}{\omega_1+\omega_3}+\frac{3\xi_1\eta_1^2\xi_{-3}}{\omega_3-\omega_1}+\frac{\eta_1^3\xi_{-3}}{-3\omega_1+\omega_3}\\ +\quad&\frac{\xi_1^3\eta_{-3}}{3\omega_1-\omega_3}+\frac{3\xi_1^2\eta_1\eta_{-3}}{\omega_1-\omega_3}+\frac{3\xi_1\eta_1^2\eta_{-3}}{-\omega_3-\omega_1}+\frac{\eta_1^3-\eta_{-3}}{-3\omega_1-\omega_3}\\ +\quad&\frac{\xi_{-1}\xi_3}{3\omega_1+\omega_3}+\frac{3\xi_{-1}^2\eta_{-1}\xi_3}{\omega_1+\omega_3}+\frac{3\xi_{-1}\eta_{-1}^2\xi_3}{\omega_3-\omega_1}+\frac{\eta_{-1}^3\xi_3}{-3\omega_1+\omega_3}\\ +\quad&\frac{\xi_{-1}^3\eta_3}{3\omega_1-\omega_3}+\frac{3\xi_{-1}^2\eta_{-1}\eta_3}{\omega_1-\omega_3}+\frac{3\xi_{-1}\eta_{-1}^2\eta_3}{-\omega_3-\omega_1}+\frac{\eta_{-1}^3-\eta_3}{-3\omega_1-\omega_3}\end{aligned}$$

which is easily seen to generate a smooth (actually analytic) vector field from $\mathcal{P}_\alpha$ to $\mathcal{P}_\alpha$ for any $\alpha\geq0$. In order to prove theorem 3.1 we have just to estimate the normal form transformation, i.e. the time ϵ^2 flow of the vector field of χ_0 and the higher order terms (45). This is completely standard; we will report here the estimates from [11] for the sake of completeness. Actually these estimates depend only on the smoothness properties of the vector field of χ_0, so we will state them making reference to a general function χ with analytic vector field. More precisely we fix ρ and we assume that the Hamiltonian vector field X_χ of χ is analytic as an application from $\mathcal{P}_\alpha\supset B_\alpha(\rho)$ to $\mathcal{P}_\alpha$. We will denote by $\mathcal{T}^t$ the time t flow of such a vector field

Lemma 6.1. *There exists ϵ_* such that, for any t with $|t|\leq\epsilon_*$, one has*

$$\sup_{\|z\|_\alpha\leq2\rho/3}\left\|\mathcal{T}^t(z)-z\right\|_\alpha\leq|t|\left\|X_\chi\right\|_{\alpha,\alpha}^\rho\ . \tag{46}$$

Proof. It is just an application of the equality
$$z(t)-z(0)=\int_0^t \frac{dz}{dt}(s)ds=\int_0^t X_\chi(z(s))ds \ .$$
□

LEMMA 6.2. *Let h be a function with Hamiltonian vector field that is C^∞ as a map from $B_\alpha(\rho)$ to $\mathcal{P}_\beta$, then for $|t|\leq \epsilon^2\leq\epsilon_*$ (see previous lemma), one has*
$$\|X_{h\circ\mathcal{T}^t}\|^{2\rho/3}_{\alpha,\beta}\preceq 1$$

Proof. First remark that, since $\mathcal{T}^t$ is a canonical transformation one has
$$X_{h\circ\mathcal{T}^t}(z)=d\mathcal{T}^{-t}(\mathcal{T}^t(z))X_h(\mathcal{T}^t(z)) \ , \tag{47}$$
from which
$$X_{h\circ\mathcal{T}^t}(z)=\left(d\mathcal{T}^{-t}(\mathcal{T}^t(z))-\mathbb{1}\right)X_h(\mathcal{T}^t(z))+X_h(\mathcal{T}^t(z)) \ .$$
To estimate the first term remark that
$$\sup_{\|z\|_\alpha\leq 2\rho/3}\left\|d\mathcal{T}^{-t}(\mathcal{T}^t(z))-\mathbb{1}\right\|\leq\sup_{\|z\|_\alpha\leq\rho}\left\|d\mathcal{T}^{-t}(z)-\mathbb{1}\right\|\preceq\epsilon^2 \tag{48}$$
where the last equality is due to the fact that $d\mathcal{T}^t$ is Lipschitz in time. □

LEMMA 6.3. *Let h be as above and assume that X_χ is smooth also as a map from $\mathcal{P}_\beta$ to itself, then, for $|t|\leq\epsilon^2\leq\epsilon_*$ one has*
$$\left\|X_{h\circ\mathcal{T}^t-h}\right\|^{2\rho/3}_{\alpha,\beta}\preceq\epsilon^2 \tag{49}$$

Proof. One has
$$X_{h\circ\mathcal{T}^t-h}(z)=\left(d\mathcal{T}^{-t}(\mathcal{T}^t(z))-\mathbb{1}\right)X_h(\mathcal{T}^t(z))+\left[X_h(\mathcal{T}^t(z))-X_h(z)\right] \ .$$
The norm of the square bracket is estimated using Lagrange mean value theorem and (46). The other term was already estimated in (48), so we have the thesis. □

Finally we have to estimate the term containing the transformation of $H_{0,L}$, to this end it is useful to remark that χ_0 is constructed in such a way that it fulfills the homological equation
$$\left\{H_{0,L},\chi_0\right\}=\tilde{f}_0 \ , \tag{50}$$
where $\tilde{f}_0$ is the part of f_0 that does not contain terms linear in the variables ξ_{k_0},η_{k_0} with $k_0\neq\pm1$.

LEMMA 6.4. *Denote*
$$\ell:=H_{0,L}\circ\mathcal{T}_0^{\epsilon^2}-H_{0,L}-\epsilon^2\left\{\chi_0,H_{0,L}\right\} \ ,$$
then, $\|X_\ell\|^{2\rho/3}_{\alpha,\alpha}\preceq\epsilon^4$.

Proof. One has
$$H_{0,L}(\mathcal{T}_0^{\epsilon^2}(z))-H_{0,L}(z)=\int_0^{\epsilon^2}\frac{d}{dt}H_{0,L}(\mathcal{T}_0^t(z))dt=\int_0^{\epsilon^2}\tilde{f}_0(\mathcal{T}_0^t(z))dt \ ,$$
where we used the homological equation to compute $\left\{H_{0,L},\chi_0\right\}$. One has
$$\ell(z)=\int_0^{\epsilon^2}(\tilde{f}_0(\mathcal{T}_0^t(z))-\tilde{f}_0(z))dt \ ;$$
Using (49) one gets the result. □

Collecting these results one easily gets theorem 3.1.

We come to theorem 3.2.

From now on we will restrict to the manifold $\mathcal{M}_{\pm1}$. By abuse of notation we will denote by $\mathcal{P}_\alpha$ the Banach space $\mathcal{P}_\alpha\cap\mathcal{M}_{\pm1}$ endowed by the natural topology and the natural symplectic form. So we will consider the restriction of the above Hamiltonians to this manifolds. The restricted function will be denoted by the same symbol as the original one. We will construct canonical transformations on $\mathcal{M}_{\pm1}$, they can be extended to the whole phase space by the choice that they act as the identity on the variables ξ_{k_0}, η_{k_0} with $k_0\neq\pm1$.

The proof of theorem 3.2 is absolutely identical to the previous one, one just has to substitute χ_0 with χ_1 defined by

$$\chi_1(z) := \frac{1}{T}\int_0^T t\left[f^{(1)}(\Phi_0^t(z)) - \langle f^{(1)}\rangle(\Phi_0^t(z))\right] dt \ ,$$

where $T := 2\pi/\omega_1$, and $\langle f^{(1)}\rangle$ defined by

$$\left\langle f^{(1)}\right\rangle (z) := \frac{1}{T}\int_0^T f^{(1)}(\Phi_0^t(z))dt \ .$$

A simple computation shows that the function χ_1 defined in (6) is such that

$$\left\{\chi_1, H_{0,L}\right\} + f^{(1)} = \left\langle f^{(1)}\right\rangle ,$$

and that such a χ_1 has an Hamiltonian vector field which is smooth as a map for $\mathcal{P}_\alpha$ to itself. For more details on the construction see [1] lemma 8.4. Here we just point out that in our case we have

$$\chi_1(\xi,\eta) = \frac{-i3\sigma}{8\omega_1^3 2\pi}\int_{\Re} G_1(\xi,\eta)dx_1$$

where

$$\begin{aligned} G_1(\xi,\eta) \quad &= \quad \frac{\xi_1^2\xi_{-1}^2 - \eta_1^2\eta_{-1}^2}{4} + \xi_1^2\xi_{-1}\eta_{-1} + \xi_1\eta_1\eta_{-1}^2 \\ &- \quad \xi_1\eta_1\eta_{-1}^2 - \eta_1^2\xi_1\eta_{-1} \ . \end{aligned}$$

Finally we come to theorem 3.3. First remark that, if one is given a function χ_2 such that

$$\left\{\chi_2, H_{1,L}\right\} + N_1 = N_2 \ ,$$

and such that its vector field is smooth, then the proof can be achieved exactly as before, just substituting the spaces $\mathcal{F}_{s,\alpha}$ to the spaces $\mathcal{P}_\alpha$. It is easy to see that the wanted function is given by

$$\chi_2(z) := -\frac{1}{2}\int_{\Re} \mathrm{sgn}\,(\tau)\tilde{N}_1\left(\Phi_1^\tau(z)\right)d\tau \ ,$$

where $\tilde{N}_1 := N_1 - N_2$, and we denoted by Φ_1^τ the flow generated by the Hamiltonian vector field of $H_{1,L}$ (and by $\mathrm{sgn}\,(\tau)$ the signum of τ, which is one for positive τ and minus one for negative τ). The proof of the regularity of the vector field is more difficult here. To this end we explicitly compute X_{χ_2}. To this end consider the map $Y^t(\xi,\eta)$ defined by

$$\begin{pmatrix} \xi_1(x_1) \\ \eta_1(x_1) \\ \xi_{-1}(x_1) \\ \eta_{-1}(x_1) \end{pmatrix} \mapsto \begin{pmatrix} i\xi_1(x_1)\xi_{-1}(x_1-2tv_1)\eta_1(x_1-2tv_1) \\ -i\xi_1(x_1+2tv_1)\eta_{-1}(x_1+2tv_1)\eta(x_1) \\ i\xi_1(x_1+2v_1t)\xi_{-1}(x_1)\eta_{-1}(x_1+2v_1t) \\ -i\xi_1(x_1-2v_1t)\eta_1(x_1-2v_1t_1)\eta_{-1}(x_1) \end{pmatrix}$$

then one sees that

$$X_{\chi_2}(\xi,\eta) := -\frac{1}{2}\int_{\Re}\mathrm{sgn}\,(\tau)Y^\tau(\xi,\eta)d\tau \ ;$$

moreover, since X_{χ_2} is a polynomial map, in order to show that it is analytic it is enough to show that it is bounded as a polynomial on $\mathcal{F}_s$. So, one has to show that it each component of Y^t considered as a function in $L^1(\Re, W^{s,1})$ is bounded by the third power of the norm of (ξ,η). This is an immediate consequence of the fact that $W^{s,1}$ is an algebra. So theorem 3.3 follows.

7. **Some technical lemmas.**

LEMMA 7.1. *Denote*

$$\begin{aligned} u_a(x_0,x_1) \quad &:= \quad \frac{1}{2\pi}\frac{1}{\sqrt{2}}\sum_{k_0\in\mathbf{Z}}\int_{\Re}\frac{\hat{\xi}_{k_0,k_1}+\hat{\eta}_{k_0,k_1}}{\sqrt{\omega_{k_0}}}\mathrm{e}^{ik_0x_0+ik_1x_1}dx_1 \ , \\ p_a(x_0,x_1) \quad &:= \quad \frac{1}{2\pi}\frac{1}{\sqrt{2}}\sum_{k_0\in\mathbf{Z}}\int_{\Re}\sqrt{\omega_{k_0}}\frac{\hat{\eta}_{k_0,k_1}-\hat{\xi}_{k_0,k_1}}{i}\mathrm{e}^{ik_0x_0+ik_1x_1}dx_1 \ , \end{aligned}$$

and

$$
\begin{aligned}
u(x_0,x_1) &:= \frac{1}{2\pi}\frac{1}{\sqrt{2}}\sum_{k_0\in\mathbf{Z}}\int_{\Re}\frac{\hat\xi_{k_0,k_1}+\hat\eta_{k_0,k_1}}{\sqrt{\omega_{k_0}(\epsilon k_1)}}\mathrm{e}^{ik_0x_0+ik_1x_1}dx_1 \ , \\
p(x_0,x_1) &:= \frac{1}{2\pi}\frac{1}{\sqrt{2}}\sum_{k_0\in\mathbf{Z}}\int_{\Re}\sqrt{\omega_{k_0}(\epsilon k_1)}\frac{\hat\eta_{k_0,k_1}-\hat\xi_{k_0,k_1}}{i}\mathrm{e}^{ik_0x_0+ik_1x_1}dx_1 \ ,
\end{aligned}
$$

which correspond to (9); then one has

$$
\begin{aligned}
\|u-u_a\|_{S^\alpha} &\preceq \epsilon\left\|(\hat\xi,\hat\eta)\right\|_{\alpha+1} \\
\|p-p_a\|_{S^\alpha} &\preceq \epsilon\left\|(\hat\xi,\hat\eta)\right\|_{\alpha+1}
\end{aligned}
\tag{51}
$$

where the norm of $u-u_0$ and of $p-p_0$ has to be computed using its Fourier transform.

Proof. Simply remark that one has

$$
\begin{aligned}
E_{k_0,k_1} &:= \frac{1}{\sqrt{\omega_{k_0}(\epsilon k_1)}}-\frac{1}{\sqrt{\omega_{k_0}}} \\
&= -\epsilon k_1\frac{2k_0+\epsilon k_1}{\sqrt{\omega_{k_0}\omega_{k_0}(\epsilon k_1)}(\sqrt{\omega_{k_0}}+\sqrt{\omega_{k_0}(\epsilon k_1)})(\omega_{k_0}+\omega_{k_0}(\epsilon k_1))}
\end{aligned}
$$

and that the Fourier transform of $u-u_a$ is given by

$$
\frac{1}{\sqrt{2}}E_{k_0,k_1}\left(\hat\xi_{k_0,k_1}+\hat\eta_{k_0,k_1}\right) \ ,
$$

which gives (51). □

Lemma 7.2. *One has*

$$
\left\|X_{H_L}-X_{H_{0,L}}-\epsilon X_{H_{1,L}}-\epsilon^2X_{H_{2,L}}\right\|^{\rho}_{\alpha+3,\alpha}\preceq\epsilon^3
$$

Proof. Applying the (linear) operator to be estimated to a vector $(\hat\xi,\hat\eta)$ one gets a vector with $\hat\xi_{k_0,k_1}$ component given by

$$
\left[\omega_{k_0}(\epsilon k_1)-(\omega_{k_0}+v_{k_0}\epsilon k_1+\hbar_{k_0}\epsilon^2k_1^2)\right]\hat\xi_{k_0,k_1} \ .
$$

Recalling the definition of $v_{k_0},\hbar_{k_0}$ one immediately obtains that the square bracket is estimated by a constant times $\epsilon^3k_1^3$. Proceeding in the same way for the $\hat\eta_{k_0,k_1}$ component one obtains the thesis. □

Given a sequence of function $\hat g_{k_0,k_1}$ such that

$$
\sum_{k_0\in\mathbf{Z}}(1+|k_0|^\alpha)\int_{\Re}(1+|k_1|^\alpha)|\hat g_{k_0,k_1}|dk_1\leq\infty \ ,
$$

consider

$$
g(x):=\sum_{k_0\in\mathbf{Z}}\int_{\Re}g_{k_0,k_1}e^{ik_0x+\epsilon xk_1} \ ,
$$

and

$$
\tilde g(k):=\int_{\Re}g(x)e^{-ikx}dx
$$

Lemma 7.3. *There exists a positive finite C such that*

$$
\int_{\Re}(1+|k|^\alpha)|\tilde g(k)|dk\leq C\sum_{k_0\in\mathbf{Z}}(1+|k_0|^\alpha)\int_{\Re}(1+|k_1|^\alpha)|\hat g_{k_0,k_1}|dk_1
$$

Proof. It is a straightforward computation and it is omitted. □

8. **Appendix.** Consider the FPU system

$$\ddot{u}(j) = -2u(j) + u(j+1) + u(j-1) - \beta\left[(u(j) - u(j-1))^3 + (u(j) - u(j+1))^3\right]$$

and introduce a new unknown $u(j_0, x_1)$ which is periodic of period $n \geq 1$ in j_0, namely

$$u(j_0 + n, x_1) = u(j_0, x_1) \ , \quad x_1 \in \Re$$

and impose that it fulfills the equations

$$\begin{aligned} \ddot{u}(j_0, x_1) &= -2u(j_0, x_1) + u(j_0 + 1, x_1 + \epsilon) + u(j_0 - 1, x_1 - \epsilon) \\ &- \beta\left[(u(j_0, x_1) - u(j_0 - 1, x_1 - \epsilon))^3 + (u(j_0, x_1) - u(j_0 + 1, x_1 + \epsilon))^3\right] \end{aligned}$$

in such a way that $u(j, \epsilon j)$ fulfills (8).

It is useful to introduce Fourier coordinates by

$$\begin{aligned} u(j_0, x_1) &= \sum_{k_0=1}^{n} e^{i\frac{2\pi j_0 k_0}{n}} \int_{\Re} e^{ik_1 x_1} \hat{q}_{k_0,k_1} dk_1 \\ p(j_0, x_1) &= \sum_{k_0=1}^{n} e^{i\frac{2\pi j_0 k_0}{n}} \int_{\Re} e^{ik_1 x_1} \hat{p}_{k_0,k_1} dk_1 \end{aligned}$$

(where p is the momentum conjugated to u). Then a straightforward computation shows that in Fourier coordinates the finite difference operator

$$u(j_0, x_1) \mapsto u(j_0, x_1) - u(j_0 - 1, x_1 - \epsilon) \ . \tag{52}$$

acts as the multiplication of $\hat{q}_{k_0,k_1}$ by $\omega_{k_0}(\epsilon k_1)\delta_{k_0,k_1}$ where

$$\begin{aligned} \omega_{k_0}(\epsilon k_1) &:= 2\sin\left(\frac{2\pi k_0}{n} + \epsilon k_1\right) \ , \\ \delta_{k_0,k_1} &:= \sin\left(\frac{2\pi k_0}{n} + \epsilon k_1\right) + i\cos\left(\frac{2\pi k_0}{n} + \epsilon k_1\right) \ , \end{aligned} \tag{53}$$

and that in such coordinates the quadratic part of the Hamiltonian is diagonal. Again it is useful to introduce complex coordinates as in (10), so that the quadratic part of the Hamiltonian takes the form (11) with index k_0 running from 1 to n, and $\hat{\xi}_{-k_0,-k_1} := \hat{\xi}_{n-k_0,-k_1}$, and $\omega_{k_0}(\epsilon k_1)$ given by (53). Concerning the nonlinearity it is easy to see that its main part has again the form of the function f_0 of sect. 4 but with a different coefficient. So the system has a form suitable for the application of our theory, and one can prove that NLS appears also here as a resonant normal form.

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