Exam of "Pseudodifferential operators, dynamics and applications"

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The goal of the exam is to develop some paradifferential calculus for the Bony-Weyl quantization. You will work with a special class of symbols, defined as the product of a function in $H^s(\mathbb{T}^d)$, $d \geq 1$, and a symbol which depends only on ξ.

Definition 0.1 (Symbols)**.** *Given $\rho \geq 0$ and $m \in \mathbb{R}$, we say that $a(x,\xi) \in \mathcal{N}_\rho^m \equiv \mathcal{N}_\rho^m(\mathbb{T}^d)$ if there exist $\varphi(x) \in H^\rho(\mathbb{T}^d)$ and $d(\xi) \in \mathcal{S}^m$ (classical symbols on $\mathbb{R}^d$), such that*

$$a(x,\xi) := \varphi(x)\, d(\xi).$$

If $a \in \mathcal{N}_\rho^m$, we define the seminorm

$$|a|_{m,\rho,n} := \|\varphi\|_\rho \sup_{|\alpha|\leq n} \sup_{\xi} \| \langle \xi \rangle^{-m+|\alpha|} \, \partial_\xi^\alpha d(\xi) \|_{L^\infty} \,, \tag{1}$$

where $a(x,\xi) = \varphi(x) d(\xi)$. Here we put $\|\varphi\|_\rho \equiv \|\varphi\|_{H^\rho(\mathbb{T}^d)}$.

Next we define the Bony-Weyl quantization. Let $0 < \epsilon_1 < \epsilon_2 \ll 1$ and consider a smooth, even function $\chi \colon \mathbb{R} \to [0,1]$ such that

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq \epsilon_1 \\ 0 & \text{if } |\xi| \geq \epsilon_2 \end{cases}.$$

Given $a \in \mathcal{N}_\rho^m$, the Bony-Weyl quantization of a is the operator

$$\mathrm{Op}^{BW}(a)[u] := \sum_{j\in\mathbb{Z}^d} \Big(\sum_{k\in\mathbb{Z}^d} \widehat{a}\Big(j-k, \frac{j+k}{2}\Big) \chi\Big(\frac{|j-k|}{\langle j+k \rangle}\Big) u_k \Big) e^{\mathrm{i} j\cdot x} \,, \tag{2}$$

where $\widehat{a}(j,\cdot)$ is the Fourier transform of a with respect to the first variable, i.e. $\widehat{a}(j,\xi) := \int_{\mathbb{T}^d} a(x,\xi)\, e^{-\mathrm{i} j\cdot x}\, \mathrm{d}x$.

Fix an arbitrary

$$s_0 > \frac{d}{2}.$$

You are asked to prove the following statements:

1. The seminorm (1) is well defined (note that the choice of $\varphi(x)$ and $d(\xi)$ is not unique), so is the operator in (2).
2. **Continuity:** For any $a \in \mathcal{N}^m_{s_0}$, $\mathrm{Op}^{BW}(a)$ extends to a bounded operator from $H^s \to H^{s-m}$ with the quantitative bound

$$\|\mathrm{Op}^{BW}(a)u\|_{s-m} \lesssim |a|_{m,s_0,0}\, \|u\|_s.$$

Hint: prove that if $\chi\Big(\frac{|k-j|}{\langle k+j\rangle}\Big) \neq 0$ then

$$|j| \sim |j+k| \sim |k|, \quad \forall j,k \in \mathbb{Z}^d\,. \tag{3}$$

3. **Bony paraproduct formula:** $u \in H^s(\mathbb{T}^d)$, $v \in H^r(\mathbb{T}^d)$ with $s + r \geq s_0$. Then one has

$$uv = \mathrm{Op}^{BW}(u)\, v + \mathrm{Op}^{BW}(v)\, u + R(u,v)$$

where the bilinear operator $R\colon H^s \times H^r \to \dot{H}^{s+r-s_0}$ fulfills the estimate

$$\|R(u,v)\|_{s+r-s_0} \lesssim \|u\|_s\, \|v\|_r$$

Hint: decompose

$$1 = \chi_\epsilon\Big(\frac{|j-k|}{\langle j+k\rangle}\Big) + \chi_\epsilon\Big(\frac{|k|}{\langle 2j-k\rangle}\Big) + \theta_\epsilon(j,k).$$

and show that if $\theta_\epsilon(j,k) \neq 0$ then

$$|j| \leq C_\epsilon \min(|j-k|,\ |k|), \qquad \forall j,k \in \mathbb{Z}^d.$$

4. **Symbolic calculus (at first order):** Let $a \in \mathcal{N}^m_{\rho+s_0}$, $b \in \mathcal{N}^{m'}_{\rho+s_0}$ with $m, m' \in \mathbb{R}$ and $\rho > 0$. Then one has

$$\mathrm{Op}^{BW}(a)\,\mathrm{Op}^{BW}(b) = \mathrm{Op}^{BW}\left(ab + \frac{1}{2\mathrm{i}}\{a,b\}\right) + R^{m+m'-\rho}(a,b) \tag{4}$$

where the linear operator $R^{m+m'-\rho}(a,b)\colon H^s \to H^{s-(m+m')+\rho}$, $\forall s \in \mathbb{R}$, with the quantitative estimate

$$\|R(a,b)u\|_{s-(m+m')+\rho} \lesssim \Big(|a|_{m,\rho+s_0,2}\ |b|_{m',s_0,2} + |a|_{m,s_0,2}\ |b|_{m',\rho+s_0,2}\Big)\, \|u\|_s. \tag{5}$$