1 Sharp paradifferential calculus

For any function $u \in \dot{H}^s(\mathbb{T}^d;\mathbb{C})$ we define the Sobolev homogeneous norm

$$\|u\|^2_{\dot{H}^s} := \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^{2s}|\hat{u}_k|^2 \,. \tag{1}$$

Definition 1.1 (Symbols)**.** *Given $\rho \geq 0$ and $m \in \mathbb{R}$, we say that $a(x,\xi) \in \mathcal{N}^m_\rho \equiv \mathcal{N}^m_\rho(\mathbb{T}^d)$ if there exist $\varphi(x) \in H^\rho(\mathbb{T}^d)$ and $d(\xi) \in \mathcal{S}^m$ (classical symbols on $\mathbb{R}^d$), such that*

$$a(x,\xi) := \varphi(x)\, d(\xi). \tag{2}$$

If $a \in \mathcal{N}^m_\rho$, we define the seminorm

$$|a|_{m,\rho,n} := \|\varphi\|_\rho \sup_{|\alpha|\leq n} \sup_\xi \| \langle\xi\rangle^{-m+|\alpha|} \, \partial^\alpha_\xi d(\xi)\|_{L^\infty} \,, \tag{3}$$

where $a(x,\xi) = \varphi(x)d(\xi)$. Here we put $\|\varphi\|_\rho \equiv \|\varphi\|_{H^\rho(\mathbb{T}^d)}$.

Remark 1.2. *Given $a \in \mathcal{N}^m_\rho$, the functions $\varphi(x)$ and $d(\xi)$ are defined up to a multiplicative constant.*

Let $0 < \epsilon < 1$ and consider a smooth function $\chi\colon \mathbb{R} \to [0,1]$ such that

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq \frac{5}{8} \\ 0 & \text{if } |\xi| \geq \frac{8}{10} \end{cases}, \qquad \chi_\epsilon(\xi) := \chi\left(\frac{|\xi|}{\epsilon}\right) .$$

Consider the associated paradifferential operator

$$\mathrm{Op}^{BW}(a)[u] := \sum_{j\in\mathbb{Z}^d} \Big(\sum_{k\in\mathbb{Z}^d} \widehat{a}\Big(j-k, \frac{j+k}{2}\Big)\, \chi\Big(\frac{|j-k|}{\langle j+k\rangle}\Big)\, u_k\Big) e^{\mathrm{i} j\cdot x} \,, \tag{4}$$

where $\widehat{a}(j,\cdot)$ is the Fourier transform of a with respect to the first variable, i.e. $\widehat{a}(j,\xi) := \int_{\mathbb{T}^d} a(x,\xi)\, e^{-\mathrm{i} j\cdot x}\, \mathrm{d}x$.

Remark 1.3. *For a symbol $a \in \mathcal{N}^m_{s_0}$ of the form* (2)*, we have that*

$$\widehat{a}\Big(j-k, \frac{j+k}{2}\Big) = \widehat{\varphi}_{j-k}\; d\Big(\frac{j+k}{2}\Big) \tag{5}$$

Note that if $\chi_\epsilon\Big(\frac{|k-j|}{\langle k+j\rangle}\Big) \neq 0$ then

$$|k-j| \leq \epsilon\langle j+k\rangle \tag{6}$$

and therefore, for $\epsilon \in (0,1)$,

$$\frac{1-\epsilon}{1+\epsilon}|k| \leq |j| \leq \frac{1+\epsilon}{1-\epsilon}|k| \,, \quad \forall j,k \in \mathbb{Z}^d \,. \tag{7}$$

This relation shows in particular that $\mathrm{Op}^{\mathrm{BW}}$ sends a constant function into a constant function and therefore that $\mathrm{Op}^{\mathrm{BW}}$ sends homogenous spaces into homogenous spaces.

Lemma 1.4. *Let $\epsilon \in (0, 1/4)$. If $\chi_\epsilon\Big(\frac{|k-j|}{\langle k+j\rangle}\Big) \neq 0$ then*

$$|j| \leq |j+k| \leq 3|j|, \quad \forall j, k \in \mathbb{Z}^d. \tag{8}$$

Proof. The second inequality in (8) follows by (7). Let us prove the first one. By (7), if $j = 0$ then $k = 0$. Then we suppose that $j \neq 0$, and therefore also $k \neq 0$. Moreover we also have that $j + k \neq 0$. If otherwise $j + k = 0$ then, by (6), we get $|2j| \leq \epsilon$ which implies $j = 0$. By (6) we have

$$|j| \leq \frac{1}{2}|j-k| + \frac{1}{2}|j+k| \leq \frac{\epsilon}{2}|j+k| + \frac{1}{2}|j+k| \leq \frac{5}{8}|j+k|$$

which implies (8). □

Fix an arbitrary

$$s_0 > \frac{d}{2}.$$

Lemma 1.5. *If $a(x,\xi) = m(x)d(\xi) \in \mathcal{N}^m_{s_0}$ and for any $\alpha \in \mathbb{N}^\alpha$ we define the sequence*

$$\mathtt{a}^\alpha_{j_1} := \sup_j \left|\widehat{(\partial^\alpha_\xi a)}_{j_1}(j)\right| \langle j\rangle^{-m+|\alpha|}, \tag{9}$$

then $\mathtt{a}^\alpha_{j_1} \in \ell^1$ with estimate

$$\|\mathtt{a}^\alpha_{j_1}\|_{\ell^1} \leq C(d, s_0)|a|_{m,s_0,|\alpha|}. \tag{10}$$

Proof. We note that thanks to the structure of the symbols we have that

$$\mathtt{a}^\alpha_{j_1} = |\widehat{m}_{j_1}| \sup_j \left|\partial^\alpha_\xi d(j)\right| \langle j\rangle^{-m+|\alpha|}.$$

Defining the constant

$$C(d, s_0) := \|\langle j_1\rangle^{-s_0}\|_{\ell^2} < +\infty,$$

and applying Holder inequality we obtain

$$\begin{aligned}\|\mathtt{a}^\alpha_{j_1}\|_{\ell^1} &= \sup_j \left|\partial^\alpha_\xi d(j)\right| \langle j\rangle^{-m+|\alpha|} \|\widehat{m}_j\|_{\ell^1} \\ &\leq C(d, s_0) \sup_j \left|\partial^\alpha_\xi d(j)\right| \langle j\rangle^{-m+|\alpha|} \|\langle j_1\rangle^{s_0}\widehat{m}_j\|_{\ell^2} = C(d, s_0)|a|_{m,s_0,|\alpha|}.\end{aligned}$$

□

Theorem 1.6. (Continuity) *Let $a \in \mathcal{N}^m_{s_0}$ with $m \in \mathbb{R}$. Then $\mathrm{Op}^{BW}(a)$ extends to a bounded operator $\dot{H}^s \to \dot{H}^{s-m}$ for any $s \in \mathbb{R}$ with the quantitative estimate*

$$\|\mathrm{Op}^{BW}(a)\, u\|_{\dot{H}^{s-m}} \leq C_{m,d,s_0}\ |a|_{m,s_0,0}\ \|u\|_{\dot{H}^s}. \tag{11}$$

Proof. We have that

$$\begin{aligned}\|\mathrm{Op}^{BW}(a)\,[u]\|^2_{\dot{H}^{s-m}} &\leq \sum_{j\in\mathbb{Z}^d\setminus\{0\}} |j|^{2(s-m)} \Big(\sum_{|j-k|\leq\epsilon\langle j+k\rangle} \Big|\hat{a}\Big(j-k, \frac{j+k}{2}\Big)\Big| |u_k|\Big)^2 \\ &\overset{(7),(8)}{\lesssim} \sum_{j\in\mathbb{Z}^d\setminus\{0\}} \Big(\sum_{|j-k|\leq\epsilon\langle j+k\rangle} \langle j-k\rangle^{s_0} \Big|\hat{a}\Big(j-k, \frac{j+k}{2}\Big)\Big| |j+k|^{-m}|u_k||k|^s \frac{1}{\langle j-k\rangle^{s_0}}\Big)^2\end{aligned}$$

where $k \neq 0$ and $k+j \neq 0$, since $j \neq 0$ and by (7), (8). Using Cauchy-Schwartz

$$\begin{aligned}\|\mathrm{Op}^{BW}(a)\,[u]\|^2_{\dot H^{s-m}} &\lesssim \sum_{j\in\mathbb{Z}^d\setminus\{0\}}\ \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \langle j-k\rangle^{2s_0}\Big|\hat a\Big(j-k,\frac{j+k}{2}\Big)\Big|^2\langle j+k\rangle^{-2m}|u_k|^2|k|^{2s}\sum_k\frac{1}{\langle k-j\rangle^{2s_0}}\\ &\lesssim_{s_0}\sum_{k\in\mathbb{Z}^d\setminus\{0\}}|u_k|^2|k|^{2s}\sum_{j\in\mathbb{Z}^d\setminus\{0\}}\langle j-k\rangle^{2s_0}\Big|\hat a\Big(j-k,\frac{j+k}{2}\Big)\Big|^2\langle j+k\rangle^{-2m}\,.\end{aligned}$$

In conclusion, for a symbol a of the form (2) we have

$$\begin{aligned}\|\mathrm{Op}^{BW}(a)\,[u]\|^2_{\dot H^{s-m}} &\lesssim_{s_0}\sum_{k\in\mathbb{Z}^d\setminus\{0\}}|u_k|^2|k|^{2s}\ \|\varphi\|^2_{\dot H^{s_0}}\sup_{\xi\in\mathbb{R}^d}\big(|d(\xi)|\langle\xi\rangle^{-m}\big)^2\\ &\le C(s_0)\|u\|^2_{\dot H^s}\ |a|^2_{m,s_0,0}\end{aligned}$$

which proves (11). □

We now prove the Bony-paraproduct decomposition

Lemma 1.7 (Bony paraproduct decomposition)**.** *for any functions $u, v \in H^s(\mathbb{T}^d)$, $s > s_0$, one has*

$$uv = \mathrm{Op}^{BW}(u)\,v + \mathrm{Op}^{BW}(v)\,u + R(u,v)$$

where the bilinear operator $R(u,v)$ fulfills, $\forall\, 0<\rho<s-s_0$, the estimate

$$\|R(u,v)\|_{s+\rho}\lesssim\|u\|_s\,\|v\|_{s_0+\rho}$$

The composition of paradifferential operators is a paradifferential operator plus a regularizing remainder. We treat first the case of paraproducts, namely when the symbols are simply functions in Sobolev space. Then we treat the case of general symbols.

In both cases an important role is played by the function

$$\phi:(\mathbb{Z}^d)^3\to\mathbb{R},\qquad \phi(j_1,j_2,j_3):=\chi(j_1,j_2+j_3)\chi(j_2,j_3)-\chi(j_1+j_2,j_3) \tag{12}$$

Lemma 1.8. *Consider the function ϕ defined in* (12)*. Then on its support:*

(i) the maximum frequency among them has modulus comparable with the second largest frequency; in formula

$$\max_2(|j_1|,|j_2|,|j_3|)\sim\max(|j_1|,|j_2|,|j_3|),\qquad \forall j_1,j_2,j_3\in\text{supp }(12). \tag{13}$$

(ii) Let $2\epsilon_2(1+\epsilon_2)<\delta<1$, where ϵ_2 defined in (**??**)*; then*

$$|j_1+j_2|\le\delta|j_3|,\qquad\forall j_1,j_2,j_3\in\text{supp }(12) \tag{14}$$

and

$$(1-\delta)\,|j_3|\le|j_1+j_2+j_3|\le(1+\delta)\,|j_3|\,,\qquad\forall j_1,j_2,j_3\in\text{supp }(12) \tag{15}$$

Proof. Let us prove first item (i).

- If $|j_1|\gg|j_2|,|j_3|$, then $\chi(j_1,j_2+j_3)=\chi(j_1+j_2,j_3)=0$, so there are such interactions in (19).

- If $|j_2| \gg |j_1|, |j_3|$, then $\chi(j_2, j_3) = \chi(j_1 + j_2, j_3) = 0$, so no such terms.
- If $|j_3| \gg |j_1|, |j_2|$, then $\chi(j_1, j_2 + j_3) = \chi(j_2, j_3) = \chi(j_1 + j_2, j_3) = 1$, so again there are no such terms.

Of course one can make (13) quantitative in terms of ϵ_1,ϵ_2, but this is not required.

We come to item (*ii*). To prove it, we show the implication

$$|j_1 + j_2| > \delta |j_3| \quad \Rightarrow \quad \phi(j_1, j_2, j_3) = 0. \tag{16}$$

So first note that $|j_1 + j_2| > \delta|j_3| > \epsilon_2 |j_3|$ implies that $\chi(j_1 + j_2, j_3) = 0$.
There are two cases: either
(a) $|j_1| > \frac{\delta}{2}|j_3|$
or
(b) $|j_2| > \frac{\delta}{2}|j_3|$;
otherwise $|j_1 + j_2| \le |j_1| + |j_2| \le \delta|j_3|$ contradicting the assumption (16).
In case (b), $|j_2| > \frac{\delta}{2}|j_3| > \epsilon_2|j_3|$ which implies $\chi(j_2, j_3) = 0$, proving (16).
In case (a), assume that both $|j_1| \le \epsilon_2|j_2 + j_3|$ and $|j_2| \le \epsilon_2|j_3|$; then

$$\frac{\delta}{2}|j_3| < |j_1| \le \epsilon_2|j_2 + j_3| \le (\epsilon_2 + \epsilon_2^2)|j_3|$$

which contradicts the assumption $2\epsilon_2(1 + \epsilon_2) < \delta$. Then either $|j_1| \le \epsilon_2|j_2 + j_3|$ or $|j_2| \le \epsilon_2|j_3|$; in both cases $\chi(j_1, j_2 + j_3)\chi(j_2, j_3) = 0$ and (16) follows. □

We are ready to prove a result of paraproducts.

Theorem 1.9 (Composition of paraproducts)**.** *Let $a, b \in H^{\rho+s_0}$ with $\rho > 0$. Then*

$$\mathrm{Op}^B(a)\,\mathrm{Op}^B(b) = \mathrm{Op}^B(ab) + R(a, b) \tag{17}$$

where the linear operator $R(a, b)\colon H^s \to H^{s+\rho}$ $\forall s \in \mathbb{R}$, with the quantitative estimate

$$\|R(a, b)u\|_{s+\rho} \le C_s \left(|a|_{0,s_0,0}\, |b|_{0,\rho+s_0,0} + |a|_{0,\rho+s_0,0}\, |b|_{0,s_0,0} \right) \|u\|_s. \tag{18}$$

Proof. Using (**??**) and the fact that a, b are functions (so are symbols independent of ξ), we get

$$\mathrm{Op}^B(a)\,\mathrm{Op}^B(b)\, u = \sum_{j_1, j_2, j_3 \in \mathbb{Z}^d} \chi(j_1, j_2 + j_3)\chi(j_2, j_3)\, \widehat{a}_{j_1}\, \widehat{b}_{j_2}\, \widehat{u}_{j_3}\, e^{\mathrm{i}(j_1+j_2+j_3)\cdot x}$$

and

$$\mathrm{Op}^B(ab)\, u = \sum_{j_1, j_2, j_3 \in \mathbb{Z}^d} \chi(j_1 + j_2, j_3)\, \widehat{a}_{j_1}\, \widehat{b}_{j_2}\, \widehat{u}_{j_3}\, e^{\mathrm{i}(j_1+j_2+j_3)\cdot x};$$

therefore we obtain that

$$\begin{aligned} R(a, b)u &= \mathrm{Op}^B(a)\,\mathrm{Op}^B(b)\, u - \mathrm{Op}^B(ab)\, u \\ &= \sum_{j_1, j_2, j_3 \in \mathbb{Z}^d} \Big(\chi(j_1, j_2 + j_3)\chi(j_2, j_3) - \chi(j_1 + j_2, j_3) \Big)\, \widehat{a}_{j_1}\, \widehat{b}_{j_2}\, \widehat{u}_{j_3}\, e^{\mathrm{i}(j_1+j_2+j_3)\cdot x} \\ &= \sum_{j_1, j_2, j_3 \in \mathbb{Z}^d} \phi(j_1, j_2, j_3)\, \widehat{a}_{j_1}\, \widehat{b}_{j_2}\, \widehat{u}_{j_3}\, e^{\mathrm{i}(j_1+j_2+j_3)\cdot x} \end{aligned} \tag{19}$$

We exploit the property of the function ϕ, described in Lemma 1.8 to estimate the norm of the operator $R(a,b)$.

First, using (15), we have that $\forall s \in \mathbb{R}$,

$$\langle j_3 \rangle^{s+\rho} \lesssim \langle j_1 + j_2 + j_3 \rangle^{s+\rho} \lesssim \langle j_3 \rangle^{s+\rho}.$$

$$\begin{aligned} \|R(a,b)u\|_{s+\rho}^2 &\leq \sum_{j \in \mathbb{Z}^d} \langle j \rangle^{2s+2\rho} \left| \sum_{j_1+j_2+j_3=j} \phi(j_1,j_2,j_3)\, \widehat{a}_{j_1}\, \widehat{b}_{j_2}\, \widehat{u}_{j_3} \right|^2 \\ &\overset{(15)}{\lesssim} \sum_j \left| \sum_{j_1+j_2+j_3=j} |\phi(j_1,j_2,j_3)|\, \langle j_3 \rangle^\rho\, |\widehat{a}_{j_1}|\, \left|\widehat{b}_{j_2}\right|\, \langle j_3 \rangle^s\, |\widehat{u}_{j_3}| \right|^2. \end{aligned}$$

Now we split the internal sum according to which frequency is the largest: denoting

$$f(j_1,j_2,j_3) := |\phi(j_1,j_2,j_3)|\, \langle j_3 \rangle^\rho\, |\widehat{a}_{j_1}|\, \left|\widehat{b}_{j_2}\right|\, \langle j_3 \rangle^s\, |\widehat{u}_{j_3}|,$$

we put

$$\begin{aligned} R_1 &:= \sum_j \Big(\sum_{\substack{j_1+j_2+j_3=j \\ |j_1| \geq |j_2| \geq |j_3|}} f(j_1,j_2,j_3) \Big)^2, \qquad & R_2 &:= \sum_j \Big(\sum_{\substack{j_1+j_2+j_3=j \\ |j_1| \geq |j_3| \geq |j_2|}} f(j_1,j_2,j_3) \Big)^2 \\ R_3 &:= \sum_j \Big(\sum_{\substack{j_1+j_2+j_3=j \\ |j_2| \geq |j_1| \geq |j_3|}} f(j_1,j_2,j_3) \Big)^2, \qquad & R_4 &:= \sum_j \Big(\sum_{\substack{j_1+j_2+j_3=j \\ |j_2| \geq |j_3| \geq |j_1|}} f(j_1,j_2,j_3) \Big)^2 \\ R_5 &:= \sum_j \Big(\sum_{\substack{j_1+j_2+j_3=j \\ |j_3| \geq |j_2| \geq |j_1|}} f(j_1,j_2,j_3) \Big)^2, \qquad & R_6 &:= \sum_j \Big(\sum_{\substack{j_1+j_2+j_3=j \\ |j_3| \geq |j_1| \geq |j_2|}} f(j_1,j_2,j_3) \Big)^2 \end{aligned}$$

Clearly we have

$$\|R(a,b)u\|_{s+\rho}^2 \lesssim R_1 + \ldots + R_6,$$

so we proceed estimating each term. Let us first consider R_1. For this term $|j_3|$ is the smallest frequency, so we have

$$f(j_1,j_2,j_3) \leq |\widehat{a}_{j_1}|\, \langle j_2 \rangle^\rho\, \left|\widehat{b}_{j_2}\right|\, \langle j_3 \rangle^s\, |\widehat{u}_{j_3}| \tag{20}$$

and conclude by Young's convolution inequality

$$\begin{aligned} R_1 &\leq \sum_j \Big(\sum_{j_1+j_2+j_3=j} |\widehat{a}_{j_1}|\, \langle j_2 \rangle^\rho\, \left|\widehat{b}_{j_2}\right|\, \langle j_3 \rangle^s\, |\widehat{u}_{j_3}| \Big)^2 \\ &= \|(\widehat{a}_j)_j * (\langle j \rangle^\rho\, \widehat{b}_j)_j * (\langle j \rangle^s\, \widehat{u}_j)_j\|_{\ell^2(\mathbb{Z}^d)}^2 \\ &\leq \|(\widehat{a}_j)_j\|_{\ell^1(\mathbb{Z}^d)}^2\, \|(\langle j \rangle^\rho\, \widehat{b}_j)_j\|_{\ell^1(\mathbb{Z}^d)}^2\, \|(\langle j \rangle^s\, \widehat{u}_j)_j\|_{\ell^2(\mathbb{Z}^d)}^2 \\ &\leq (\|a\|_{s_0}\, \|b\|_{\rho+s_0}\, \|u\|_s)^2 \end{aligned} \tag{21}$$

One proceeds similarly for R_2, R_3, R_4, getting

$$R_2,\, R_3 \leq (\|a\|_{\rho+s_0}\, \|b\|_{s_0}\, \|u\|_s)^2, \qquad R_4 \leq (\|a\|_{s_0}\, \|b\|_{\rho+s_0}\, \|u\|_s)^2. \tag{22}$$

We come to R_5. In this case we exploit that the largest frequency is comparable with the second largest frequency according to (13). Thus, for this term, we have that $|j_3| \le C|j_2|$ for C sufficiently large. So again $f(j_1, j_2, j_3)$ is estimated as in (20), and thus R_5 fulfills an estimate as in (21). One proceeds analogously for R_6, proving that it fulfills an estimate as the first one of (22). Now recall that $\|a\|_\mu = |a|_{0,\mu,0}$ and collect all the estimates to get (18). □

Next we consider the case when a, b are symbols depending in a nontrivial way from ξ.

Theorem 1.10 (Composition of paradifferential operators)**.** *Let $a \in \mathcal{N}^m_{\rho+s_0}$, $b \in \mathcal{N}^{m'}_{\rho+s_0}$ with $m, m' \in \mathbb{R}$ and $\rho > 0$. Define the symbol*

$$a \#_\rho b := \sum_{|\alpha|<\rho} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi) \quad \in \quad \sum_{k<\rho} \Gamma^{m+m'-k}_{\rho+s_0-k}. \tag{23}$$

Then one has

$$\mathrm{Op}^B(a)\,\mathrm{Op}^B(b) = \mathrm{Op}^B(a\#_\rho b) + R^{m+m'-\rho}(a,b) \tag{24}$$

where the linear operator $R^{m+m'-\rho}(a,b)\colon H^s \to H^{s-(m+m')+\rho}$, $\forall s \in \mathbb{R}$, with the quantitative estimate

$$\|R(a,b)u\|_{s-(m+m')+\rho} \lesssim \Big(|a|_{m,\rho+s_0,\rho}\, |b|_{m',s_0,0} + |a|_{m,s_0,\rho}\, |b|_{m',\rho+s_0,0} \Big) \|u\|_s. \tag{25}$$

Proof. Because $H^s(\mathbb{T}^d)$ is an algebra for $s > s_0$, we first note that

$$a \in \mathcal{N}^m_{\rho+s_0}, \quad b \in \mathcal{N}^{m'}_{\rho+s_0} \quad \Rightarrow \quad ab \in \mathcal{N}^{m+m'}_{\rho+s_0}.$$

Now, for $|\alpha| \le \rho$

$$\partial_\xi^\alpha a \in \mathcal{N}^{m-|\alpha|}_{\rho+s_0}, \qquad D_x^\alpha b \in \mathcal{N}^{m'}_{\rho-|\alpha|+s_0} \quad \Rightarrow \quad \partial_\xi^\alpha a\; D_x^\alpha b \in \mathcal{N}^{m+m'-|\alpha|}_{\rho-|\alpha|+s_0}.$$

This proves that $a\#_\rho b$ (defined in (40)) is a symbol in $\sum_{k<\rho} \mathcal{N}^{m+m'-k}_{\rho+s_0-k}$.

Next we compute $\mathrm{Op}^B(a)\,\mathrm{Op}^B(b)$. Using formula (**??**) for $\mathrm{Op}^B(a)$ and formula (**??**) for $\mathrm{Op}^B(b)$, we write

$$\begin{aligned}
\mathrm{Op}^B(a)\,\mathrm{Op}^B(b)\,u &= \frac{1}{(2\pi)^{2d}} \sum_{j_2,j_3} a_\chi(x, j_2+j_3)\, \widehat{b}_{j_2}(j_3) \chi(j_2,j_3)\, \widehat{u}_{j_3}\, e^{\mathrm{i}(j_2+j_3)\cdot x} \\
&= \frac{1}{(2\pi)^{2d}} \sum_{j_1,j_2,j_3} \chi(j_1, j_2+j_3)\, \chi(j_2,j_3)\, \widehat{a}_{j_1}(j_2+j_3)\, \widehat{b}_{j_2}(j_3)\, \widehat{u}_{j_3}\, e^{\mathrm{i}(j_1+j_2+j_3)\cdot x}
\end{aligned}$$

Now we consider $\widehat{a}_{j_1}(j_2+j_3)$ and perform a Taylor expansion around j_3 with increment j_2 up to order $n \le \rho$, getting

$$\widehat{a}_{j_1}(j_2+j_3) = \sum_{|\alpha|<\rho} \frac{1}{\alpha!} \partial_\xi^\alpha \widehat{a}_{j_1}(j_3) j_2^\alpha + \mathtt{R}_\rho(a; j_1, j_2, j_3) \tag{26}$$

$$\mathtt{R}_\rho(a; j_1, j_2, j_3) := \sum_{|\alpha|=\lfloor\rho\rfloor+1} j_2^\alpha \frac{|\alpha|}{\alpha!} \int_0^1 (1-t)^{|\alpha|-1} \partial_\xi^\alpha \widehat{a}_{j_1}(j_3 + t j_2) \mathrm{d}t \tag{27}$$

Consider now the product $\widehat{a}_{j_1}(j_2+j_3)\,\widehat{b}_{j_2}(j_3)$; substituiting the Taylor expansion of $\widehat{a}_{j_1}(j_2+j_3)$ and using that

$$\partial_\xi^\alpha \widehat{a}_{j_1}(j_3) = \widehat{(\partial_\xi^\alpha a)}_{j_1}(j_3), \qquad j_2^\alpha\, \widehat{b}_{j_2}(j_3) = \widehat{(D_x^\alpha b)}_{j_2}(j_3),$$

we write

$$\widehat{a}_{j_1}(j_2+j_3)\,\widehat{b}_{j_2}(j_3) = \sum_{|\alpha|<\rho} \frac{1}{\alpha!} \widehat{(\partial_\xi^\alpha a)}_{j_1}(j_3)\,\widehat{(D_x^\alpha b)}_{j_2}(j_3) + \mathtt{R}_n(a; j_1, j_2, j_3)\widehat{b}_{j_2}(j_3)$$

and finally get

$$\begin{aligned} &\mathrm{Op}^B(a)\,\mathrm{Op}^B(b)\,u \\ &= \frac{1}{(2\pi)^{2d}} \sum_{|\alpha|<\rho} \frac{1}{\alpha!} \sum_{j_1,j_2,j_3} \chi(j_1, j_2+j_3)\,\chi(j_2,j_3)\,\widehat{(\partial_\xi^\alpha a)}_{j_1}(j_3)\,\widehat{(D_x^\alpha b)}_{j_2}(j_3)\,\widehat{u}_{j_3}\, e^{\mathrm{i}(j_1+j_2+j_3)\cdot x} \\ &\quad + \frac{1}{(2\pi)^{2d}} \sum_{j_1,j_2,j_3} \chi(j_1, j_2+j_3)\,\chi(j_2,j_3)\,\mathtt{R}_\rho(a; j_1,j_2,j_3)\,\widehat{b}_{j_2}(j_3)\,\widehat{u}_{j_3}\, e^{\mathrm{i}(j_1+j_2+j_3)\cdot x} \end{aligned} \tag{28}$$

We come to the term $\mathrm{Op}^B(a\#_\rho b)\,u$. We compute

$$\begin{aligned} \mathrm{Op}^B(a\#_\rho b)\,u &= \frac{1}{(2\pi)^d} \sum_{j_3} (a\#_\rho b)_\chi(x, j_3)\,\widehat{u}_{j_3}\, e^{\mathrm{i}j_3\cdot x} \\ &= \frac{1}{(2\pi)^d} \sum_{j,j_3} \chi(j,j_3)\,\widehat{(a\#_\rho b)}_j(j_3)\,\widehat{u}_{j_3}\, e^{\mathrm{i}(j+j_3)\cdot x} \\ &= \frac{1}{(2\pi)^{2d}} \sum_{|\alpha|<\rho} \frac{1}{\alpha!} \sum_{j_1,j_2,j_3} \chi(j_1+j_2, j_3)\,\widehat{(\partial_\xi^\alpha a)}_{j_1}(j_3)\,\widehat{(D_x^\alpha b)}_{j_2}(j_3)\,\widehat{u}_{j_3}\, e^{\mathrm{i}(j_1+j_2+j_3)\cdot x} \end{aligned}$$

where in the last passage we used that, by the definition (40),

$$\widehat{(a\#_\rho b)}_j(j_3) = \frac{1}{(2\pi)^d} \sum_{|\alpha|<\rho} \frac{1}{\alpha!} \sum_{j_1+j_2=j} \widehat{(\partial_\xi^\alpha a)}_{j_1}(j_3)\,\widehat{(D_x^\alpha b)}_{j_2}(j_3).$$

Therefore we found that, with $\phi(j_1, j_2, j_3)$ defined in (12),

$$\begin{aligned} &\big(\mathrm{Op}^B(a)\,\mathrm{Op}^B(b) - \mathrm{Op}^B(a\#_\rho b)\big)\,u = R_I(a,b)u + R_{II}(a,b)u \\ &R_I(a,b)u := \frac{1}{(2\pi)^{2d}} \sum_{|\alpha|<\rho} \frac{1}{\alpha!} \sum_{j_1,j_2,j_3} \phi(j_1,j_2,j_3)\widehat{(\partial_\xi^\alpha a)}_{j_1}(j_3)\,\widehat{(D_x^\alpha b)}_{j_2}(j_3)\,\widehat{u}_{j_3}\, e^{\mathrm{i}(j_1+j_2+j_3)\cdot x} \\ &R_{II}(a,b)u := \frac{1}{(2\pi)^{2d}} \sum_{j_1,j_2,j_3} \chi(j_1, j_2+j_3)\,\chi(j_2,j_3)\,\mathtt{R}_\rho(a; j_1,j_2,j_3)\,\widehat{b}_{j_2}(j_3)\,\widehat{u}_{j_3}\, e^{\mathrm{i}(j_1+j_2+j_3)\cdot x} \end{aligned} \tag{29}$$

We show now that the opeartors R_I and R_{II} fulfills estimate (42). We begin with $R_I(a,b)u$.

Using Lemma 1.8 (ii), we have that

$$\begin{aligned} &\|R_I(a,b)u\|^2_{s-(m+m')+\rho} \\ &\lesssim \sum_{|\alpha|<\rho} \sum_{j\in\mathbb{Z}^d} \langle j\rangle^{2(s-(m+m')+\rho)} \Bigg| \sum_{j_1+j_2+j_3=j} \phi(j_1,j_2,j_3)\,\widehat{(\partial_\xi^\alpha a)}_{j_1}(j_3)\,\widehat{(D_x^\alpha b)}_{j_2}(j_3)\,\widehat{u}_{j_3} \Bigg|^2 \\ &\overset{(15)}{\lesssim} \sum_{j,\alpha} \Bigg| \sum_{j_1+j_2+j_3=j} |\phi(j_1,j_2,j_3)|\,\langle j_3\rangle^\rho \left|\widehat{(\partial_\xi^\alpha a)}_{j_1}(j_3)\right| \langle j_3\rangle^{-m} \left|\widehat{(D_x^\alpha b)}_{j_2}(j_3)\right| \langle j_3\rangle^{-m'}\,\langle j_3\rangle^s\,|\widehat{u}_{j_3}| \Bigg|^2 . \end{aligned}$$

We split the internal sum according to which frequency is the largest: denoting

$$f^\alpha(j_1,j_2,j_3) := |\phi(j_1,j_2,j_3)|\ \langle j_3\rangle^\rho \left|\widehat{(\partial_\xi^\alpha a)}_{j_1}(j_3)\right| \langle j_3\rangle^{-m}\ \left|\widehat{(D_x^\alpha b)}_{j_2}(j_3)\right| \langle j_3\rangle^{-m'}\ \langle j_3\rangle^s\,|\widehat{u}_{j_3}|\,, \tag{30}$$

we put

$$R_1 := \sum_{\alpha,j}\Big(\sum_{\substack{j_1+j_2+j_3=j\\ |j_1|\ge|j_2|\ge|j_3|}} f^\alpha(j_1,j_2,j_3)\Big)^2, \qquad R_2 := \sum_{\alpha,j}\Big(\sum_{\substack{j_1+j_2+j_3=j\\ |j_1|\ge|j_3|\ge|j_2|}} f^\alpha(j_1,j_2,j_3)\Big)^2$$

$$R_3 := \sum_{\alpha,j}\Big(\sum_{\substack{j_1+j_2+j_3=j\\ |j_2|\ge|j_1|\ge|j_3|}} f^\alpha(j_1,j_2,j_3)\Big)^2, \qquad R_4 := \sum_{\alpha,j}\Big(\sum_{\substack{j_1+j_2+j_3=j\\ |j_2|\ge|j_3|\ge|j_1|}} f^\alpha(j_1,j_2,j_3)\Big)^2$$

$$R_5 := \sum_{\alpha,j}\Big(\sum_{\substack{j_1+j_2+j_3=j\\ |j_3|\ge|j_2|\ge|j_1|}} f^\alpha(j_1,j_2,j_3)\Big)^2, \qquad R_6 := \sum_{\alpha,j}\Big(\sum_{\substack{j_1+j_2+j_3=j\\ |j_3|\ge|j_1|\ge|j_2|}} f^\alpha(j_1,j_2,j_3)\Big)^2$$

Clearly we have

$$\|R_I(a,b)u\|^2_{s-(m+m')+\rho} \lesssim R_1+\ldots+R_6,$$

so we proceed estimating each term. Let us first consider R_1. For this term $|j_3|$ is the smallest frequency, so $\langle j_3\rangle^\rho = \langle j_3\rangle^{\rho-|\alpha|}\,\langle j_3\rangle^{|\alpha|} \le \langle j_2\rangle^{\rho-|\alpha|}\,\langle j_3\rangle^{|\alpha|}$, and we get

$$\begin{aligned} f^\alpha(j_1,j_2,j_3) &\le \left|\widehat{(\partial_\xi^\alpha a)}_{j_1}(j_3)\right| \langle j_3\rangle^{-m+|\alpha|}\ \langle j_2\rangle^{n-|\alpha|} \left|\widehat{(D_x^\alpha b)}_{j_2}(j_3)\right| \langle j_3\rangle^{-m'}\ \langle j_3\rangle^s\,|\widehat{u}_{j_3}| \\ &\le \mathtt{a}^\alpha_{j_1}\,\mathtt{b}^\alpha_{j_2}\ \langle j_3\rangle^s\,|\widehat{u}_{j_3}| \end{aligned} \tag{31}$$

where the sequences $\mathtt{a}^\alpha = (\mathtt{a}^\alpha_j)_j$ and $\mathtt{b}^\alpha = (\mathtt{b}^\alpha_j)_j$ are defined by

$$\mathtt{a}^\alpha_{j_1} := \sup_{j_3}\left|\widehat{(\partial_\xi^\alpha a)}_{j_1}(j_3)\right| \langle j_3\rangle^{-m+|\alpha|}\,, \qquad \mathtt{b}^\alpha_{j_2} := \langle j_2\rangle^{n-|\alpha|}\sup_{j_3}\left|\widehat{(D_x^\alpha b)}_{j_2}(j_3)\right| \langle j_3\rangle^{-m'}\,.$$

By Young's convolution inequality we deduce

$$R_1 \lesssim \sum_\alpha \|\mathtt{a}^\alpha * \mathtt{b}^\alpha * (\langle j\rangle^s\,\widehat{u}_j)_j\|^2_{\ell^2(\mathbb{Z}^d)} \lesssim \sum_\alpha \|\mathtt{a}^\alpha\|^2_{\ell^1(\mathbb{Z}^d)}\ \|\mathtt{b}^\alpha\|^2_{\ell^1(\mathbb{Z}^d)}\ \|(\langle j\rangle^s\,\widehat{u}_j)_j\|^2_{\ell^2(\mathbb{Z}^d)}, \tag{32}$$

so we need only to show that the sequences $\mathtt{a}^\alpha, \mathtt{b}^\alpha$ are both in $\ell^1(\mathbb{Z}^d)$. To prove this, we apply Lemma 1.5 to a, getting

$$\|\mathtt{a}^\alpha\|_{\ell^1(\mathbb{Z}^d)} \lesssim |a|_{m,s_0,|\alpha|}\,, \tag{33}$$

and we apply the same Lemma to b, getting

$$\|\mathtt{b}^\alpha\|_{\ell^1(\mathbb{Z}^d)} \lesssim |D_x^\alpha b|_{m',s_0,0} \lesssim |b|_{m',\rho+s_0,0}\,. \tag{34}$$

Combining (32) with the estimates (33), (34) we find

$$R_1 \lesssim \Big(|a|_{m,s_0,n}\ |b|_{m',\rho+s_0,0}\ \|u\|_s\Big)^2\ . \tag{35}$$

Consider now R_2. In this case $|j_2| \le |j_3| \le |j_1|$. Moreover by Lemma 1.8 (*ii*) the largest and second largest frequency are equivalent, so $|j_3| \sim |j_1|$. It follows that $f^\alpha(j_1,j_2,j_3)$ in (30) is estimated by

$$\begin{aligned} f^\alpha(j_1,j_2,j_3) &\le \langle j_1\rangle^\rho\ \left|\widehat{(\partial_\xi^\alpha a)}_{j_1}(j_3)\right| \langle j_3\rangle^{-m+|\alpha|}\ \langle j_2\rangle^{-|\alpha|} \left|\widehat{(D_x^\alpha b)}_{j_2}(j_3)\right| \langle j_3\rangle^{-m'}\ \langle j_3\rangle^s\,|\widehat{u}_{j_3}| \\ &\le \widetilde{\mathtt{a}}^\alpha_{j_1}\,\widetilde{\mathtt{b}}^\alpha_{j_2}\ \langle j_3\rangle^s\,|\widehat{u}_{j_3}| \end{aligned} \tag{36}$$

where now the sequences $\widetilde{\mathtt{a}}^\alpha = (\widetilde{\mathtt{a}}^\alpha_j)_j$ and $\widetilde{\mathtt{b}}^\alpha = (\widetilde{\mathtt{b}}^\alpha_j)_j$ are defined by

$$\widetilde{\mathtt{a}}^\alpha_{j_1} := \langle j_1\rangle^n \sup_{j_3} \left|\widehat{(\partial_\xi^\alpha a)}_{j_1}(j_3)\right| \langle j_3\rangle^{-m+|\alpha|}, \qquad \widetilde{\mathtt{b}}^\alpha_{j_2} := \langle j_2\rangle^{-|\alpha|} \sup_{j_3} \left|\widehat{(D_x^\alpha b)}_{j_2}(j_3)\right| \langle j_3\rangle^{-m'}.$$

Again we need to estimate the $\ell^1(\mathbb{Z}^d)$ norm of $\widetilde{\mathtt{a}}^\alpha$ and $\widetilde{\mathtt{b}}^\alpha$. We apply Lemma 1.5 , getting

$$\|\widetilde{\mathtt{a}}^\alpha\|_{\ell^1(\mathbb{Z}^d)} \lesssim |a|_{m,\rho+s_0,n} ,$$
$$\|\widetilde{\mathtt{b}}^\alpha\|_{\ell^1(\mathbb{Z}^d)} \lesssim |b|_{m',s_0,0} .$$

We are ready to estimate R_2. We apply Young's convolution inequality and obtain that

$$R_2 \lesssim \Big(|a|_{m,\rho+s_0,n}\ |b|_{m,s_0,0}\ \|u\|_s\Big)^2 \tag{37}$$

The terms R_3, R_4 are estimated in an analogous way. Concerning R_5, R_6, one proceeds similarly exploiting that, according to (13), the largest frequency is comparable with the second largest frequency. Collecting all the estimates one obtains that $R_I(a,b)$ fulfills estimate (42).

We come to $R_{II}(a,b)$ defined in (29). First note that on the support of this term we have

$$\langle j_1\rangle \le \epsilon_2 \langle j_2 + j_3\rangle, \qquad \langle j_2\rangle \le \epsilon_2 \langle j_3\rangle ;$$

in particular we have that

$$|j_1 + j_2| \le \epsilon_2 |j_2| + 2\epsilon_2 |j_3| \le (\epsilon_2^2 + 2\epsilon_2)|j_3|$$

and provided $\epsilon_2^2 + 2\epsilon_2 < 1$ we have that

$$\langle j_1 + j_2 + j_3\rangle \sim \langle j_3\rangle .$$

With this information we compute

$$\begin{aligned}
&\|R_{II}(a,b)u\|^2_{s-(m+m')+\rho} \\
&\lesssim \sum_j \langle j\rangle^{2(s-(m+m')+\rho)} \left| \sum_{j_1+j_2+j_3=j} \chi(j_1, j_2+j_3)\, \chi(j_2,j_3)\, \mathtt{R}_\rho(a; j_1,j_2,j_3)\, \widehat{b}_{j_2}(j_3)\, \widehat{u}_{j_3} \right|^2 \\
&\lesssim \sum_j \left| \sum_{j_1+j_2+j_3=j} |\mathtt{R}_\rho(a;j_1,j_2,j_3)|\ \langle j_3\rangle^{-m+\rho}\ \left|\widehat{b}_{j_2}(j_3)\right|\ \langle j_3\rangle^{-m'}\ \langle j_3\rangle^s\, |u_{j_3}| \right|^2
\end{aligned}$$

Denote

$$f_\rho(j_1,j_2,j_3) := \chi(j_1, j_2+j_3)\, \chi(j_2,j_3)\, |\mathtt{R}_\rho(a;j_1,j_2,j_3)|\ \langle j_3\rangle^{-m+\rho}\ \left|\widehat{b}_{j_2}(j_3)\right|\ \langle j_3\rangle^{-m'}\ \langle j_3\rangle^s\, |u_{j_3}| ; \tag{38}$$

now we use that on the support of $\chi(j_2,j_3)$ one has $|j_2| \le \epsilon_2 |j_3|$, to estimate

$$\langle j_3\rangle^{-m+\rho} \le \langle j_3\rangle^{-m+\lfloor\rho\rfloor+1} \langle j_3\rangle^{\rho-\lfloor\rho\rfloor-1} \lesssim \langle j_3\rangle^{-m+\lfloor\rho\rfloor+1} \langle j_2\rangle^{\rho-\lfloor\rho\rfloor-1} ;$$

we also use that, exploiting definition (27),

$$|\mathtt{R}_\rho(a;j_1,j_2,j_3)| \lesssim |j_2|^{\lfloor\rho\rfloor+1} \sum_{|\alpha|=\lfloor\rho\rfloor+1} \left| \int_0^1 (1-t)^{|\alpha|-1} \partial_\xi^\alpha \widehat{a}_{j_1}(j_3 + t j_2) \mathrm{d}t \right| .$$

So we bound

$$f_\rho(j_1,j_2,j_3) \lesssim \mathtt{r}_{j_1}\ \mathtt{b}_{j_2}\ \langle j_3\rangle^s |u_{j_3}|$$

where the sequences $\mathtt{r} = (\mathtt{r}_j)_j$ and $\mathtt{b} = (\mathtt{b}_j)_j$ are

$$\mathtt{r}_{j_1} := \sum_{|\alpha|=\lfloor\rho\rfloor+1} \sup_{j_2,j_3} \chi(j_2,j_3)\ \langle j_3\rangle^{-m+\lfloor\rho\rfloor+1} \left| \int_0^1 (1-t)^{|\alpha|-1} \widehat{\left(\partial_\xi^\alpha a\right)}_{j_1}(j_3+tj_2)\mathrm{d}t \right|$$
$$\mathtt{b}_{j_2} := \langle j_2\rangle^\rho \sup_{j_3} \left|\widehat{b}_{j_2}(j_3)\right|\ \langle j_3\rangle^{-m'}.$$

By Lemma 1.5 and by the cut-off support on frequencies $j_3 \sim j_3 + tj_2$ one deduces that

$$\|\mathtt{r}\|_{\ell^1(\mathbb{Z}^d)} \lesssim |a|_{m,s_0,n}\ ,$$
$$\|\mathtt{b}\|_{\ell^1(\mathbb{Z}^d)} \lesssim |b|_{m',\rho+s_0,0}\,.$$

Thus we conclude using Young's convolution inequality that

$$\begin{aligned}\|R_{II}(a,b)u\|_{s-(m+m')+n} &\le \|\mathtt{r} * \mathtt{b} * (\langle j\rangle^s\, \widehat{u}_j)\|_{\ell^2(\mathbb{Z}^d)} \le \|\mathtt{r}\|_{\ell^1(\mathbb{Z}^d)}\|\mathtt{b}\|_{\ell^1(\mathbb{Z}^d)}\|(\langle j\rangle^s\, \widehat{u}_j)\|_{\ell^2(\mathbb{Z}^d)}\\ &\lesssim |a|_{m,s_0,n}\ |b|_{m',\rho+s_0,0}\ \|u\|_s.\end{aligned}$$

This proves the claim. □

A corollary of this result is the following:

Corollary 1.11. *Given $a(x,\xi), c(x,\xi) \in \mathcal{N}^0_{s_0+2}$ and $b(x,\xi)\in\mathcal{N}^2_{s_0+2}$ we have that*

$$\mathrm{Op}^{BW}(a)\circ\mathrm{Op}^{BW}(b)\circ\mathrm{Op}^{BW}(c) = abc + R^1(a,b,c) + R^0(a,b,c), \tag{39}$$

with $R^1(a,b,c) = -R^1(c,b,a)$ and $R^0(a,b,c)$ is a bounded operator in H^s for every $s\in\mathbb{R}$ such that there exist a constant $C_s>0$ depending on s such that

$$\|R^0(a,b,c)\|_{\mathcal{L}(H^s,H^s)} \le C_s |a|_{0,s_0+2,2}|b|_{2,s_0+2,2}|c|_{0,s_0+2,2}.$$

Proof. Applying Theorem 1.10 we have that

$$\mathrm{Op}^{BW}(b)\circ\mathrm{Op}^{BW}(c) = \mathrm{Op}^{BW}(bc) + \mathrm{Op}^{BW}\left(\frac{1}{2\mathrm{i}}\{b,c\}\right) + R^0(b,c).$$

Applying $\mathrm{Op}^{BW}(a)$ we have

$$\mathrm{Op}^{BW}(a)\circ\mathrm{Op}^{BW}(b)\circ\mathrm{Op}^{BW}(c) = \mathrm{Op}^{BW}(a)\circ\mathrm{Op}^{BW}(bc)+\mathrm{Op}^{BW}(a)\circ\mathrm{Op}^{BW}\left(\frac{1}{2\mathrm{i}}\{b,c\}\right)+\mathrm{Op}^{BW}(a)\circ R^0(b,c).$$

Applying again Theorem 1.10 we obtain

$$\mathrm{Op}^{BW}(a)\circ\mathrm{Op}^{BW}(bc) = \mathrm{Op}^{BW}(abc) + \mathrm{Op}^{BW}\left(\frac{1}{2\mathrm{i}}\{a,bc\}\right) + R^0(a,b,c)$$

and

$$\mathrm{Op}^{BW}(a)\circ\mathrm{Op}^{BW}\left(\frac{1}{2\mathrm{i}}\{b,c\}\right) = \frac{1}{2\mathrm{i}}\{b,c\}a + R^0(a,b,c).$$

Collecting all the terms we obtain (39) with

$$R^1(a,b,c) = \mathrm{Op}^{BW}\left(\frac{1}{2\mathrm{i}}(\{a,bc\} + \{b,c\}a\right) = \mathrm{Op}^{BW}\left(\frac{1}{2\mathrm{i}}(\{a,c\}b + \{b,c\}a + \{a,b\}c\right),$$

that satisfies $R^1(a,b,c) = -R^1(c,b,a)$. The estimates on the remainder $R^0(a,b,c)$ follows from Theorem 1.10 and Theorem 1.6. □

Corollary 1.12 (Commutator)**.** *With the same assumptions of Theorem 1.10, define the symbol*

$$\begin{aligned}\{a,b\}_\rho &:= \mathrm{i}(a\#_\rho b - b\#_\rho a) \qquad (40)\\ &= \mathrm{i}\sum_{1\le|\alpha|<\rho}\frac{1}{\alpha!}\left(\partial_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi) - \partial_\xi^\alpha b(x,\xi)\, D_x^\alpha a(x,\xi)\right) \in \sum_{1\le k<\rho}\mathcal{N}^{m+m'-k}_{\rho+s_0-k}.\end{aligned}$$

Then one has

$$\mathrm{i}[\mathrm{Op}^B(a)\,,\mathrm{Op}^B(b)] = \mathrm{Op}^B(\{a,b\}_\rho) + R^{m+m'-\rho}(a,b) \qquad (41)$$

where the linear operator $R^{m+m'-\rho}(a,b)\colon H^s \to H^{s-(m+m')+\rho}$, $\forall s\in\mathbb{R}$, *with the quantitative estimate*

$$\|R(a,b)u\|_{s-(m+m')+\rho} \lesssim \left(|a|_{m,\rho+s_0,n}\;|b|_{m,s_0,0} + |a|_{m,s_0,n}\;|b|_{m,\rho+s_0,0}\right)\|u\|_s. \qquad (42)$$

We will use the following Moser estimates for composition.

Theorem 1.13 (Moser estimates)**.** *Let* $\Omega\subset\mathbb{C}^2$ *an open and* $\sigma > \frac{d}{2}$. *Let* $F\in C^\infty(\Omega;\mathbb{C})$ *a smooth function in the real sense and such that* $F(0) = 0$ *and* $K\subset\Omega$ *compact, then for any function* $U\in H^\sigma(\mathbb{T}^d;\mathbb{C}^2)\cap\mathcal{U}$ *such that*

$$U(x)\in K, \quad \forall x\in\mathbb{T}^d,$$

we have

$$\|F(U)\|_{H^\sigma} \le C_\sigma \sup_{z\in K}|F'(z)|\|U\|_{H^\sigma}. \qquad (43)$$

Corollary 1.14. *Suppose* $F\in C^\infty(\Omega;\mathbb{C})$ *and* $U,W\in B^1_{s_0+\delta}([-T,T];r)$ *with* $\delta\ge 0$ *such that*

$$W(t,x), U(t,x)\in K, \quad \forall (t,x)\in[-T,T]\times\mathbb{T}^d,$$

with Ω, K *like in Theorem 43. Let* $d(\xi)\in\mathcal{N}^m_{s_0+\delta}$ *and consider the time dependent symbol* $a(U;\xi) := F(U(x,t))d(\xi)$. *Then we have*

- $a\in\mathcal{N}^m_{s_0+\delta}$ *and there exist a constant* $C_r>0$ *which depends on* r *and* K *such that*

$$|a|_{m,s_0+\delta,n} \le C_r; \qquad (44)$$

- *If* $\delta\ge 2$ *then* $\partial_t a\in\mathcal{N}^m_{s_0+\delta-2}$ *and there exist a constant* $C_r>0$ *which depends on* r *and* K *such that*

$$|\partial_t a|_{m,s_0+\delta-2,n} \le C_r; \qquad (45)$$

- *If* K *is convex then* $a(U;\xi) - a(W;\xi)\in\mathcal{N}^m_{s_0+\delta}$ *and there exist a constant* $C_r>0$ *which depends on* r *and* K *such that*

$$|a(U;\xi) - a(W;\xi)|_{m,s_0+\delta,n} \le C_r\|U-W\|_{H^{s_0+\delta}}; \qquad (46)$$