Exam of “Pseudodifferential operators, dynamics and applications”

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The goal of the exam is to prove the following Strichartz estimates for the linear Klein-Gordon equation:

Theorem 0.1. *Any solution of*

$$\partial_{tt}u - \Delta u + u = 0, \qquad u(0) = u_0, \quad \dot{u}(0) = u_1 \tag{1}$$

in $\mathbb{R}^1_t \times \mathbb{R}^3_x$ *satisfies the estimates*

$$\|u\|_{L^3_t L^6_x} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2}. \tag{2}$$

The proof is more involved than the case of Schrödinger, and it makes use of Littlewood-Paley decomposition, oscillatory integrals, Besov spaces and the TT^* argument. You will sketch the proof of the theorem following the scheme below.

The idea is to split the frequency domain in diadic shells, estimates the norm of each localized term (essentially as for Schrödinger), sup up the terms in Besov spaces and use embeddings theorems to go back to Sobolev spaces. For some results, we will work in $\mathbb{R}^d$. Let's start.

1. **Prove** that any solution of (1) is given by the Fourier multiplier

   $$u(t) = \cos(t\langle D\rangle)u_0 + \frac{\sin(t\langle D\rangle)}{\langle D\rangle}u_1$$

   where $\langle D\rangle = \sqrt{1-\Delta}$. Deduce that it is sufficient to prove the Strichartz estimate

   $$\|e^{\pm \mathrm{i} t\langle D\rangle} f\|_{L^3_t L^6_x} \lesssim \|f\|_{H^1_x}$$

   for the Fourier multiplier $e^{\pm \mathrm{i} t\langle D\rangle} f := \int_{\mathbb{R}^d} e^{\mathrm{i}(\pm t\langle \xi\rangle + x\cdot\xi)}\, \widehat{f}(\xi)\, \mathrm{d}\xi$.
   Remark that, as a convolution kernel

   $$e^{\pm \mathrm{i} t\langle D\rangle} f = K_t * f, \qquad K_t(x) := \int_{\mathbb{R}^d} e^{\mathrm{i}(\pm t\langle \xi\rangle + x\cdot\xi)} \mathrm{d}\xi$$

   (it is not necessary to prove this last formula)

2. Denote by χ_0 a cut-off function equal to one when ξ is close to zero and χ a cut-off function equal to one on $\frac{1}{2} < |\xi| < 2$ and identically zero in a neighbourhood of 0. Define

   $$\Phi^{\pm}_{\lambda}(t,x) := \int_{\mathbb{R}^d} e^{\mathrm{i}(\pm t\langle \xi\rangle + x\cdot\xi)}\, \chi\left(\frac{\xi}{\lambda}\right) \mathrm{d}\xi, \qquad \text{for } \lambda \geq 1,$$
   $$\Phi^{\pm}_{0}(t,x) := \int_{\mathbb{R}^d} e^{\mathrm{i}(\pm t\langle \xi\rangle + x\cdot\xi)}\, \chi_0(\xi)\, \mathrm{d}\xi.$$

Prove that

$$\sup_{x\in\mathbb{R}^d} \left|\Phi_0^{\pm}(t,x)\right| \lesssim \langle t\rangle^{-\frac{d}{2}}. \tag{3}$$

Hint: show that the phase function has only one stationary point ξ_0. When ξ_0 is outside the support of χ_0 apply rapid decay estimates. When ξ_0 belongs to the support of χ_0 apply stationary phase (you need a variant of the result that we proved in class: see e.g. Lemma 3.14, pag 35, in `https://math.berkeley.edu/~evans/semiclassical.pdf`)

It's a bit more involved to prove that

$$\sup_{x\in\mathbb{R}^d} \left|\Phi_\lambda^{\pm}(t,x)\right| \lesssim \lambda^{\frac{d}{2}+1} t^{-\frac{d}{2}} \tag{4}$$

for all $t > 0$ and $\lambda \geq 1$. (It is not necessary for you to prove it)

3. Let us go back to Klein-Gordon equation. Put

$$U_\lambda(t) := e^{\mathrm{i}t\langle D\rangle}\,\chi\left(\frac{D}{\lambda}\right), \qquad \text{for } \lambda \geq 1, \tag{5}$$

$$U_0(t) := e^{\mathrm{i}t\langle D\rangle}\,\chi_0\left(D\right). \tag{6}$$

Prove that for any $d \geq 1$, $2 < p \leq \infty$, $2 \leq q \leq \infty$ with $\frac{1}{p} + \frac{d}{2q} = \frac{d}{4}$ one has the bounds

$$\|U_\lambda f\|_{L^p_t L^q_x} \leq C\,\langle\lambda\rangle^{\beta}\,\|f\|_{L^2_x} \tag{7}$$

where $\beta = \frac{1}{2}\left(\frac{d}{2}+1\right)\left(\frac{1}{q'} - \frac{1}{q}\right)$ and either $\lambda = 0$ or $\lambda \geq 1$.
Hint: prove first an $L^1 \to L^\infty$ decay estimate; interpolate with the $L^2 \to L^2$ estimate to obtain $L^{q'} \to L^q$ estimates. Then use a TT^* argument.

4. It is time to sum up the bounds obtained over a a geometric sequence of λ by means of Littlewood–Paley theory. Recall the definition of Besov norms

$$\|f\|_{B^\sigma_{r,2}} := \|P_0 f\|_{L^r_x} + \left(\sum_{j\geq 0} 2^{2\sigma j}\|P_j f\|^2_{L^r_x}\right)^{\frac{1}{2}}$$

where P_0, P_j are the standard Littlewood-Paley projectors.
Prove that

$$\|e^{\mathrm{i}t\langle D\rangle} f\|_{L^p_t B^0_{q,2}} \lesssim \|f\|_{H^\beta}, \qquad \|e^{\mathrm{i}t\langle D\rangle} f\|_{L^p_t B^{1-\beta}_{q,2}} \lesssim \|f\|_{H^1}$$

5. We are almost done: let us go back to more useful spaces via embedding.
Prove that

$$B^\sigma_{r,2} \hookrightarrow B^0_{q,2} \hookrightarrow L^q$$

for any $2 \leq r \leq q < \infty$ and $\sigma = d\left(\frac{1}{r} - \frac{1}{q}\right)$, $\sigma = 0$. Deduce (2).