Exam of “Pseudodifferential operators, dynamics and applications”

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July 10, 2020

The irrotational Euler-Korteweg equation

$$\begin{cases} \partial_t \rho = -\mathrm{div}(\rho \nabla \phi) \\ \partial_t \phi = -\frac{1}{2}|\nabla \phi|^2 - g(\rho) + K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2 \end{cases} \tag{1}$$

is a modification of the Euler equations for compressible fluids to include capillary effects. The scalar variable $\rho(t,x) > 0$ is the density of the fluid, whereas $\phi(t,x)$ is a scalar potential function. They are both real valued. The functions $K(\rho)$, $g(\rho)$ are defined on $\mathbb{R}^+$, smooth, bounded with all their derivatives, $g(0) = 0$ and $K(\rho)$ is positive. We will consider this system on $H^\sigma := H^\sigma(\mathbb{R}^d) \times H^\sigma(\mathbb{R}^d)$, $\sigma \geq 0$.

The system is known to admit stationary solutions of the form

$$\begin{pmatrix} \rho(t,x) \\ \phi(t,x) \end{pmatrix} = \begin{pmatrix} a(x) \\ b(x) \end{pmatrix} \tag{2}$$

where $a(x), b(x)$ are smooth, real valued functions, bounded with their derivatives, and such that

$$a(x) \geq c > 0 \text{ for any } x \in \mathbb{R}^d. \tag{3}$$

The goal of the exam is to prove an energy estimate for the system obtained linearizing (1) **around the stationary solution** (2)**.**

This requires to write the linear system in a pseudodifferential form, pass to complex coordinates, introduce a modified energy equivalent to the norm H^σ, and bound the variation of the modified energy. You will need to use symbolic calculus in the Weyl quantization. Let's start.

1. Linearize system (1) around the solution (2), and write it as

$$\partial_t \begin{pmatrix} \rho \\ \phi \end{pmatrix} = \mathrm{Op}^W \left(\begin{pmatrix} \nabla b \cdot \mathrm{i}\xi & a|\xi|^2 \\ -K(a)|\xi|^2 & \nabla b \cdot \mathrm{i}\xi \end{pmatrix} \right) \begin{pmatrix} \rho \\ \phi \end{pmatrix} + \mathrm{Op}^W(A_0) \begin{pmatrix} \rho \\ \phi \end{pmatrix} \tag{4}$$

   where A_0 is a matrix of symbols in $\mathcal{S}^0$ (symbols of order 0). Here and below we use the notation

$$\mathrm{Op}^W\left(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \right) := \begin{pmatrix} \mathrm{Op}^W(a_1) & \mathrm{Op}^W(a_2) \\ \mathrm{Op}^W(a_3) & \mathrm{Op}^W(a_4) \end{pmatrix}.$$

2. Pass to complex coordinates $\begin{pmatrix} u \\ \overline{u} \end{pmatrix}$, obtaining the system

$$\partial_t \begin{pmatrix} u \\ \overline{u} \end{pmatrix} = \mathbb{J}\mathrm{Op}^W(A_2 + A_1 + A_0) \begin{pmatrix} u \\ \overline{u} \end{pmatrix} \tag{5}$$

where $\mathbb{J} := \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}$,

$$A_2 := \begin{pmatrix} a_+ & a_- \\ a_- & a_+ \end{pmatrix} |\xi|^2, \qquad A_1 := \begin{pmatrix} \nabla b \cdot \xi & 0 \\ 0 & -\nabla b \cdot \xi \end{pmatrix}$$

and $A_0 \in \mathcal{S}^0$. You need to compute $a_\pm$ and to prove that there are matrices $F^\pm$ such that

$$F^{-1}\, \mathbb{J} \begin{bmatrix} a_+ & a_- \\ a_- & a_+ \end{bmatrix} F = \mathbb{J}\lambda(x)\,, \tag{6}$$

where $\lambda(x)$ is positive and bounded away from 0.

Now it is time for the core argument. You need to prove energy estimates for (5). In particular the goal is to prove that every solution fulfills the estimate

$$\|u(t)\|_\sigma^2 \le C_1 \|u(0)\|_\sigma^2 + C_2 \int_0^t \|u(\tau)\|_\sigma^2 \,\mathrm{d}\tau, \tag{7}$$

for some constants $C_1, C_2 > 0$. A direct estimate fails (why?). The strategy is to construct a modified energy, which is equivalent to the norm H^σ, and bound its time variation. The *modified energy* is, for $\sigma > 0$

$$\mathcal{E}_\sigma(U)^2 := \langle \mathrm{Op}^W\left(\lambda^\sigma(x)|\xi|^{2\sigma}\right) \mathrm{Op}^W\left(F^{-1}\right) U, \mathrm{Op}^W\left(F^{-1}\right) U \rangle, \quad U = \begin{pmatrix} u \\ \overline{u} \end{pmatrix} \tag{8}$$

where we introduce the *real* scalar product

$$\langle V, W \rangle := 2\Re \int_{\mathbb{R}^d} v(x)\overline{w}(x)\,\mathrm{d}x, \qquad V = \begin{bmatrix} v \\ \overline{v} \end{bmatrix}, \quad W = \begin{bmatrix} w \\ \overline{w} \end{bmatrix}.$$

4. Prove that $\mathcal{E}_\sigma(U)$ is equivalent to the H^σ norm, in particular there exists $C > 0$ s.t.

   $$C^{-1}\|V\|_\sigma^2 - \|V\|_0^2 \le \mathcal{E}_\sigma(V)^2 \le C\|V\|_\sigma^2, \quad \forall V \in H^\sigma$$

   [Hint: you will probably need to use that $\mathrm{Op}^W(F)\,\mathrm{Op}^W\left(F^{-1}\right) = \mathrm{Id} + \mathrm{Op}^W\left(S^{-1}\right)$.]

5. Prove that

   $$\frac{d}{dt}\mathcal{E}_\sigma(U(t))^2 \le C\|U(t)\|_\sigma^2$$

   and deduce the energy inequality (7).

   [Hint: quantizing (6) you get

   $$\mathrm{Op}^W\left(F^{-1}\right)\, \mathbb{J}\mathrm{Op}^W(A_2 + A_1)\, \mathrm{Op}^W(F) = \mathbb{J}\mathrm{Op}^W\left(\lambda(x)|\xi|^2 + \nabla b \cdot \xi\right) + \mathrm{Op}^W(A_0) \tag{9}$$

   where $F^\pm, A_0 \in \mathcal{S}^0$ (for the moment just use this)].

6. (Bonus) Prove (9).