The Fidelity Approach, criticality, and boundary-CFT

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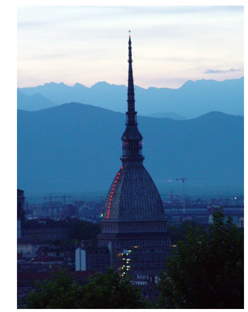

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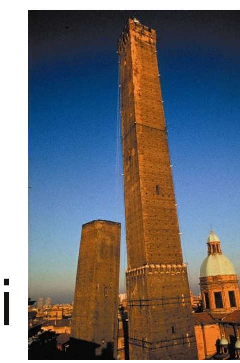

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Obergurgl, ESF Conference
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Approaches to criticality

Standard

$$f = \frac{1}{\beta L^d} \log tr\left(e^{-\beta H}\right)$$

$$e = \frac{\langle \Psi | H | \Psi \rangle}{L^d}$$

$$H = H_0 + \lambda V$$

$$\chi_{VV} = \frac{\partial e}{\partial \lambda^2} = \frac{1}{L^d} \sum_{n \neq 0} \frac{\left|\langle n | V | 0 \rangle\right|^2}{E_n - E_0}$$

Fidelity

$$F = \left| \langle \Psi | \Psi ' \rangle \right|$$

Why the fidelity?

How do we distinguish states?

$|\Psi_P\rangle \qquad |\Psi_Q\rangle$

by measuring! $A \rightarrow \{a_i, |i\rangle\}$

Observable

$$p_i = |\langle i|\Psi_P\rangle|^2 \quad q_i = p_i + \delta p_i$$

distinguishing experiments

distance in probability space

Fisher (infinitesimal) distance

$$d_F(\{p_i\},\{q_i\}) = \sum_i \frac{(\delta p_i)^2}{p_i}$$

Problem: each preparation $|\psi_P\rangle$ defines infinitely many probability distribution p_i, one for each observable.

One has to maximize over all possible experiments!

Surprise!

$$\max_{Exps} d_F(P,Q) =$$ *Projective Hilbert space distance!*

Q-fidelity!

$$D_{FS}(P,Q) \equiv \cos^{-1}\frac{|\langle Q|P\rangle|}{\|Q\|\cdot\|P\|} \equiv \cos^{-1}(F)$$

(Wootters 1981)

Statistical distance and geometrical one collapse*:*

Hilbert space geometry ***is (quantum) information geometry…..***

Question: What about non pure preparations ?!? ρ_P

Answer: Bures metric & Uhlmann fidelity!
(Braunstein Caves 1994)

$$d(\rho_0,\rho_1) = \cos^{-1}(F)$$

$$F(\rho_0,\rho_1) = tr\sqrt{\rho_1^{1/2}\rho_0\rho_1^{1/2}}$$

Remark: In the classical case one has commuting objects, simplification ….

Idea:

Distance → **Metric**

$$d_{FS}(\lambda,\lambda+\delta\lambda)=\arccos(F)$$
$$=\sqrt{G\,\delta\lambda^2}\equiv ds$$
$$ds^2=G_{\mu,\nu}\,\delta\lambda^{\mu}\,\delta\lambda^{\nu}$$

defines a metric in the space of states

$$F=1-\frac{1}{2}G_{\mu,\nu}\,\delta\lambda^{\mu}\,\delta\lambda^{\nu}+O\left(\delta\lambda^3\right)$$

if Ψ, H are C^1 at λ

$$G_{\mu,\nu}=\sum_{n>0}\frac{\langle n|\partial_{\mu}H|0\rangle\langle 0|\partial_{\nu}H|n\rangle}{\left(E_n(\lambda)-E_0(\lambda)\right)^2}$$

At (Quantum) Critical Points ***F*** **→ sharp drop**
induced metric G → sharp increase

Comparison

$$H = H_0 + \lambda V$$

$$\chi_{VV} = \frac{1}{L^d} \sum_{n \neq 0} \frac{\left| \langle n | V | 0 \rangle \right|^2}{E_n - E_0}$$

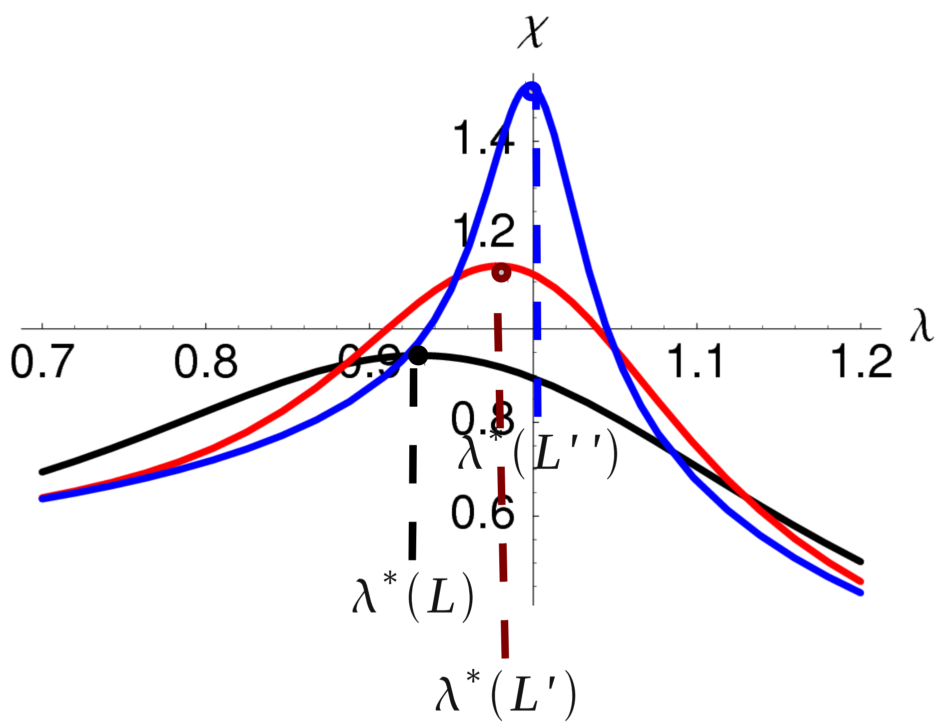
$$g = \frac{1}{L^d} \sum_{n \neq 0} \frac{\left| \langle n | V | 0 \rangle \right|^2}{\left(E_n - E_0 \right)^2}$$

$\chi_{VV}(\lambda \approx \lambda_c, L) \sim L^{(\zeta + d - 2\Delta_V)}$ $\quad \xi \gg L \quad$ $g(\lambda \approx \lambda_c, L) \sim L^{(2\zeta + d - 2\Delta_V)}$

$\chi_{VV}(\lambda) \sim \left| \lambda - \lambda_c \right|^{-\nu(\zeta + d - 2\Delta_V)}$ $\quad \xi \ll L \quad$ $g(\lambda) \sim \left| \lambda - \lambda_c \right|^{-\nu(2\zeta + d - 2\Delta_V)}$

More divergent

The search for critical point



$$\lambda^*(L) = \lambda_c + \frac{A}{L^{\theta}}$$

Shift exponent

Good method

Large Θ

Generally

(but not always)

$$\theta = \frac{1}{\nu}$$

Efficiency of known methods

Name	Eqn	θ
Binder (4th order reduced) cumulant	$\partial_L B_L^{(4)}(\lambda^*)=0$	$1/\nu$
PRG	$\partial_L [L^\zeta \Delta_L(\lambda^*)]=0$	$1/\nu+\epsilon$
FSCM	$\partial_L [\langle V\rangle_L / L^d]=0$	$2/\nu$
Fidelity	$\partial_\lambda g_L(\lambda^*)=0$	$1/\nu$

Accelerated methods

Roncaglia, LVC, Degli Esposti Boschi, PRB (2008)

$$\tilde{\lambda}=\lambda-\lambda_c\,, \quad z=\tilde{\lambda}\,L^{1/\nu}$$

$$e_{sing}=\underbrace{L^{-(d+\zeta)}\left[\phi_0(z)+L^{-\omega}h(\tilde{\lambda})\phi_1(z)+\ldots\right]}_{\textbf{Scaling (RG) terms}}+\underbrace{F(\tilde{\lambda})L^{-(d+\zeta)}}_{\textbf{Off-Scaling}}+\ldots$$

$$e_L=e_\infty-\frac{\pi c\,v(\lambda)}{6}\frac{1}{L^2}+o(L^{-2})$$

$$e_L\equiv\langle H\rangle/L^d \quad b_L\equiv\langle V\rangle/L^d$$

Name	Eqn	θ
HCM (Homogeneity condition method)	$(d+\zeta+1)\,\partial_L e_L+L\,\partial_L^2 e_L=0$	$2/\nu+\omega$
MHC (Modified homogeneity condition)	$\dfrac{\partial_L b_L}{\partial_L e_L}=\dfrac{\partial_L^2 b_L}{\partial_L^2 e_L}$	$2/\nu+\omega$

Tests

XY model in transverse field

$$H_{XY} = -\sum_i \left[\frac{(1+\gamma)}{2} \sigma_i^x \sigma_{i+1}^x + \frac{(1-\gamma)}{2} \sigma_i^y \sigma_j^y + h \sigma_i^z \right]$$

$$e_L(h,\gamma) = e_\infty(1,\gamma) - \frac{\pi|\gamma|}{6} L^{-2} - \frac{7\pi^3(4\gamma^2-3)}{1440|\gamma|} L^{-4}$$
$$+(h-1)\left[b_\infty(1,\gamma) - \frac{\pi}{12|\gamma|} L^{-2} - \frac{7\pi^3(3-2\gamma^2)}{2880|\gamma|^3} L^{-4} \right]$$
$$-(h-1)^2 \frac{\log(L)+\gamma_E+\log(8|\gamma|/\pi)-1}{2\pi|\gamma|} + O[(h-1)^3]$$

$$d = \zeta = \nu = 1$$

$$\theta_{standard} = 1$$

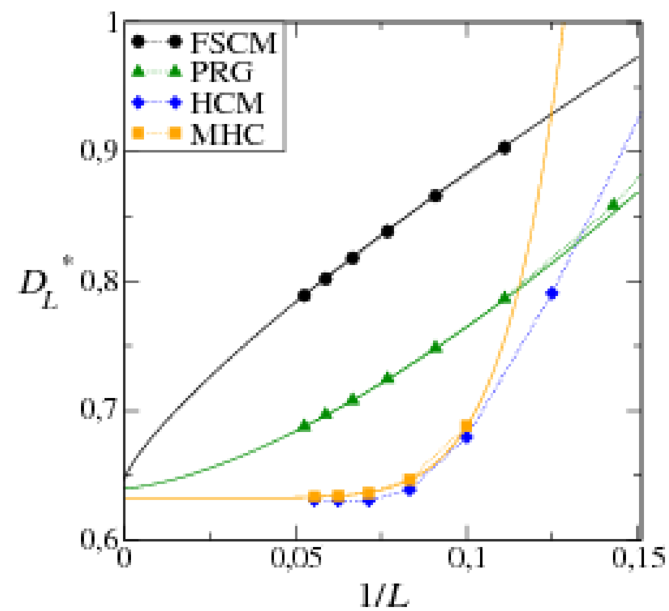
$$\theta_{PRG} = 3$$

$$\theta_{HCM} = 4$$

$$\theta_{MHC} = 4$$

Anysotropic spin-1

$$H = J \sum_{\langle i,j \rangle} S_i^x S_j^x + S_i^y S_j^y + \lambda S_i^z S_j^z + D \left(S_i^z \right)^2$$



	FSCM	PRG	HCM	MHC
D_c	0.647	0.640	0.630	0.633
fitted θ	0.79	1.50	7.6	7.6

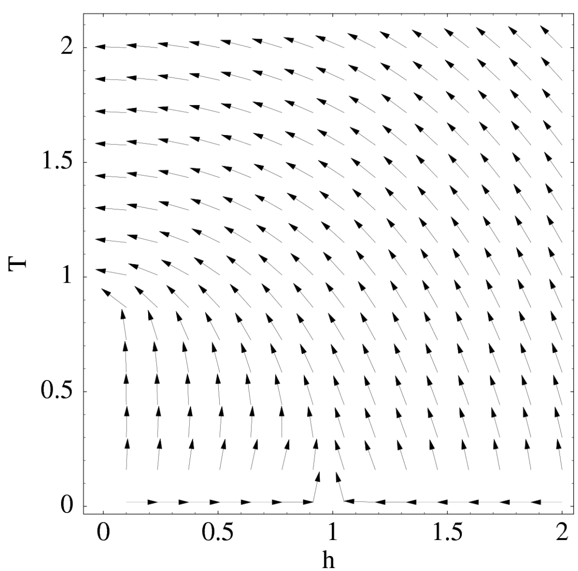
But: fidelity good for crossover

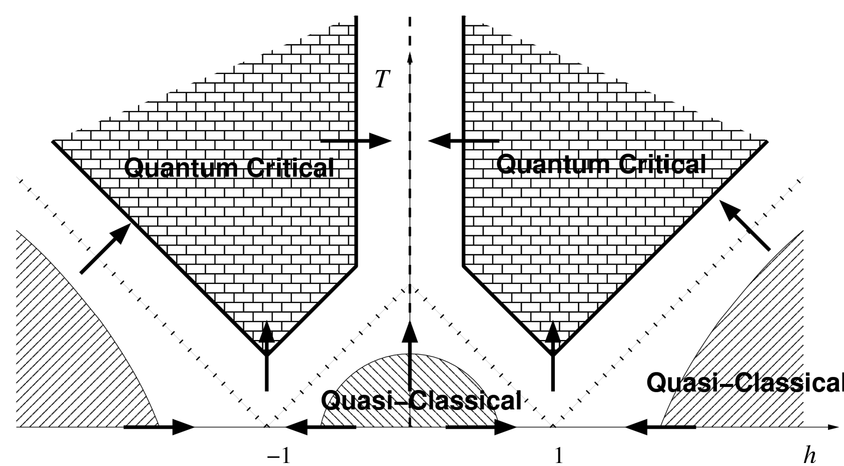
Zanardi, LVC, Giorda, PRA (2007)

Eigenvector of $g_{\mu,\nu}$ → direction of maximal distinguishability

Consider (h,T) phase diagram

$$H_{Q-Ising} = -\sum_i \left[\sigma_i^x \sigma_{i+1}^x + h\sigma_i^z\right]$$





See also A. Kahn, P. Pieri PRA (2009) for BEC-BCS crossover

Fidelity also good for various other purposes

Estimation theory: best measurement to *estimate* a given parameter

M.G.A. Paris, P. Zanardi,
LVC, Giorda, ...
R. Floreanini (yesterday's talk)
A. Smerzi (today's talk)

Adiabatic theorem and adiabatic quantum computation

D. Lidar, A. Rezakhani,
A. Polkovnikov,...

New universal quantities

LCV, H. Saluer, P. Zanardi

Universal quantities

		bulk	finite-size
specific heat	$C^{\sin}$	$t^{-\alpha}$	$L^{\alpha/\nu}$
magnetic susceptibility	χ	$t^{-\gamma}$	$L^{\gamma/\nu}$
correlation length	ξ	$t^{-\nu}$	L
free energy density	$f^{\sin}$	$t^{2-\alpha}$	L^{-d}
order parameter[(a)]	M	t^{β}	$L^{-\beta/\nu}$
latent heat[(a)]	ℓ_h	$t^{1-\alpha}$	$L^{(\alpha-1)/\nu}$

Critical Point → **Scale invariance** → **Universal Critical Exponents**

Q: What about universal factors?

Energy of 1D Quantum Critical systems with PBC:

I. Affleck, J. Cardy, PRL's 1986

$$E_L = e\,L - \frac{\pi c v}{6}\frac{1}{L} + o(L^{-1})$$

von Neumann entropy of topologically ordered states of a region of boundary length *L*

$$S_L = s\,L - \gamma + o(L^0)$$

A. Hamma, R. Ionicioiu, P. Zanardi, PRA (2005)
A. Kitaev and J. Preskill, PRL (2006)
M. Levin and X. -G. Wen, PRL (2006)

Universal Factors

Fidelity: universal factor

$$F = \left| \langle \Psi(\lambda), \Psi(\lambda') \rangle \right|$$

bulk term

boundary terms

$$\ln F = -f L^d - f_b L^{d-1} + \cdots + \ln g_0 + \text{smaller terms}$$

"Residual distinguishability"

$$F = g_0 e^{-f L^d + \cdots}$$

Universal order one factor

&

Depends only on the boundary conditions!

Q: when do we expect *g* to be present?
Consider $\lambda \to \lambda'$

$$\langle \Psi | \Psi' \rangle \approx e^{-\delta \lambda^2 G/2} \qquad G_{sing} \sim L^{2(d+\zeta-\Delta_v)}$$

zero order term
for **marginal** perturbation

$$\Delta_v = d + \zeta$$

Fidelity and Stat-Mech

LVC, H. Saleur, P. Zanardi, PRB (2009)

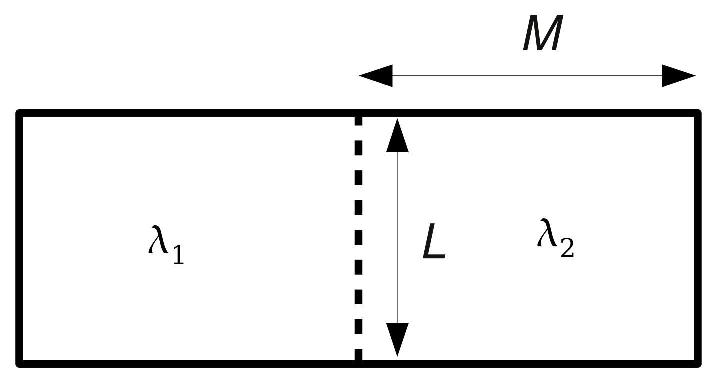
***d*-dim Quantum Mechanics. Hamiltonian:** $H(\lambda)$

***(d+1)* Statistical Mechanics. Transfer matrix:** $T(\lambda)$

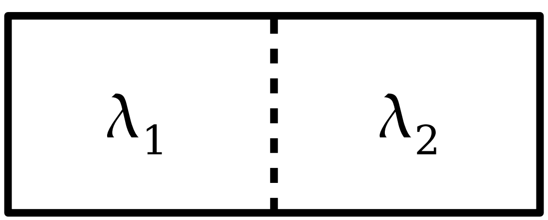
$$|\Psi(\lambda)\rangle = \lim_{M\to\infty} T(\lambda)^M |\phi_0\rangle / \sqrt{Z(\lambda)}$$

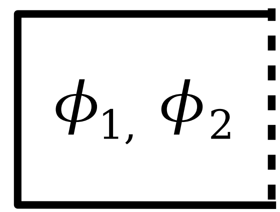
$$\langle\Psi(\lambda_1)|\Psi(\lambda_2)\rangle = \frac{Z(\lambda_1,\lambda_2)}{\sqrt{Z(\lambda_1)Z(\lambda_2)}}$$

$Z(\lambda_1,\lambda_2) =$



"Folding"





$= \langle \Omega_c | B \rangle$

Boundary state

From BCFT:
Affleck, Ludwig, Saleur, Oshikawa...

Universal order-one factor (boundary degeneracy)

$$\langle \Omega_c | B \rangle = g_0 \times \text{Bulk term}$$

Universal term in the fidelity g_0 depends only on the universality class and boundary conditions

Example: Luttinger liquid

consider (1+1) CFT:
example $c=1$ “free” Boson theory

$$S = \frac{\lambda}{2}\int dx^2 \left(\nabla\phi\right)^2$$

direct calculation in the PBC case

M.-F. Yang (PRB 2007),
J. V. Fjaerestad JSTAT (2008)

$$F(\lambda_{1,}\lambda_2) = \frac{Z\left(\frac{\lambda_1+\lambda_2}{2}\right)}{\sqrt{Z(\lambda_1)Z(\lambda_2)}} = \prod_{k\neq 0}\sqrt{\frac{2\sqrt{\lambda_1\lambda_2}}{\lambda_1+\lambda_2}} = \left(\sqrt{\frac{2\sqrt{\lambda_1\lambda_2}}{\lambda_1+\lambda_2}}\right)^{L-1} = g_0 f^{L}$$

$$g = \sqrt{\frac{\lambda_1+\lambda_2}{2\sqrt{\lambda_1\lambda_2}}}$$

agrees with BCFT result

I. Affleck, M. Oshikawa and H. Saleur
Nucl. Phys. B594 (2001)

Zero-order term: “experimental” check

(1+1) c=1 Universality class

XXZ 1D lattice quantum theory
(critical $|\Delta|<1$)

$$H=\sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$

(1+1) continuum field theory
c=1 CFT free boson λ

$$S=\frac{\lambda}{2}\int dx^2 (\nabla\phi)^2$$

Bethe-Ansazt + Bosonization:

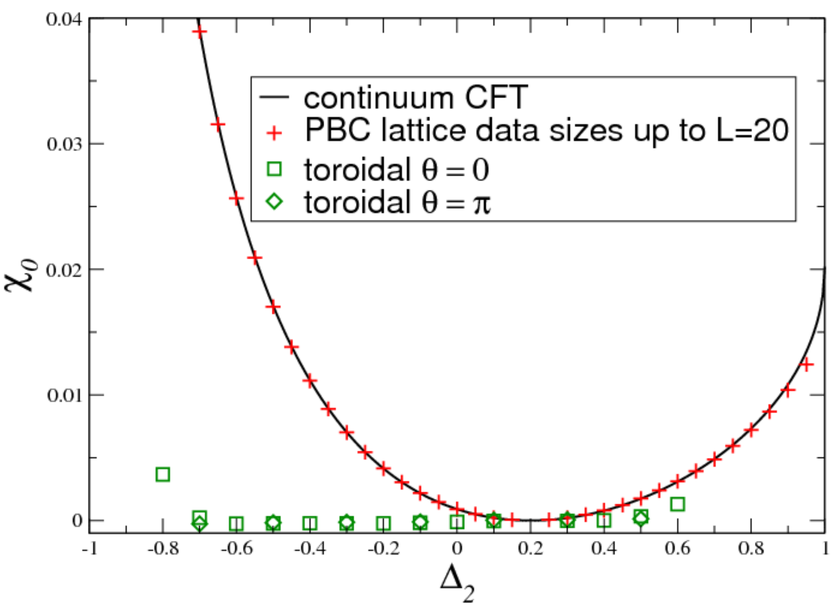
$$\lambda=\frac{\pi-\arccos(\Delta)}{2\pi^2}$$

calculation for **PBC**

$$F(\Delta_1,\Delta_2)=\left|\langle\Psi(\Delta_1)|\Psi(\Delta_2)\rangle\right|$$

$\chi_0 \equiv -\log(g_0)$ $\quad \Delta_1 = 0.20$

$$g_0=\sqrt{\frac{\lambda_1+\lambda_2}{2\sqrt{\lambda_1\lambda_2}}}$$



The **same** g appears also for excited states and/or free BCs!
instead **toroidal** BCs induce anti-periodic BCs in the CFT => $\ln g = 0$!

toroidal BCs: $\quad \sigma_{L+1}^{\pm} = e^{\pm i\theta}\sigma_1^{\mp}\ ,\ \ \sigma_{L+1}^{z} = -\sigma_1^{z}$

Generalization

$$H=\sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$

$$F\left(\Delta_1, \Delta_2\right) = \left|\langle \Psi(\Delta_1) | \Psi(\Delta_2) \rangle\right|$$

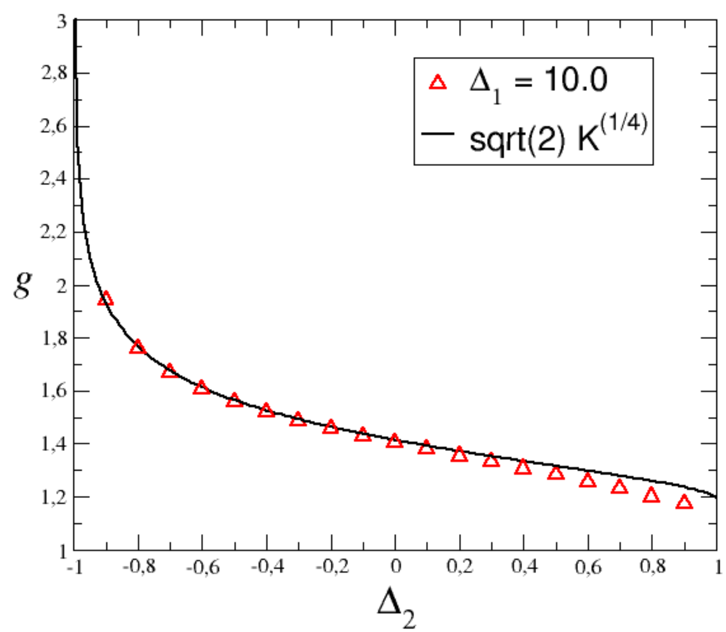
$$\Delta_1 \gg 1 \ , \quad |\Psi(\Delta_1)\rangle \simeq \frac{1}{\sqrt{2}}\left(|Neél\rangle + |ANeél\rangle\right)$$

Induces **Dirichlet** BCs on the field φ

$$g_D = K^{1/4}$$

Luttinger Liquid Parameter

$$g_{tot} = 2 \times \frac{1}{\sqrt{2}} \times K^{1/4} = \sqrt{2} K^{1/4}$$



Summary

- Fidelity: geometric approach, operational meaning. Adiabatic quantum computation, Estimation theory, Equilibration...

- Want to spot the critical point?
  $\rightarrow$ Use accelerated methods

- Universal *prefactors*

Dankeschön!

Fidelity 2). Example in d=2

Quantum eight vertex Model

$$H = \sum_{i,j} h_{i,j}(c)$$

critical for $c^2 \leq 2$

(2+1) continuum field theory
Lifshitz

$$S = \frac{1}{2}\int dx^3 \left[(\partial_\tau \phi)^2 + \kappa^2 (\nabla^2 \phi)^2\right]$$

$$|\Psi(c)\rangle = \sum_{\mathcal{C}} c^{n_c(\mathcal{C})} |\mathcal{C}\rangle / \sqrt{Z_{2D}(c^2)}$$

Partition function for 2D classical 8 vertex Model

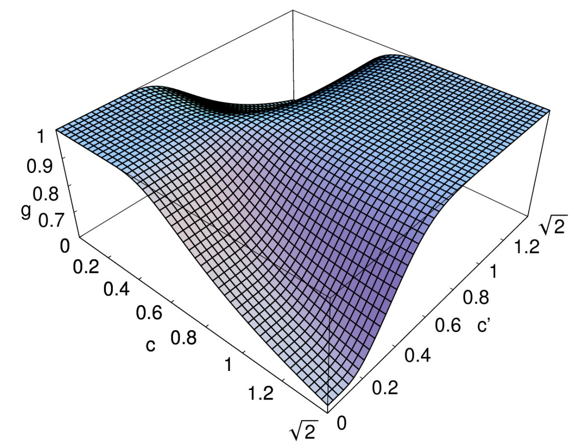
$$F = \langle \Psi(c) | \Psi(c') \rangle = \frac{Z_{2D}(c\,c')}{\sqrt{Z_{2D}(c^2)\, Z_{2D}(c'^2)}}$$

Fidelity 2). Example in d=2 (II)

Periodic BCs:

modular invariant c=1 partition function

$$Z_{2D} = e^{-f L_1 L_2} Z_{CFT}\left(L_1/L_2, \lambda\right)$$



Free BCs:

non-universal terms

$$\ln Z_{2D} = -f L_1 L_2 - f_b (L_1 + L_2) + \frac{1}{8} \ln L_1 L_2$$

$$+ \frac{1}{8} \ln \frac{L_1}{L_2} - \frac{1}{2} \ln \eta(q)$$

instead for **free** BCs ln*g* =0 !

$$q = e^{-2\pi L_1 L_2}, \quad \eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

However:

$$F_L(\lambda,\lambda') \to \begin{cases} 0, & L\to\infty \\ 1, & \delta\lambda\to 0 \end{cases}$$ (Anderson orthogonality catastrophe)

Prescription (A)

1) Keep *L* finite

$$\lambda' = \lambda + \delta\lambda$$

$$F = 1 - \frac{1}{2} G_{\mu,\nu}\, \delta\lambda^{\mu}\, \delta\lambda^{\nu} + O\left(\delta\lambda^{3}\right)$$

Theorem: at regular points (massive) then $G_{\mu,\nu} \propto L^{d}$

$$\Rightarrow \quad g_{\mu,\nu} \equiv \frac{G_{\mu,\nu}}{L^{d}}$$

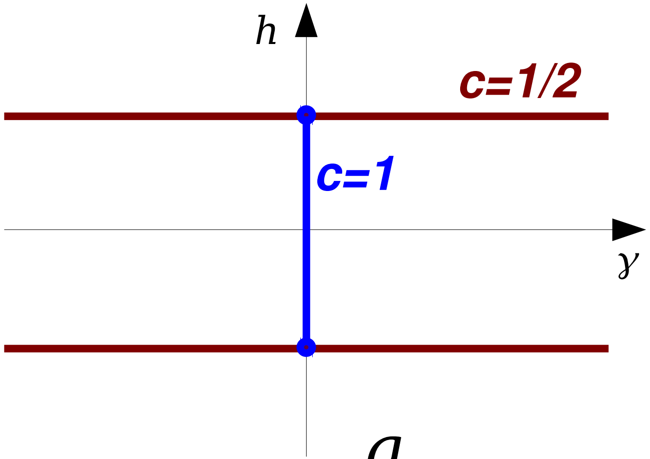
2) Send *L* to infinity

$$g(\lambda) \equiv \lim_{L\to\infty} \lim_{\delta\lambda\to 0} \frac{2(1-F)}{L^{d}\,\delta\lambda^{2}}$$

$\Rightarrow$ **QPTs are characterized by singularities of $g(\lambda)$!**

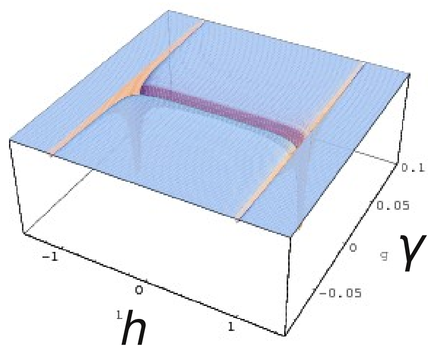
Examples

P. Zanardi, N. Paunkovic PRE (2006)

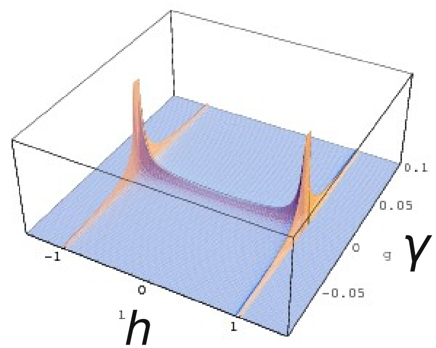


$$H=-\sum_i \frac{(1+\gamma)}{2}\sigma_i^x\sigma_{i+1}^x+\frac{(1-\gamma)}{2}\sigma_i^y\sigma_{i+1}^y+h\sigma_i^z$$

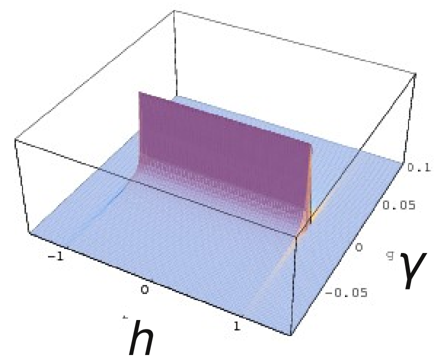
F



$g_{h,h}$



$g_{\gamma,\gamma}$



Generalization: All quasi-free fermions

polar decomposition

$$H=\sum_{ij} c_i^+ A_{ij} c_j+\frac{1}{2}\sum_{ij}\left(c_i^+ B_{ij} c_j^+ +\mathrm{h.c.}\right) \qquad A=A^+,\ B=-B^+,\ Z\equiv A-B,\ T=Z|Z|^{-1}$$

$$|\langle\Psi_1|\Psi_2\rangle|=\sqrt{|det[(T_1+T_2)/2]|}$$

P. Zanardi, M. Cozzini, P. Giorda, JSTAT (2007)
M. Cozzini, P. Giorda, P. Zanardi, PRB (2007)

Scaling of the metric tensor

in most solvable examples $g(\lambda_c) \propto L$ divergence

$$H = H_0 + \lambda V \quad V = \sum_x v(x)$$

$$V(\tau) = e^{\tau H_0} V e^{-\tau H_0}$$

$$g = \frac{1}{L^d} \int_0^\infty \tau \langle V(\tau) V(0) \rangle_c d\tau$$

$$= \sum_x \int_0^\infty \tau \langle v(\tau, x) v(0,0) \rangle_c d\tau$$

quasi-critical region

$$g(\lambda \approx \lambda_c) \sim L^{-\Delta_g} \quad \xi \gg L$$

$$\xi \sim |\lambda - \lambda_c|^{-\nu}$$

$$g(\lambda) \sim |\lambda - \lambda_c|^{\nu \Delta_g} \quad \xi \ll L$$

off-critical region (TDL)

spatial dimension

$$\Delta_g = 2\Delta_v - 2\zeta - d$$

scaling dimension of *v(x)*

dynamical exponent

in quasi-free fermions $\Delta_v = \zeta = d = 1 \Rightarrow \Delta_g = -1$

$$g(\lambda_c) \sim L$$

$$g(\lambda) \sim |\lambda - \lambda_c|^{-1}$$

divergence of *g* when $\Delta_g < 0$

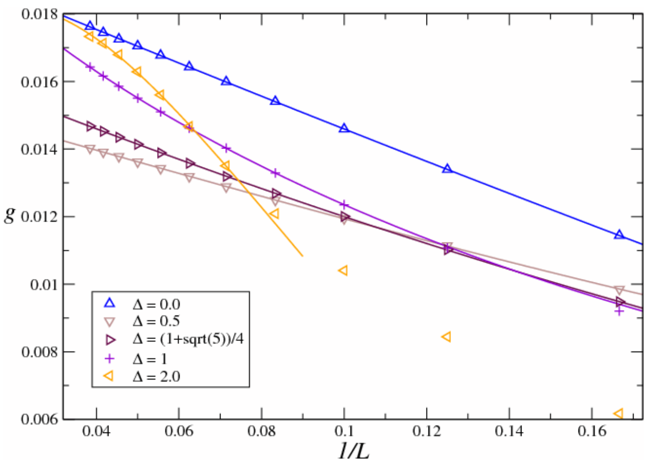
say $\Delta_v < 1.5$ ($\zeta = d = 1$)

$v(x)$ **sufficiently relevant!**

Scaling of the metric tensor (II)

XXZ model spin 1/2

$$H=\sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$



on the critical region $|\Delta|<1$ scaling predicts

$$g=A_1+A_2 L^{-1}+A_3 L^{3-2\Delta_v}\,, \quad \Delta_v=4K$$

on the massive region (here $\Delta>1$)

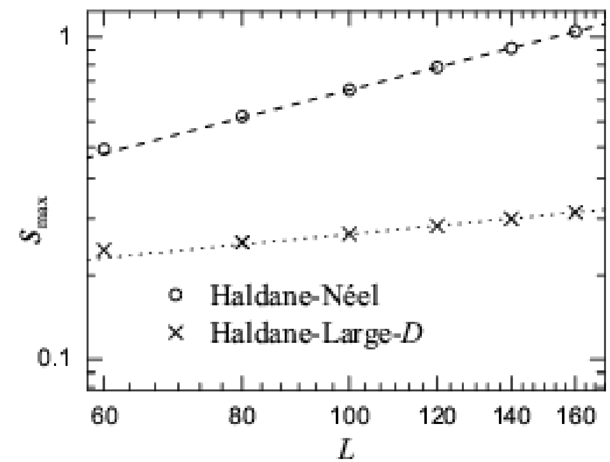
$$g=A_1+A_2 L^{-1/2} e^{-L/\xi}$$

LCV, P. Zanardi, PRL 2007

Luttinger parameter obtained from Bethe-Ansatz

λ-D model spin 1

$$H=\sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \lambda S_i^z S_{i+1}^z + D(S_i^z)^2$$



scaling formula in accordance with previous data

Y.-C. Tzeng, M.-F. Yang PRA 2008
Y.-C. Tzeng et al. arXiv:0804.0537

Connection with estimation theory

Hamiltonian

$$H = H(\lambda, h)$$

unknown parameter
(e.g. coupling constant)

tunable parameter
(e.g. external field)

seek for λ

- Measure observable A →{a_1, a_2,..}
- Build estimator $\hat{\lambda} = \hat{\lambda}(a_{1,} a_{2,} \ldots)$

Quantum Cramer-Rao Bound

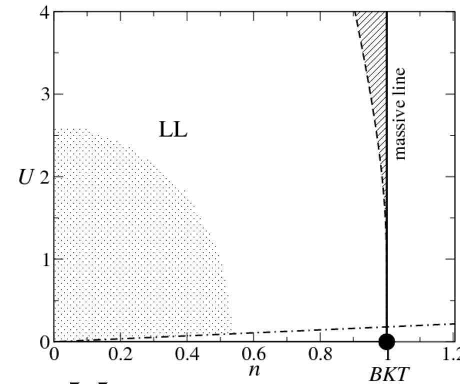
$$Var_\lambda[\hat{\lambda}] \geqslant \boldsymbol{H}^{-1} \qquad \boldsymbol{H}_{\mu,\nu} = 4\, G_{\mu,\nu}$$

!

Optimize over all possible measures. In the one-parameter case the bound can be attained: Measure the SLD!

Fidelity scaling near Mott transition

LCV, Cozzini, Buonsante, Massel, Bray-Ali, Zanardi
PRB (2008)



Effective theory in the Metal

$$H = -\sum_i \left(c^+_{i+1} c_i + h.c. \right) + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

$$H_{eff} = H_{charge} + H_{spin}$$

$$g = g_{charge} + g_{spin}$$

$$H_\nu = \text{free Boson theory,}$$

$$\rightarrow \{ v_\nu , K_\nu \}$$

velocity

Luttinger parameter

$$g_\nu = \frac{1}{8} \left(\frac{d \log K_\nu}{dU} \right)^2$$

dressed charge
from Bethe-Ansatz

(for Charge stiffness: Stafford, Millis PRB (1993))

$$g = g_{charge} = \frac{1}{2} \left(\frac{d \log Z}{dU} \right)^2$$

density

$$\delta = 1 - n$$

Correlation length
at half-filling

conjecture: hyper-scaling for Z

$$Z(U, \delta) = \Phi_Z \left(\xi(U) \delta \right)$$

Scaling functions

$$g = \left(\frac{d \log \xi}{dU} \right)^2 \Phi_g \left(\xi(U) \delta \right)$$

Fidelity scaling near Mott transition (II)

conjecture can be proven for $x \ll 1$ and $x \gg 1$ with $x = \xi\delta$

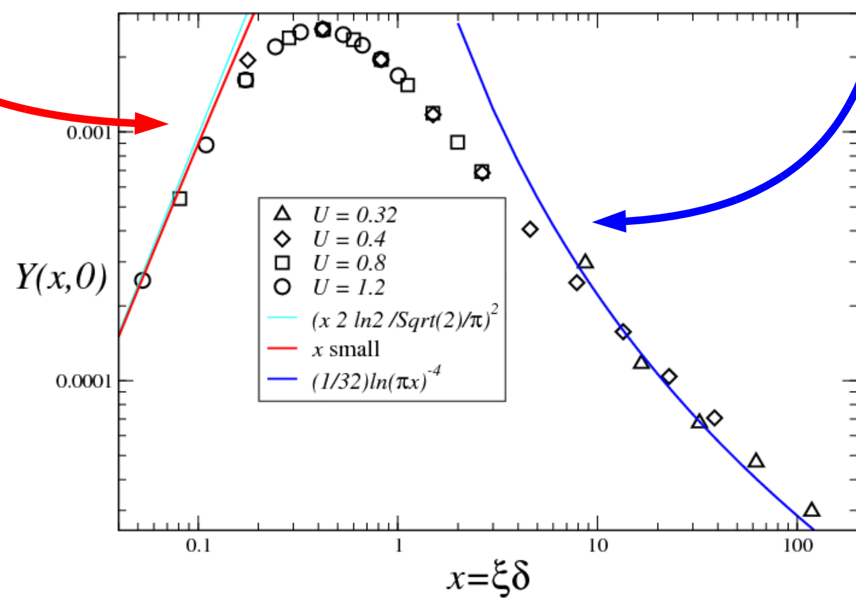
generalize:

$$g\left(U,\delta,L\right) = \left(\frac{d\log\xi}{dU}\right)^2 Y\left(\xi(U)\delta, \xi(U)/L\right)$$

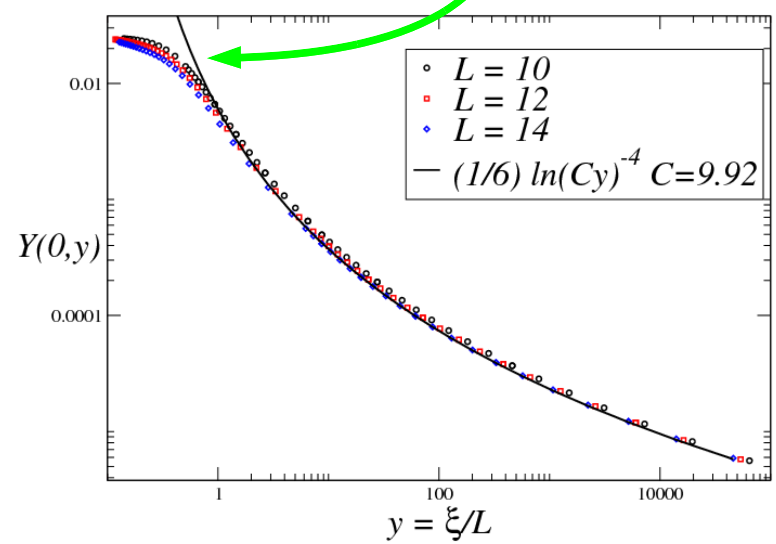
small x: Bethe-Ansatz

large x: RG

Consistency with free (U=0) value



numerical integration of BA equations away from half-filling



exact (Lanczos) diagonalization at half-filling

Surprise!

$$\max_{Exps} d_F(P,Q) =$$ *Projective Hilbert space distance!*

***Q-fidelity*!**

$$D_{FS}(P,Q) \equiv \cos^{-1} \frac{|\langle Q | P \rangle|}{\| Q \| \cdot \| P \|} \equiv \cos^{-1}(F)$$

(Wootters 1981)

Statistical distance and geometrical one collapse*:*

Hilbert space geometry ***is*** ***(quantum) information geometry…..***

Question: What about non pure preparations ?!? ρ_P

Answer: Bures metric & Uhlmann fidelity!
(Braunstein Caves 1994)

$$d(\rho_0, \rho_1) = \cos^{-1}(F)$$

$$F(\rho_0, \rho_1) = tr\sqrt{\rho_1^{1/2} \rho_0 \rho_1^{1/2}}$$

emark: In the classical case one has commuting objects, simplification ….