Quantum Metrology with Identical Particles

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Outline

- The usual notion of separability has to be reconsidered when applied to states describing identical particles
- A definition of separability not related to any *a priori* Hilbert space tensor product structure is needed: it can be given in terms of commuting algebras of observables
- This generalized notion of entanglement, based on a dual description in terms of *operators* rather than *states*, will be applied to the case of a ultracold gas confined in a double-well trap
- The theoretical results concerning the use of the notion of quantum Fisher information in getting sub-shot-noise accuracies in quantum metrological phase estimation need to be generalized and physically reinterpreted

N-particle entanglement

The usual notion of entanglement for states of a system of N distinguishable particles makes use of the natural tensor product structure of the N-body system:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$$

A state for the N-body system, represented by a density matrix ρ acting on $\mathcal{H}$, is said to be separable if it can be written as a convex combination of single-particle states

$$\rho = \sum_k p_k \, \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \ldots \otimes \rho_k^{(N)} \; , \qquad p_k \geq 0 \; , \quad \sum_k p_k = 1$$

In the case of identical particles, these are not allowed quantum states for the system!

Identical particles entanglement

According to the standard rules of quantum mechanics:

- A pure state $|\psi\rangle$ of N identical particles must be a symmetric or antisymmetric combination of tensor products of N-single particle vector states
- A mixed state, *i.e.* a density matrix, must be a linear convex combination of projections $|\psi\rangle\langle\psi|$ onto such symmetrized or antisymmetrized vectors

Not even the so-called symmetric states of the form

$$\rho = \sum_k p_k \, \rho_k \otimes \rho_k \otimes \ldots \otimes \rho_k$$

are in general admissible states for a system of identical particles

Example: two qubits

The Hilbert space of two distinguishable qubits is four-dimensional, $\mathbf{C}^2 \otimes \mathbf{C}^2 \equiv \mathbf{C}^4$, spanned by the basis vectors:

$$|+,+\rangle \quad |+,-\rangle \quad |-,+\rangle \quad |-,-\rangle$$

Instead, the Hilbert space for two *identical* qubits is a symmetric three-dimensional subspace of $\mathbf{C}^4$ in the case of bosons, spanned by

$$|+,+\rangle \quad |-,-\rangle \quad \frac{|+,-\rangle + |-,+\rangle}{\sqrt{2}}$$

or an antisymmetric one-dimensional subspace for fermions, spanned by

$$\frac{|+,-\rangle - |-,+\rangle}{\sqrt{2}}$$

States of the type $\rho = \sum_k p_k\, \rho_k^{(1)} \otimes \rho_k^{(2)}$ or even $\rho = \sum_k p_k\, \rho_k \otimes \rho_k$, can not be written in general as a convex combination of solely projections onto symmetric states or the antisymmetric one!

Separability for identical particles

- An *algebraic bipartition* of the algebra of observables $\mathcal{O}$ is any pair $(\mathcal{O}_1, \mathcal{O}_2)$ of *commuting subalgebras* of $\mathcal{O}$

- An element (operator) of $\mathcal{O}$ is said to be *local* with respect to the bipartition $(\mathcal{O}_1, \mathcal{O}_2)$ if it is the product $O_1 O_2$ of an element O_1 of $\mathcal{O}_1$ and another O_2 of $\mathcal{O}_2$

- A state ω on the algebra $\mathcal{O}$ will be called *separable* with respect to the bipartition $(\mathcal{O}_1, \mathcal{O}_2)$ if the expectation $\omega(O_1 O_2)\big(\equiv \mathrm{Tr}[\omega\, O_1 O_2]\big)$ of any local operator $O_1 O_2$ can be decomposed into a linear convex combination of products of expectations:

$$\omega(O_1 O_2) = \sum_k \lambda_k\, \omega_k^{(1)}(O_1)\, \omega_k^{(2)}(O_2) \qquad \lambda_k \geq 0\ , \quad \sum_k \lambda_k = 1$$

where $\omega_k^{(1)}$ and $\omega_k^{(2)}$ are states on $\mathcal{O}$; otherwise the state ω is said to be *entangled* with respect the bipartition $(\mathcal{O}_1, \mathcal{O}_2)$

Example: two qubits

In the case of distinguishible particles, this definition reduces to the usual notion of separability!

For the two qubit system:

- choose the subalgebras $\mathcal{O}_1$ and $\mathcal{O}_2$ to coincide with the 2×2 matrix algebras of the single-qubits:

$$\mathcal{O}_1 = \{O_1 \otimes \mathbf{1}\} \qquad \mathcal{O}_2 = \{\mathbf{1} \otimes O_2\}$$

- take as operation of expectation the usual trace operator over the corresponding density matrix:

$$\omega_\rho(O_1 O_2) = \mathrm{Tr}[\rho\, O_1 \otimes O_2]$$

This mean value can be written as a sum of products of expectations if

$$\rho = \sum_k p_k\, \rho_k^{(1)} \otimes \rho_k^{(2)}$$

Identical bosons in a double-well trap

In a suitable approximation, the dynamics of cold atoms in an double-well potential can be described by a two-mode Bose-Hubbard hamiltonian:

$$H = E\left[a_1^\dagger a_1 + a_2^\dagger a_2\right] + U\left[(a_1^\dagger a_1)^2 + (a_2^\dagger a_2)^2\right] - J\left[a_1^\dagger a_2 + a_2^\dagger a_1\right]$$

- Trapping potential term $\propto E$;
- On-site boson-boson repulsive interaction term $\propto U$
- Hopping term $\propto J$;

The total number N of particles is conserved: the Hilbert space is thus $(N+1)$-dimensional.

The many-body model

Introduce a complete set of single-particle atom states

$$\{|w_i\rangle\}_{i=1}^{\infty}\,, \qquad |w_i\rangle = a_i^\dagger|0\rangle$$

The bosonic creation operator can then be decomposed as

$$\psi^\dagger(x) = \sum_i w_i^*(x)\, a_i^\dagger$$

$$[a_i^\dagger,\, a_j] = \langle w_i|w_j\rangle = \delta_{ij}$$
$$[\psi^\dagger(x),\, \psi(y)] = \delta(x-y)$$

where $w_i(x) = \langle x|w_i\rangle$ are the corresponding wavefunctions

The Bose-Hubbard Hamiltonian results from a *tight binding approximation*, where only the first two of the basis vector are relevant; in this case $w_{1,2}(x)$ are orthogonal functions, w_1 localized within the first well, w_2 within the second one.

Number states

The $N+1$-dimensional Hilbert space can be spanned by Fock states

$$|k,\, N-k\rangle = \frac{(a_1^\dagger)^k (a_2^\dagger)^{N-k}}{\sqrt{k!(N-k)!}}\, |0\rangle$$

with k particles in the first well and $N-k$ in the second.

In this (second-quantized) formalism, symmetrization of the elements of the Hilbert space, as required by the identity of the particles filling the two wells, is automatically guaranteed by the commutativity of the two creation operators

Commuting subalgebras of observables

All polynomials in a_1, $a_1^\dagger$ and similarly all polynomials in a_2, $a_2^\dagger$ (together with their respective norm closures) form two commuting subalgebras of the algebra $\mathcal{A}$ of all operators on the Fock space, $\mathcal{A}_1,\, \mathcal{A}_2 \subset \mathcal{A}$:

$$[A_1,\, A_2] = 0 \quad \text{for any} \;\; A_1(a_1, a_1^\dagger) \in \mathcal{A}_1,\;\; A_2(a_2, a_2^\dagger) \in \mathcal{A}_2$$

They define a bipartition $(\mathcal{A}_1,\, \mathcal{A}_2)$ of $\mathcal{A}$ and therefore can be used to provide the notion of separability for the states describing the identical atoms in the trap

Separable states

With respect to this natural mode bipartition, $(\mathcal{A}_1, \mathcal{A}_2)$, the Fock states turn out to be separable

$$\langle k, N-k | A_1 A_2 | k, N-k \rangle = \langle k | A_1 | k \rangle \, \langle N-k | A_2 | N-k \rangle$$

in terms of single-mode Fock states

$$|k\rangle := \frac{(a_1^\dagger)^k}{\sqrt{k!}} |0\rangle \qquad |N-k\rangle := \frac{(a_2^\dagger)^{N-k}}{\sqrt{(N-k)!}} |0\rangle$$

All states separable with respect to the bipartition $(\mathcal{A}_1, \mathcal{A}_2)$ must be in diagonal form with respect to the Fock basis:

$$\rho = \sum_{k=0}^{N} p_k \, |k, N-k\rangle\langle k, N-k| \ , \qquad p_k \geq 0 \ , \quad \sum_{k=0}^{N} p_k = 1$$

Local and non-local observables

Most observables of physical interest are non-local with respect to the bipartition $(\mathcal{A}_1, \mathcal{A}_2)$.

Take the following collective bilinear $su(2)$ operators:

$$J_x = \frac{1}{2}\left(a_1^\dagger a_2 + a_1 a_2^\dagger\right) \qquad J_y = \frac{1}{2i}\left(a_1^\dagger a_2 - a_1 a_2^\dagger\right) \qquad J_z = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right)$$

whose exponentials measure phase accumulation inside the interferometer

While $e^{i\theta J_x}$ and $e^{i\theta J_y}$, $\theta \in [0, 2\pi]$, are non-local, the exponential of J_z turns out to be local:

$$e^{i\theta J_z} = e^{i\theta a_1^\dagger a_1/2} \cdot e^{-i\theta a_2^\dagger a_2/2}\ , \qquad e^{i\theta a_1^\dagger a_1/2} \in \mathcal{A}_1\ , \quad e^{-i\theta a_2^\dagger a_2/2} \in \mathcal{A}_2\ .$$

Changing the bipartition

Introduce a new set of creation and annihilation operators $b_i^\dagger$, b_i, $i = 1, 2$:

$$b_1 = \frac{a_1 + a_2}{\sqrt{2}} \qquad b_2 = \frac{a_1 - a_2}{\sqrt{2}}$$

so that

$$J_x = \frac{1}{2}\left(b_1^\dagger b_1 - b_2^\dagger b_2\right) \qquad J_y = \frac{1}{2i}\left(b_1 b_2^\dagger - b_1^\dagger b_2\right) \qquad J_z = \frac{1}{2}\left(b_1 b_2^\dagger + b_1^\dagger b_2\right)$$

Using the operators $b_i^\dagger$, b_i, one can define a new bipartition $(\mathcal{B}_1, \mathcal{B}_2)$ of the full algebra $\mathcal{A}$, so that it is now the exponential of J_x that turns out to be local:

$$e^{i\theta J_x} = e^{i\theta b_1^\dagger b_1} \cdot e^{-i\theta b_2^\dagger b_2} \,, \qquad e^{i\theta b_1^\dagger b_1} \in \mathcal{B}_1 \,, \quad e^{-i\theta b_2^\dagger b_2} \in \mathcal{B}_2 \,.$$

Therefore, an operator which is local with respect to a given bipartition, can result non-local in different one

Changing the basis states

The above Bogolubov transformation corresponds to a change of basis in the Hilbert space; for instance

$$b_1^\dagger|0\rangle = \frac{\left[a_1^\dagger|0\rangle + a_2^\dagger|0\rangle\right]}{\sqrt{2}} \qquad b_2^\dagger|0\rangle = \frac{\left[a_1^\dagger|0\rangle - a_2^\dagger|0\rangle\right]}{\sqrt{2}}$$

which are energy eigenstates of the Bose-Hubbard Hamiltonian in the limit of a highly penetrable barrier.

As a consequence, the Fock states result entangled with respect to this new bipartition $(\mathcal{B}_1, \mathcal{B}_2)$

$$|k, N-k\rangle \sim \sum_{r=0}^{k} \sum_{s=0}^{N-k} \binom{k}{r} \binom{N-k}{s} (-1)^{N-k-s} \left(b_1^\dagger\right)^{r+s} \left(b_2^\dagger\right)^{N-r-s} |0\rangle$$

so that $|k, N-k\rangle$ is a combination of $(\mathcal{B}_1, \mathcal{B}_2)$-separable states.

Quantum metrology

The state transformation $\rho_{\rm in} \mapsto \rho_\theta$ inside the interferometer results as a pseudo-spin rotation along a given unit vector $\vec{n} = (n_x, n_y, n_z)$

$$\rho_{\rm in} \mapsto \rho_\theta = U_\theta \, \rho_{\rm in} \, U_\theta^\dagger \ , \qquad U_\theta = e^{i\theta \, J_n} \ , \quad J_n = \vec{J} \cdot \vec{n}$$

The accuracy $\Delta\theta$ with which the phase θ can be obtained in a measurement involving the operator J_n and the initial state $\rho_{\rm in}$ is limited by

$$\Delta\theta \geq \frac{1}{\sqrt{F[\rho_{\rm in}, J_n]}}$$

Given the interferometer, *i.e.* J_n, $\Delta\theta$ can be minimized by choosing an initial state that maximizes the quantum Fisher information $F[\rho_{\rm in}, J_n]$

Distinguishable particles

In this case, for any separable state ρ_{sep} the quantum Fisher information is bounded by N:

$$F\left[\rho_{\mathrm{sep}}, J_{n}\right] \leq N$$

thus, the best achievable precision is bounded by the shot-noise-limit

$$\Delta\theta \geq \frac{1}{\sqrt{N}}$$

But in general,

$$F\left[\rho, J_{n}\right] \leq N^{2}$$

so that using entangled initial states:

$$\Delta\theta \geq 1/N$$

eventually reaching the Heisenberg limit

Identical particles

The notion of separability requires the choice of an algebraic bipartition

Select the spatial bipartition $(\mathcal{A}_1, \mathcal{A}_2)$

In the case of the separable pure state $\rho_k = |k, N-k\rangle\langle k, N-k|$:

$$F\left[\rho_k, J_n\right] = (n_x^2 + n_y^2)\left[N + 2k(N-k)\right]$$

and can always be made greater than N with a suitable choice of k, thus beating the shot-noise-limit

Actually, for $\rho_{N/2} = |N/2, N/2\rangle\langle N/2, N/2|$, one can even get close to the Heisenberg limit:

$$F\left[\rho_{N/2}, J_n\right] \simeq N^2/2$$

Identical particles

This result suggests a different experiment and the use of a different bipartition

Take $\vec{n}$ along the x direction; in the energy bipartition $(\mathcal{B}_1, \mathcal{B}_2)$

$$J_n = \frac{1}{2}\left(b_1^\dagger b_1 - b_2^\dagger b_2\right)$$

and the rotation around $\vec{n}$ is local

$$e^{i\theta J_n} = e^{i\theta b_1^\dagger b_1/2}\ e^{-i\theta b_2^\dagger b_2/2} \qquad e^{i\theta b_1^\dagger b_1/2} \in \mathcal{B}_1 \quad e^{-i\theta b_2^\dagger b_2/2} \in \mathcal{B}_2$$

but $|N/2, N/2\rangle$ is no longer separable

$$|N/2, N/2\rangle \sim \sum_{k,r=0}^{N/2} \binom{N/2}{k}\binom{N/2}{r}(-1)^{N/2-r}\left(b_1^\dagger\right)^{k+r}\left(b_2^\dagger\right)^{N-k-r}|0\rangle$$

Outlook

- The standard notion of separability becomes meaningless when applied to systems of identical particles; it can be replaced by a generalized one, that makes use of a "dual" language, focusing on the algebra $\mathcal{O}$ of operators of the system instead of the set of its quantum states
- Sub-shot-noise phase estimation accuracy in quantum metrology can be achieved either by acting with a non-local operation on separable states, or by devising a local measuring procedure on an entangled initial state
- There is always a limit in accuracy due to decohering effects induced by the environment:

$$\frac{\Gamma_{\text{Fock}}}{\Gamma_{\text{semiclassical}}} \simeq N$$

References

- F. Benatti, R. Floreanini, U. Marzolino, *Sub-shot-noise quantum metrology with entangled identical particles*, Ann. Phys. **325** (2010) 924
- S. Anderloni, F. Benatti, R. Floreanini and A. Trombettoni, *Quantum measuring processes for trapped ultracold bosonic gases*, J. Phys. A **42** (2009) 035306
- R. Alicki, F. Benatti and R. Floreanini, *Charge oscillations in superconducting nanodevices coupled to external environments*, Phys. Lett. A **372** (2008) 1968