Topological Renyi Entropy

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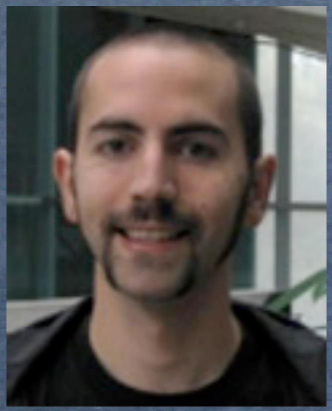

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What is topological Order?

- A system with a gap in the bulk
- A degenerate ground state whose degeneracy depends on topology
- the degenerate ground states are locally indistinguishable

symmetry vs non symmetry breaking

$$H_{ising} = -\lambda_1 \sum_i \sigma_i^z \sigma_{i+1}^z - \lambda_2 \sum_i \sigma_i^x$$

For $\lambda_1 \gg \lambda_2$ the GS is Z_2 symmetry broken

$$|\uparrow\uparrow\uparrow \ldots \uparrow\uparrow\rangle \qquad |\downarrow\downarrow\downarrow \ldots \downarrow\downarrow\rangle$$

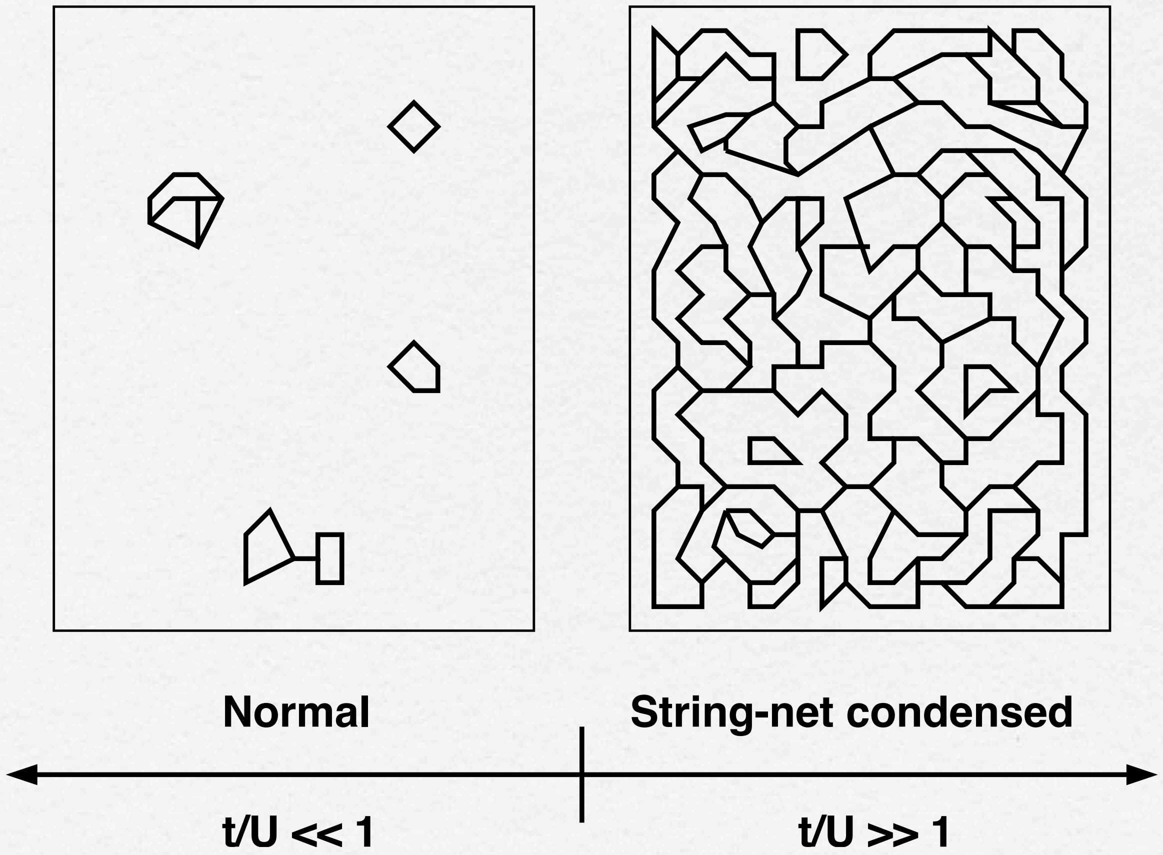
They can be distinguished locally $\rightarrow$ local order parameter

$$\begin{aligned} \langle\uparrow \ldots \uparrow|\sigma_i^z|\uparrow \ldots \uparrow\rangle &= +1 \\ \langle\downarrow \ldots \downarrow|\sigma_i^z|\downarrow \ldots \downarrow\rangle &= -1 \end{aligned}$$

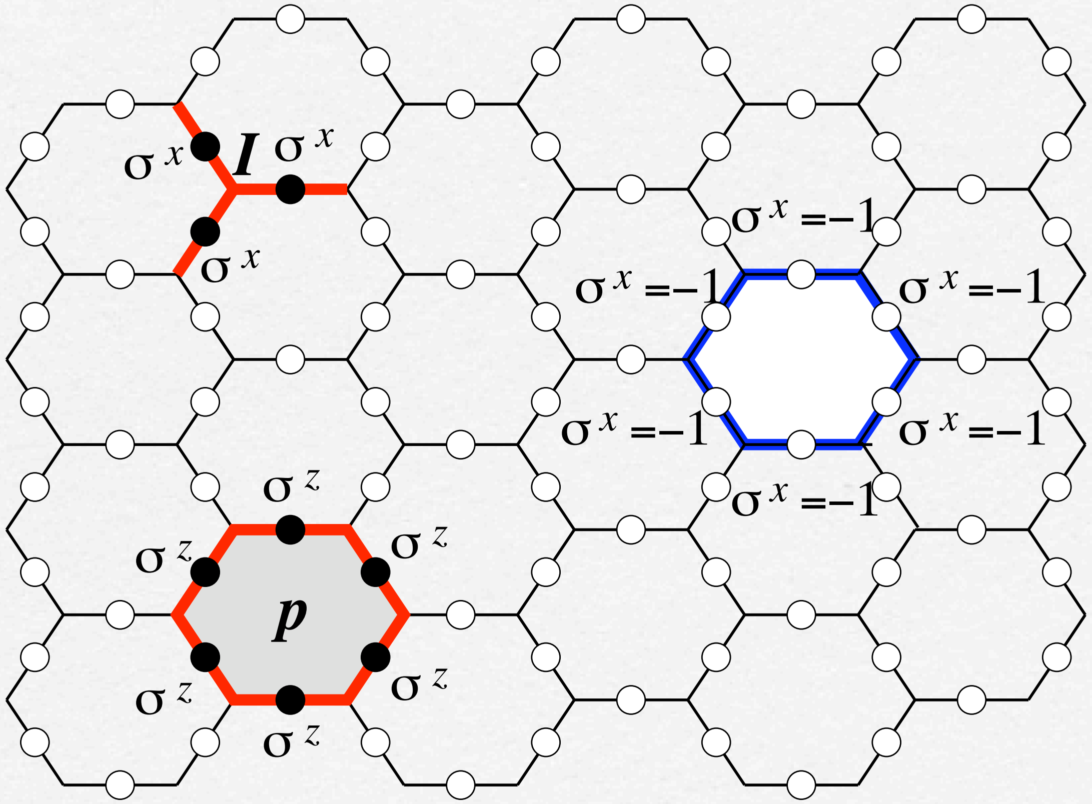
- Topologically ordered states cannot be described by a local order parameter: no “magnetization”
- Perturbing with any symmetry breaking term does not select any ground state
- Indeed, the splitting of degenerate ground states due to perturbations is exponentially small!

- The mechanism for TO is the condensation of closed string-like objects
- The kinetic term makes strings fluctuate, the potential term is a string tension
- the ground state is a quantum liquid of large strings

$$H = U H_U + t H_t$$



Example, Z2 lattice gauge theory



The Hilbert space is such that

$$\prod_{\text{legs of } I} \sigma_i^x |\Phi\rangle = |\Phi\rangle$$

$$H_{Z_2} = -U \sum_i \sigma_i^x + t \sum_p \prod_{\text{edges of } p} \sigma_j^z$$

This is the low energy theory of the toric code on the full Hilbert space

$$H_{TC} = -J \sum_I \prod_{\text{legs of I}} \sigma_i^x - U \sum_i \sigma_i^x + t \sum_p \prod_{\text{edges of } p} \sigma_j^z$$

topological entanglement in Z2 theory

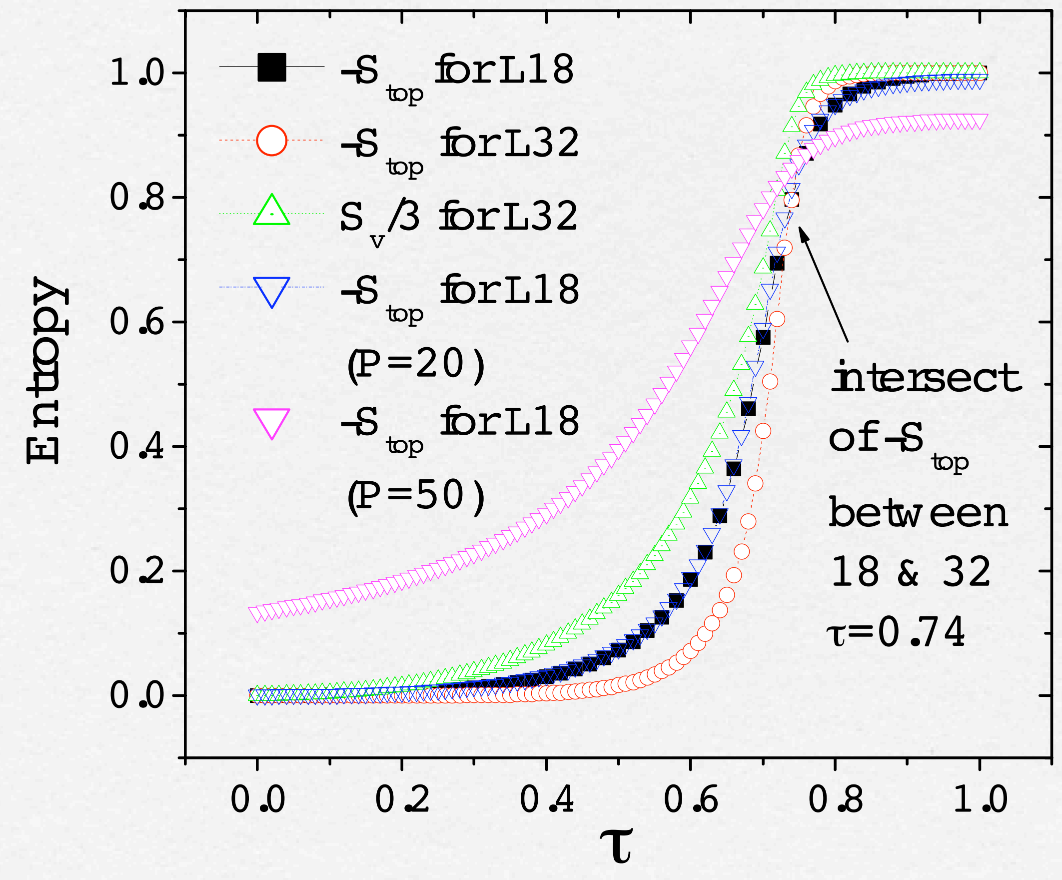
- We need something that is nonlocal
- Perhaps topological order is a pattern of long-range entanglement
- We can compute the entanglement entropy of a subsystem A

$$\rho_A = Tr_B \rho$$

$$S(\rho_A) = -Tr \rho_A \log \rho_A$$

$$S(\rho_A) = L - \log_2 |Z_2|$$

A. H., R. Ionicioiu, and P. Zanardi, PRA 71, 022315 (2005)



A.H., W. Zhang, S. Haas, D. Lidar PRB 77, 155111 (2008)

General lattice gauge theories

- the N types of strings correspond to the different elements of the gauge group G
- the quantum dimension D is the order of the group

$$D^2 = \sum_i d_i^2$$

- S_top = log D

$$H_{QD} = \sum_s (1 - A(s)) + \sum_p (1 - B(p))$$

$$A(s) = \sum_{g \in G} \prod_{j \in s} L^g(j, s)$$

$$B_e(p) = \sum_{h_1 h_2 h_3 h_4 = e} \prod_{m=1}^{4} T^{h_m}(j_m, p)$$

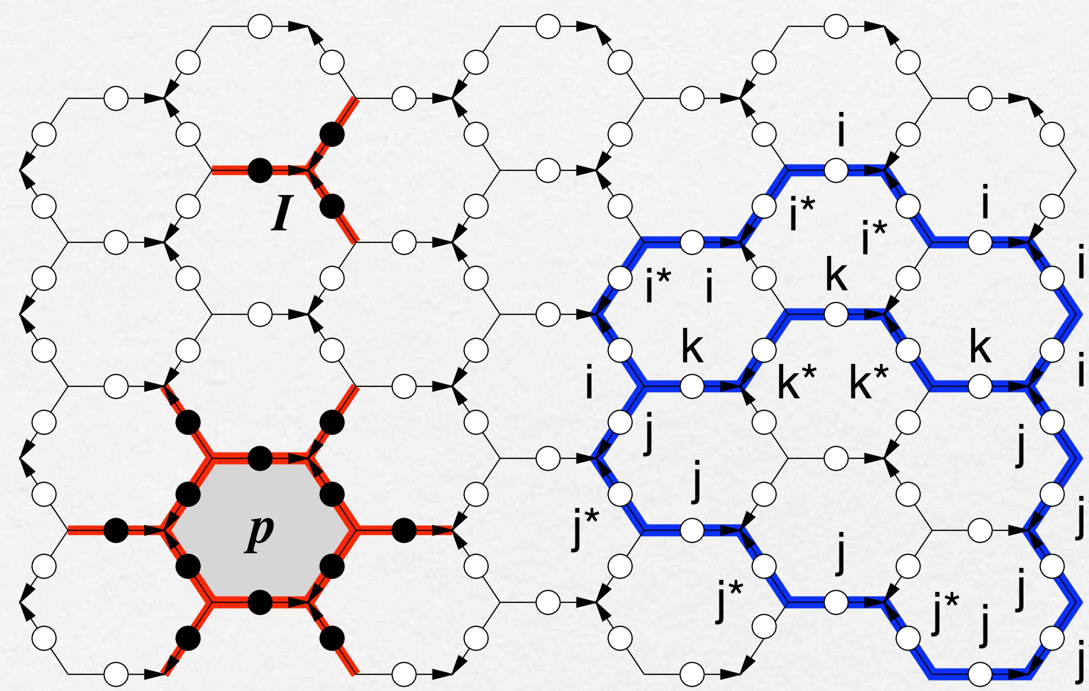
$$L_+^g |z\rangle = |gz\rangle$$

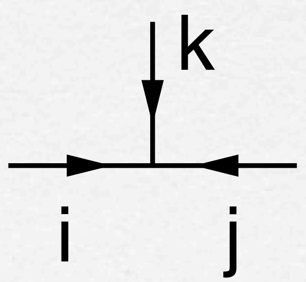
$$T_+^h |z\rangle = \delta_{h,z} |z\rangle$$

A. Kitaev and J. Preskill, PRL 96, 110404 (2006).
M. Levin and X.-G. Wen, PRL 96, 110405 (2006).

General String-Nets in 2+1D

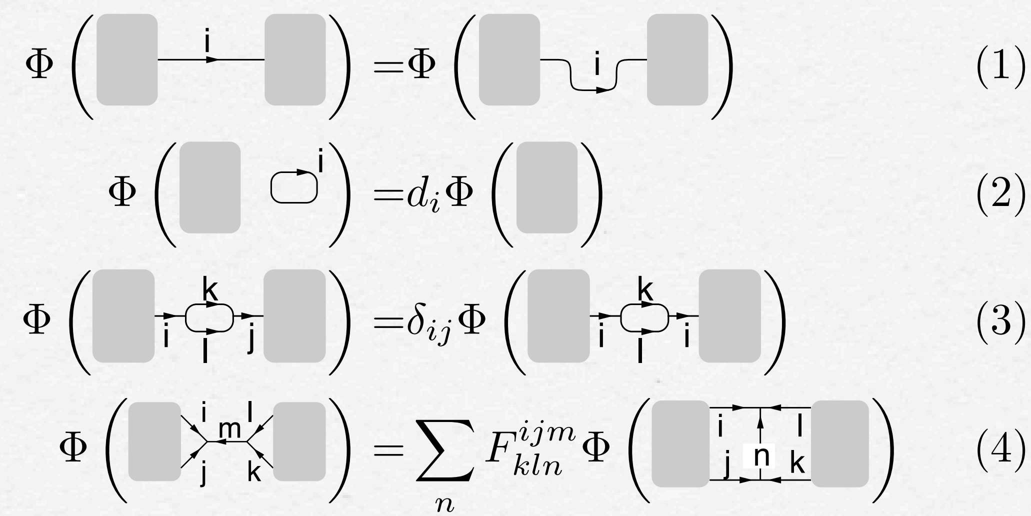
- We describe parity and time reversal topologically ordered states: string-nets
- We have a trivalent lattice
- We specify the number N of different string types and the "branching rules"
- the Hilbert space is made of string configurations





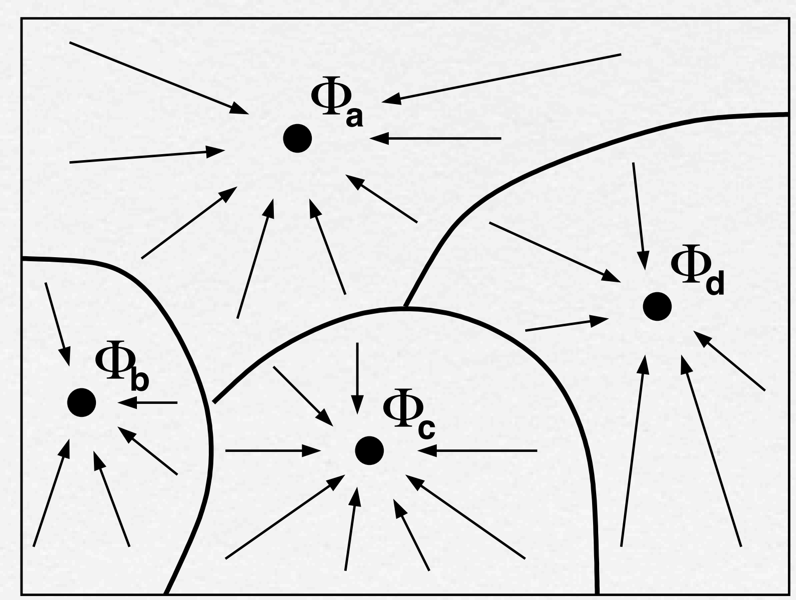
M. Levin, X.G. Wen, Phys.Rev. B71 (2005) 045110

The amplitudes respect some topological rules



This tensor F is the 6j symbol in the gauge theory

These rules for the amplitudes specify a fixed-point wavefunction for some RG procedure



M. Levin, X.G. Wen, Phys.Rev. B71 (2005) 045110

- In this setting, we have $D = \sum_i d_i^2$
- this number does not classify all the topological phases
- we need finer invariants
- Idea: we could use the Renyi entropy
- This is a family of quantities -> entanglement spectrum

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log_2[Tr(\rho^\alpha)]$$

$$U\rho U^{-1} = diag(\lambda_1 \dots \lambda_K)$$

Let's first look at the gauge theory models

$$\rho_A = \kappa \Pi$$

- We find that the reduced density matrix is proportional to a projector

$$\lambda_1 = \ldots = \lambda_{\kappa^{-1}} = \frac{1}{\kappa}$$

- Then its entanglement spectrum is flat!

$$S_\alpha^{(top)}(\rho_A) = \log D$$

What happens in string-net models?

- powers of the reduced density matrix are obtained using the fusion rules F

$$Tr(\rho_A^\alpha) = \frac{D^\alpha}{D^{\alpha n}} \sum_{\{q\}} F_{\{q\}} \prod_m d_{q_m}^\alpha$$

- lengthy calculation of the Renyi entropy yields

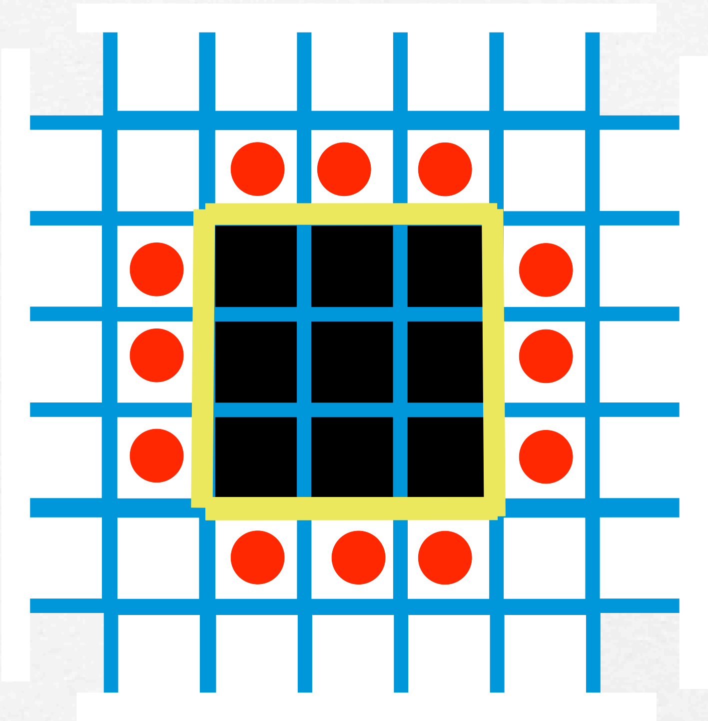
$$S_\alpha(\rho_A) = \frac{n}{1-\alpha} \log\left(\frac{\langle d^\alpha | d \rangle}{d} D^\alpha \right) - \log D$$

S.Flammia, A.H., T. Hughes, X.G. Wen PRL 103, 261601 (2009)

The topological part is independent of α

The origin of the topological term

- ❑ the entanglement is completely contained in the boundary
- ❑ the area law is corrected by a topological constraint



$$\rho \equiv Q_A Q_B \tilde{\rho} \otimes \rho^{(\text{bulk})} Q_A Q_B$$

$$\rho_A^{(\text{area})} \equiv \tilde{\rho}_A \otimes \rho_A^{(\text{top})} = \otimes_{j=1}^{n} \rho_j \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

S.Flammia, A.H., T. Hughes, X.G. Wen PRL 103, 261601 (2009)

Conclusions

- Topological order can be detected by entanglement
- All the entanglement lives on the boundary
- TO is a constraint on the states on the boundary
- The entanglement spectrum is flat, we need finer invariants to classify all TO