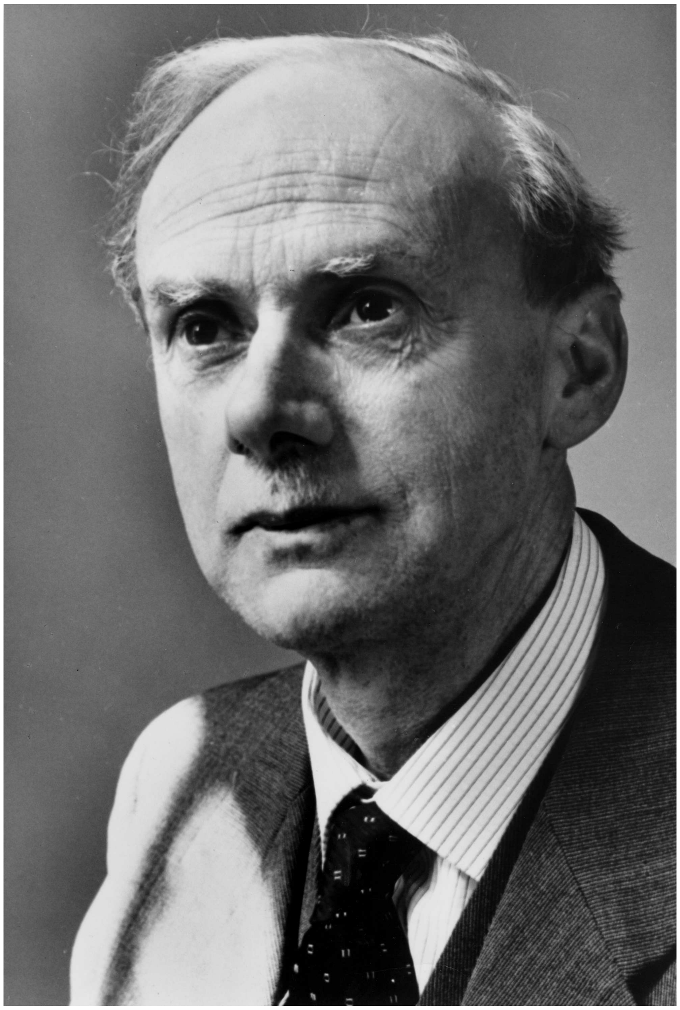
Prerequisites for Fractional Charge



Dirac Matrix Equation

$$(\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\psi = i\,\frac{\partial}{\partial t}\,\psi$$

ψ complex (charged excitations)

$\boldsymbol{\alpha}, \beta \Rightarrow$ matrices

$$\mathbf{p} \equiv \frac{1}{i}\,\boldsymbol{\nabla}$$

$m \equiv$ mass parameter

Dirac Equations
(First-order matrix equations)

$$[\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m]\psi = E\psi$$

Usual (mass term position-independent, homogenous)
Dirac equation: continuum solutions $E > 0$ and $E < 0$

"vacuum":

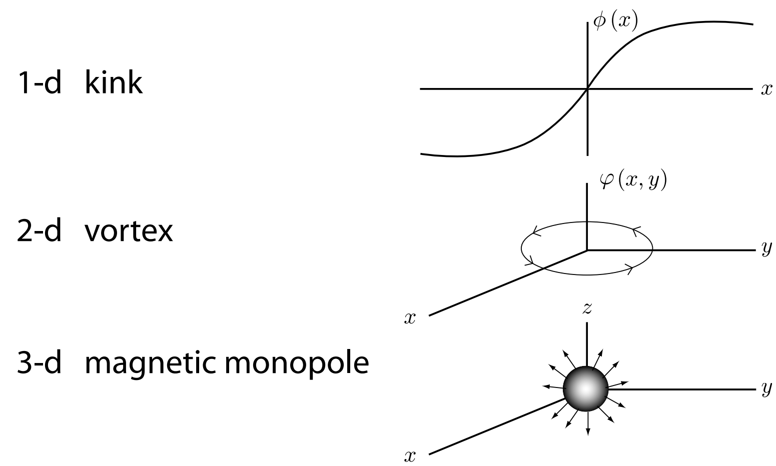
Particle interpretation:	$E < 0$ states filled (antiparticles)
	$E > 0$ states empty (particles)
condensed matter:	$E < 0$ states filled (valence band)
interpretation:	$E > 0$ states empty (conduction band)
	m produces gap

"vacuum" carries no net charge

Dirac equation in the presence of a defect

(mass term position-dependent, soliton)

$$[\boldsymbol{\alpha} \cdot \mathbf{p} + \beta\, m\,(\mathbf{r})]\psi = E\psi$$



continuum solutions $E > 0, E < 0$

AND isolated, normalizable $E = 0$ solution

"mid-gap" state is found by explicit calculation

is guaranteed by index theorems

CENTRAL QUESTION: in "vacuum" is mid-gap state empty or filled, what is its charge?

Fractional Charge (Analytic Derivation)

Vacuum charge density:

$$\rho(\mathbf{r}) = \int_{-\infty}^{0} dE\, \rho_E\,(\mathbf{r}) \qquad \rho_E = \psi_E^\dagger \psi_E$$

renormalized charge in soliton background

$$Q = \int d\mathbf{r} \int_{-\infty}^{0-} dE \Big(\rho_E^s(\mathbf{r}) - \rho_E^0(\mathbf{r}) \Big)$$

Evaluation simple in the presence of an energy reflection symmetry

(charge conjugation + time reflection)

≡ "chiral" symmetry

$$U \psi_E = \psi_{\text{-}E}$$

$$\rho_{\text{-}E} = \psi_{\text{-}E}^\dagger\, \psi_{\text{-}E} = \psi_E^\dagger\, U^\dagger\, U\, \psi_E = \rho_E$$

(Guarantees single state in gap is at mid-gap, $E = 0$)

Fractional Charge Calculation

completeness: $$\int_{-\infty}^{\infty} dE\, \psi_E^\dagger(\mathbf{r})\psi_E(\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$

$$\Rightarrow \int_{-\infty}^{\infty} dE[\rho_E^s(\mathbf{r}) - \rho_E^0(\mathbf{r})] = 0$$

Conjugation ($\rho_E = \rho_{-E}$) and zero mode $\Rightarrow$

$$\int_{-\infty}^{0-} dE\Big(2\rho_E^s(\mathbf{r}) - 2\rho_E^0(\mathbf{r})\Big) + \psi_{E=0}^\dagger(\mathbf{r})\psi_{E=0}(\mathbf{r}) = 0$$

$$\int_{-\infty}^{0-} dE\Big(\rho_E^s(\mathbf{r}) - \rho_E^0(\mathbf{r})\Big) = -\frac{1}{2}\,\psi_{E=0}^\dagger(\mathbf{r})\psi_{E=0}(\mathbf{r})$$

$$Q = -\frac{1}{2}$$

Any dimension!

Empty mid-gap state: $Q = -\frac{1}{2}$
Filled mid-gap state: $Q = +\frac{1}{2}$

Eigenvalue, not expectation value!

1-d (Polyacetylene)

Rebbi & RJ *PRD* **13**, 3398 (76): Zero mode (math)
Su, Schrieffer & Heeger, *PRL* **42**, 1698 (79):
linearization at Fermi surface $\Rightarrow$ Dirac equation,
Peierls' instability, kink

2-d (Graphene)

Wallace, *PR* **71**, 662 (47): linearization at Fermi surface
Rossi & RJ *NPB* **190**, 681 (81): Zero mode (math)
Semenoff, *PRL* **53**, 2449 (84): Dirac equation
Hou, Chamon & Mudry *PRL* **98**, 186809 (07):
Kekulé distortion

Elaborations

(a) No energy reflection symmetry
Goldstone & Wilczek, *PRL* **47**, 987 (81)
Semenoff & RJ, *PRL* **50**, 439 (83)
Niemi & Semenoff, *Phys. Rep.* **135**, 100 (86)
(η invariant, spectral asymmetry)
Chamon et al., *PRL* **100**, 110405 (08),
PRB **77**, 235432 (08)

(b) Gauging the graphene model (depinning vortices)
SY Pi & RJ *PRL* **98**, 266402 (07)
Chamon et al. *op. cit* (erases irrational charge)

Fractional Charge
Second Quantized Description

Expansion of quantum Fermi field in presence of defect & zero mode

$$\Psi = \sum_{E>0} (b_E \, \psi_E^s + d_E^\dagger \psi_E^{*s}) + a\psi_{E=0}$$

$$\Psi^\dagger = \sum_{E>0} (b_E^\dagger \, \psi_E^{s*} + d_E \, \psi_E^s) + a^\dagger \, \psi_{E=0}$$

$$a^\dagger a + a a^\dagger = 1$$

How to realize on states:

$$a \,|+>=|->, a^\dagger \,|+>= 0, a \,|->= 0, a^\dagger \,|->=|+>$$

$Q = \int dr \Psi^\dagger \Psi$ needs proper definition (regularization) as in vacuum sector use (Schwinger)

$$Q = \frac{1}{2} \int dr (\Psi^\dagger \Psi - \Psi \Psi^\dagger)$$

$$= \frac{1}{2} \sum_{E>0} (b_E^\dagger b_E + d_E d_E^\dagger - b_E b_E^\dagger - d_E^\dagger d_E) + \frac{1}{2}(a^\dagger a - a a^\dagger)$$

$$= \sum_{E>0} (b_E^\dagger b_E - d_E^\dagger d_E) + a^\dagger a - \frac{1}{2}$$

$Q \,|- > = -\frac{1}{2}\,|- >,\ Q\,|+ > = \frac{1}{2}\,|+ >$ eigenvalue !

Majorana Matrix Equation

$$(\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\psi = i \frac{\partial}{\partial t} \psi$$

ψ real (neutral excitations)

$\boldsymbol{\alpha}$ real, β imaginary

$$\mathbf{p} = \frac{1}{i} \boldsymbol{\nabla}$$

Majorana Fermions

- Charge neutral fermion (neutrino?)
- Excitations in super-conductor described by Bogoliubov – DeGennes eqn.
  Charge conservation disappears owing to interaction with charge 2 Cooper pair
  – symmetry breaking

2-d Example

$$H = \psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{p}\, \psi + (\varphi\, \psi^\dagger\, i\sigma^2\, \psi^* + h.c.)$$

$\psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}$ electron operators, φ Cooper pair

φ constant $\varphi_0 \Rightarrow$ mass gap in spectrum

φ soliton/vortex $\varphi_s \Rightarrow$ real mid-gap state

Rossi & RJ *NPB* **190**, 681 (81)

Fu & Kane, *PRL* **100**, 096407 (08) [0707.1692]

Expansion as before with $b = d$

$$\Psi = \Psi^\dagger = \sum_{E>0} \left(b_E \, \psi_E^s + b_E^\dagger \, \psi_E^{*s} \right) \; + a \, \psi_{E=0}$$

$$a^\dagger = a$$

$$Q = \sum_{E>0} \; \left(b_E^\dagger \, b_E - b_E^\dagger \, b_E \right) + \frac{1}{2} \; (aa - aa) = 0$$

$a^\dagger a + a a^\dagger = 2a^2 = 1$ How to realize on states?

$1d : \quad a \, |x\rangle = \pm \frac{1}{\sqrt{2}} \; |x\rangle$ no fermion parity

$2d : \quad a \, |x\rangle = \frac{1}{\sqrt{2}} \; |y\rangle$ fermion parity

$a \, |y\rangle = \frac{1}{\sqrt{2}} \; |x\rangle$

Chamon et al. PRB **81**, 224515 (10) [1001.2760]