FORCED VIBRATIONS OF A NONHOMOGENEOUS STRING

PIETRO BALDI† AND MASSIMILIANO BERTI‡

Abstract. We prove existence of vibrations of a nonhomogeneous string under a nonlinear time periodic forcing term in the case in which the forcing frequency avoids resonances with the vibration modes of the string (nonresonant case). The proof relies on a Lyapunov–Schmidt reduction and a Nash–Moser iteration scheme.

Key words. nonlinear wave equations, periodic solutions, small divisors, Nash–Moser iteration scheme

AMS subject classifications. 35B10, 35L70, 58C15

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1. Introduction. In this paper we study forced vibrations of a nonhomogeneous string,

\begin{equation}
\begin{aligned}
\rho(x)u_{tt} - (p(x)u_x)_x &= \mu f(x, \omega t, u), \\
u(0,t) &= u(\pi, t) = 0,
\end{aligned}
\end{equation}

where \(\rho(x) > 0\) is the mass per unit length, \(p(x) > 0\) is the modulus of elasticity multiplied by the cross-sectional area (see [15, p. 291]), \(\mu > 0\) is a parameter, and the nonlinear forcing term \(f(x, \omega t, u)\) is \((2\pi/\omega)\)-periodic in time (i.e., \(f(x, \cdot, u)\) is \(2\pi\)-periodic).

Equation (1) is a nonlinear model also for propagation of waves in nonisotropic media describing seismic phenomena; see, e.g., [2].

We look for \((2\pi/\omega)\)-time periodic solutions \(u(x, t)\) of (1).

This problem has received wide attention since the pioneering paper of Rabinowitz [26] dealing with the weakly nonlinear homogeneous string with \(\rho(x) = p(x) = 1\), \(\mu\) small, and \(\omega = 1\). In this case the forcing frequency \(\omega\) enters in resonance with the proper eigenfrequencies \(\omega_j = j \in \mathbb{N}\) of the string.

For functions \(2\pi\)-periodic in time and satisfying spatial Dirichlet boundary conditions, the spectrum \([-l^2 + j^2, l \in \mathbb{Z}, j \geq 1\) of the D’Alembertian operator \(\partial_{tt} - \partial_{xx}\) possesses the zero eigenvalue with infinite multiplicity (for \(|l| = j\)), but the other eigenvalues are well separated. The corresponding infinite-dimensional bifurcation problem is solved in [26] for nonlinearities \(f\) which are monotone in \(u\); see [7] for nonmonotone \(f\).

Subsequently many other results, both of bifurcation and of a global nature (\(\mu = 1\), have been obtained, still for rational forcing frequencies \(\omega \in \mathbb{Q}\), relying on the separation properties of the spectrum; see, e.g., [27, 28, 14, 31, 4].

When the forcing frequency \(\omega \in \mathbb{R} \setminus \mathbb{Q}\) is irrational (nonresonant case) the situation is completely different. Indeed the wave operator \(\omega^2 \partial_{tt} - \partial_{xx}\) does not possess the
zero eigenvalue, but its spectrum \(-\omega^2 l^2 + j^2, l \in \mathbb{Z}, j \geq 1\) accumulates to zero for almost every \(\omega\). This is a “small divisors problem.”

We underline that this “small divisors” phenomenon arises naturally for more realistic model equations like (1) where the density \(\rho(x)\) and the modulus of elasticity \(p(x)\) are not constant. Indeed in this case the eigenfrequencies \(\omega_j\) of the string are no longer integer numbers, having the asymptotic expansion

\[\omega_j^2 = \frac{j^2}{c^2} + b + O\left(\frac{1}{j}\right)\]

with suitable constants \(c, b\) depending on \(\rho, p\); see (66).

If \(\omega = m/n \in \mathbb{Q}\), good separation properties of the spectrum can be recovered when \(p(x) = \rho(x)\) (so \(c = 1\)) and assuming the extra condition \(b \neq 0\); see [3, 29]. Indeed in this case the linear spectrum

\[-\omega^2 l^2 + \omega_j^2 = -\omega^2 l^2 + j^2 + b + O\left(\frac{1}{j}\right)\]

possesses at most finitely many zero eigenvalues and the remaining part of the spectrum is far away from zero. On the other hand, if \(b = 0\), the eigenvalues with \((l, j) \in (n, m)\mathbb{N}\) tend to zero (also in the case \(\omega \in \mathbb{Q}\)!

Existence of weak solutions in the nonresonant case was proved by Acquistapace [1] for \(\rho = 1, \mu\) small, and for a zero measure set of forcing frequencies \(\omega\) for which the eigenvalues \(-\omega^2 l^2 + \omega_j^2\) are far away from zero. These frequencies are essentially the numbers whose continued fraction expansion is bounded; see [30].

For a similar zero measure set of frequencies, McKenna [23] has obtained some result when \(\mu = 1, \rho = p = 1, f(x, t, u) = g(u) + h(x, t)\) with \(g\) uniformly Lipschitz, via a fixed point argument; see also [5]; for related results using variational methods see [18, 10].

Existence of classical solutions of (1) for a positive measure set of frequencies was proved by Plotnikov and Yungerman [24] for the homogeneous string \(\rho = p = 1, \mu\) small, and \(f\) monotone in \(u\). This monotonicity condition allows one to control the constant coefficient in the asymptotic expansion of the eigenvalues (like \(b\) in (2)) of some perturbed linearized operator.

Recently Fokam [19] has proved existence of classical periodic solutions for large frequencies \(\omega\) in a set of asymptotically full measure for the homogeneous string \(\rho = p = 1, \mu\) small, and \(f = u^3 + h(x, t)\) with \(h\) a trigonometric polynomial odd in time and space.

In the present paper we prove existence of classical solutions of the nonhomogeneous string (1) for every \(\rho(x), p(x) > 0\) for general nonlinear terms \(f(x, \omega t, u)\), and for \((\mu, \omega)\) belonging to a large measure Cantor set \(B_\gamma\), when the ratio \(\mu/\omega\) is small. Our Theorem 1 covers both the case \(\mu \to 0\) (weak forcing) and the case \(\omega \to +\infty\) (rapid forcing).

In the limit \(\mu/\omega \to 0\) the solution we find tends to a static equilibrium \(v(x)\) with smaller, zero average oscillations \(w(x, t)\) of amplitude \(O(\mu/\omega)\); see (13), (14), and Figure 1. The nonlinearity \(f\) selects such \(v\) through the infinite-dimensional bifurcation equation (10), which possesses nondegenerate solutions under natural assumptions on \(f\); see Hypothesis (V). This problem is not present in [19], where, thanks to the symmetry assumptions on \(f\), there is no bifurcation equation.

Considering the structure of the expected solution, it is natural to attack the problem via a Lyapunov–Schmidt decomposition.
In the range equation (to find $w$) a small divisors problem arises, and we solve it with a Nash–Moser-type iterative scheme. The inversion of the “linearized operators”—which is the core of any Nash–Moser scheme—is obtained adapting the techniques of [8] to the present time-dependent case (section 6). This method is also reminiscent of the approach of Kuksin (unpublished) explained by Bourgain in [13, pp. 90–94]. See also the works of Craig and Wayne [16, 17] and Bourgain [11, 12] for related techniques.

It is in the solution of the range equation where the interaction between the forcing frequency $\omega$ and the normal modes of oscillations of the string linearized at different positions (approximating better and better the final string configuration) appears.

The set $B_\gamma$ of “nonresonant” parameters $(\mu, \omega)$ for which we find a solution of the range equation (and then of (1)) is constructed avoiding these primary resonances. In particular the forcing frequency $\omega$ must not enter in resonance with the normal frequencies of oscillations of the string linearized at the limit solution; see (15). At the end of the construction we obtain a large measure Cantor set $B_\gamma$ which looks like Figure 2. Outside this set the effect of resonance phenomena shall in general destroy the existence of periodic solutions like those found in Theorem 1.
Finally we recall that related existence results of periodic and quasi-periodic solutions for autonomous Hamiltonian PDEs have been obtained via KAM-type techniques since the pioneering works of Kuksin [21] and Wayne [32]; see also [22] and the references therein.

We now present rigorously our results.

1.1. Main result. After a time rescaling we look for $2\pi$-periodic solutions of

$$\begin{cases} 
\omega^2 \rho(x) u_{tt} - (p(x) u_x)_x &= \mu f(x, t, u), \\
u(0, t) = u(\pi, t) &= 0,
\end{cases}$$

where $\mu \in [0, \bar{\mu}]$ for some given $\bar{\mu} > 0$, under the $2\pi$-periodic forcing term

$$f(x, t, u) = \sum_{l \in \mathbb{Z}} f_l(x, u) e^{ilt} = f_0(x, u) + \bar{f}(x, t, u),$$

where

$$\bar{f}(x, t, u) := \sum_{l \neq 0} f_l(x, u) e^{ilt}.$$ 

We suppose that $f$ is analytic in $(t, u)$; more precisely,

$$f(x, t, u) = \sum_{l \in \mathbb{Z}, k \in \mathbb{N}} f_{lk}(x) u^k e^{ilt},$$

where $f_{lk}(x) \in H^1((0, \pi); \mathbb{C})$, $f_{-l,k} = f^\ast_{lk}$ (the symbol $z^\ast$ denotes the complex conjugate of $z \in \mathbb{C}$), and we assume the following hypothesis on the decay of the coefficients $\|f_l\|_{H^1}$.

Hypothesis (F). There exist $2\sigma_0 > 0$, $r > 0$ such that

$$\sum_{l \in \mathbb{Z}} \|f_l\|_{H^1}^2 (1 + l^2) e^{2\sigma_0 |l|} := C_k^2(f) < \infty \quad\text{and}\quad \sum_{k=0}^{+\infty} C_k(f) r^k < \infty.$$

For example, trigonometric polynomials in $t$ and polynomials in $u$, namely,

$$f(x, t, u) = \sum_{|l| \leq L, 0 \leq k \leq K} f_{lk}(x) u^k e^{ilt}$$

for some $L, K \in \mathbb{N}$, satisfy Hypothesis (F) for every $\sigma_0, r$.

Remark 1. We notice that if $f(x, t, 0) = \sum_{l \in \mathbb{Z}} f_0(x) e^{ilt} \neq 0$, then (3) does not possess the trivial solution $u = 0$.

We look for periodic solutions of (3) in the Hilbert space

$$X_{\sigma,s} := \left\{ u : \mathbb{T} \rightarrow H^1_0((0, \pi); \mathbb{R}), \ u(x, t) = \sum_{l \in \mathbb{Z}} u_l(x) e^{ilt}, \ u_l \in H^1_0((0, \pi); \mathbb{C}), \ u_{-l} = u_l^\ast, \ ||u||_{\sigma,s}^2 := \sum_{l \in \mathbb{Z}} ||u_l||_{H^1}^2 (1 + l^2) e^{2\sigma |l|} < \infty \right\}$$

of $2\pi$-periodic-in-time functions valued in $H^1((0, \pi); \mathbb{R})$ which have a bounded analytic extension on the complex strip $|\text{Im } t| < \sigma$ with trace function on $|\text{Im } t| = \sigma$ belonging to $H^s(\mathbb{T}; H^1((0, \pi); \mathbb{C}))$. 

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For $s > 1/2$, $\mathcal{X}_{\sigma,s}$ is a multiplicative Banach algebra:

\begin{equation}
\|uv\|_{\sigma,s} \leq c_s \|u\|_{\sigma,s} \|v\|_{\sigma,s} \quad \forall u, v \in \mathcal{X}_{\sigma,s}
\end{equation}

with

\begin{equation}
c_s := 2^s \left( \sum_{n \in \mathbb{Z}} \frac{1}{1 + n^{2s}} \right)^{1/2};
\end{equation}

see, e.g., [6]. We shall use the notation $\mathcal{X}_{\sigma}$, resp., $\|\cdot\|_{\sigma}$, for $\mathcal{X}_{\sigma,1}$, resp., $\|\cdot\|_{\sigma,1}$.

1.2. The Lyapunov–Schmidt reduction. To find solutions of (3) we implement the Lyapunov–Schmidt reduction according to the decomposition

$$
\mathcal{X}_{\sigma,s} = V \oplus (W \cap \mathcal{X}_{\sigma,s}),
$$

where

$$
V := H^1_0(0, \pi), \quad W := \left\{ w = \sum_{l \neq 0} w_l(x) e^{ilt} \in \mathcal{X}_{0,s} \right\},
$$

writing every $u \in \mathcal{X}_{\sigma,s}$ as $u = u_0(x) + \sum_{l \neq 0} u_l(x) e^{ilt}$.

Projecting (3) with $u = v + w$, $v \in V$, $w \in W$, yields

\begin{equation}
\begin{cases}
-(pv')' = \mu \Pi_V f(v + w) & \text{(bifurcation equation)}, \\
L_\omega w = \mu \Pi_W f(v + w) & \text{(range equation)},
\end{cases}
\end{equation}

where $\Pi_V$, $\Pi_W$ denote the projectors, $f(u)(x, t) := f(x, t, u(x, t))$, and

$$
L_\omega u := \omega^2 \rho(x) u_{tt} - (p(x) u_x)_x.
$$

We shall find solutions of (8) when $\mu/\omega$ is small. In this limit $w$ tends to 0 and the bifurcation equation reduces to the time-independent equation

\begin{equation}
-(pv')' = \mu f_0(v)
\end{equation}

because, by (4), for $w = 0$

$$
\Pi_V f(v) = \Pi_V f_0(x, v(x)) + \Pi_V \tilde{f}(x, t, v(x)) = f_0(v).
$$

The infinite-dimensional “0th order bifurcation equation” (9) is a second order ODE, which, under natural conditions on $f_0$, possesses nondegenerate solutions satisfying the boundary conditions $v(0) = v(\pi) = 0$.

**HYPOTHESIS (V).** The problem

\begin{equation}
\begin{cases}
-(p(x)v'(x))' = \mu f_0(x, v(x)), \\
v(0) = v(\pi) = 0
\end{cases}
\end{equation}

admits a solution $\bar{v} \in H^1_0(0, \pi)$ which is nondegenerate, namely, the linearized equation

$$
-(ph')' = \mu f_0'(\bar{v}) h
$$

possesses in $H^1_0(0, \pi)$ only the trivial solution $h = 0$. 

We note that, for $\mu = 0$, the trivial solution $\bar{v} = 0$ is nondegenerate, so, by the implicit function theorem, Hypothesis (V) is automatically satisfied for $\mu$ small. We deal also with the case $\mu$ not small; see, for example, Lemmas 2 and 3.

By the implicit function theorem, Hypothesis (V) implies the existence of a smooth map

$$(\mu, w) \mapsto v(\mu, w) \in V$$

such that $v(\mu, w)$ solves the bifurcation equation in (8); see Lemma 4.

Remark 2. For a discussion about the difficulties caused by a degenerate solution, we refer to [9].

Let $\lambda_j$ denote the eigenvalues of the Sturm–Liouville problem

$$-(p(x)y'(x))' = \lambda \rho(x)y(x),$$

and $\omega_j := \sqrt{\lambda_j}$. These are the frequencies of the free vibrations of the string (note that all the eigenvalues $\lambda_j$ are positive). Physically, it is the sequence of the fundamental tone $\omega_1$ and all its overharmonics $\omega_2, \omega_3, \ldots$ which compose the musical note of the string.

For $\gamma \in (0, 1)$ we define

$$A_\gamma := \left\{(\mu, \omega) \in (\mu_1, \mu_2) \times (\gamma, +\infty) : \frac{\mu}{\omega} < C'\gamma^5, \ |\omega_l - \omega_j| > \frac{\gamma}{l\pi}, \ \forall l = 1, \ldots, N_0, \ j \geq 1\right\},$$

where $\omega_j$ are given by (11), and $(\mu_1, \mu_2), N_0 \in \mathbb{N}$, $C' > 0$ shall be fixed in the next theorem.

Theorem 1 (existence). Suppose $p(x), \rho(x) > 0$ are of class $H^3(0, \pi)$, $f$ satisfies Hypothesis (F), and Hypothesis (V) holds for some $\mu_0 \in [0, \bar{\mu}]$.

Fix $\tau \in (1, 2), \gamma \in (0, 1)$. There exist a neighborhood $(\mu_1, \mu_2)$ of $\mu_0$, $N_0 \in \mathbb{N}$, positive constants $C, C'$ (depending on $\rho, p, f, \bar{\mu}, \bar{v}, \tau$), a map

$$\tilde{w} \in C^\infty(A_\gamma, X_{\sigma_0/2} \cap W),$$

and a Cantor set $B_\gamma \subset A_\gamma$ of positive measure such that, for all $(\mu, \omega) \in B_\gamma$,

$$(\mu, \omega) := v(\mu, \bar{w}(\mu, \omega)) + \tilde{w}(\mu, \omega) \in V \oplus (W \cap X_{\sigma_0/2})$$

is a classical solution of (3) and satisfies

$$\tilde{u}(\cdot, t) \in H^3(0, \pi) \cap H^1_0(0, \pi) \ \forall t \in \mathbb{R}. $$

The Cantor set $B_\gamma$ is defined in (15) and satisfies the measure estimate (56).

Furthermore, for all $(\mu, \omega) \in A_\gamma$, the following estimates hold:

$$(14) \quad ||\tilde{w}(\mu, \omega)||_{\sigma_0/2} \leq C\frac{\mu}{\gamma \omega}, \quad ||v(\mu, \bar{w}(\mu, \omega)) - v(\mu, 0)||_{H^1} \leq C\frac{\mu}{\gamma \omega},$$

and $||v(\mu, 0) - \bar{v}||_{H^1} \leq C|\mu - \mu_0|$.

The neighborhood $(\mu_1, \mu_2)$ of $\mu_0$ is fixed in Lemma 4, the integer $N_0$ is fixed in Lemma 9, and the constant $C'$ is fixed in Lemma 13.
Estimate (14) shows how close the solution \( \tilde{u} \) is to the static configuration \( v(\mu, 0) \); see Figure 1.

**Remark 3.** We underline that the function \( \tilde{w}(\mu, \omega) \), as well as \( \tilde{u}(\mu, \omega) \), is defined for all the values of the parameters \( (\mu, \omega) \in A_\gamma \) and not only on the Cantor set \( B_{\gamma} \) (\( \tilde{w}(\mu, \omega) \) is introduced in Lemma 11). What is true is that if \( (\mu, \omega) \in B_{\gamma} \), then \( \tilde{w}(\mu, \omega) \) solves the range equation; see Theorem 3. As a consequence, if \( (\mu, \omega) \in B_{\gamma} \), then \( \tilde{u}(\mu, \omega) \) solves (3).

The Cantor set \( B_{\gamma} \) is explicitly defined by

\[
B_{\gamma} := \left\{ (\mu, \omega) \in (\mu_1, \mu_2) \times (2\gamma, +\infty) : \left| \omega l - \omega_j \right| > \frac{2\gamma}{l^\gamma} \quad \forall \ l = 1, \ldots, N_0, \ j \geq 1, \right\}
\]

where

\[
e := \frac{1}{\pi} \int_0^\pi \left( \frac{p(x)}{p(x)} \right)^{1/2} dx
\]

and \( \tilde{\lambda}_j(\mu, \omega) := \tilde{\omega}_j^2(\mu, \omega) \) denote the eigenvalues of the Sturm–Liouville problem

\[
\left\{ \begin{array}{ll}
-(py')' - \mu \Pi_V f'(\tilde{u}(\mu, \omega)) y &= \lambda \rho y, \\
y(0) &= y(\pi) = 0.
\end{array} \right.
\]

Note that \( B_{\gamma} \) is constructed by means of the function \( \tilde{u}(\mu, \omega) \), which is defined for all \( (\mu, \omega) \in A_\gamma \); see Remark 3.

**Remark 4.** If some \( \tilde{\lambda}_j(\mu, \omega) \) is negative, then \( \tilde{\omega}_j(\mu, \omega) = i \sqrt{\tilde{\lambda}_j(\mu, \omega)} \) is a purely imaginary complex number and the nonresonance conditions in (15) are trivially satisfied.

The Cantor set \( B_{\gamma} \) is large in a measure theoretical sense; see section 4.3. In particular, for all \( \mu \in (\mu_1, \mu_2) \),

\[
S(\mu) := \{ \omega : (\mu, \omega) \in \bigcup_{\gamma \in (0, 1)} B_{\gamma} \}
\]

has asymptotically full measure at \( \omega \to +\infty \), i.e.,

\[
\lim_{\omega \to +\infty} |S(\mu) \cap (\omega, \omega + 1)| = 1.
\]

Analogously,

\[
S(\omega) := \{ \mu : (\mu, \omega) \in \bigcup_{\gamma \in (0, 1)} B_{\gamma} \}
\]

satisfies, for all \( \omega' > 0 \) and for all \( \gamma' \in (0, 1) \),

\[
\lim_{\mu \to 0} \left\{ \omega \in (\omega', \omega' + 1): \left| \frac{S(\omega) \cap (\omega, \omega + 1)}{\mu} \right| \geq 1 - \gamma' \right\} = 1.
\]

Finally we discuss the regularity of the solution \( \tilde{u}(x, t) \) found in Theorem 1 with respect to \( x \) (by construction \( \tilde{u} \) is analytic with respect to \( t \)).

**Theorem 2 (regularity).** Assume the hypotheses of Theorem 1. In addition, suppose that, for some \( m \geq 3 \),

\[
\rho(x) \in H^m(0, \pi), \quad p(x) \in H^{m+1}(0, \pi), \quad f_{ik}(x) \in H^m(0, \pi) \quad \forall \ l, k
\]

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and, for some \( r_m > 0 \),

\[
\sum_{l \in \mathbb{Z}, k \geq 0} \| f_{lk} \|_{H^{r_m} r_m^k} < +\infty.
\]

If \( \| \tilde{u}(\cdot, t) \|_{H^1 r_m^{-1}} \) is small enough, then

\[
\tilde{u}(\cdot, t) \in H^{m+2} (0, \pi) \cap H^1_0 (0, \pi).
\]

This conclusion holds true, for example, when \( f_0(x, 0) = d_u f_0(x, 0) = 0 \) and \( \mu/\gamma \omega \) is small enough.

Note that the regularity (22) requires no skewsymmetry assumptions on \( f \) and requires just a smallness condition for the \( H^1 \) norm of \( \tilde{u}(\cdot, t) \).

Remark 5. If \( f(x, t, u) \) is a trigonometric polynomial in \( t \) and a polynomial in \( u \) as in (5), then the series in (21) is a finite sum. Therefore the conclusion (22) is true without smallness conditions for \( \tilde{u} \).

In particular, if \( \rho(x), p(x), f_{lk}(x) \) are \( C^\infty \) (for example, \( f = \cos x \cos t (1 + u^2) \)), then the solution \( \tilde{u} \) is \( C^\infty \) also in the variable \( x \) (the above \( f \) does not satisfy the skewsymmetry assumption (24)).

The subtle problem to prove Theorem 2 is that, because of Dirichlet boundary conditions, the Sobolev regularity of a function with respect to \( x \) is not characterized by the rapid decaying properties of the Fourier coefficients (unless we assume skewsymmetry assumptions on the nonlinearity and restrict solutions to \( u(x, t) \) odd in \( x \); see Remark 6). Theorem 2 is proved in section 5 via bootstrap arguments.

1.3. Outline of the proof. In section 2 we prove that, under assumption (F), the composition operator induced by the nonlinearity \( f \) on \( X_{\sigma, s} \) is an analytic map.

In section 3, we find a solution \( v(\mu, w) \) of the infinite-dimensional bifurcation equation in (8). Thanks to assumption (V) (which is verified on several examples in Lemmas 2 and 3), \( v(\mu, w) \) is obtained in Lemma 4 by a standard implicit function theorem.

In section 4 we solve the range equation by means of an iterative Nash–Moser implicit function theorem. The final theorem, Theorem 3, is proved in several steps.

In section 4.1 we find inductively a sequence of approximate solutions \( w_n(\mu, \omega) \) defined on smaller and smaller subsets \( A_n \) of the parameters \( (\mu, \omega) \) (see (39)). The reason for these “excisions” is to avoid resonance phenomena in order to prove the invertibility of the linearized operators obtained at each step of the iteration; see conditions (33)–(34) in Lemma 7.

In section 4.2 we extend these approximate solutions \( w_n(\mu, \omega) \) to \( C^\infty \)-functions \( \tilde{w}_n(\mu, \omega) \) defined for all the values of the parameters \( (\mu, \omega) \) and converging (superexponentially fast) to a \( C^\infty \) map \( \tilde{w} \) defined for all \( (\mu, \omega) \); see Lemma 11. It is in proving the regularity of \( w_n \) with respect to the parameters \( (\mu, \omega) \) that we find it convenient to define the approximate solutions \( w_n \) as exact solutions of (41) (with \( k = n \)); see Remark 9.

In Lemma 12, we prove that the Cantor set \( B_{\gamma} \), defined in (15) by means of \( \tilde{w} \), is contained in \( A_n \) (which depends on \( w_{n-1} \)) for each \( n \). This is a consequence of the superexponentially fast convergence of \( \tilde{w}_n \) to \( \tilde{w} \); see (52).

In section 4.3 we prove that \( B_{\gamma} \) is a large set in a measure theoretical sense.

In all the previous steps we have to assume smallness conditions for \( \mu/\omega \). The most restrictive one is \( \mu/\omega < C' \gamma^5 \) in the definition (12) of \( A_\gamma \).

In section 5 we conclude the proof of the existence Theorem 1, and we prove the regularity Theorem 2.
In section 6 we study the key step for the inversion of the linearized operators. Lemma 7 is obtained by a variant of the techniques developed in [8]. In particular, the key estimate on the small divisors of Lemma 18 is reminiscent of the method of Kuksin explained in [13, pp. 90–94].

Notation. The symbols $K, K_i, K'_i$ shall denote positive constants depending only on $p, p, f, \bar{u}, \bar{v}, \tau$.

2. Regularity of the composition operator. We first prove the analyticity of the composition operator

$$u(x, t) \mapsto f(x, t, u(x, t))$$

induced by $f$ on $X_{\sigma, s}$.

By the Banach algebra property (6) of $X_{\sigma, s}$ the composition operator

$$u \mapsto u^k \quad \forall k \in \mathbb{N}$$

is an analytic map from $X_{\sigma, s}$ into itself. Thanks to the rapid decay of the coefficients $\|f_{lk}\|_{H^s}$ assumed in Hypothesis (F), this property holds true also for the composition operator $f(u)$.

**Lemma 1.** Let $f$ satisfy assumption (F). For every $\sigma \in [0, \sigma_0], s > 1/2$, the composition operator $f$ is analytic on the ball $\{u \in X_{\sigma, s} : \|u\|_{\sigma, s} < r/c_s\}$, where $c_s$ is defined in (7).

**Proof.** First note that

$$\sum_{l \in \mathbb{Z}} \|u_l\|_{\sigma, s} \leq \sqrt{\frac{\pi}{2}} \sum_{l \in \mathbb{Z}} \|u_l\|_{H^s} \leq \sqrt{\frac{\pi}{2}} \left(\sum_{l \in \mathbb{Z}} \|u_l\|_{H^s}^2 (1 + l^{2s})\right)^{1/2} \left(\sum_{l \in \mathbb{Z}} \frac{1}{1 + l^{2s}}\right)^{1/2}$$

so $\|u\|_{\sigma, s} \leq c_s \|u\|_{\sigma, s}$ for all $u \in X_{\sigma, s}, \sigma \geq 0, s > 1/2$, and $f(x, t, u(x, t))$ is well-defined.

By definition of the norm $\|\cdot\|_{\sigma, s}$, there exists $C := C(\sigma_0, s) > 0$ such that for all $\sigma \in [0, \sigma_0]$ and for all $k \in \mathbb{N}$,

$$\|\sum_{l \in \mathbb{Z}} f_{lk}(x) e^{ilt}\|_{\sigma, s} \leq C \|\sum_{l \in \mathbb{Z}} f_{lk}(x) e^{ilt}\|_{2\sigma_0, 1}.$$  

Next

$$\|\sum_{l \in \mathbb{Z}} f_{lk}(x) e^{ilt}\|_{2\sigma_0, 1}^2 = \sum_{l \in \mathbb{Z}} \|f_{lk}\|_{H^s}^2 (1 + l^2) e^{(2\sigma_0)|t|} =: C_k^2(f) < +\infty$$

by assumption (F). Therefore $\sum_{l \in \mathbb{Z}} f_{lk}(x) e^{ilt} \in X_{\sigma, s}$ and

$$\|\sum_{l \in \mathbb{Z}} f_{lk}(x) e^{ilt}\|_{\sigma, s} \leq C C_k(f).$$  

Using the algebra property (6) of $X_{\sigma, s}$ and (23)

$$\|f(u)\|_{\sigma, s} \leq \sum_{k=0}^{\infty} \left(\sum_{l \in \mathbb{Z}} \|f_{lk}(x) e^{ilt}\|_{\sigma, s} \right) u^k \|_{\sigma, s}$$

$$\leq C_s \sum_{k=0}^{\infty} C_k(f) (c_s \|u\|_{\sigma, s})^k \leq C \sum_{k=0}^{\infty} C_k(f) r^k < +\infty$$

for $c_s \|u\|_{\sigma, s} < r$, by (F) again.
The analyticity of the composition operator $f$ with respect to $\|\cdot\|_{\sigma,s}$ follows from the properties of the power series as explained in [25, Appendix A].

We emphasize that the analyticity of $f$ as a map in $X_{\sigma,s}$ is not an assumption but follows from (F).

Remark 6. If $f(x, t, u)$ admits an analytic extension, which is $2\pi$-periodic in $x$ and skewsymmetric, namely,

$$f(-x, t, -u) = -f(x, t, u),$$

then the Dirichlet problem on $[0, \pi]$ is equivalent to the $2\pi$-periodic problem within the space of all functions odd in $x$. In this case also the spatial regularity is characterized by the decay properties of the Fourier coefficients. Therefore we could look for analytic solutions of (3),

$$u(x, t) = \sum_{l \in \mathbb{Z}} u_l(x) e^{ilt},$$

which are periodic and odd in $x$, more precisely with

$$u_l(x) \in Y := \left\{ y(x) = \sum_{j \geq 1} y_j \sin(jx) : \sum_{j \geq 1} |y_j|^2 j^{2b} e^{2aj} < +\infty \right\}$$

for some $a \geq 0$, $b > 1/2$. Without the oddness assumption (24) the composition operator $f$ does not map this subspace into itself. It is for this reason that we consider the space $X_{\sigma,s}$ of functions valued in $H^1_0(0, \pi)$: also without (24), $f$ sends $X_{\sigma,s}$ into itself (Lemma 1).

Throughout this paper we shall use spaces $X_{\sigma,s}$ with $\sigma \in [\sigma_0/2, \sigma_0]$ and $s \in S := \{1, 1 - \frac{1}{2} - \frac{1}{2}, 1 + \frac{(\tau-1)\tau}{2-\tau}\}$. So we choose $\bar{c} := \max_{s \in S} c_s$ as a multiplicative algebra constant for all spaces $X_{\sigma,s}$.

By Lemma 1, $f$ is analytic in the ball

$$\left\{ u \in X_{\sigma,s} : \|u\|_{\sigma,s} < R_0 := \frac{r}{\bar{c}} \right\}$$

and $f, f', f'', \ldots$ are bounded, uniformly in $\sigma, s$.

3. The bifurcation equation. Now we give some examples in which Hypothesis (V) holds.

Lemma 2. Suppose $f_0(x, u) = u^m$ for $m \geq 3$ odd and $p(x) \equiv 1$. Then, for all $\mu$, there exists an unbounded sequence of nondegenerate solutions $v_n$ of (10).

Proof. All the solutions of the autonomous equation $-v'' = \mu v^m$ are periodic and can be parametrized by their energy

$$E = \frac{1}{2} v'^2 + \frac{\mu}{m+1} v^{m+1}.$$ 

Let $T_E$ denote the period of the solution $v_E$. We can suppose $v_E(0) = 0$, so $v'_E(0) = \sqrt{2E}$. The other boundary condition $v_E(\pi) = 0$ is satisfied iff

$$\frac{T_E}{2} = \pi \quad \text{for some } k \in \mathbb{N}.$$
By symmetry and energy conservation \( v_E(T_E/4) = [(m+1)E/\mu]^{1/2} \). So

\[
T_E = 4 \int_0^{[\frac{(m+1)E}{\mu}]^{1/2}} \left[ 2 \left( E - \frac{\mu x^{m+1}}{m+1} \right) \right]^{-1/2} dx \\
= \frac{4(m+1/\mu)^{1/2}}{E^{1/2} - m} \int_0^1 \frac{dy}{\sqrt{2(1-y^{m+1})}} =: C(m,\mu)
\]

by the change of variable \( y = x [E(m+1)/\mu]^{-1/2} \), and (25) is satisfied at infinitely many energy levels. Let \( \bar{E} > 0 \) such that \( T_E = 2\pi/k \) and denote the solution \( \bar{v} := v_{\bar{E}}. \)

Let us prove that \( \bar{v} \) is nondegenerate. Any solution \( h \) of the linearized equation at \( \bar{v} \),

\[
(26) \quad -h''(x) = \mu m \bar{v}^{m-1}(x) h(x),
\]

can be written as \( h = A\bar{v} + B\beta \), \( A, B \in \mathbb{R} \), because \( \bar{v}(x) \) and \( \beta(x) := (\partial_E v_{\bar{E}})_{|E=\bar{E}}(x) \) are solutions of (26); they are independent because \( \bar{v}'(0) \neq 0 \) while \( \beta(0) = 0 \). If \( h(0) = 0 \), then \( A = 0 \). We claim that \( \beta(\pi) \neq 0 \); as a consequence, if \( h(\pi) = 0 \), then \( B = 0 \), and so \( h = 0 \), i.e., \( \bar{v} \) is nondegenerate. To prove that \( \beta(\pi) \neq 0 \), we differentiate at \( \bar{E} \) the identity \( v_{\bar{E}}(kT_E/2) = 0, \)

\[
\beta(\pi) + \bar{v}'(\pi)(\partial_E T_E)_{|E=\bar{E}} = 0.
\]

Since \( \bar{v}'(\pi) = (-1)^k \sqrt{2\bar{E}} \neq 0 \) and \( \partial_E T_E \neq 0 \), we get \( \beta(\pi) \neq 0 \).

**Lemma 3.** If \( f_0(x,0) = d_u f_0(x,0) = 0 \), then \( \bar{v} = 0 \) is a nondegenerate solution of (10) for every \( \mu \).

**Proof.** The linearized equation \( -(ph')' = 0 \), \( h(0) = h(\pi) = 0 \) has only the trivial solution.

When Hypothesis (V) holds at some \((\mu_0, \bar{v})\), we solve first the bifurcation equation in (8) using the standard implicit function theorem. We find, for every \( w \) small enough and \( \mu \) in a neighborhood of \( \mu_0 \), a unique solution \( v(\mu, w) \) of the bifurcation equation.

**Lemma 4** (solution of the bifurcation equation). There exist \( 0 < R < R_0 \), a neighborhood \([\mu_1, \mu_2]\) of \( \mu_0 \), and a \( C^\infty \) map

\[
[\mu_1, \mu_2] \times \{ w \in W \cap X_{\sigma,s} : \|w\|_{\sigma,s} < R \} \to V, \quad (\mu, w) \mapsto v(\mu, w)
\]

such that \( v(\mu, w) \) solves the bifurcation equation in (8).

**Proof.** The linear operator

\[
h \mapsto -(ph')' - \mu_0 d_v \Pi V f(v) [h] = -(ph')' - \mu_0 f'_0(v) h
\]

is invertible on \( H^2_0(0, \pi) \) by Hypothesis (V). Then we apply the implicit function theorem.

**Remark 7.** The solutions of the 0th order bifurcation equation (10) found in Lemmas 2 and 3 are nondegenerate for every \( \mu \), so, in that case, we can continue \( v(\mu, w) \) for all \([\mu_1, \mu_2] = [0, \bar{\mu}]\).

We denote by \( \lambda_j(\mu, w) := \omega_j^2(\mu, w) \) the eigenvalues of the Sturm–Liouville problem

\[
(27) \quad \begin{cases} 
-(py')' - \mu \Pi V f'(v(\mu, w) + w) y = \lambda y, \\
y(0) = y(\pi) = 0.
\end{cases}
\]
Lemma 5. The eigenvalues of (27) satisfy the continuity property

\[ |\lambda_j(\mu, w) - \lambda_j(\mu', w')| \leq K (|\mu - \mu'| + \|w - w'||_{\sigma, s}) \]

for some constant \( K > 0 \) independent of \( j \).

Proof. For the proof of the lemma, see the appendix. \( \square \)

The nondegeneracy of \( \bar{v} = v(\mu_0, 0) \) means that \( \lambda_j(\mu, 0) \neq 0 \) for all \( j \). By (28),

\[ \delta_0 := \inf \{ \| \lambda_j(\mu, w) \| : j \geq 1, \mu \in [\mu_1, \mu_2], \|w\|_{\sigma, s} \leq R \} > 0, \]

(29)

taking, if necessary, \( |\mu_2 - \mu_1| \) and \( R \) smaller in Lemma 4.

Note also that the index \( j_0 \) of the smallest positive eigenvalue is constant, independently of \( (\mu, w) \).

4. Solution of the range equation. It remains to solve the range equation

\[ L_\omega w = \mu \Pi_W F(\mu, w), \]

where

\[ F(\mu, w) := f(v(\mu, w) + w). \]

By Lemmas 1 and 4, \( F \) is \( C^\infty \) and bounded, together with its derivatives, on \([\mu_1, \mu_2] \times B_R\), where \( B_R := \{ w \in W \cap X_{\sigma, s} : \|w\|_{\sigma, s} < R \} \).

4.1. The Nash–Moser recursive scheme. We define the sequence of finite-dimensional subspaces

\[ W^{(n)} := \left\{ w = \sum_{1 \leq |l| \leq N_n} w_l(x)e^{ilt} \right\} \subset W, \]

where

\[ N_n := N_0 2^n \]

and \( N_0 \in \mathbb{N} \) will be fixed in Lemma 9. We also set

\[ W^{(n)}_\perp := \left\{ w = \sum_{|l| > N_n} w_l(x)e^{ilt} \in W \right\} \]

and denote by \( P_n \), resp., \( P_\perp \), the projection on \( W^{(n)} \), resp., \( W^{(n)}_\perp \). Note that \( P_n \circ \Pi_W = P_n \).

Lemma 6 (smoothing estimate). For \( w \in W^{(n)}_\perp \), if \( 0 < \sigma'' < \sigma' \),

\[ \|w\|_{\sigma'', s} \leq \exp[-(\sigma' - \sigma'')N_n]\|w\|_{\sigma', s}. \]

Proof. It follows from the definition of the norms \( \| \cdot \|_{\sigma, s} \) and \( W^{(n)}_\perp \); see, e.g., [16, 8]. \( \square \)

The key property for the construction of the iterative sequence is the invertibility of the linear operator

\[ L_n(w)h := -L_\omega h + \mu P_n [d_w F(\mu, w)h] \]

\[ = -L_\omega h + \mu P_n [f'(v(\mu, w) + w)(h + d_w v(\mu, w)[h])] \quad \forall h \in W^{(n)}. \]
LEMMA 7 (inversion of the linear problem). Let $\omega > 0$, $\tau \in (1, 2)$, $\gamma \in (0, 1)$, $\gamma < \omega$, and $\sigma \in (0, \sigma_0]$. Assume the “Melnikov” nonresonance conditions
\begin{equation}
|\omega - \frac{2}{c}| > \frac{\gamma \omega}{t - 1} \quad \forall l = 1, 2, \ldots, N_n, \quad \forall j \geq 1,
\end{equation}
where $c$ is defined in (16), and
\begin{equation}
|\omega^2 t^2 - \lambda_j(\mu, w)| > \frac{\gamma \omega}{t - 1} \quad \forall l = 1, 2, \ldots, N_n, \quad j \geq 1,
\end{equation}
where $\lambda_j(\mu, w)$ are the eigenvalues of (27).

Let $u := v(\mu, w) + w$. There exist $K_1, K_1'$ such that if
\begin{equation}
\frac{\mu}{3\gamma \omega} \|W f'(u)\|_{\sigma, 1 + \frac{\alpha - \mu}{2}} < K_1',
\end{equation}
then $L_n(w)$ is invertible and
\begin{equation}
\|L_n(w)^{-1} h\|_{\sigma} \leq \frac{K_1 N_n^{\gamma - 1}}{\gamma \omega} \|h\|_{\sigma} \quad \forall h \in W^{(n)}.
\end{equation}

Proof. For the proof of the lemma, see section 6.

Remark 8. The condition $\omega > 0$ means that (1) is nonautonomous. Indeed, if $\omega = 0$, the nonlinearity $f(x, \omega t, u) = f(x, 0, u)$ is independent of $t$.

For $\vartheta := 3\sigma_0/\pi^2$ we define the sequence
\begin{equation}
\sigma_{n+1} := \sigma_n - \frac{\vartheta}{(n + 1)^2}, \quad \sigma_0 > \sigma_1 > \sigma_2 > \cdots > \sigma_0/2.
\end{equation}

Let $A_0$ denote the open set
\begin{align*}
A_0 := \{(\mu, \omega) \in (\mu_1, \mu_2) \times (\gamma, +\infty) : |\omega l - \omega_j| > \frac{\gamma}{t - 1} \quad \forall l = 1, \ldots, N_0, \quad j \geq 1\},
\end{align*}
where $\omega_j$ are defined by (11).

LEMMA 8 (approximate solution). There exist $K_2, K_2'$ such that if $(\mu, \omega) \in A_0$ and $\mu N_0^{\gamma - 1}/\gamma \omega < K_2'$, then there exists a solution $w_0 := w_0(\mu, \omega) \in W^{(0)}$ of
\begin{equation}
L_\omega w_0 = \mu P_0 F(\mu, w_0)
\end{equation}
satisfying $\|w_0\|_{\sigma_0} \leq \mu K_2 N_0^{\gamma - 1}/\gamma \omega$.

Proof. By definition of $A_0$, the eigenvalues of $(1/\rho)L_\omega$ satisfy
\begin{align*}
|\omega^2 t^2 - \lambda_j| > \frac{\gamma \omega}{t - 1} \quad \forall l = 1, 2, \ldots, N_0, \quad j \geq 1,
\end{align*}
so $L_\omega$ is invertible on $W^{(0)}$ and, for some $K$,
\begin{equation}
\|L_\omega^{-1} h\|_{\sigma_0} \leq \frac{K N_0^{\gamma - 1}}{\gamma \omega} \|h\|_{\sigma_0} \quad \forall h \in W^{(0)}.
\end{equation}

Then we look for a solution $w_0 \in W^{(0)}$ of $w_0 = \mu L_\omega^{-1} P_0 F(\mu, w_0)$. The right-hand side term is a contraction in $\{\|w_0\|_{\sigma_0} < R\}$ if $\mu N_0^{\gamma - 1}/\gamma \omega$ is sufficiently small.
Given \( w_n \in W^{(n)}, \|w_n\|_{\sigma_n} < R, \) and \( A_n \subseteq A_0, \) we define

\[
A_{n+1} := \left\{ (\mu, \omega) \in A_n : |\omega - \omega_j(\mu, w_n)| > \frac{\gamma}{l}, \ |\omega - \frac{j}{c}| > \frac{\gamma}{l} \right\},
\]

\[\forall l = 1, 2, \ldots, n+1, \ j \geq 1 \right\} \subseteq A_n,
\]

where \( \lambda_j(\mu, w_n) = \omega_j^2(\mu, w_n) \) are defined in (27) with \( w = w_n. \)

In Lemma 8 we have constructed \( h_0 := w_0 \) for \( (\mu, \omega) \in A_0. \) Next, we proceed by induction. By means of \( w_0 \) we define the set \( A_1 \) as above, and we find \( w_1 := h_0 + h_1 \in W^{(1)} \) for every \( (\mu, \omega) \in A_1 \) by Lemma 9 below. Then we define \( A_2, \) we find \( w_2 \in W^{(2)} \), and so on. The main goal of the construction is to prove that, at the end of the recurrence, the set of parameters \( (\mu, \omega) \in \cap_n A_n \) is actually a large set (see Lemmas 12 and 13).

**Lemma 9** (inductive step). Fix \( \chi := 3/2. \) There exist \( N_0 \in \mathbb{N} \) (depending only on \( \rho, p, f, \bar{v}, r, \tau) \) and \( K_3 \leq K_2/N_0^{\gamma-1} \) with the following property.

Suppose that \( h_i \in W^{(i)} \) for all \( i = 0, \ldots, n \) satisfy

\[
\|h_i\|_{\sigma_i} < \frac{\mu K_3 N_0^{\gamma-1}}{\gamma \omega} \exp(-\chi),
\]

where \( K_3 := \varepsilon K_2 \) and \( K_2 \) is the constant in Lemma 8; for all \( k = 0, \ldots, n, \) let \( w_k := h_0 + \cdots + h_k \) satisfy \( \|w_k\|_{\sigma_k} < R \) and

\[
L_\omega w_k = \mu P_{k,F}(\mu, w_k)
\]

and suppose that \( (\mu, \omega) \in A_n, \) where \( A_{n+1} \) is constructed by means of \( w_i \) as shown above.

If \( (\mu, \omega) \in A_{n+1} \) and \( \mu/\gamma^3 \omega < K_3, \) then there exists \( h_{n+1} \in W^{(n+1)} \) satisfying

\[
\|h_{n+1}\|_{\sigma_{n+1}} < \frac{\mu K_3 N_0^{\gamma-1}}{\gamma \omega} \exp(-\chi^{n+1}),
\]

such that \( w_{n+1} = w_n + h_{n+1} \) verifies \( \|w_{n+1}\|_{\sigma_{n+1}} < R \) and

\[
L_\omega w_{n+1} = \mu P_{n+1,F}(\mu, w_{n+1}).
\]

**Proof.** In short \( \mathcal{F}(w) := \mathcal{F}(\mu, w) \) and \( \mathcal{DF}(w) := d_\omega \mathcal{F}(\mu, w) \). Equation (43) for \( w_{n+1} = w_n + h_{n+1} \) is \( L_\omega [w_n + h_{n+1}] = \mu P_{n+1,F}(w_n + h_{n+1}) \).

By assumption, \( w_n \) satisfies (41) for \( k = n \), namely, \( L_\omega w_n = \mu P_n F(w_n) \), so the equation for \( h_{n+1} \) can be written as

\[
\mathcal{L}_{n+1}(w_n) h_{n+1} + \mu (P_{n+1} - P_n) \mathcal{F}(w_n) + \mu P_{n+1} Q = 0,
\]

where, as defined in (32), \( \mathcal{L}_{n+1}(w_n) h_{n+1} := -L_\omega h_{n+1} + \mu P_{n+1} \mathcal{D}(w_n) h_{n+1}, \) and \( Q \) denotes the quadratic remainder

\[
Q = Q(w_n, h_{n+1}) := \mathcal{F}(w_{n+1}) - \mathcal{F}(w_n) - D_\omega \mathcal{F}(w_n) h_{n+1}.
\]

**Step 1. Inversion of \( \mathcal{L}_{n+1}(w_n). \)** We verify the assumptions of Lemma 7. By definition of \( A_{n+1}, \) \( \omega \) satisfies (33). If \( \lambda_j(\mu, w_n) < 0, \) then \( |\omega^2 l^2 - \lambda_j(\mu, w_n)| \geq \omega^2 l^2 > \gamma \omega / l^{\gamma-1} \) because \( \omega > \gamma / l^{\gamma-1} \) because \( (\mu, \omega) \in A_{n+1}. \) In both cases the nonresonance condition (34) holds.
To verify (35) we need an estimate for $w_n$. Let $\eta := \tau(\tau - 1)/(2 - \tau)$ and $\alpha > 0$. Using the elementary inequality

$$\frac{1 + l^2(1 + l)}{1 + l^2} \cdot \frac{e^{2(l - 2\alpha)l}}{e^{2\alpha l}} \leq \frac{2l^2\eta}{e^{2\alpha l}} \leq 2\max_{y > 0}(y^2\eta e^{-2\alpha y}) = 2\left(\frac{\eta}{\alpha e}\right)^2 \forall l \neq 0,$$

we deduce

$$\|h_i\|_{\sigma_{n+1},1+\eta} \leq \frac{C_\eta}{(\sigma_i - \sigma_{n+1})^\theta} \|h_i\|_{\sigma_i},$$

where $C_\eta := \sqrt{2}(\eta/e)^\eta$. Since $\sigma_i - \sigma_{n+1} \geq \sigma_i - \sigma_{i+1}$ for every $i \leq n$,

$$\|w_n\|_{\sigma_{n+1},1+\eta} \leq \sum_{i=0}^n \|h_i\|_{\sigma_{n+1},1+\eta} \leq C_\eta \sum_{i=0}^n \|h_i\|_{\sigma_i} \leq C_\eta \frac{\mu K_3 N_0^{\tau-1}}{\gamma \omega} \sum_{i=0}^n (\sigma_i - \sigma_{i+1})^\theta,$$

using (40) where $S_\eta := (C_\eta/\theta \eta) \sum_{i=0}^{+\infty} (i + 1)^{2\eta} \exp(-\chi^i) < +\infty$. If

$$\frac{S_\eta \mu K_3 N_0^{\tau-1}}{\gamma \omega} < R,$$

then

$$\|f'(u_n)\|_{\sigma_{n+1},1+\eta} \leq K$$

for some $K$, where $u_n := v(\mu, w_n) + w_n$. Hence Hypothesis (35) is verified for $\mu/\gamma^3 \omega$ sufficiently small.

Analogously we get $\|w_n\|_{\sigma_n} < R$ if $\mu N_0^{\tau-1}/\gamma \omega$ is small enough.

By Lemma 7 the operator $\mathcal{L}_{n+1}(w_n)$ is invertible on $W^{(n+1)}$ and

$$\|\mathcal{L}_{n+1}(w_n)^{-1}h\|_{\sigma_{n+1}} \leq \frac{K_1 N_0^{\tau-1}}{\gamma \omega} \|h\|_{\sigma_{n+1}} \forall h \in W^{(n+1)}.$$

Equation (44) amounts to the fixed point problem

$$h_{n+1} = -\mu \mathcal{L}_{n+1}(w_n)^{-1} \left[ (P_{n+1} - P_n)\mathcal{F}(w_n) + P_{n+1}Q \right] := \mathcal{G}(h_{n+1})$$

for $h_{n+1} \in W^{(n+1)}$.

**Step 2. $\mathcal{G}$ is a contraction.** We prove that $\mathcal{G}$ is a contraction on the ball $B_{n+1} := \{|\|\|_{\sigma_{n+1}} < r_{n+1}\}$, where $r_{n+1} := (\mu K_3 N_0^{\tau-1}/\gamma \omega) \exp(-\chi^{n+1})$, implying (42). By (31)

$$\|P_{n+1} - P_n\| \mathcal{F}(w_n)\|_{\sigma_{n+1}} \leq \|\mathcal{F}(w_n)\|_{\sigma_n} \exp[-(\sigma_n - \sigma_{n+1})N_n].$$

Since $\|w_n\|_{\sigma_n} < R$, we have $\|Q\|_{\sigma_{n+1}} \leq K \|h_{n+1}\|_{\sigma_{n+1}}^2$. Hence, by (45),

$$\|\mathcal{G}(h_{n+1})\|_{\sigma_{n+1}} \leq K \frac{\mu N_0^{\tau-1}}{\gamma \omega} \left( \exp[-(\sigma_n - \sigma_{n+1})N_n] + \|h_{n+1}\|_{\sigma_{n+1}}^2 \right).$$

Therefore $\mathcal{G}(B_{n+1}) \subseteq B_{n+1}$ if

$$\frac{\mu K N_0^{\tau-1}}{\gamma \omega} \exp[-(\sigma_n - \sigma_{n+1})N_n] < \frac{r_{n+1}}{2}, \quad \frac{\mu K N_0^{\tau-1}}{\gamma \omega} r_{n+1} < \frac{r_{n+1}}{2}.$$
By the definition of $\sigma_n$ in (37) and $N_n := N_02^n$, the first inequality is verified for every $n \geq 0$ if $\sigma_0N_0$ is greater than a constant depending only on $\chi, K, K_3$. The second inequality is verified for every $n \geq 0$ if $\mu N_0^{-1} / \gamma \omega$ is small enough.

The estimate for $\|Gh - Gk\|$, $h, k \in B_{n+1}$, is similar. The lemma now follows from the contraction mapping theorem. \hfill \square

Remark 9. In the previous scheme $h_{n+1}$ is found as an exact solution of (44). We find this convenient to prove the regularity of $h_{n+1}$ with respect to the parameters $(\mu, \omega)$ in Lemma 10. However, other schemes are possible. For example, we could define $h_{n+1}$ as a solution of the linearized equation $L_{n+1}(w_n)h + \mu(P_{n+1} - P_n)f(w_n) = 0$.

Corollary 1 (existence). Suppose $A_\infty := \cap_{n \geq 0} A_n \neq \emptyset$. If $(\mu, \omega) \in A_\infty$ and $\mu / \gamma 3\omega < K'_3$, then

$$w_\infty(\mu, \omega) := \sum_{n \geq 0} h_n(\mu, \omega) \in W \cap X_{\sigma_0/2}$$

is a solution of the range equation (30) satisfying $\|w_\infty\|_{\sigma_0/2} \leq K_{\infty}\mu / \gamma \omega$ for some $K_{\infty}$.

Proof. Since $w_n$ solves (41) for $k = n$,

$$-L_\omega w_n + \mu \Pi_w f(u_n) = \mu P_n f(u_n) \in W(\eta),$$

where $u_n := v(\mu, w_n) + w_n$. By (31)

$$\lim_{n \to +\infty} \| - L_\omega w_n + \mu f(u_n)\|_{\sigma_0/2} \leq \lim_{n \to +\infty} K \exp[-(\sigma_n - \sigma_0/2)N_n] = 0.$$ 

Since $w_n \to w_\infty$ in $\| \cdot \|_{\sigma_0/2}$ also $f(u_n) \to f(u_\infty)$ in the same norm, while $L_\omega w_n \to L_\omega w_\infty$ in the sense of distributions. So $w_\infty$ is a weak solution of the range equation (30).

Remark 10. We shall prove, as a consequence of Lemma 12 and section 4.3, that $A_\infty$ is actually a positive measure set. One possible way to prove it uses the Whitney extension of $w_\infty$ of section 4.2.

4.2. Whitney $C^\infty$ extension. The functions $h_n$ constructed in Lemmas 8 and 9 depend smoothly on the parameters $(\mu, \omega)$.

Lemma 10. There exist $K_4$ and $K'_4 \leq K'_3$ such that the maps

$$h_i : A_i \cap \{\mu / \gamma 3\omega < K'_4\} \to W^{(i)}$$

are $C^\infty$ and

$$\|\partial_\omega h_i(\mu, \omega)\|_{\sigma_i} \leq \frac{K_4 \mu}{\gamma^4 \omega} \exp(-\chi_0^i), \quad \|\partial_\mu h_i(\mu, \omega)\|_{\sigma_i} \leq \frac{K_4}{\gamma \omega} \exp(-\chi_0^i),$$

where $\chi_0 := (1 + \chi) / 2 = 5/4$.

Proof. Since $v_0 = \mu L_\omega^{-1} P_0 f(\mu, w_0)$, by the implicit function theorem the map $w_0$ is $C^\infty$. Differentiating the identity $L_\omega(L_\omega^{-1} h) = h$ with respect to $\omega$, by (38) we get $\|\partial_\omega L_\omega^{-1} h\|_{\sigma_0} \leq (K / \gamma 2\omega) \|h\|_{\sigma_0}$. For $\mu / \gamma \omega$ small,

$$\|\partial_\omega w_0\|_{\sigma_0} \leq \frac{K_4 \mu}{\gamma^2 \omega}.$$ 

Differentiating with respect to $\mu$ we get $\|\partial_\mu w_0\|_{\sigma_0} \leq K'/\gamma \omega$ for some $K'$. 

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By induction, suppose that \( h_i \) depends smoothly on \((\mu, \omega) \in A_i\) for every \( i = 0, \ldots, n\). For \((\mu, \omega) \in A_{n+1}\), by (43), \( h_{n+1} \) is a solution of

\[
-L_\omega h_{n+1} + \mu P_{n+1}[F(w_n + h_{n+1}) - F(w_n)] + \mu(P_{n+1} - P_n)F(w_n) = 0.
\]

By the implicit function theorem \( h_{n+1} \in C^\infty \) once we prove that

\[
\mathcal{L}_{n+1}(w_{n+1})[z] := -L_\omega z + \mu P_{n+1}D\mathcal{F}(w_n + h_{n+1})[z]
\]

is invertible. By (45), \( \mathcal{L}_{n+1}(w_n) \) is invertible. Hence it is sufficient that

\[
\left\| \mathcal{L}_{n+1}^{-1}(w_n)(\mathcal{L}_{n+1}(w_{n+1}) - \mathcal{L}_{n+1}(w_n)) \right\|_{\sigma_{n+1}} < \frac{1}{2},
\]

which holds true for \( \mu^2/\gamma \omega \) small enough; indeed, by (42),

\[
\|\mathcal{L}_{n+1}(w_{n+1}) - \mathcal{L}_{n+1}(w_n)\|_{\sigma_{n+1}} \leq K\mu\|h_{n+1}\|_{\sigma_{n+1}} \leq \frac{\mu^2K'N_0^{r_{n+1}}}{\gamma \omega} \exp(-\chi_{n+1}).
\]

Finally (45) implies

\[
\|\mathcal{L}_{n+1}(w_{n+1})^{-1}\|_{\sigma_{n+1}} \leq \frac{2K_1N_0^{r_{n+1}}}{\gamma \omega}.
\]

Differentiating (47) with respect to \( \omega \)

\[
\mathcal{L}_{n+1}(w_{n+1})[\partial_\omega h_{n+1}] = 2\omega p(x)(h_{n+1})_\ell - \mu(P_{n+1} - P_n)D\mathcal{F}(w_n)\partial_\omega w_n
\]

and, using (48) and (31),

\[
\|\partial_\omega h_{n+1}\|_{\sigma_{n+1}} \leq \frac{KN_0^{r_{n+1}}}{\gamma \omega} \left( \omega N_0^{r_{n+1}}\|h_{n+1}\|_{\sigma_{n+1}} + \frac{\mu\|\partial_\omega w_n\|_{\sigma_n}}{\exp((\sigma_n - \sigma_{n+1})N_0)} \right)
\]

We note that \( \|\partial_\omega w_n\|_{\sigma_n} \leq \sum_{i=0}^{n} \|\partial_\omega h_i\|_{\sigma_i} \). Using (46) the sequence \( a_n := \|\partial_\omega h_n\|_{\sigma_n} \) satisfies

\[
a_{n+1} \leq \frac{KN_0^{r_{n+1}}}{\gamma \omega} \left( \omega^2 N_0^{r_{n+1}} + \frac{\omega \gamma r_{n+1}}{N_0^{r_{n+1}}} \sum_{i=0}^{n} a_i + \mu r_{n+1} \sum_{i=0}^{n} a_i \right)
\]

\[
\leq b_{n+1} \left( 1 + \sum_{i=0}^{n} a_i \right), \quad \text{where} \quad b_{n+1} := \frac{K\mu}{\gamma^2 \omega} N_0^{r_{n+1}} \exp(-\chi_{n+1}),
\]

recalling that \( r_{n+1} = (\mu K/\gamma \omega) \exp(-\chi_{n+1}) \). By induction, for \( K\mu/\omega \gamma^2 < 1 \), we have \( a_n \leq 2b_n \) and

\[
\|\partial_\omega h_{n+1}\|_{\sigma_{n+1}} \leq \frac{K\mu}{\gamma^2 \omega} N_0^{r_{n+1}} \exp(-\chi_{n+1}) \leq \frac{K'\mu}{\gamma^2 \omega} \exp(-\chi_0),
\]

where \( \chi_0 := (1 + \chi)/2 \). It follows that \( \|\partial_\omega w_{n+1}\|_{\sigma_{n+1}} \leq K\mu/\gamma \omega \).
Differentiating (47) with respect to $\mu$ we obtain the estimate for $\partial_\mu h_{n+1}$. \hfill \Box

Define, for $\nu_0 > 0$,

\begin{equation}
A^*_n := \left\{ (\mu, \omega) \in A_n : \text{dist}((\mu, \omega), \partial A_n) > \frac{\nu_0 \gamma^4}{N_n^3} \right\},
\end{equation}

\begin{equation}
\tilde{A}_n := \left\{ (\mu, \omega) \in A_n : \text{dist}((\mu, \omega), \partial A_n) > \frac{2\nu_0 \gamma^4}{N_n^3} \right\} \subset A^*_n.
\end{equation}

**Lemma 11** (Whitney extension). There exists a $C^\infty$ map

$\hat{w} : A_0 \cap \left\{ (\mu, \omega) : \frac{\mu}{\gamma^3 \omega} < K_4 \right\} \to W \cap X_{\sigma_0/2}$

satisfying

\begin{equation}
\| \hat{w}(\mu, \omega) \|_{\sigma_0/2} \leq \frac{K_5 \mu}{\gamma \omega},
\end{equation}

\begin{equation}
\| \partial_\omega \hat{w}(\mu, \omega) \|_{\sigma_0/2} \leq \frac{C(\nu_0) \mu}{\gamma^3 \omega}, \quad \| \partial_\mu \hat{w}(\mu, \omega) \|_{\sigma_0/2} \leq \frac{C(\nu_0)}{\gamma^3 \omega}
\end{equation}

for some $K_5$ and for some $C(\nu_0) > 0$, such that, for $(\mu, \omega) \in \tilde{A}_\infty := \cap_{n \geq 0} A_n$, $\hat{w}(\mu, \omega)$ solves the range equation (30).

Moreover there exists a sequence of $C^\infty$ maps

$\hat{w}_n : A_0 \cap \left\{ (\mu, \omega) : \frac{\mu}{\gamma^3 \omega} < K_4 \right\} \to W^{(n)}$

such that $\hat{w}_n(\mu, \omega) = w_n(\mu, \omega)$ for $(\mu, \omega) \in \tilde{A}_n$, and

\begin{equation}
\| \hat{w}(\mu, \omega) - \hat{w}_n(\mu, \omega) \|_{\sigma_0/2} \leq \frac{K_5 \mu}{\gamma \omega} \exp(-\chi^n).
\end{equation}

**Proof.** Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^+$ be a $C^\infty$-function supported in the open ball $B(0,1)$ of center 0 and radius 1 and with $\int_{\mathbb{R}^2} \varphi = 1$. Let $\varphi_n : \mathbb{R}^2 \to \mathbb{R}$ be the mollifier

\[ \varphi_n(x) := \frac{N_n^6}{\nu_0 \gamma^8} \varphi \left( \frac{N_n^3}{\nu_0 \gamma^4} x \right). \]

Supp$(\varphi_n) \subset B(0, \nu_0 \gamma^4/N_n^3)$ and $\int_{\mathbb{R}^2} \varphi_n = 1$. We define $\psi_n : \mathbb{R}^2 \to \mathbb{R}$ as

\[ \psi_n(x) := (\varphi_n * \chi_{A^*_n})(x) = \int_{\mathbb{R}^2} \varphi_n(y - x) \chi_{A^*_n}(y) \, dy, \]

where $\chi_{A^*_n}$ is the characteristic function of the set $A^*_n$. $\psi_n$ is $C^\infty$,

\begin{equation}
|D\psi_n(x)| \leq \int_{\mathbb{R}^2} |D\varphi_n(x - y)| \chi_{A^*_n}(y) \, dy \leq \frac{N_n^3}{\nu_0 \gamma^4} C,
\end{equation}

where $C := \int_{\mathbb{R}^2} |D\varphi| \, dy$.

\[ 0 \leq \psi_n(x) \leq 1, \quad \text{supp}(\psi_n) \subset A_n, \quad \psi_n(x) = 1 \quad \forall \, x \in \tilde{A}_n. \]
We define, for \((\mu, \omega) \in A_0\), the \(C^\infty\)-functions

\[
\tilde{h}_n(\mu, \omega) := \begin{cases} 
\psi_n(\mu, \omega)h_n(\mu, \omega) & \text{if } (\mu, \omega) \in A_n, \\
0 & \text{if } (\mu, \omega) \notin A_n,
\end{cases}
\]

and

\[
\tilde{w}_n(\mu, \omega) := \sum_{i=0}^n \tilde{h}_i, \quad \tilde{w}(\mu, \omega) := \sum_{i \geq 0} \tilde{h}_i,
\]

which is a series if \((\mu, \omega) \in A_\infty := \cap_{n \geq 0} A_n\).

The estimate for \(\|\tilde{w}\|_{\sigma_{n/2}}\) follows by \(\|\tilde{h}_i\|_{\sigma_1} \leq \|h_i\|_{\sigma_1}\) (because \(0 \leq \psi_i \leq 1\)) and (40). The estimates for the derivatives in (51) follow by differentiating the product \(\tilde{h}_i = \psi_i h_i\) and using (53), (40), and Lemma 10. Similarly it follows that \(\tilde{w}\) is in \(C^\infty\); see [8] for details.

For \((\mu, \omega) \in \tilde{A}_n\), \(\psi_n(\mu, \omega) = 1\), implying \(\tilde{w}_n = w_n\). As a consequence, for \((\mu, \omega) \in \tilde{A}_\infty := \cap_{n \geq 0} \tilde{A}_n\), by Corollary 1, \(\tilde{w} = \tilde{w}_\infty\) solves (30).

Finally, using (40),

\[
\|\tilde{w} - \tilde{w}_n\|_{\sigma_{n/2}} \leq \sum_{i \geq n+1} \|\tilde{h}_i\|_{\sigma_i} \leq \sum_{i \geq n+1} \frac{K\mu}{\gamma \omega} \exp(-\chi^i) \leq \frac{K'\mu}{\gamma \omega} \exp(-\chi^n). \tag*{\Box}
\]

In the next lemma we fix the constant \(\nu_0\) introduced in (50).

**Lemma 12.** There exist \(\nu_0 > 0\) and \(K'_\gamma \leq K'_5\) such that if \(\mu/\gamma^3 \omega < K'_5\), then

\[
B_\gamma \subset A_n \subset A_0 \quad \forall n \geq 0,
\]

where \(B_\gamma\) is defined in (15) taking \(C' \leq K'_5\).

**Proof.** The proof is by induction. Let \((\mu, \omega) \in B_\gamma\). Then \((\mu, \omega) \in \tilde{A}_0\) if \(A_0\) contains the closed ball of center \((\mu, \omega)\) and radius \(2\nu_0 \gamma^4 / N_0^3\). Let \((\omega', \mu')\) belong to such a ball. Then, for all \(l = 1, \ldots, N_0\),

\[
|\omega' l - \omega_j| \geq |\omega l - \omega_j| - |\omega - \omega'| l > \frac{2\gamma}{l^\tau} - \frac{2\nu_0 \gamma^4}{N_0^3} l \geq \frac{\gamma}{l^\tau}
\]

if \(\nu_0 \leq 1/2\).

Suppose now that \(B_\gamma \subset \tilde{A}_n\) and let \((\mu, \omega) \in B_\gamma\). To prove that \((\mu, \omega) \in \tilde{A}_{n+1}\), we have to show that the closed ball of center \((\mu, \omega)\) and radius \(2\nu_0 \gamma^4 / N_{n+1}^3\) is contained in \(A_{n+1}\). Let \((\mu', \omega')\) belong to such a ball. The nonresonance condition on \(|\omega' l - j/c|\) is verified, as above, for \(\nu_0 \leq 1/2\). For the other condition, we denote in short \(w_j^\omega(\mu', \omega') := \omega_j(\mu', \omega)\) (see (27) for the definition of \(\omega_j(\mu, w)\)). It results, for all \(l = 1, \ldots, N_{n+1}\), in

\[
|\omega' l - w_j^\omega(\mu', \omega')| \geq |\omega l - \omega_j(\mu, \omega)| - |\omega - \omega'| l - |w_j^\omega(\mu', \omega') - \omega_j(\mu, \omega)|
\]

\[
> \frac{2\gamma}{l^\tau} - \frac{2\nu_0 \gamma^4}{N_{n+1}^3} - |w_j^\omega(\mu', \omega') - \omega_j(\mu, \omega)|
\]

\[
> \frac{3\gamma}{2l^\tau} - |w_j^\omega(\mu', \omega') - \omega_j(\mu, \omega)|
\]

(54)
if \( \nu_0 \leq 1/4 \). Now we estimate the last term

\[
|\omega^n_j(\mu', \omega') - \bar{\omega}_j(\mu, \omega)| = \frac{|\lambda^n_j(\mu', \omega') - \tilde{\lambda}_j(\mu, \omega)|}{|\omega_j(\mu, \omega)| + |\omega^n_j(\mu', \omega')|} \leq \frac{|\lambda^n_j(\mu', \omega') - \tilde{\lambda}_j(\mu, \omega)|}{\sqrt{\delta_0}}
\]

by (29), both for \( j < j_0 \) and for \( j \geq j_0 \). By the comparison principle (28)

\[
\delta_0^{-1/2}|\lambda^n_j(\mu', \omega') - \tilde{\lambda}_j(\mu, \omega)| \leq K|\mu - \mu'| + K\|w_n(\mu', \omega') - \bar{w}(\mu, \omega)\|_{\sigma_0/2}.
\]

By Lemma 10, \( \|\partial_\omega w_n\|_{\sigma_0/2}, \|\partial_\mu w_n\|_{\sigma_0/2} \leq K/\gamma^2 \omega \) for some other \( K \), and being \( \omega, \omega' > \gamma \),

\[
K\|w_n(\mu', \omega') - w_n(\mu, \omega)\|_{\sigma_0/2} \leq \frac{K'\mu}{\gamma^2} \exp(-\lambda^n) < \frac{\gamma}{8l^\tau} \quad \forall l = 1, \ldots, N_{n+1}
\]

if \( \nu_0 \) is small enough \( (1 < \tau < 2) \). On the other hand, since \( (\mu, \omega) \in \bar{A}_n \) we have \( w_n(\mu, \omega) = \bar{w}_n(\mu, \omega) \) (Lemma 11) and, by (52),

\[
K\|w_n(\mu, \omega) - \bar{w}(\mu, \omega)\|_{\sigma_0/2} \leq \frac{K'\mu}{\gamma^2} \exp(-\lambda^n) < \frac{\gamma}{8l^\tau} \quad \forall l = 1, \ldots, N_{n+1}
\]

for \( \mu/\gamma^2 \omega \) sufficiently small. By (54), collecting the previous estimates,

\[
|\omega'^l - \omega_j^n(\mu', \omega')| > \frac{\gamma}{l^\tau} \quad \forall l = 1, \ldots, N_{n+1}
\]

and \( (\mu', \omega') \) belongs to \( A_{n+1} \).

---

**4.3. Measure of the Cantor set \( B_\gamma \).** In the following \( R := (\mu', \mu'') \times (\omega', \omega'') \) denotes a rectangle contained in the region \( \{ (\mu, \omega) \in [\mu_1, \mu_2] \times (2\gamma, +\infty) : \mu < K_6' \gamma^5 \omega \} \). Furthermore we consider \( \omega'' - \omega' \) as a fixed quantity ("of order 1").

**Lemma 13.** There exist \( K_6 \) and \( K_6' \leq K_6 \) such that, taking \( C' \leq K_6' \) in the definition (15) of \( B_\gamma \), for all \( \mu \in (\mu_1, \mu_2) \) the section

\[
S_\gamma(\mu) := \{ \omega : (\mu, \omega) \in B_\gamma \}
\]

satisfies the measure estimate

\[
|S_\gamma(\mu) \cap (\omega', \omega'')| \geq (1 - K_6' \gamma)(\omega'' - \omega').
\]

As a consequence, for every \( R := (\mu', \mu'') \times (\omega', \omega'') \)

\[
|B_\gamma \cap R| \geq |R| (1 - K_6' \gamma).
\]

**Proof.** We consider the inequalities \( |\omega l - \bar{\omega}_j(\mu, \omega)| > 2\gamma/l^\tau \) in the definition of \( B_\gamma \). The analogous inequalities for \( |\omega l - \omega_j| \) and \( |\omega l - j/c| \) are simpler because \( j/c \) and \( \omega_j \) do not depend on \( (\mu, \omega) \).

The complementary set we have to estimate is

\[
\mathcal{C} := \bigcup_{l, j \geq 1} \mathcal{R}_{ij},
\]

where \( \mathcal{R}_{ij} := \{ \omega \in (\omega', \omega'') : |\omega - \bar{\omega}_j(\mu, \omega)| \leq 2\gamma/l^\tau \} \).
We claim that
\begin{equation}
|\partial_\omega \tilde{w}_j(\mu, \omega)| \leq \frac{K\mu}{\gamma\omega}.
\end{equation}
Indeed, by the same arguments as in the proof of Lemma 12 and the comparison principle (28), we have
\begin{equation}
|\tilde{w}_j(\mu, \omega) - \tilde{w}_j(\mu, \omega')| \leq K\|\tilde{w}(\mu, \omega) - \tilde{w}(\mu, \omega')\|_{\sigma_0/2} \leq \frac{K\mu}{\gamma\omega} |\omega - \omega'|
\end{equation}
using (51). As a consequence of (57)
\begin{equation}
\partial_\omega (l\omega - \tilde{w}_j(\mu, \omega)) \geq l - \frac{K\mu}{\gamma\omega} \geq \frac{l}{2} \quad \forall l \geq 1
\end{equation}
for $\mu/\gamma\omega$ small enough; we deduce $|\mathcal{R}_{ij}| \leq 4\gamma/l^{r+1}$.

Furthermore the set $\mathcal{R}_{ij}$ is nonempty only if
\begin{equation}
\omega' l - \frac{2\gamma}{l^r} < \tilde{w}_j(\mu, \omega) < \omega'' l + \frac{2\gamma}{l^r}.
\end{equation}
So, for every fixed $l$, the number of indices $j$ such that $\mathcal{R}_{ij} \neq \emptyset$ is
\begin{equation}
\sharp\{j\} \leq \frac{1}{\delta} \left( l(\omega'' - \omega') + \frac{4\gamma}{l^r} \right) + 1 \leq Kl(\omega'' - \omega'),
\end{equation}
where
\begin{equation}
\delta := \inf \left\{ |\tilde{w}_{j+1}(\mu, \omega) - \tilde{w}_j(\mu, \omega)| : j \geq 1, (\mu, \omega) \in B_\gamma \right\}.
\end{equation}
For $\|\tilde{w}\|_{\sigma_0/2} \leq K'\mu/\gamma\omega < R$ we have $\delta \geq \delta_1$, where
\begin{equation}
\delta_1 := \inf \left\{ |\omega_{j+1}(\mu, w) - \omega_j(\mu, w)| : j \geq 1, \mu \in [\mu_1, \mu_2], \|w\|_{\sigma_0/2} \leq R \right\} > 0,
\end{equation}
as proved in the appendix.

In conclusion, the measure of the complementary set is
\begin{equation}
|\mathcal{C}| \leq \sum_{l=1}^{+\infty} \frac{4\gamma}{l^{r+1}} Kl(\omega'' - \omega') \leq K'(\omega'' - \omega')\gamma
\end{equation}
and (55) is proved. Integrating on $(\mu', \mu'')$ we obtain (56). \qed

By Fubini’s theorem also the section $S_\gamma(\omega)$ is large for $\omega$ in a large set.

**LEMMA 14.** Let
\begin{equation}
S_\gamma(\omega) := \{ \mu : (\mu, \omega) \in B_\gamma \}.
\end{equation}
For every $R := (\mu', \mu'') \times (\omega', \omega'')$, $\gamma' \in (0, 1)$ we obtain
\begin{equation}
\left\{ \omega \in (\omega', \omega'') : \frac{|S_\gamma(\omega) \cap (\mu', \mu'')|}{\mu'' - \mu'} \geq 1 - \gamma' \right\} \supseteq (\omega'' - \omega')\left( 1 - K_6 \frac{\gamma}{\gamma'} \right).
\end{equation}

**Proof.** Consider
\begin{equation}
\Omega^+ := \{ \omega \in (\omega', \omega'') : |S_\gamma(\omega) \cap (\mu', \mu'')| \geq (\mu'' - \mu')(1 - \gamma') \},
\end{equation}
\begin{equation}
\Omega^- := \{ \omega \in (\omega', \omega'') : |S_\gamma(\omega) \cap (\mu', \mu'')| < (\mu'' - \mu')(1 - \gamma') \}.
\end{equation}
Using Fubini’s theorem
\[
|B_\gamma \cap R| = \int_\omega'' |S_\gamma_\omega(\omega) \cap (\mu', \mu'')| \, d\omega \\
= \int_{\Omega'} |S_\gamma_\omega(\omega) \cap (\mu', \mu'')| \, d\omega + \int_{\Omega''} |S_\gamma_\omega(\omega) \cap (\mu', \mu'')| \, d\omega \\
\leq (\mu'' - \mu')|\Omega^+| + (\mu'' - \mu')(1 - \gamma')|\Omega^-|. 
\] (60)

By (56), \(|B_\gamma \cap R| \geq (\omega'' - \omega')(\mu'' - \mu')(1 - K_6\gamma)\) and therefore, by (60),
\[
(\omega'' - \omega')(1 - K_6\gamma) \leq |\Omega^+| + (1 - \gamma')|\Omega^-| = (\omega'' - \omega') - \gamma'|\Omega^-| 
\] (61)

because \(|\Omega^+| + |\Omega^-| = \omega'' - \omega'.\) Then

\[
|\Omega^-| \leq (\omega'' - \omega')K_6\gamma/\gamma'.
\]

and, by the first inequality in (61), \(|\Omega^+| \geq (\omega'' - \omega')(1 - K_6\gamma/\gamma'),\) which is (59).

Inequalities (55) and (59) imply the measure estimates (18)–(19).

The main conclusions of this section are summarized in the following theorem, which follows by Lemmas 11, 12, and 13.

**Theorem 3** (solution of the range equation). There exist \(\tilde{w} \in C^\infty(A, W \cap X_{\sigma_0/2})\) satisfying (51) and the large (see (56)) Cantor set \(B_\gamma\) defined in (15) such that, for every \((\mu, \omega) \in B_\gamma\), the function \(\tilde{w}(\mu, \omega)\) solves the range equation (30).

**5. Proof of Theorems 1 and 2.**

**Proof of Theorem 1.** By Theorem 3 for all \((\mu, \omega) \in B_\gamma\) the function \(\tilde{w}(\mu, \omega) \in X_{\sigma_0/2}\) solves the range equation (30). By Lemma 4, \(v(\mu, \tilde{w}(\mu, \omega))\) solves the bifurcation equation in (8), and therefore

\[
\tilde{u} := v(\mu, \tilde{w}(\mu, \omega)) + \tilde{w}(\mu, \omega) \in X_{\sigma_0/2}
\]

is a solution of (3). Estimates (14) follow by (51).

Since \(\tilde{u}\) solves
\[
-(p(x)\tilde{u}_x)_x = \mu f(x, t, \tilde{u}) - \omega^2 p(x)\tilde{u}_{tt}
\] (62)
we deduce
\[
-(p(x)\tilde{u}_x(t, x))_x \in H^1(0, \pi) \quad \forall t \in \mathbb{R}.
\]
This implies, since \(p(x) \in H^3(0, \pi),\) that
\[
\tilde{u}(t, x) \in H^3(0, \pi) \cap H^1_0(0, \pi) \subset C^2(0, \pi) \quad \forall t \in \mathbb{R}. \quad \square
\]

**Proof of Theorem 2.** For every fixed \(t,\) by the algebra property of \(H^m\)
\[
\|f(x, t, u(x, t))\|_{H^m} \leq \sum_{l,k} \|f_{lk}(x)w^k(x)\|_{H^m} \leq K \sum_{l,k} \|f_{lk}\|_{H^m} \|w^k\|_{H^m}
\]
for some \(K > 0.\)
Using the Gagliardo–Nirenberg-type inequality
\[ \|u^k\|_{H^m} \leq (C_m \|u\|_{H^1})^{k-1} \|u\|_{H^m} \]
valid for every \( u \in H^1_0 \cap H^m \) (see, e.g., [26, 20]), we get
\[ \|f(x, t, u(x, t))\|_{H^m} \leq K \|u\|_{H^m} \sum_{l,k} \|f_{lk}\|_{H^m} (C_m \|u\|_{H^1})^{k-1}, \]
which is convergent for \( \|u\|_{H^1} < r_m/C_m \) by (21).

The solution \( \tilde{u} \) satisfies (62) and \( \tilde{u}(\cdot, t) \in H^3(0, \pi) \) for all \( t \).
By assumption \( \|\tilde{u}\|_{H^1} < r_m/C_m \). By induction, assume \( \tilde{u}(\cdot, t) \in H^k \) for \( k = 3, \ldots, m \). Hence \( \tilde{u}_k(\cdot, t) \in H^k \) and \( \rho(x)\tilde{u}_k(\cdot, t) \in H^k \) because \( \rho \in H^m \). Furthermore, by (63), \( f(x, t, \tilde{u}) \in H^k \). We deduce, by (62), that \( p(x)\tilde{u}_x \in H^{k+1} \). Finally \( \tilde{u} \in H^{k+2} \) because \( p \in H^{m+1} \).
If \( f_0(x, 0) = a_w f_0(x, 0) = 0 \), then, by Lemma 3, we can take \( v(\mu, 0) = 0 \) for all \( \mu \).
Therefore, by (14),
\[ \|\tilde{u}(t, \cdot)\|_{H^1} \leq \|\tilde{u}\|_{\sigma_0/2} \leq \frac{2C\mu}{\gamma \omega} \forall t, \]
and, for \( \mu/\gamma \omega \) small enough, we deduce the regularity in (22). □

6. Inversion of the linearized problem. Here we prove Lemma 7. Decomposing in Fourier series
\[ f'(u) = \sum_{k \in \mathbb{Z}} a_k(x)e^{ikt} \]
we write, for all \( h = \sum_{1 \leq |l| \leq N_n} h_l(x)e^{ilt} \in W(n), \)
\[ -L_{\omega}h + \mu P_n[f'(u)h] = \sum_{1 \leq |l| \leq N_n} \left[ \omega^2 l^2 \rho h_l + \partial_x (\rho \partial_x h_l) \right] e^{ilt} + \mu P_n \left[ \left( \sum_{k \in \mathbb{Z}} a_k e^{ikt} \right) \left( e^{ILT} \right) \right] \]
\[ = \sum_{1 \leq |l| \leq N_n} \left[ \omega^2 l^2 \rho h_l + \partial_x (\rho \partial_x h_l) + \mu a_0 h_l \right] e^{ilt} + \mu \sum_{|l|, |k|+l| \in \{1, \ldots, N_n\}, k \neq 0} a_k h_l e^{i(k+l)t}. \]
Hence \( L_n(w) \) defined in (32) can be decomposed as
\[ L_n(w)h = \rho \left( Dh + M_1 h + M_2 h \right), \]
where
\[ Dh := \frac{1}{\rho} \sum_{|l| = 1}^{N_n} \left[ \omega^2 l^2 \rho h_l + (p h_l)' + \mu a_0 h_l \right] e^{ilt}, \]
\[ M_1 h := \frac{\mu}{\rho} \sum_{|l|, |k| \in \{1, \ldots, N_n\}, l \neq k} a_{k-l} h_l e^{ikt}, \]
\[ M_2 h := \frac{\mu}{\rho} P_n \left[ f'(u) d_w v(\mu, w)[h] \right]. \]
Note that $D$ is a diagonal operator in time Fourier basis. To study the eigenvalues of $D$, we use Sturm–Liouville-type techniques.

**Lemma 15 (Sturm–Liouville).** The eigenvalues $\lambda_j(\mu, w)$ of the Sturm–Liouville problem (27) form a strictly increasing sequence which tends to $+\infty$. Every $\lambda_j(\mu, w)$ is simple and the following asymptotic formula holds:

$$\lambda_j(\mu, w) = \frac{j^2}{c^2} + b + M(\mu, w) + r_j(\mu, w), \quad |r_j(\mu, w)| \leq \frac{K}{j}$$

for all $j \geq 1$, $(\mu, w) \in [\mu_1, \mu_2] \times B_R$, where

$$c := \frac{1}{\pi} \int_0^\pi \left( \frac{\rho}{\rho} \right)^{1/2} dx, \quad b := \frac{1}{4\pi c} \int_0^\pi \left[ \frac{(\rho \rho')'}{\rho \sqrt{\rho}} \right]' \frac{1}{\sqrt{\rho}} dx,$$

$$M(\mu, w) := -\frac{\mu}{c^2} \int_0^\pi \Pi_V f'(v(\mu, w) + w) \sqrt{\rho} dx.$$

The eigenfunctions $\varphi_j(\mu, w)$ of (27) form an orthonormal basis of $L^2(0, \pi)$ with respect to the scalar product $(y, z)_{L^2} := c^{-1} \int_0^\pi yz \rho dx$. For $K$ big enough

$$(y, z)_{\mu, w} := \frac{1}{c} \int_0^\pi p y' z' + [K\rho - \mu \Pi_V f'(v(\mu, w) + w)] yz \rho dx$$

defines an equivalent scalar product on $H^1_0(0, \pi)$ and

$$K' \|y\|_{H^1} \leq \|y\|_{\mu, w} \leq K'' \|y\|_{H^1} \quad \forall y \in H^1_0.$$ 

$\varphi_j(\mu, w)$ is also an orthogonal basis of $H^1_0(0, \pi)$ with respect to the scalar product $(\cdot, \cdot)_{\mu, w}$ and, for $y = \sum_{j \geq 1} \hat{y}_j \varphi_j(\mu, w)$,

$$\|y\|^2_{L^2} = \sum_{j \geq 1} \hat{y}_j^2, \quad \|y\|^2_{\mu, w} = \sum_{j \geq 1} \hat{y}_j^2 (\lambda_j(\mu, w) + K).$$

**Proof.** For the proof of the lemma, see the appendix. □

We develop

$$Dh = \sum_{1 \leq |l| \leq N_n} D_l h_l e^{il},$$

where

$$D_l z := \frac{1}{\rho} \left[ \omega^2 l^2 \rho z + (p z')' + \mu a_0 z \right] \quad \forall z \in H^1_0(0, \pi)$$

and $a_0 = \Pi_V f(v(\mu, w) + w)$.

By Lemma 15 each $D_l$ is diagonal with respect to the basis $\varphi_j(\mu, w)$:

$$z = \sum_{j \geq 1} \hat{z}_j \varphi_j(\mu, w) \in H^1_0(0, \pi) \Rightarrow D_l z = \sum_{j \geq 1} (\omega^2 l^2 - \lambda_j(\mu, w)) \hat{z}_j \varphi_j(\mu, w).$$

**Lemma 16.** Suppose all the eigenvalues $\omega^2 l^2 - \lambda_j(\mu, w)$ are not zero. Then

$$|D_l|^{-1/2} z := \sum_{j \geq 1} \frac{\hat{z}_j \varphi_j(\mu, w)}{\sqrt{\omega^2 l^2 - \lambda_j(\mu, w)}}$$
satisfies
\[(69) \quad \| |D|^{-1/2} z \|_{\dot{H}^1} \leq \frac{K}{\sqrt{\alpha_1}} \| z \|_{\dot{H}^1} \quad \forall z \in H^1_0(0, \pi), \]

where \( \alpha_1 := \min_{j \geq 1} \left| \omega_j^2 l^2 - \lambda_j(\mu, w) \right| > 0. \)

**Proof.** By (68) \( \| |D|^{-1/2} z \|_{\mu, w}^2 \leq (1/\alpha_1) \| z \|_{\mu, w}^2. \) Hence (69) follows by the equivalence of the norms (67).

**Lemma 17** (inversion of \(D\)). Assume the nonresonance condition (34). Then \( |D|^{-1/2} : W^{(n)} \to W^{(n)} \) defined by
\[
|D|^{-1/2} h := \sum_{1 \leq |l| \leq N_n} |D_l|^{-1/2} h_l e^{i\mu}\]

satisfies
\[
\| |D|^{-1/2} h \|_{\sigma, s} \leq \frac{K}{\sqrt{\gamma \omega}} \| h \|_{\sigma, s} \leq \frac{K N_{\tau}^{1/2}}{\sqrt{\gamma \omega}} \| h \|_{\sigma, s} \quad \forall h \in W^{(n)}. \]

**Proof.** By (69) and \( \alpha_{-l} = \alpha_l \geq \gamma \omega/|l|^\tau - 1 \)
\[
\| |D|^{-1/2} h \|_{\sigma, s}^2 = \sum_{1 \leq |l| \leq N_n} \| |D_l|^{-1/2} h_l \|_{\dot{H}^1}^2 (1 + l^{2s}) e^{2\sigma |l|} \]
\[
\leq \sum_{1 \leq |l| \leq N_n} \frac{K^2 |l|^{\tau - 1}}{\gamma \omega} \| h_l \|_{\dot{H}^1}^2 (1 + l^{2s}) e^{2\sigma |l|} \]
\[
\leq \frac{K'}{\gamma \omega} \| h \|_{\sigma, s}^2 \]

because \( |l|^{\tau - 1}(1 + l^{2s}) < 2(1 + |l|^{2s + \tau - 1}) \) for all \( |l| \geq 1. \)

To prove the invertibility of \(L_n(w)\) we write (64) as

\[(70) \quad L_n(w) = \rho |D|^{1/2}(U + T_1 + T_2)|D|^{1/2}, \]

where
\[
(71) \quad \left\{ \begin{array}{ll}
U := |D|^{-1/2} D |D|^{-1/2}, \\
T_i := |D|^{-1/2} M_i |D|^{-1/2}, & i = 1, 2 \end{array} \right. \]

With respect to the basis \(\varphi_j(\mu, w) e^{i\mu}\) the operator \(U\) is diagonal and its \((l, j)\)th eigenvalue is \(\text{sign}(\omega^2 l^2 - \lambda_j(\mu, w)) \in \{ \pm 1 \}\), implying that the operator norm is
\[(72) \quad \| U \|_{\sigma} := \sup_{\| h \|_{\sigma} \leq 1} \| U h \|_{\sigma} = 1. \]

The smallness of \(T_i\) requires an analysis of the small divisors. Formula (66) implies, by Taylor expansion, the asymptotic dispersion relation
\[(73) \quad \left| \omega_j(\mu, w) - \frac{j}{c} \right| \leq \frac{K}{j}, \]
and there exists $K$ such that, for every $x \geq 0$,
\begin{equation}
|x^2 - \lambda_j(\mu, w)| = \min_{j \geq 1} |x^2 - \lambda_j(\mu, w)| \Rightarrow j^* \geq Kx.
\end{equation}

**Lemma 18** (analysis of the small divisors). Assume the nonresonance conditions (33)–(34) and $\omega > \gamma$. Then for all $|k|, |l| \in \{1, \ldots, N_n\}$, $k \neq l$,
\begin{equation}
\alpha_k \alpha_l \geq \left( \frac{K \gamma \omega}{|k - l|^\nu_{2(\nu - 1)}} \right)^2,
\end{equation}
where $\alpha_l := \min_{j \geq 1} |\omega^2 l^2 - \lambda_j(\mu, w)|$.

**Proof.** Since $\alpha_{-l} = \alpha_l$ for all $l$, we can suppose $k, l \geq 1$.

We distinguish two cases, if $k, l$ are close to or far from each other. Let $\beta := (2 - \tau)/\tau \in (0, 1)$.

**Case 1.** Let $2|k - l| > (\max\{k, l\})^\beta$. By (34)
\begin{equation}
\alpha_k \alpha_l \geq \frac{(\gamma \omega)^2}{(kl)^{-\beta - 1}} \geq \frac{(\gamma \omega)^2}{(\max\{k, l\})^{2(\tau - 1)}} \geq \frac{C(\gamma \omega)^2}{|k - l|^{2(\tau - 1)}}.
\end{equation}

**Case 2.** Let $0 < 2|k - l| \leq (\max\{k, l\})^\beta$. In this case $2k \geq l \geq k/2$. Indeed, if $k > l$, then $2|k - l| \leq k^2$, so $2l \geq 2k - k^2 \geq k$ because $\beta \in (0, 1)$—analogously if $l > k$.

Let $i, j$, resp., $j$, be an integer which realizes the minimum $\alpha_k$, resp., $\alpha_l$, and write in short $\lambda_i(\mu) := \lambda_j(\mu, w), \omega_j(\mu) := \omega_j(\mu, w)$.

If both $\lambda_i(\mu), \lambda_j(\mu) < 0$, then $\alpha_i \geq \omega^2 i^2$, $\alpha_k \geq \omega^2 k^2$, $\alpha_i \alpha_k \geq \omega^4 > \gamma^2 \omega^2$.

If only $\lambda_j(\mu) < 0$, then $\alpha_i \alpha_k \geq \gamma \omega^2 k^{2(\nu - 1)} \geq 2^{1 - \tau} \gamma \omega^2 \geq 2^{1 - \tau} \gamma \omega^2$.

The really resonant cases happen if $\lambda_i(\mu), \lambda_j(\mu) > 0$. Suppose, for example, that $\max\{k, l\} = k$. By (73), $|\omega_j(\mu) - (j/c)| \leq K/j$, and, by (74), $i \geq K\omega k, j \geq K\omega l$. Hence, using also (33),
\begin{align*}
|\omega k - \omega_i(\mu)| - (\omega l - \omega_j(\mu)) &= |\omega(k - l) - (\omega_i(\mu) - \omega_j(\mu))| \\
&\geq |\omega(k - l) - \frac{i - j}{c} - \frac{K}{\omega l} - \frac{K}{\omega k}| \\
&\geq \frac{\gamma}{(k - l)^2} - \frac{3K}{\omega k} \geq \frac{2\gamma}{k^{2\beta \tau}} - \frac{3K}{\omega k}
\end{align*}
because $2(k - l) \leq k^2, 2l \geq k$. Since $\beta \tau < 1$ and $k \leq 2l$,
\begin{equation}
|(\omega k - \omega_i(\mu)) - (\omega l - \omega_j(\mu))| \geq \frac{1}{2} \left( \frac{\gamma}{k^{\beta \tau}} + \frac{\gamma}{l^{\beta \tau}} \right) \forall k \geq \left( \frac{K}{\omega \gamma} \right)^{\frac{1}{\nu - \beta \tau}} := k^*.
\end{equation}

We reach the same conclusion if $\max\{k, l\} = l$. It follows that, for $\max\{k, l\} \geq k^*$, there holds $|\omega k - \omega_i(\mu)| \geq \gamma/2k^{\beta \tau}$ or $|\omega l - \omega_j(\mu)| \geq \gamma/2l^{\beta \tau}$. Suppose $|\omega k - \omega_i(\mu)| \geq \gamma/2k^{\beta \tau}$. Then
\begin{equation}
\alpha_k = |\omega^2 k^2 - \omega_i^2(\mu)| \geq |\omega k - \omega_i(\mu)| \omega k \geq \frac{\gamma \omega}{2} k^{1 - \beta \tau}.
\end{equation}

Since $l \leq 2k$, for $\alpha_l$ we can use (34),
\begin{equation}
\alpha_k \alpha_l \geq \frac{\gamma \omega k^{1 - \beta \tau}}{2} \geq \frac{\gamma^2 \omega^2}{2} k^{2 - \tau - \beta \tau} = \frac{\gamma^2 \omega^2}{2} \frac{1}{2}
\end{equation}
because $2 - \tau - \beta \tau = 0$.  

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On the other hand, if \( \max\{k, l\} < k^* = (K/\omega \gamma)^{1/(\tau - 1)} \), we can use (34) for both \( k, l \):

\[
\alpha_k \alpha_l \geq \frac{(\gamma \omega)^2}{(kl)^{\tau - 1}} > \frac{(\omega \gamma)^2}{(k^*)^{2(\tau - 1)}} = \frac{(\omega \gamma)^2}{K^{2(\tau - 1)}} > \frac{\gamma^0 \omega^2}{K^2}
\]

(using \( \omega > \gamma \)). Since \( \gamma < 1 \), taking the minimum for all these cases concludes the proof. \( \square \)

**Lemma 19 (estimate of \( T_1 \)).** Assume the nonresonance conditions (33)–(34), \( \omega > \gamma \), and \( \Pi_W f'(u) = \sum_{t \neq 0} a_l(x)e^{ilt} \in X_{\sigma, 1 + \frac{\tau - 1}{2}} \). There exists \( K \) such that

\[
\|T_1 h\|_\sigma \leq \frac{K \mu}{\gamma^3 \omega} \|\Pi_W f'(u)\|_{\sigma, 1 + \frac{\tau - 1}{2}} \|h\|_\sigma \quad \forall h \in W^{(n)}.
\]

**Proof.** For all \( h \in W^{(n)} \), \( T_1 h = \sum_{1 \leq |k| \leq N_n} (T_1 h)_k e^{ikt} \), where

\[
(T_1 h)_k = |D_k|^{-1/2} (M_1 |D|^{-1/2} h)_k = |D_k|^{-1/2} \left[ \sum_{1 \leq |l| \leq N_n, l \neq k} \frac{\alpha_k - l}{\rho} |D_l|^{-1/2} h_l \right].
\]

Setting \( A_m := \|a_m/\rho\|_{H^1} \), using (69) and Lemma 18, we obtain

\[
(75) \quad \|T_1 h\|_{H^1} \leq K \mu \sum_{1 \leq |l| \leq N_n, l \neq k} \frac{A_k - l}{\sqrt{\alpha_k} \sqrt{\alpha_l}} \|h_l\|_{H^1} \leq \frac{K \mu}{\gamma^3 \omega} S_k,
\]

where

\[
S_k := \sum_{|l| \leq N_n, l \neq k} A_k - l |k - l|^{\tau(\tau - 1)} \|h_l\|_{H^1}.
\]

By (75) we get, defining \( S(t) := \sum_{|k| = 1}^N S_k e^{ikt} \),

\[
\|T_1 h\|_{\sigma}^2 = \sum_{|k| = 1}^N \|T_1 h_k\|_{H^1}^2 (1 + k^2) e^{2\sigma |k|} \leq \left( \frac{K \mu}{\gamma^3 \omega} \right)^2 \sum_{|k| = 1}^N S_k^2 (1 + k^2) e^{2\sigma |k|} = \left( \frac{K \mu}{\gamma^3 \omega} \right)^2 \|S\|_{\sigma}^2.
\]

Since \( S = P_n (\varphi \psi) \) with \( \varphi(t) := \sum_{l \in \mathbb{Z}} A_l |l|^{\frac{\tau(\tau - 1)}{2}} e^{ilt} \) and \( \psi(t) := \sum_{|l| = 1}^N \|h_l\|_{H^1} e^{ilt} \)

\[
\|T_1 h\|_{\sigma} \leq \frac{K \mu}{\gamma^3 \omega} \|\varphi\|_{\sigma} \|\psi\|_{\sigma} \leq \frac{K \mu}{\gamma^3 \omega} \|\Pi_W f'(u)\|_{\sigma, 1 + \frac{\tau - 1}{2}} \|h\|_{\sigma}
\]

because \( \|\varphi\|_{\sigma} \leq 2 \|\Pi_W f'(u)\|_{\sigma, 1 + \frac{\tau - 1}{2}} \) and \( \|\psi\|_{\sigma} = \|h\|_{\sigma} \). \( \square \)

**Lemma 20 (estimate of \( T_2 \)).** Suppose that \( \Pi_W f'(u) \in X_{\sigma, 1 + \frac{\tau - 1}{2}} \). Then

\[
\|T_2 h\|_{\sigma} \leq \frac{K \mu}{\gamma^3 \omega} \|\Pi_W f'(u)\|_{\sigma, 1 + \frac{\tau - 1}{2}} \|h\|_{\sigma} \quad \forall h \in W^{(n)}
\]

for some \( K \).
Proof. By the definitions (71) and (65) and by Lemma 17,
\[ \|T_2 h\|_{\sigma} \leq \frac{K}{\sqrt{\gamma \omega}} \|M_2 |D|^{-1/2} h\|_{\sigma,1+\frac{\tau-1}{2}} \]
\[ \leq \frac{K' \mu}{\sqrt{\gamma \omega}} \|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau-1}{2}} \|d_w v(\mu, w)\|_{\sigma,1+\frac{\tau-1}{2}} \]
\[ = \frac{K' \mu}{\sqrt{\gamma \omega}} \|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau-1}{2}} \|d_w v(\mu, w)\|_{H^1} \]
because \(d_w v(\mu, w)\|D|^{-1/2} h\| \in V\). By Lemmas 4 and 17
\[ \|d_w v(\mu, w)\|_{H^1} \leq K \|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau-1}{2}} \]
\[ \leq K' \mu \sqrt{\gamma \omega} \|h\|_{\sigma} \]
implying the thesis. \(\Box\)

Proof of Lemma 7. By (72), \(\|U\|_{\sigma} = 1\). If
\[ \|T_1 + T_2\|_{\sigma} < \frac{1}{2}, \tag{76} \]
then by Neumann series \(U + T_1 + T_2\) is invertible in \((W^{(n)}, \|\|_\sigma)\) and
\[ \|(U + T_1 + T_2)^{-1}\|_{\sigma} < 2. \]
By Lemmas 19 and 20, condition (76) is verified if
\[ \|T_1\|_{\sigma} \leq \frac{K \mu}{\gamma^3 \omega} \|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau-1}{2}} < \frac{1}{4} \tag{77} \]
and
\[ \|T_2\|_{\sigma} \leq \frac{K \mu}{\gamma \omega} \|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau-1}{2}} \leq \frac{K \mu}{\gamma^3 \omega} \|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau-1}{2}} < \frac{1}{4} \tag{78} \]
(we recall that \(\gamma \in (0,1)\) and \((\tau - 1)/2 < \tau(\tau - 1)/(2 - \tau)\) because \(\tau > 1\)). Both conditions (77) and (78) are satisfied if
\[ \frac{\mu}{\gamma^3 \omega} \|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau-1}{2}} < \frac{1}{4K} =: K'_1, \]
which is condition (35). Hence, inverting (70)
\[ \mathcal{L}_n(w)^{-1} h = |D|^{-1/2} (U + T_1 + T_2)^{-1} |D|^{-1/2} \left( \frac{h}{\rho} \right), \]
which, using Lemma 17, yields (36). \(\Box\)

7. Appendix.

Proof of Lemma 15. Let \(a(x) \in L^2(0, \pi)\). Under the “Liouville change of variable”
\[ x = \psi(\xi) \Leftrightarrow \xi = g(x), \quad g(x) := \frac{1}{c} \int_{0}^{x} \left( \frac{\rho(s)}{p(s)} \right)^{1/2} ds, \tag{79} \]
we have that \((\lambda, y(x))\) satisfies
\[
(80) \quad \begin{cases}
-(p(x)y'(x))' + a(x)y(x) = \lambda \rho(x)y(x), \\
y(0) = y(\pi) = 0
\end{cases}
\]
iff \((\nu, z(\xi))\) satisfies
\[
(81) \quad \begin{cases}
-z''(\xi) + [q(\xi) + \alpha(\xi)]z(\xi) = \nu z(\xi), \\
z(0) = z(\pi) = 0,
\end{cases}
\]
where
\[
\nu = c^2 \lambda, \quad r(x) = \sqrt{p(x)} \rho(x), \quad z(\xi) = y(\psi(\xi)) r(\psi(\xi)),
\]
\[
\alpha(\xi) = c^2 \frac{a(\psi(\xi))}{\rho(\psi(\xi))}, \quad q(\xi) = c^2 Q(\psi(\xi)), \quad Q = \frac{p}{\rho} r'' + \frac{1}{2} \left( \frac{p}{\rho} \right)' \frac{r'}{r}.
\]
By [25, Theorem 4 in Chapter 2, p. 35], the eigenvalues of (81) form an increasing sequence \(\nu_j\) satisfying the asymptotic expansion
\[
\nu_j = j^2 + \frac{1}{\pi} \int_0^\pi (q + \alpha) \, d\xi - \frac{1}{\pi} \int_0^\pi \cos(2j\xi)(q(\xi) + \alpha(\xi)) \, d\xi + r_j, \quad |r_j| \leq \frac{C}{j},
\]
where \(C := C(\|q + \alpha\|_{L^2})\) is a positive constant. Moreover every \(\nu_j\) is simple [25, Theorem 2, p. 30].

Since \(p, \rho\) are positive and belong to \(H^3\), if \(a \in H^1\), then \(q, \alpha \in H^1\). Integrating by parts, \(|\int_0^\pi \cos(2j\xi)(q + \alpha) \, d\xi| \leq \|q + \alpha\|_{H^1}/j\) and so
\[
\nu_j = j^2 + \frac{1}{\pi} \int_0^\pi (q + \alpha) \, d\xi + r_j', \quad |r_j'| \leq \frac{C'}{j}
\]
for some \(C' := C'(\|q + \alpha\|_{H^1}).\) Dividing by \(c^2\) and using the inverse Liouville change of variable, we obtain the formula for the eigenvalues \(\lambda_j(a)\) of (80),
\[
(82) \quad \lambda_j(a) = \frac{j^2}{c^2} + \frac{1}{\pi c} \int_0^\pi \frac{Q\sqrt{p}}{\sqrt{\rho}} \, dx + \frac{1}{\pi c} \int_0^\pi \frac{a}{\sqrt{\rho p}} \, dx + r_j(a), \quad |r_j(a)| \leq \frac{C}{j}
\]
for some \(C(p, \rho, \|a\|_{H^1}) > 0\). Formula (66) follows for \(a(x) = -\mu \Pi V f'(v(\mu, w) + w)(x)\) and some algebra.

By [25, Theorem 7, p. 43], the eigenfunctions of (81) form an orthonormal basis for \(L^2\). Applying the Liouville change of variable (79) in the integrals, the eigenfunctions \(\varphi_j(a)\) of (80) form an orthonormal basis for \(L^2\) with respect to the scalar product \((\cdot, \cdot)_{L^2}\).

Finally, since \(\varphi_j := \varphi_j(a)\) solves
\[
-(p\varphi_j')' + (K\rho + a)\varphi_j = (\lambda_j(a) + K)\rho\varphi_j,
\]
multiplying by \(\varphi_i\) and integrating by parts gives
\[
(\varphi_j, \varphi_i)_{\mu, w} = \delta_{i,j}(\lambda_j(a) + K),
\]
and (68) follows (note that \(\lambda_j(a) + K > 0\) for all \(j\) and for \(K\) large enough). \(\square\)
Proof of Lemma 5. Let $a, b \in H^1(0, \pi)$ and consider $\alpha := \frac{c^2 a(\psi)}{\rho(\psi)}$, $\beta := \frac{c^2 b(\psi)}{\rho(\psi)}$ constructed as above via the Liouville change of variable (79). By [25, p. 34], for every $j$

$$|\lambda_j(a) - \lambda_j(b)| = \frac{1}{c^2} |H_j(\alpha) - H_j(\beta)| \leq \frac{1}{c^2} \|\alpha - \beta\|_{\infty} \leq K\|a - b\|_{H^1},$$

and (28) follows by the mean value theorem because $\mu \Pi f(\nu(\mu) + w)$ has bounded derivatives on bounded sets. \qed

Proof of (58). By the asymptotic formula (73)

$$\min_{j \geq 1} \left| \omega_{j+1}(\mu, w) - \omega_j(\mu, w) \right| \geq \frac{1}{c} - \frac{2K}{j} > \frac{1}{2c}$$

if $j > 4Kc$, uniformly in $\mu \in [\mu_1, \mu_2]$, $w \in B_R$. For $1 \leq j \leq 4Kc$ the minimum

$$m_j := \min_{(\mu, w) \in [\mu_1, \mu_2] \times B_R} \left| \omega_{j+1}(\mu, w) - \omega_j(\mu, w) \right|$$

is attained because $a \mapsto \lambda_j(a)$ is a compact function on $H^1$ by the compact embedding $H^1(0, \pi) \hookrightarrow L^\infty(0, \pi)$ and by (83) (see also [25, Theorem 3, pp. 31 and 34]). Each $m_j > 0$ because all the eigenvalues $\lambda_j$ are simple. \qed

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