10 Supersymmetric gauge dynamics: $\mathcal{N} = 1$

The very basic questions one should ask about a quantum field theory regard the way its symmetries are realized in its vacua, and what the dynamics around such vacua is.

- Given a QFT with gauge group $G$ and global symmetry group $G_F$, how are these realized in the vacuum?
- Which phases may enjoy such a theory?
- Are there tools to give not only qualitative but also quantitative answers to these questions?

It is very difficult to fully or even partially answer these questions, in general. However, as we will discuss in this lecture, for supersymmetric theories this is possible, sometime.

10.1 Confinement and mass gap in QCD, YM and SYM

Asymptotically free gauge theories are expected to enjoy many interesting and fascinating phenomena at low energy, like confinement, the generation of mass gap, chiral symmetry breaking etc... However, such phenomena may be realized very differently, for different theories. Below we want to consider three specific theories, namely QCD, YM and SYM, which are all UV-free and all said to be confining, and show how different the IR dynamics of these theories actually is.

- QCD, the theory of strong interactions.

  At high energy QCD is a weakly coupled theory, a $SU(3)$ gauge theory of weakly interacting quarks and gluons. It grows, through renormalization effects, to become strong in low energy processes. So strong so to bind quarks into nucleons. The strong coupling scale of QCD is

  \[ \Lambda_{QCD} \sim 300 \text{ MeV} . \]  

Note that as compared to protons and neutrons, constituent quarks are relatively light (the $u$ and $d$ quarks are order of a few MeV; the $s$ quark is order 100 MeV). Most of the mass of nucleons comes from quark kinetic energy and the interactions binding quarks together.
The reason why we cannot see free quarks, we usually say, is confinement: quarks are bound into nucleons and cannot escape. In fact, this statement is not completely correct: if we send an electron deep into a proton, we can make the quark escape! The processes is summarized in Figure 10.1.

Figure 10.1: Charge screening: confinement in QCD.

If the electron is energetic enough, a large amount of energy, in the form of chromoelectric field, appears in the region between the escaping quark and the rest of the proton. When the field becomes strong enough, of order \( \Lambda_{\text{QCD}}^4 \sim (300 \text{ MeV})^4 \), flux lines can break and produce \( q - \bar{q} \) pairs (this is a familiar phenomenon also in electromagnetism: electric fields beyond a certain magnitude cannot survive; strong fields with energy density bigger that \( m_e^4 \sim (1 \text{ Mev})^4 \) decay by producing \( e^+ - e^- \) pairs). The \( \bar{q} \) quark binds to the escaping quark while the \( q \) quark binds to the other two quarks in the proton. Therefore, the original quark does escape, the force between it and the remaining proton constituent drops to zero. Just, the escaping quark is not alone, it is bound into a meson. This phenomenon should be better called charge screening, rather than confinement.

Can we have confinement in a more strict sense? Suppose that quarks were much more massive, say \( m_q \sim 1 \text{ TeV} \). Now proton mass would be order the TeV. The dynamics drastically changes, now. Repeating the previous experiment, when the chromoelectric field becomes order \( \Lambda_{\text{QCD}}^4 \), there is not enough energy now to produce \( q - \bar{q} \) pairs. The force between the escaping quark and the proton goes to a constant: a string, or a tube of chromoelectric flux of thickness \( \sim \Lambda_{\text{QCD}}^{-1} \) and tension (energy per unit length) of order \( \Lambda_{\text{QCD}}^2 \) connects the two, see Figure 10.2. Not only the quark is confined, it is the flux itself which is confined. This is certainly a more precise definition of confinement: it holds regardless of quarks, in the sense that it holds also in the limit \( m_q \to \infty \), namely when the quarks disappear from the spectrum. It is a property of the pure glue. In summary, strict confinement would be a property of QCD only if the quarks were very massive, more precisely in the limit \( F/N << 1 \), where \( F \) is the number of light quarks and \( N \) the number of colors. Real-life quarks are light enough to let the chromoelectric flux tube break. Hence, actual QCD does
not confine in the strict sense.

Let us discuss the structure of QCD vacua in more detail, looking at how the global symmetry group of QCD is realized in the vacuum. In what follows, we consider only the three light quarks, $u, d$ and $s$ and forget the other ones (which are dynamically less important). So we have $F = 3$ flavors. Moreover, we will first put ourselves in the limit where the light quarks are massless, which is approximately true for $u, d$ and $s$ quarks constituting ordinary matter (protons and neutrons). Only later we will consider the effect of the small quark masses. In this massless limit the QCD Lagrangian reads

$$\mathcal{L} = -\frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \sum_i \bar{q}_L^i \gamma^\mu \gamma^5 q_L^i + \sum_i \bar{q}_R^i \gamma^\mu \gamma^5 q_R^i, \quad i = 1, 2, 3.$$  \hspace{1cm} \text{(10.2)}

Quark quantum numbers under the global symmetry group are

<table>
<thead>
<tr>
<th>SU(3)$_L$</th>
<th>SU(3)$_R$</th>
<th>U(1)$_A$</th>
<th>U(1)$_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>q$_L$</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>q$_R$</td>
<td>1</td>
<td>$\bar{3}$</td>
<td>1</td>
</tr>
</tbody>
</table>

(10.3)

As well known there is an axial anomaly, in the sense that the $U(1)_A$ symmetry is broken to $\mathbb{Z}_2$ at the quantum level. Therefore, the continuous global symmetries at the quantum level are just

$$G_r = SU(3)_L \times SU(3)_R \times U(1)_B.$$  \hspace{1cm} \text{(10.4)}

Experimental and theoretical considerations plus several numerical simulations on the lattice lead to a definite picture of the realization of the global symmetry, as well as the $SU(3)$ gauge symmetry, at low energy. As we have already discussed, the theory undergoes confinement (or better charge screening) and quarks and gluons are bounded into color singlet states. What about the global symmetry group? It is believed that at low energy only a subgroup survives

$$SU(3)_D \times U(1)_B.$$  \hspace{1cm} \text{(10.5)}
under which hadrons are classified: $SU(3)_D$ gives the flavor quantum numbers and $U(1)_B$ is the baryon charge. The remaining generators must be broken, somehow. The intuitive picture is as follows. Due to confinement, at strong coupling quarks and anti-quarks are bound into pairs, and the vacuum is filled by a condensate of these color singlet quark bilinears

$$
\langle q_L^i q_R^j \rangle = \Delta \delta^{ij}, \quad (10.6)
$$

where $\Delta \sim \Lambda^3_{\text{QCD}}$. This condensate is invariant under a diagonal $SU(3)$ subgroup of the original $SU(3)_R \times SU(3)_L$ group and is then responsible for the spontaneous breaking of the chiral symmetry of the original symmetry group $G_F$

$$
SU(3)_L \times SU(3)_R \times U(1)_B \rightarrow SU(3)_D \times U(1)_B . \quad (10.7)
$$

Eight global symmetries are broken by the quark condensates and hence we would expect eight Goldstone bosons. The latter are indeed observed experimentally, and correspond to the eight pseudoscalar mesons, the pions, $\pi^0, K^0, \bar{K}^0, \eta$.

$$
\begin{align*}
\pi^+ &= u\bar{d} , & \pi^- &= d\bar{u} , & \pi^0 &= d\bar{d} - u\bar{u} , & \eta &= u\bar{u} + d\bar{d} - 2s\bar{s} \\
K^0 &= s\bar{d} , & K^- &= \bar{u}s , & \bar{K}^0 &= s\bar{d} , & K^+ &= s\bar{u} \\
\end{align*} \quad (10.8)
$$

Let us first notice that if $U(1)_A$ were not anomalous, we would have had a ninth meson, the so-called $\eta'$ meson, which would have corresponded to a shift in the phase of the condensate (10.6). The condensate breaks spontaneously the $\mathbb{Z}_{2F}$ symmetry down to $\mathbb{Z}_2$, but this does not give rise to any massless particle. The $\eta'$ has a periodic potential with $F$ minima, each of them being $\mathbb{Z}_2$ invariant, and related one another by $\mathbb{Z}_F$ rotations. These minima are not isolated, though, since they are connected via $SU(F)_L \times SU(F)_R$ rotations. This means that there is a moduli space of vacua. Via a $SU(3)$ rotation acting separately on $q_L$ and $q_R$, the condensate (10.6) can be put in the form

$$
\langle q_L^i q_R^j \rangle = \Delta U^{ij} \quad (10.9)
$$

where $U^{ij}$ is a $SU(3)$ matrix on which a $SU(3)_L \times SU(3)_R$ rotation acts as

$$
U \rightarrow A_L^j U A_R^i , \quad (10.10)
$$

which shows there exists a $SU(3)_D$ rotation ($A_L = A_R$) under which the matrix $U$ is invariant. So the moduli space of vacua is a $SU(3)$ manifold.

The quantum fluctuations of the entries of this matrix hence represent the massless excitations around the vacua of massless QCD. An effective Lagrangian for such
excitations (the pions) can be written in terms of $U(x)$ and its derivatives. This Lagrangian should be invariant under the full global symmetry group $G_F$, hence non-derivative terms are not allowed (the only $G_F$-invariant function would be $U^\dagger U = 1$), which is simply saying that the pions are massless in the massless QCD limit we are considering. The structure of the effective Lagrangian hence reads

$$L_{eff} = f_\pi^2 \left( \partial_\mu U^\dagger \partial^\mu U \right) + \kappa \partial_\mu U^\dagger \partial^\mu U \partial_\nu U^\dagger \partial^\nu U + \ldots,$$  \hfill (10.11)

where $\kappa \sim 1/M^2$, with $M$ being some intrinsic mass scale of the theory, and traces on flavor indices are understood. At low momenta only the first term contributes and we then get a definite prediction for mesons scattering amplitudes, in terms of a single parameter $f_\pi$.

Now, quarks are not exactly massless and therefore the above picture is only approximate. In reality, the $SU(3)_L \times SU(3)_R$ symmetry is only approximate since quark masses correspond to (weak) $G_F$-breaking terms. This has the effect to make the pions be only pseudo-Goldstone bosons. Hence, one would expect them to be massive, though pretty light, and this is indeed what we observe in Nature.

In principle, one should have gotten the chiral Lagrangian (10.11) from the UV Lagrangian (10.2) by integrating out high momentum modes. This is difficult (next-to-impossible, in fact). However, we know in advance the expression (10.11) to be right, since that is the most general Lagrangian one can write describing pion dynamics and respecting the original symmetries of the problem. Combining the expression (10.11) with weak $G_F$-breaking terms induced by actual quark masses, one gets a Lagrangian which, experimentally, does a good job.

Summarizing, we see that combining symmetry arguments, lattice simulations, experimental observations and some physical reasoning, we can reach a rather reasonable understanding of the low energy dynamics of QCD. This is all very nice but one would like to gain, possibly, a theoretical (i.e. more microscopic) understanding of QCD phenomena. As of today, this is still an open question for QCD. And, more generally, it is so for any generic gauge theory. As we will later see, supersymmetry lets one have more analytical tools to answer this kind of questions, having sometime the possibility to derive strong coupling phenomena like confinement, chiral symmetry breaking and the generation of a mass gap, from first principles.

• Yang-Mills (YM) theory, gauge interactions without matter fields.

Let us focus, for definiteness, on YM theory with gauge group $SU(N)$. This is
again a UV-free theory, the one-loop $\beta$-function being

$$\beta_g = \frac{g^3}{16\pi^2} \left( -\frac{11}{3} N \right). \quad (10.12)$$

There are two claims about this theory, coming from a combination of experimental and theoretical reasoning, analytic and lattice calculations.

1. The theory has a mass gap, i.e. there are no massless fields in the spectrum. Rather, there is a discrete set of states with masses of order $\Lambda$, the scale where the one-loop gauge coupling diverges (higher loop and non-perturbative effects do not change the actual value of $\Lambda$ in any sensible way)

$$\Lambda = \mu e^{-\frac{8\pi^2}{\sigma^2(\nu)b_1}} \text{ where } b_1 = \frac{11}{3} N. \quad (10.13)$$

The low energy spectrum consists of glueballs. These are sort of gluons bound states which however do not consist of a fixed number of gluons (gluon number is not a conserved quantum number in strong interactions), but rather of a shifting mass of chromoelectric flux lines. Unlike gluons, for which a mass term is forbidden (because they have only two polarizations), glueballs include scalars and vectors with three polarizations (as well as higher spin particle states), for which a mass term is allowed. Such mass should clearly be of order of the dynamical scale, $m \sim \Lambda$, so not to contradict perturbation theory.

The low energy spectrum is very different from QCD. In QCD there is a mass gap just because quarks are massive. If $u, d$ and $s$ quarks were massless, we would not have had a mass gap in QCD since pions would have been exact Goldstone bosons and hence massless. Here instead there is a genuine mass gap, see Figure 10.3.

2. The theory undergoes confinement (now in the strict sense). The chromoelectric flux is confined, it cannot spread out in space over regions larger than about $\Lambda^{-1}$ in radius. How can we see confinement, namely the presence of strings which contain the chromoelectric flux? Let us add some heavy quarks to the theory and let us see whether these quarks are confined, as it was the case for very massive QCD. The Lagrangian would read

$$\mathcal{L} = -\frac{1}{4} \text{Tr} F^{\mu\nu} F_{\mu\nu} + i \bar{\psi} D\psi - M \bar{\psi} \psi \quad , \quad M >> \Lambda. \quad (10.14)$$

In the limit $M \to \infty$ the test particles become chromoelectrostatic sources, and play no role in the dynamics.
Figure 10.3: The spectrum of pure YM theory. Classically the theory is a theory of weakly coupled massless gluons. At the quantum level, a mass gap is generated, and the physical spectrum consists of a discrete set of glueball states, whose masses are order $\Lambda$.

If confinement occurs, we would expect a linear potential between the two quarks. Indeed, in an unconfined theory, the electric flux is uniformly distributed over a sphere surrounding a charge, and falls-off as $1/r^2$. In a confining theory with flux tubes, the flux tube has a fixed cross-sectional area $\sim \Lambda^{-2}$. Thus, for any sphere of radius $r \gg R \equiv \Lambda^{-1}$, the flux is zero except in a region of area $\Lambda^{-2}$, see Figure 10.4.

Hence, the electric field in that region has a magnitude which is $r$-independent, which implies that the force it generates on a test charge is also $r$-independent, and so the potential $V$ between charges would grow linearly in $r$. The force goes to a constant, it never drops to zero, see Figure 10.5 (that this is the behavior of YM theory it is something which has not been proven analytically, but via numerical lattice simulations).
Let us take a closer look to the potential, which generically reads

$$V(r) = T_r r.$$  \hspace{1cm} (10.15)

The proportionality coefficient has dimension of an energy per unit length, and it is the so-called string tension. On general ground, one would expect the string tension to depend in some way on the gauge group representation the test charges transform. This is pretty obvious since, e.g., for the singlet representation $T_r$ is clearly zero, while for actual quarks, which transform in the fundamental representation, it is not. In fact, as we are going to show below, the string tension does not depend on the representation itself, but actually on what is called the $N$-ality of the representation.

Let us consider a (gauge) group $G$. Its center, $C_G$ is defined as the part of $G$ which commutes with all generators. For $G = SU(N)$, we have that

$$C_G = \left\{ U_\alpha^\beta = e^{2\pi i k/N} \delta_\alpha^\beta ; \ k = 0, 1, \ldots, N - 1 \mod N \right\},$$ \hspace{1cm} (10.16)

where $\alpha$ an index in the fundamental and $\bar{\beta}$ in the anti-fundamental of the gauge group. Hence, in this case, $C_G = \mathbb{Z}_N$. Let us now consider some representation $R$. An element $\rho$ of this representation is labeled by $n$ upper indices $\alpha_i$ and $\bar{n}$ lower indices $\bar{\beta}_i$, each upper index transforming in the fundamental and each lower index transforming in the anti-fundamental representation. If one acts with the center of the group on $\rho$ one gets

$$C_G : \rho \to e^{2\pi i (n-\bar{n})/N} \rho.$$ \hspace{1cm} (10.17)

The coefficient $n - \bar{n}$ is called the $N$-ality of the representation $\rho$. If $\rho$ has $N$-ality $p$, then the complex conjugate representation $\rho^C$ has $N$-ality $N - p$ (from eq. (10.17) it follows that the $N$-ality is defined modulo $N$). For instance, while the
adjoint representation and the trivial representation have \( p = 0 \), the fundamental representation has \( p = 1 \) and the anti-fundamental has \( p = N - 1 \).

Clearly, representations break into equivalence classes under the center of the gauge group. It turns out that the string tension \( T_R \) is not a function of the representation but actually of the N-ality (the basic reason for this is that gluon number is not a conserved quantity in YM theories, while N-ality is). Let us consider heavy test particles transforming either in the anti-symmetric representation or in the symmetric representation of the gauge group, \( \psi_S \) and \( \psi_A \), respectively. Each of them will have its own string tension, \( T_S \) and \( T_A \) (but same N-ality, \( p = 2 \)).

Suppose that \( T_S > T_A \). Since gluon number is not a conserved quantity, we can add a gluon \( A_\mu \) coming from the chromoelectric flux tube next to \( \psi_S \). The charge of the bound state \( \psi_S A_\mu \) is \( \text{Adj} \otimes \text{Symmetric} = \oplus \text{Representations} \), where all representations entering the sum have the same N-ality (the same as the symmetric representation, in fact, since the N-ality is an additive quantity, and that of the adjoint representation is zero). For \( G = SU(3) \) we have

\[
6 \otimes 8 = 3 + 6 + 15 + 24 ,
\]

(10.18)

where the first representation on the r.h.s. is the anti-symmetric representation. Since we have assumed that \( T_S > T_A \) it is energetically favored to pop a gluon out of the vacuum and put it near to \( \psi_S \) (and another one near to \( \bar{\psi}_S \)) since this has an energy cost (of order \( \Lambda \)) which is lower than the energy gain, proportional to \((T_S - T_A)r\), which for sufficiently large \( r \) always wins. In other words, in YM theory the representation of a chromoelectric source is not a conserved quantum number; only the N-ality is. Therefore, for all representations with the same N-ality, there is only one stable configuration of strings, the one with lowest tension, as shown in Figure 10.6. In summary, the tension of stable flux tubes are labeled by \( p \), the N-ality, not by \( R \), the representation, as anticipated.

Notice that charge conjugation symmetry ensures that \( T_p = T_{N-p} \). Therefore, there are order \( N/2 \) stable flux tube configurations for \( SU(N) \). For \( SU(3) \) there is only one single confining string, that with N-ality \( p = 1 \), since \( T_0 = 0 \) and \( T_2 = T_1 \). Multiple flux tubes only arise for larger gauge groups (for instance, for \( G = SU(4) \) we have two string tensions, with N-ality \( p = 1 \) and \( p = 2 \), respectively).

All what we said let us also understand how to classify gauge singlets bound states. While gluons are not confined by flux tubes, since \( T_{\text{Adj}} = T_0 = 0 \), any heavy quark \( \psi \) with N-ality \( \neq 0 \) will experience a linear potential and a constant force
Figure 10.6: The string tensions corresponding to the antisymmetric and symmetric representations. All strings have the same N-ality, $p = 2$. The flux tube in the symmetric representation decays into that of the anti-symmetric one (which has lower energy) by popping-up a gluon out of the vacuum.

which will confine it to an antiquark $\bar{\psi}$ (these are the mesons) or, more generally, to some combination of quarks and anti-quarks with opposite N-ality. For instance, such combination can be made of $N - 1$ quarks and the bound state is called a baryon.

In passing, let us finally notice that fully analogous statements can be made without introducing heavy quarks, but rather computing Wilson loops (in different representations). In a confining theory, the Wilson loop should follow the area law

$$W_C = T_R e^{-AC}, \quad (10.19)$$

while for unconfined theories (including those enjoying charge screening!) it follows the perimeter law.

• SYM, supersymmetric gauge interactions without matter fields.

We will study this theory in detail later. Here, we would just like to emphasize similarities and differences with respect to pure YM theories and QCD.

Similarly to YM, SYM enjoys strict confinement, a mass gap and no pions. Similarly to QCD, it has a sort of chiral symmetry breaking and an anomaly, which makes the corresponding $\eta'$-like particle being massive. Finally, it differs from both since it has multiple isolated vacua.
Let us recall the structure of the (on-shell) SYM Lagrangian

\[ L_{\text{SYM}} = -\text{Tr} \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\lambda} \gamma^\mu \partial_\mu \lambda \right] \]  \hspace{1cm} (10.20)

First notice that gauginos do not break flux tubes since they transform in the adjoint representation, which is in the same N-ality class of the singlet representation. So gauginos behave very differently from QCD quarks, in this respect. Basically, the presence of these fermion fields does not change the confining behavior of pure YM glue, since gauginos cannot break flux tubes. That is why SYM enjoys strict confinement, differently from QCD. On the other hand, the $U(1)_R$ symmetry resembles the axial symmetry of QCD, since it is anomalous and it is broken to $\mathbb{Z}_{2N}$ at the quantum level (recall gauginos have R-charge equal to one). Finally, also SYM enjoys chiral symmetry breaking, in the sense that gaugino bilinears acquire a non-vanishing VEV in the vacuum. More precisely, we have

\[ \langle \lambda \lambda \rangle \sim \Lambda^3 e^{2\pi ik/N}, \quad k = 0, 1, \ldots, N - 1, \]  \hspace{1cm} (10.21)

which breaks $\mathbb{Z}_{2N} \rightarrow \mathbb{Z}_2$. Hence there are $N$ isolated vacua, each of them $\mathbb{Z}_2$ symmetric, related by $\mathbb{Z}_N$ rotations, as shown in Figure 10.7.

![Figure 10.7: The vacuum structure of pure $\mathcal{N} = 1$ SYM. The $N$ vacua are isolated, and related by $\mathbb{Z}_N$ rotations.](image)

The $\eta'$ is the phase of the condensate (10.21) (similarly to QCD), but the vacua are isolated, so there is a dynamical mass gap (unlike QCD and like YM).

### 10.2 Phases of gauge theories: examples

We would like now to consider, in more general terms, which kind of phases a generic gauge theory can enjoy. Roughly, there are basically three different such phases
• Higgs phase: the gauge group $G$ is spontaneously broken, all vector bosons obtain a mass.

• Coulomb phase: vector bosons remain massless and mediate $1/r$ long range interactions (here we are not making any distinction between proper Coulomb phase and free phase, see below).

• Wilson or confining phase: color sources, like quarks, gluons, etc..., are bound into color singlets (here we are using the word confinement in its weakest sense, i.e., we are not distinguishing between charge screening and strict confinement).

Notice that the Coulomb phase is not specific to abelian gauge theories, as QED. A non-abelian gauge theory with enough matter fermion content may become IR-free, giving a long-range potential between color charges $V(r) \sim a(r) \times 1/r$, with $a(r)$ a coefficient decreasing logarithmically with $r$.

There can of course be intermediate situations, where for instance the original gauge group is Higgsed down to a subgroup $H$, which then confines, or is in a Coulomb phase (this is what happens in the SM), etc... In these cases the phase of the gauge theory is defined by what happens to $H$ in the vacua, regardless of the fate of the original gauge group $G$.

Below we consider two examples which will clarify the meaning of some of above statements, but also point out some subtleties one could encounter when dealing with concrete models.

### 10.2.1 Coulomb phase and free phase

Let us consider SQED. The scalar potential reads

\[
V = m^2|\phi_-|^2 + m^2|\phi_+|^2 + \frac{1}{2} e^2 (|\phi_+|^2 - |\phi_-|^2)^2 ,
\]

where $\phi_-$ and $\phi_+$ are the scalar fields belonging to the two chiral superfields $\Phi_-$ and $\Phi_+$ with electric charge $\pm 1$, and a superpotential mass term $W = m\Phi_-\Phi_+$ has been allowed. Let us consider massive and massless cases separately.

• $m \neq 0$. In this case the vacuum is at $\langle \phi_- \rangle = \langle \phi_+ \rangle = 0$. Heavy static probe charges would experience a potential

\[
V \sim \frac{\alpha(r)}{r} , \quad \alpha(r) \sim \frac{1}{\log r} .
\]
However, the logarithmic fall-off is frozen at distance $r = m^{-1}$: for larger distances $\alpha$ stops running. Hence, the asymptotic potential reads

$$V(r) \sim \frac{\alpha_*}{r} \quad , \quad \alpha_* = \alpha(r = m^{-1}) ,$$

(10.24)

which simply says that massive SQED is in a Coulomb phase.

- $m = 0$. In this case the potential gets contributions from D-terms only. Now there are more vacua, actually a moduli space of vacua. Besides the origin of field space, also any $\langle \phi_- \rangle = \langle \phi_+ \rangle \neq 0$ satisfies the D-equations. One can parameterize the supersymmetric vacua in terms of the gauge invariant combination $u = \langle \phi_- \phi_+ \rangle$. We have then two options. When $u \neq 0$ we are in a Higgs phase, the gauge group $U(1)$ is broken and the photon becomes massive (the theory is described by a massive vector multiplet and a massless chiral multiplet). When instead $u = 0$ the gauge group remains unbroken. Still, we are in a different phase with respect to the massive case. The basic difference is that the coupling $\alpha(r)$ does not stop running, now, since $m = 0$, and hence it ends-up vanishing at large enough distances. In other words, the potential again reads

$$V(r) \sim \frac{\alpha(r)}{r} \quad , \quad \alpha(r) \sim \frac{1}{\log r}$$

(10.25)

but now $\alpha = 0$ for $r \to \infty$. This is not really a Coulomb phase but actually what is called a free phase. At low energy (large enough distances) the theory is a theory of free massless particles.

Let us emphasize again that both the Coulomb phase and the free phase are not specific to abelian gauge theories, and can be enjoyed even by non-abelian (IR-free) theories. We will see examples of this phenomenon later in this lecture.

### 10.2.2 Continuously connected phases

Sometime there is no gauge-invariant distinction between phases. Let us show this non-trivial fact with a simple, though instructive, example.

Let us consider a $SU(2)$ gauge theory with a $SU(2)$ scalar doublet $\phi$ (a Higgs field), a $SU(2)$ singlet $e_R$ and a $SU(2)$ doublet $L = (\nu_L, e_L)$, with interaction Lagrangian

$$\mathcal{L}_{\text{int}} = a\bar{\nu}_L \phi e_R + h.c.$$

(10.26)
This is nothing but a one-family EW theory model. As it happens in standard EW theory, this theory can be realized in the Higgs phase, where the field $\phi$ gets a non-vanishing VEV

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \quad (10.27)$$

In this phase all three gauge bosons get a mass, the neutrino $\nu_L$ remains massless while the electron gets a mass $m_e = av/v_2$.

Suppose instead that the theory were realized in a different phase, a confinement phase. In such phase one would not observe massless gauge bosons (as above) while fermions and Higgs bosons would bind into $SU(2)$ singlet combinations

$$E_L = \phi^\dagger L, \quad N_L = \epsilon_{ab} \phi_a L_b, \quad e_R \quad (10.28)$$

in terms of which the interaction Lagrangian becomes

$$\mathcal{L}_{\text{int}} = m \bar{E}_L e_R + h.c. , \quad (10.29)$$

where $m \sim a \Lambda$. So we see that $E_L$ and $e_R$ pair-up and become massive while $N_L$ remains massless. The spectrum in this phase is the same as that of the Higgs phase: there is no gauge-invariant distinction between the two phases! Obviously, they differ quantitatively: for instance, $E_L$ is a composite field and its pair production would be suppressed by a form factor which is not observed. Still, we need experiments to discern between the two phases and understand which one is actually realized in Nature.

The general lesson we want to convey here is that by adjusting some parameter of a gauge theory, sometime one can move continuously from one type of phase to another. In this non-abelian example, there is no invariant distinction between Higgs and Wilson phase. We will encounter many such situations when studying supersymmetric gauge theory dynamics.

### 10.3 N=1 SQCD: perturbative analysis

In what follows we will consider SQCD and its quantum behavior (including non-perturbative effects) and try to answer the basic questions about its dynamical properties in the most analytical possible way. Let us first summarize what we already know about SQCD, which is its classical and quantum, though perturbative only, behavior. SQCD is a renormalizable supersymmetric gauge theory with gauge
group $SU(N)$, $F$ flavors $(Q, \tilde{Q})$ and no superpotential. Interaction terms are present and come from D-terms. The group of continuous global symmetries at the quantum level is $G_v = SU(F)_L \times SU(F)_R \times U(1)_B \times U(1)_R$ with charge assignment

\[
\begin{array}{cccc}
SU(F)_L & SU(F)_R & U(1)_B & U(1)_R \\
Q^i_a & F & \bullet & 1 & \frac{F-N}{F} \\
\tilde{Q}^b_j & \bullet & F & -1 \frac{F-N}{F} \\
\lambda & \bullet & \bullet & 0 & 1 \\
\end{array}
\]

As already emphasized, for pure SYM the R-symmetry is anomalous, and only a $\mathbb{Z}_{2N}$ subgroup of $U(1)_R$ survives at the quantum level.

What do we know, already, about the quantum properties of SQCD? We know there is a huge moduli space of supersymmetric vacua, described by the D-term equations

\[
D^A = Q^i_b (T^A)^{bc} Q^j_c - \tilde{Q}^b_i (T^A)^{bc} \tilde{Q}^i_c = 0 ,
\]

where $A = 1, 2, \ldots, N^2 - 1$ is an index in the adjoint representation of $SU(N)$.

Up to flavor and global gauge rotations, a solution of the above equations can be found for both $F < N$ and $F \geq N$. For $F < N$ one can show that on the moduli space (10.30) the matrices $Q$ and $\tilde{Q}$ can be put, at most, in the following form

\[
Q = \begin{pmatrix}
v_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & v_2 & \ldots & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & 0 & \ldots & 0 \\
0 & 0 & \ldots & v_F & 0 & \ldots & 0 \\
\end{pmatrix} = \tilde{Q}^T
\]

Hence, at a generic point of the moduli space the gauge group is broken to $SU(N - F)$. The (classical) moduli space can be parameterized in terms of mesons fields

\[
M^i_j = Q^i_a \tilde{Q}^a_j
\]

without the need of considering any classical constraint since the meson matrix has maximal rank for $F < N$.

For $F \geq N$ the matrices $Q$ and $\tilde{Q}$ can also be brought to a diagonal form on the
moduli space

\[
\begin{pmatrix}
  v_1 & 0 & \ldots & 0 \\
  0 & v_2 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & v_N \\
  0 & 0 & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & 0 \\
\end{pmatrix}
, \quad
\begin{pmatrix}
  \tilde{v}_1 & 0 & \ldots & 0 \\
  0 & \tilde{v}_2 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & \tilde{v}_N \\
  0 & 0 & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(10.33)

where \(|v_i|^2 - |\tilde{v}_i|^2 = a|, with a a \(i\)-independent number. At a generic point of the
moduli space the gauge group is now completely broken. The moduli space is effi-
ciently described in terms of mesons and baryons, now, modulo classical constraints
between them, which are now non-trivial and increase with \(F\), at fixed \(N\). The
mesons are again defined as in eq. (10.32) but the meson matrix does not have max-
imal rank anymore. Baryons are gauge invariant single trace operators made out of
\(N\) fields \(Q\) respectively \(N\) fields \(\tilde{Q}\), with fully anti-symmetrized indices and read

\[
B_{i_1 \ldots i_{F-N}} = \epsilon_{i_1 i_2 \ldots i_{F-N} j_1 \ldots j_N} \epsilon^{\alpha_1 \alpha_2 \ldots \alpha_N} Q_{\alpha_1} Q_{\alpha_2} \cdots Q_{\alpha_N},
\]

\[
\tilde{B}^{i_1 \ldots i_{F-N}} = \epsilon^{i_1 i_2 \ldots i_{F-N} j_1 \ldots j_N} \epsilon_{\alpha_1 \alpha_2 \ldots \alpha_N} \tilde{Q}^{\alpha_1} \tilde{Q}^{\alpha_2} \cdots \tilde{Q}^{\alpha_N},
\]

(10.34)

where \(a_i\) are gauge indices and \(i_l, j_l\) are flavor indices.

As far as quantum correction are concerned, we know the exact (perturbative)
expression for the gauge coupling which, in the holomorphic scheme, reads

\[
\tau = \frac{\theta_{\text{YM}}}{2\pi} + i \frac{4\pi}{g^2(\mu)} = \frac{b_1}{2\pi} \log \frac{\Lambda}{\mu}, \quad b_1 = 3N - F \quad \text{and} \quad \Lambda = \mu e^{2\pi i \tau/b_1}.
\]

(10.35)

### 10.4 N=1 SQCD: non-perturbative dynamics

Our goal is to understand how this whole picture is modified once non-perturbative
corrections are taken into account. Notice that for a UV-free theory, as SQCD is
for \(F < 3N\), quantum corrections are expected to modify the perturbative analysis
only near the origin of field space. Indeed, for large value of scalar field VEVs, the
gauge group gets broken (and the gauge coupling hence stops running) for small
values of the gauge coupling constant

\[
e^{-\frac{\theta_{\text{YM}}^2}{g^2(\langle Q \rangle)} + i \theta_{\text{YM}}} = \left( \frac{\Lambda}{\langle Q \rangle} \right)^{3N-F} \to 0 \quad \text{for} \quad \langle Q \rangle \to \infty.
\]

(10.36)
This implies that for large field VEVs the gauge coupling freezes at a value $g_*$ where classical analysis works properly. The smaller the field VEV the more important are quantum corrections. Hence, generically, we expect non-perturbative dynamics to modify the perturbative answer mostly near the origin of field space.

![Diagram](image.png)

Figure 10.8: The gauge coupling running of a UV-free theory. The large $\langle Q \rangle$ region is a weakly coupled region where classical analysis is correct, since the value at which the gauge coupling stops running, $g = g_*$, is small.

Not surprisingly, for any fixed value of $N$, several non-perturbative dynamical properties change with the number of flavors, $F$. Hence, in what follows, we will consider different cases separately.

### 10.4.1 Pure SYM: gaugino condensation

This case, $F = 0$, was already analyzed, at a qualitative level. Let us first recall that this is the only case in which there does not exist an anomaly-free R-symmetry. At the quantum level, only a discrete $Z_{2N}$ R-symmetry survives. Using holomorphy arguments, it is easy to see what the structure of the non-perturbative generated superpotential should be. Let us first notice that the operator $e^{2\pi i \tau/N}$ has R-charge $R = 2$. In other words, due to the transformation properties of $\theta_{\text{YM}}$ under R-symmetry transformations, $\theta_{\text{YM}} \rightarrow \theta_{\text{YM}} + 2N\alpha$, we have

$$e^{2\pi i \tau/N} \rightarrow e^{2\pi i \tau/N}.$$

Because of confinement, assuming a mass gap, the effective Lagrangian should depend only on $\tau$, and hence $W_{\text{eff}}$, if any, should also depend only on $\tau$. Imposing R-symmetry, by dimensional analysis the only possible term reads

$$W_{\text{eff}} = c \mu^3 e^{2\pi i \tau/N} = c \Lambda^3.$$
where \(c\) is an undetermined coefficient (which in principle could also be zero, of course). This innocent-looking constant superpotential contribution contains one crucial physical information. Given the presence of massless strong interacting fermion fields (the gaugino) one could wonder whether in SYM theory gauginos undergo pair condensation, as it happens in QCD, where quark bilinears condense. Looking at the SYM Lagrangian

\[
\mathcal{L} = \frac{1}{32\pi} \text{Im} \left[ \int d^2 \theta \tau \text{Tr} W^\alpha W_\alpha \right],
\]

we see that \(\lambda^\alpha \lambda_\alpha\) is the scalar component of \(W^\alpha W_\alpha\) and (minus) \(F_\tau\) acts as a source for it (recall we are thinking of \(\tau\) as a spurion superfield, \(\tau = \tau + \sqrt{2} \theta \psi - \theta F_\tau\)). Therefore, in order to compute the gaugino condensate one should just differentiate the logarithm of the partition function \(Z = \int DVe^{i\int \mathcal{L}}\) with respect to \(F_\tau\). In fact, under the assumption of a mass gap, the low energy effective action depends only on \(\tau\), since gauge fields have been integrated out, and it coincides with the effective superpotential (10.38), giving for the gaugino condensate

\[
\langle \lambda \lambda \rangle = -16\pi i \frac{\partial}{\partial F_\tau} \log Z = -16\pi i \frac{\partial}{\partial F_\tau} \int d^2 \theta W_{\text{eff}}(\tau) = 16\pi i \frac{\partial}{\partial \tau} W_{\text{eff}}(\tau)
\]

where in doing the second step we have used the fact that

\[
W_{\text{eff}} = w_{\text{eff}}(\tau) + \sqrt{2} \frac{\partial W_{\text{eff}}}{\partial \tau} \theta \psi - \theta \left( \frac{\partial W_{\text{eff}}}{\partial \tau} F_\tau + \frac{1}{2} \frac{\partial^2 W_{\text{eff}}}{\partial \tau^2} \psi^2_\tau \right).
\]

The upshot is that we can compute eq. (10.40) for the effective superpotential (10.38). We get

\[
\langle \lambda \lambda \rangle = -\frac{32\pi^2}{N} c \mu^3 e^{2\pi i \alpha/N} \equiv \alpha \Lambda^3.
\]

which means that if \(c \neq 0\) indeed gauginos condense in SYM. Since gauginos have \(R = 1\), this implies that in the vacua the \(\mathbb{Z}_{2N}\) symmetry is broken to \(\mathbb{Z}_2\) and that there are in fact \(N\) distinct (and isolated, in this case) vacua. All these vacua appear explicitly in the above formula since the transformation

\[
\theta_{YM} \to \theta_{YM} + 2\pi k,
\]

which is a symmetry of the theory, sweeps out \(N\) distinct values of the gaugino condensate

\[
\langle \lambda \lambda \rangle \to e^{2\pi i \alpha} \langle \lambda \lambda \rangle, \quad \theta_{YM} \to \theta_{YM} + 2N \alpha \simeq \theta_{YM} + 2\pi k
\]
where \( k = 0, 1, \ldots, 2N - 1 \), and \( k = i \) and \( k = i + N \) give the same value of the gaugino condensate. In other words, we can label the \( N \) vacua with \( N \) distinct phases of the gaugino condensate \((0, 2\pi \frac{1}{N}, 2\pi \frac{2}{N}, \ldots, 2\pi \frac{N-1}{N})\), recall Figure 10.7.

This ends our discussion of pure SYM. It should be stressed that to have a definitive picture we should find independent ways to compute the constant \( c \) in eq. (10.38), since if it were zero, then all our conclusions would have been wrong (in particular, there would not be any gaugino condensate, and hence we would have had a unique vacuum preserving the full \( \mathbb{Z}_{2N} \) symmetry). We will come back to this important point later.

### 10.4.2 SQCD for \( F < N \): the ADS superpotential

For \( F < N \) classical analysis tells that there is a moduli space of complex dimension \( F^2 \), parameterized by meson field VEVs. The question, again, is whether an effective superpotential is generated due to strong coupling dynamics. Let use again holomorphy, and the trick of promoting coupling constants to spurion superfields. The effective superpotential could depend on meson fields and on the complexified gauge coupling, through \( \Lambda \). The quantum numbers of (well educated functions of) these two basic objects are

\[
\begin{pmatrix}
U(1)_B & U(1)_A & U(1)_R \\
\det M & 0 & 2F & 2(F - N) \\
\Lambda^{3N-F} & 0 & 2F & 0
\end{pmatrix}
\]

Both above objects are invariant under the non-abelian part of the global symmetry group (notice that \( \det M \) is the only \( SU(F)_L \times SU(F)_R \) invariant one can make out of \( M \)). From the table above it follows that the only superpotential term which can be generated should have the following form

\[
W_{\text{eff}} = c_{N,F} \left( \frac{\Lambda^{3N-F}}{\det M} \right)^{\frac{1}{N-F}},
\]

where, again, the overall constant, which generically will be some function of \( N \) and \( F \), is undetermined. That (10.44) is the only possible term can be understood as follows. The effective superpotential should have R-charge two, \( U(1)_A \) and \( U(1)_B \) charges equal to zero, and should have dimension three. The chiral superfields that might contribute to it are \( \Lambda, W^\alpha \) and the meson matrix \( M \). Provided what we just stated about non-abelian global symmetries the generic expression for \( W_{\text{eff}} \) should
be made of terms like
\[ W_{\text{eff}} \sim \Lambda^{(3N-F)n} (W^\alpha W_\alpha)^m (\text{det} M)^p, \]  
(10.45)

where \( n, m \) and \( p \) are integer numbers. The invariance under the baryonic symmetry is guaranteed by any such term. As for the other two global symmetries we get
\[
\begin{align*}
0 &= 2nF + 2pF \\
2 &= 2m + 2p(F - N)
\end{align*}
\rightarrow
\begin{align*}
n &= -p \\
p &= (m - 1)/(N - F)
\end{align*}
\]  
(10.46)

Since \( 3N - F > 0 \), in order to have a meaningful weak coupling limit, we should have \( n \geq 0 \), which implies that \( p \leq 0 \) and \( m \leq 1 \). On the other hand, we should have \( m \geq 0 \) in order for the Wilsonian action to be local (it needs to have a sensible derivative expansion), which finally implies that \( m = 0, 1 \). The contribution \( m = 1 \) and hence \( p = n = 0 \) is the tree level result (the gauge kinetic term, in fact). The contribution \( m = 0 \) which implies \( p = -1/(N - F) \) and \( n = 1/(N - F) \) is precisely the so-called Affleck-Dine-Seiberg (ADS) superpotential contribution (10.44).

In what follows we would like to analyze several properties of the ADS superpotential, possibly understanding where it comes from, physically, and eventually determine the coefficient \( c_{N,F} \).

Let us consider again the classical moduli space. At a generic point of the moduli space the \( SU(N) \) gauge group is broken to \( SU(N - F) \). Suppose for simplicity that all scalar field VEVs are equal, \( v_i = v \), recall expression (10.31). Clearly the theory behaves differently at energies higher or lower than \( v \). At energies higher than \( v \) the gauge coupling running is that of SQCD with gauge group \( SU(N) \) and \( F \) massless flavors. At energies lower than \( v \) all matter fields become massive (and should be integrated out) while the gauge group is broken down to \( SU(N - F) \). Hence the theory runs differently and, accordingly, the dynamical generated scale, \( \Lambda_L \) is also different. More precisely we have
\[
\begin{align*}
E > v & \quad \frac{4\pi}{g^2(\mu)} = \frac{3N - F}{2\pi} \log \frac{\mu}{\Lambda} \\
E < v & \quad \frac{4\pi}{g^2_L(\mu)} = \frac{3(N - F)}{2\pi} \log \frac{\mu}{\Lambda_L}.
\end{align*}
\]  
(10.47)

If supersymmetry is preserved the two above equations should match at \( E = v \). This is known as scale matching (that there are no threshold factors reflects a choice of subtraction scheme, on which threshold factors depend; this is the correct matching
in, e.g., the \( \mathcal{DR} \) scheme. Hence we get
\[
\Lambda^3(N-F) = \Lambda^{3N-F} \frac{1}{v^{3F}} = \frac{\Lambda^{3N-F}}{\det M} \quad \rightarrow \quad \Lambda^3_L = \left( \frac{\Lambda^{3N-F}}{\det M} \right)^{\frac{1}{N-F}}.
\]
(10.48)

This implies that
\[
W_{\text{eff}} = c_{N,F} \left( \frac{\Lambda^{3N-F}}{\det M} \right)^{\frac{1}{N-F}} = c_{N,F} \Lambda^3_L,
\]
(10.49)
which means that
\[
c_{N,F} = c_{N-F,0}.
\]
(10.50)

Besides getting a relation between \( c \)'s for different theories (recall these are \( (N, F) \)-dependent constants, in general), we also get from the above analysis some physical intuition for how the ADS superpotential is generated. One can think of \( W_{\text{eff}} \) being generated by gaugino condensation of the left over \( SU(N-F) \) gauge group (recall that gaugino condensation is in \textit{one-to-one correspondence} with the very existence of an effective superpotential for pure SYM theory: the two are fully equivalent statements).

Let us now start from SQCD with a given number of flavors and suppose to give a mass \( m \) to the \( F \)-th flavor. At high enough energy this does not matter much. But below the scale \( m \) the theory behaves as SQCD with \( F - 1 \) flavors. More precisely, we have
\[
E > m \quad \frac{4\pi}{g^2(\mu)} = \frac{3N-F}{2\pi} \log \frac{\mu}{\Lambda_F},
\]
\[
E < m \quad \frac{4\pi}{g_L^2(\mu)} = \frac{3N-(F-1)}{2\pi} \log \frac{\mu}{\Lambda_{L,F-1}}.
\]
(10.51)

Matching the scale at \( E = m \) we obtain the following relation between non-perturbative scales
\[
\Lambda^{3N-F+1}_{L,F-1} = m \Lambda^3_{F}^{N-F},
\]
(10.52)
which tells that the effective superpotential for SQCD with one massive flavor and \( F - 1 \) massless ones can be written in the following equivalent ways
\[
W_{\text{eff}} = c_{N,F-1} \left( \frac{\Lambda^{3N-F+1}_{L,F-1}}{\det \tilde{M}} \right)^{\frac{1}{N-F+1}} = c_{N,F-1} \left( \frac{m \Lambda^3_F}{\det \tilde{M}} \right)^{\frac{1}{N-F+1}}.
\]
(10.53)

where \( \tilde{M} \) is the meson matrix made out of \( F - 1 \) flavors. Let us check this prediction using holomorphic decoupling. The superpotential of SQCD with \( F - 1 \) massless
flavors and one massive one reads

$$W_{\text{eff}} = c_{N,F} \left( \frac{\Lambda_F^{3N-F}}{\det M} \right)^{\frac{1}{N-F}} + m Q^F \bar{Q}_F . \quad (10.54)$$

The F-term equation for $M^i_i$ for $i \neq F$ implies $M^i_i = 0$, and similarly for $M^i_j$. So the meson matrix can be put into the form

$$M = \begin{pmatrix} \hat{M} & 0 \\ 0 & t \end{pmatrix} , \quad t \equiv M^F_F . \quad (10.55)$$

The F-term equation for $t$ gives

$$0 = -\frac{c_{N,F}}{N-F} \left( \frac{\Lambda_F^{3N-F}}{\det M} \right) \left( \frac{1}{t} \right)^{\frac{1}{N-F}} + m \quad (10.56)$$

which implies

$$t = \left[ \frac{N-F}{c_{N,F}} m \left( \frac{\Lambda_F^{3N-F}}{\det M} \right) \right]^{\frac{F-N}{N-F+1}} . \quad (10.57)$$

Plugging this back into eq. (10.54) one gets

$$W_{\text{eff}} = (N - F + 1) \left( \frac{c_{N,F}}{N-F} \right)^{\frac{F-N}{N-F+1}} \left( m \frac{\Lambda_F^{3N-F}}{\det M} \right)^{\frac{1}{N-F+1}} . \quad (10.58)$$

Comparing with the expression (10.53) one finds complete agreement and, as a bonus, the following relation between coefficients

$$c_{N,F-1} = (N - F + 1) \left( \frac{c_{N,F}}{N-F} \right)^{\frac{F-N}{N-F+1}} . \quad (10.59)$$

Combining this result with the relation we found before, eq. (10.50), one concludes that all coefficients are related one another as

$$c_{N,F} = (N - F) c^{\frac{1}{N-F}} , \quad (10.60)$$

with a unique common coefficient $c$ to be determined. This result tells that if the ADS superpotential can be computed exactly for a given value of $F$ (hence fixing $c$), then we know its expression for any other value!

Let us consider the case $F = N - 1$, which is the extreme case in the window $F < N$. In this case

$$c_{N,N-1} = c . \quad (10.61)$$
Interestingly, for $F = N - 1$ the gauge group is fully broken, so there is no left-over strong IR dynamics. In other words, any term appearing in the effective action should be visible in a weak-coupling analysis. Even more interesting, the ADS superpotential for $F = N - 1$ is proportional to $\Lambda^{2N+1}$ which is nothing but how one-instanton effects contribute to gauge theory amplitudes (recall that for $F = N - 1$ $b_1 = 2N + 1$, and $e^{-S_{\text{inst}}} \sim \Lambda^{b_1}$), suggesting that in this case the ADS superpotential is generated by instantons. At weak coupling, a reliable one-instanton calculation can indeed be done and gives $c = 1$. Via eq. (10.60) this result hence fixes uniquely $c_{N,F}$ for arbitrary values of $N$ and $F$ as
\begin{equation}
    c_{N,F} = N - F,
\end{equation}
giving finally for the ADS superpotential the following exact expression
\begin{equation}
    W_{\text{ADS}} = (N - F) \left( \frac{\Lambda^{3N-F}}{\det M} \right)^{\frac{1}{2N}}.
\end{equation}
Notice that this also fixes the coefficient of the effective superpotential of pure SYM theory which is
\begin{equation}
    W_{\text{SYM}} = N \Lambda^3,
\end{equation}
implying, via eq. (10.40), that gauginos do condense!

Let us finally see how does the ADS superpotential affect the moduli space of vacua. From the expression (10.63) we can compute the potential, which is expected not to be flat anymore, since the effective superpotential $W_{\text{ADS}}$ depends on scalar fields (through the meson matrix). The potential
\begin{equation}
    V_{\text{ADS}} = \sum_i \left[ \frac{\partial W_{\text{ADS}}}{\partial Q_i} \right]^2 + \left[ \frac{\partial W_{\text{ADS}}}{\partial \tilde{Q}_i} \right]^2
\end{equation}
is minimized at infinity in field space, namely for $Q = \tilde{Q} \to \infty$, where it reaches zero, see Figure 10.9. This can be easily seen noticing that, qualitatively, $\det M \sim M^F$, which implies that $V_{\text{ADS}} \sim |M|^{-\frac{2N}{F}}$, which is indeed minimized at infinity. This means that the theory does not admit any stable vacuum at finite distance in field space: the (huge) classical moduli space is completely lifted at the quantum level! This apparently strange behavior makes sense, in fact, if one thinks about it for a while. For large field VEVs, eventually for $v \to \infty$, we recover pure SYM which has indeed supersymmetric vacua (that is, zero energy states). This is part of the space of D-term solutions of SQCD; any other configuration would have higher energy.
and would hence be driven to the supersymmetric one. Let us suppose this picture were wrong and that SQCD had a similar behavior as QCD: confinement and chiral symmetry breaking. Then we would have expected a quark condensate to develop \( \langle \psi_Q \psi_{\tilde{Q}} \rangle \neq 0 \). Such condensate, differently from a gaugino condensate (which we certainly have), would break supersymmetry, since it is nothing but an F-term for the meson matrix \( M_i^j \). Hence this configuration would have \( E > 0 \) and thus any configuration with \( E = 0 \) would be preferred. The latter are all configurations like (10.31) which, by sending \( v_i \) all the way to infinity, reduce to SYM, which admits supersymmetry preserving vacua. The ADS superpotential simply shows this.

There is a caveat in all this discussion. In our analysis we have not included wave-function renormalization effects. The latter could give rise, in general, to non-canonical Kähler potential terms, which could produce wiggles or even local minima in the potential. However, at most this could give rise to metastable vacua (which our

![Figure 10.9: The runaway behavior of the quantum corrected potential of SU(N) SQCD with \( F < N \).](image)

![Figure 10.10: The effect of a non-canonical Kähler potential on the ADS potential. The picture on the right cannot hold if the assumption of mass gap for pure SYM is correct.](image)
holomorphic analysis cannot see), but it would not lift the absolute supersymmetric minima at infinity, a region where the Kähler potential is nearly canonical in the UV-variables $Q$ and $\tilde{Q}$. On the other hand, no supersymmetric minima can arise at finite distance in field space. These would correspond to singularities of the Kähler metric, implying that at those specific points in field space extra massless degrees of freedom show-up. This cannot be, if the assumption of mass gap for pure SYM (to which the theory reduces at low enough energy, at generic points in the classical moduli space) is correct.

10.4.3 Integrating in and out: the linearity principle

The superpotential of pure SYM is sometime written as

$$W_{\text{VY}} = NS \left(1 - \log \frac{S}{\Lambda^3}\right)$$

(10.66)

where $S = -\frac{1}{32\pi^2} \text{Tr} W^\alpha W_\alpha$ is the so-called glueball superfield and the subscript VY stands for Veneziano-Yankielowicz. Let us first notice that integrating $S$ out (recall we are supposing pure SYM has a mass gap) we get

$$\frac{\partial W_{\text{VY}}}{\partial S} = N \left(1 - \log \frac{S}{\Lambda^3}\right) + NS \left(-\frac{1}{S}\right) = 0 ,$$

(10.67)

which implies

$$\langle S \rangle = \Lambda^3 .$$

(10.68)

Plugging this back into the VY superpotential gives

$$W_{\text{VY}} = N \Lambda^3 ,$$

(10.69)

which is nothing but the effective superpotential of pure SYM we have previously derived. From this view point the two descriptions seem to be equivalent: while the above superpotential can be obtained from the VY one by integrating $S$ out, one can say that the VY superpotential is obtained from (10.69) integrating the glueball superfield $S$ in. Similarly, one can integrate in $S$ in the ADS superpotential, obtaining what is known as the TVY (Taylor-Veneziano-Yankielowicz) superpotential

$$W_{\text{TVY}} = (N - F) S \left[1 - \frac{1}{N - F} \log \left(\frac{S^{N-F} \det M}{\Lambda^{3(N-F)}}\right)\right] .$$

(10.70)

Obviously, integrating $S$ out one gets back the ADS superpotential we previously derived. And, consistently, adding mass terms for all matter fields, $\sim \text{Tr} m M$, and integrating $M$ out, one gets from the TVY superpotential the VY superpotential.
Let us try to understand better the meaning of all that (and actually the complete equivalence between these apparently different descriptions). Naively, one could imagine the VY or TVY superpotentials containing more information than the ADS superpotential, since they include one more dynamical field, the glueball superfield $S$. As we are going to discuss below, this intuition is not correct.

Let us try to be as general as possible and consider a supersymmetric gauge theory admitting also a tree-level superpotential $W_{\text{tree}}$. Given a set of chiral superfields $\Phi_i$, the generic form of such superpotential is

$$ W_{\text{tree}} = \sum_r \lambda_r X_r(\Phi_i) , \quad (10.71) $$

where $\lambda_r$ are coupling constants and $X_r$ gauge invariant combinations of the chiral superfields $\Phi_i$. In general, one would expect the non-perturbative generated superpotential $W_{\text{non-pert}}$ to be a (holomorphic) function of the couplings $\lambda_r$, the gauge invariant operators $X_r$, and of the dynamical generated scales $\Lambda_s$ (we are supposing, to be as most general as possible, the gauge group not to be simple, hence we allow for several dynamical scales). In fact, as shown by Intriligator, Leigh and Seiberg, $W_{\text{non-pert}}$ does not depend on the couplings $\lambda_r$. This fact implies that the full effective superpotential (which includes both the tree level and the non-perturbative contributions) is linear in the couplings, and hence this is sometime referred to as linearity principle. The upshot is that, in general, we have

$$ W_{\text{eff}} = \sum_r \lambda_r X_r + W_{\text{non-pert}}(X_r, \Lambda_s) . \quad (10.72) $$

Let us focus on the dependence on, say, $\lambda_1$. At low enough energy (where the superpotential piece dominates) we can integrate out the field $X_1$ by solving its F-term equation only, which, because of eq. (10.72), reads

$$ \lambda_1 = - \frac{\partial}{\partial X_1} W_{\text{non-pert}} . \quad (10.73) $$

The above equation is the same as a Legendre transform. In other words, the coupling $\lambda_r$ and the gauge invariant operator $X_r$ behave as Legendre dual variables. Solving for $X_1$ in terms of $\lambda_1$ and all other variables, and substituting in eq. (10.72), one obtains an effective superpotential with a complicated dependence on $\lambda_1$ but where $X_1$ has been integrated out. Repeating the same reasoning for all $X_r$ one can integrate out all fields and end-up with an effective superpotential written in terms
of couplings only

\[ W_{\text{eff}}(\lambda_r, \Lambda_s) = \left[ \sum_r \lambda_r X_r + W_{\text{non-pert}}(X_r, \Lambda_s) \right]_{X_r(\lambda, \Lambda)}. \]  

(10.74)

The point is that the Legendre transform is invertible. Therefore, as we can integrate out a field, we can also integrate it back in, by reversing the procedure

\[ \langle X_r \rangle = \frac{\partial}{\partial \lambda_r} W_{\text{eff}}(\lambda_r, \Lambda_s). \]  

(10.75)

The reason why the two descriptions, one in terms of the fields, one in terms of the dual couplings, are equivalent is because we have not considered D-terms. D-terms contain the dynamics (e.g. the kinetic term). Hence, if we ignore D-terms, namely if we only focus on holomorphic terms as we are doing here (which is a more and more correct thing to do the lower the energy), integrating out or in a field is an operation which does not make us lose or gain information. As far as the holomorphic part of the effective action is concerned, a field and its dual coupling are fully equivalent.

What about the dynamical scales \( \lambda_s \)? Can one introduce canonical pairs for them, too? The answer is yes, and this is where the physical equivalence between ADS and TVY superpotentials we claimed about becomes explicit. Let us consider pure SYM, for definiteness. One can write the gauge kinetic term as a contribution to the tree level superpotential in the sense of eq. (10.71)

\[ W_{\text{tree}} = \frac{\tau(\mu)}{16 \pi i} \text{Tr} W^a W_a = 3N \log \left( \frac{\Lambda}{\mu} \right) S, \]  

(10.76)

where \( S \) is a \( X \)-like field and \( 3N \log (\Lambda/\mu) \) the dual coupling. In other words, one can think of \( S \) and \( \log \Lambda \) as Legendre dual variables. From this viewpoint, the SYM superpotential (10.64) is an expression of the type (10.74), where the field \( S \) has been integrated out and the dependence on the dual coupling is hence non-linear. Indeed (10.64) can be re-written as

\[ W_{\text{SYM}} = N \Lambda^3 = N \mu^3 e^{\frac{1}{3} 3N \log \frac{\Lambda}{\mu}}, \]  

(10.77)

where the coupling appears non-linearly. Using now eq. (10.75) applied to this dual pair, one gets

\[ \langle S \rangle = \frac{1}{3N} \Lambda \frac{\partial}{\partial \Lambda} W_{\text{eff}} = \Lambda^3. \]  

(10.78)

Therefore

\[ W_{\text{non-pert}}(S) = W_{\text{eff}} - W_{\text{tree}} = NS - 3N \log \left( \frac{\Lambda}{\mu} \right) S = NS - NS \log \frac{S}{\mu^3}, \]  

(10.79)
which is correctly expressed, according to the linearity principle, in terms of $S$ only, and not the coupling, $\log \Lambda$. We can now add the two contributions, the one above and (10.76) and get for the effective superpotential an expression in the form (10.72)

$$W_{\text{eff}} = W_{\text{non-pert}} + W_{\text{tree}} = N S \left(1 - \log \frac{S}{\Lambda^3}\right)$$

which is nothing but the VY superpotential! The same reasoning can be applied to a theory with flavor and/or with multiple dynamical scales. The upshot is one and the same: integrating in (TVY) or out (ADS) fields holomorphically, are operations which one can do at no cost. The two descriptions are physically equivalent.

In the table below we summarize the relation between couplings and dual field variables for the most generic situation

<table>
<thead>
<tr>
<th>Couplings</th>
<th>$b_1 \log \frac{\Lambda_1}{\mu}$</th>
<th>$b_2 \log \frac{\Lambda_1}{\mu}$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fields</td>
<td>$S_1$</td>
<td>$S_2$</td>
<td>$X_1$</td>
<td>$X_2$</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>

Suppose that the mass spectrum of above (composite) fields is as follows

$m \uparrow$

- $S$, $\log \Lambda$
- $X', \lambda'$, massive fields
- $X, \lambda$, massless fields

where the $X'$s are a set of massless (possibly composite) fields, the $X''$s are massive ones, and the $\Lambda$'s are dynamical scales (glueball superfields are all massive because of mass gap of the pure glue theory, i.e. all $\Lambda$'s $\neq 0$). The most Wilsonian thing to do would be to describe the effective superpotential in terms of fields $X$, and couplings $\lambda'$ and $\Lambda$

$$W_{\text{eff}} = W_{\text{eff}}(X, \lambda', \Lambda).$$

In this sense, the ADS superpotential is more Wilsonian than the TVY. Seemingly, for pure SYM the most Wilsonian thing to do is to express the effective superpotential as a function of the coupling only (since the glueball superfield is massive), namely as $W_{\text{eff}} = N \Lambda^3$. However, since integrating in and out fields are equivalent.
operations, one can very well choose to write down the effective superpotential by integrating $X$ fields out and $X'$ and $S$ fields in (or anything in between these two extreme cases)

$$W_{\text{eff}} = W_{\text{eff}}(\lambda, X', S),$$

(10.82)

going a n equivalent way of describing the low energy effective theory superpotential. In fact, one should bare in mind that as far as the massless fields $X$ are concerned, there is no actual energy range for which integrating them out makes real physical sense, and this would be indicated by the Kähler potential of the effective theory being ill-defined (in other words, there is no energy range in which the kinetic term of such massless fields is negligible, since the energy is always bigger or equal than the field mass, which is vanishing). On the contrary, in presence of a mass gap, that is in the absence of $X$-like fields, the two descriptions, one in terms of couplings the other in terms of fields, are equivalent (as far as the F-term!), since now no singularities are expected in the Kähler potential. And this is a more and more exact equivalence the lower the energy.

10.4.4 SQCD for $F = N$ and $F = N + 1$

Let us now go back to our analysis of the IR dynamics of SQCD with gauge group $SU(N)$ and $F$ flavors. What about the case $F \geq N$? As we are going to see, things change drastically. For one thing, a properly defined effective superpotential cannot be generated. There is no way of constructing an object respecting all symmetries, with the correct dimension, and being vanishing in the classical limit, using couplings and fields we have (mesons, baryons and the dynamical scale $\Lambda$). This has the effect that for $F \geq N$ the classical moduli space is not lifted. This does not mean nothing interesting happens. For instance, the moduli space can be deformed by strong dynamics effects. Moreover, the perturbative analysis does not tell us what the low energy effective theory looks like; as we will see instead (mainly using holomorphy arguments), in some cases we will be able to make very non-trivial statements about the way light degrees of freedom interact, and in turn about the phase the theory enjoys.

In what follows, we will consider qualitatively different cases separately. Let us start analyzing the case $F = N$. It is easy to see that in this case all gauge invariant operators have R-charge $R = 0$, so one cannot construct an effective superpotential with $R = 2$. However, as we are going to show, something does happen due to strong dynamics.
Besides the mesons, there are now two baryons

\[
B = \epsilon^{a_1 a_2 \ldots a_N} Q_{a_1}^1 Q_{a_2}^2 \ldots Q_{a_N}^N
\]

\[
\tilde{B} = \epsilon_{a_1 a_2 \ldots a_N} \tilde{Q}_1^{a_1} \tilde{Q}_2^{a_2} \ldots \tilde{Q}_N^{a_N}.
\]

The classical moduli space is parameterized by VEVs of mesons and baryons. There is, however, a classical constraint between them

\[
\det M - B\tilde{B} = 0 \quad (10.83)
\]

(this comes because for \( N = F \) we have that \( \det Q = B \) and \( \det \tilde{Q} = \tilde{B} \) and the determinant of the product is the product of the determinants). One can ask whether this classical constraint is modified at the quantum level. In general, one could expect the quantum version of the above classical constraint to be

\[
\det M - B\tilde{B} = a \Lambda^{2N}, \quad (10.84)
\]

where \( a \) is a (undetermined for now) dimensionless and charge-less constant. Let us try to understand why this is the (only) possible modification one can have. First notice that this modification correctly goes to zero in the classical limit, \( \Lambda \to 0 \).

Second, the power with which \( \Lambda \) enters, \( 2N \), is the one-loop coefficient of the \( \beta \)-function and is exactly that associated with a one instanton correction, since for \( N = F \) we have for the instanton action

\[
e^{-S_{\text{inst}}} \sim e^{-\frac{8\pi^2}{9g^2} + i\phi_{\text{YM}}} \sim \Lambda^{2N}. \quad (10.85)
\]

Finally, there are no symmetry reasons not to allow for it (modulo the constant \( a \) which can very well be vanishing, after all). So, given that in principle a modification like (10.84) is allowed, everything boils down to determine whether the constant \( a \) is vanishing or has a finite value.

The constraint (10.84) can be implemented, formally, by means of a Lagrange multiplier, allowing a superpotential

\[
W = A \left( \det M - B\tilde{B} - a\Lambda^{2N} \right) \quad (10.86)
\]

where \( A \) is the Lagrange multiplier, whose equation of motion is by construction the constraint (10.84). The interesting thing is that one can use holomorphic decoupling to fix the constant \( a \). Adding a mass term for the \( N \)-th flavor, \( W = mM_N^N \), the low energy theory reduces to SQCD with \( F = N - 1 \). Imposing that after having
integrated out the $N$-th flavor one obtains an effective superpotential which matches the ADS superpotential for $F = N - 1$, fixes $a = 1$, that is

$$\det M - B\tilde{B} = \Lambda^{2N}.$$  \hspace{1cm} (10.87)

So the quantum constraint is there, after all. Actually, it is necessary for it to be there in order to be consistent with what we already know about the quantum properties of SQCD with $F < N$!

Several comments are in order at this point.

This is the first case where a moduli space of supersymmetric vacua persists at the quantum level. Still, the quantum moduli space is different from the classical one. The moduli space (10.83) is singular. It has a singular submanifold reflecting the fact that on this submanifold additional massless degrees of freedom arise. This is the submanifold where not only (10.83) is satisfied, but also $d(\det M - B\tilde{B}) = 0$, which makes the tangent space singular and therefore good local coordinates not being well-defined. This happens whenever baryon VEVs vanish, $B = \tilde{B} = 0$, and the meson matrix has rank $k \leq N - 2$. On this subspace a $SU(N - k)$ gauge group remains unbroken, and corresponding gluons (as well as some otherwise massive matter fields) remain massless. The quantum moduli space (10.87) is instead smooth. Basically, when $B = \tilde{B} = 0$ the rank of the meson matrix is not diminished since its determinant does not vanish, now: everywhere on the quantum moduli space the gauge group is fully broken.

Classically, the origin is part of the space of vacua. Hence, chiral symmetry can be unbroken. At the quantum level, instead, the origin is excised so in any allowed vacuum chiral symmetry is broken (like in QCD). Moreover, being the moduli space non-singular, means there are no massless degrees of freedom other than mesons and baryons. But the latter are indeed massless, since are moduli. Hence in SQCD with $N = F$ there is no mass gap (as for massless QCD). By supersymmetry, there are also massless composite fermions.

Obviously, the chiral symmetry breaking pattern is not unique. Different points on the moduli space display different patterns. At a generic point, where all gauge invariant operators get a VEV, all global symmetries are broken. But there are submanifolds of enhanced global symmetry. For instance, along the mesonic branch, defined as

$$M^i_j = \Lambda^2 \delta^i_j \ , \ B = \tilde{B} = 0,$$  \hspace{1cm} (10.88)
we have that
\[ SU(F)_L \times SU(F)_R \times U(1)_B \times U(1)_R \longrightarrow SU(F)_D \times U(1)_B \times U(1)_R , \]  
(10.89)
a chiral symmetry breaking pattern very much similar to QCD. Along the baryonic branch, which is defined as
\[ M = 0 \quad , \quad B = -\hat{B} = \Lambda^N , \]  
(10.90)
we have instead
\[ SU(F)_L \times SU(F)_R \times U(1)_B \times U(1)_R \longrightarrow SU(F)_L \times SU(F)_R \times U(1)_R , \]  
(10.91)
which is very different from QCD (the full non-abelian chiral symmetry is preserved).

Which phases does the theory enjoy? The point where all field VEVs vanish, i.e. the origin, is excised. Therefore, the gauge group is always broken and the theory is hence in a Higgs phase. Still, near the origin the theory can be better thought to be in a confined phase, since the theory is smooth in terms of mesons and baryons, and, moreover, we are in the strongly coupled region of field space, where an inherently perturbative Higgs description is not fully appropriate. In fact, there is no order parameter which can distinguish between the two phases; there is no phase transition between them (this is similar to the prototype example of one-family EW theory we discussed already). In this respect, notice that the Wilson loop is not a useful order parameter here since it follows the perimeter law, no matter where one sits on the moduli space: we do not have strict confinement but just charge screening, as in QCD, since we have (light) matter transforming in the fundamental representation of the gauge group, and therefore flux lines can (and do) break. The qualitative difference between classical and quantum moduli spaces, and their interpretation is depicted in Figure 10.11.

A non-trivial consistency check of this whole picture comes from computing ’t Hooft anomalies in the UV and in the IR. Let us consider, for instance, the mesonic branch. The charges under the unbroken global symmetries, \( SU(F)_D \times U(1)_B \times \)
Figure 10.11: Classical picture (left): at the origin gauge symmetry is recovered, and chiral symmetry is not broken. Quantum picture (right): the (singular) origin has been replaced by a circle of theories where chiral symmetry is broken (rather than Higgs phase, this resembles more closely the physics of a confining vacuum).

$U(1)_{R}$ of the UV (fundamental) and IR (composite) degrees of freedom are as follows

\[
\begin{array}{ccc}
SU(F)_D & U(1)_B & U(1)_R \\
\psi_Q & F & 1 \\
\bar{\psi}_Q & \bar{F} & -1 \\
\lambda & \bullet & 0 \\
\psi_M & \text{Adj} & 0 \\
\psi_B & \bullet & F \\
\bar{\psi}_{B} & \bullet & -F \\
\end{array}
\]

where we have being using the constraint (10.87) to eliminate the fermionic partner of Tr $M$, so that $\psi_M$ transforms in the Adjoint of $SU(F)_D$. We can now compute diverse triangular anomalies and see whether computations done in terms of UV and IR degrees of freedom agree. We get

\[
\begin{align*}
SU(F)_D^2 & U(1)_R & 2N_F^2(-1) = -N & F(-1) = -F \\
U(1)_B^2 & U(1)_R & -2NF & -2F^2 \\
U(1)_R^3 & & -2NF + N^2 - 1 & -(F^2 - 1) - 1 - 1 = -F^2 - 1
\end{align*}
\]

Since (crucially!) $F = N$ we see that ’t Hooft anomaly matching holds. A similar computation can be done for the baryonic branch finding again perfect agreement.
between the UV and IR 't Hooft anomalies. This rather non-trivial agreement ensures that our low energy effective description in terms of mesons and baryons, subject to the constraint (10.87), is most likely correct.

Let us move on and consider the next case, \( F = N + 1 \). The moduli space is again described by mesons and baryons. We have \( N + 1 \) baryons of type \( B \) and \( N + 1 \) baryons of type \( \tilde{B} \) now

\[
B_i = \epsilon_{ij_1...j_N} \epsilon^{a_1a_2...a_N} Q_{a_1}^{j_1} Q_{a_2}^{j_2} \cdots Q_{a_N}^{j_N} \\
\tilde{B}^i = \epsilon^{i j_1...j_N} \epsilon_{a_1a_2...a_N} \tilde{Q}_{j_1}^{a_1} \tilde{Q}_{j_2}^{a_2} \cdots \tilde{Q}_{j_N}^{a_N}.
\]

As we are going to show, differently from the previous case, the classical moduli space not only is uplifted, but is quantum exact, also. In other words, there are no quantum modifications to it.

This apparently surprising result can be proved using holomorphic decoupling. The rationale goes as follows. As proposed by Seiberg, this system can be described, formally, by the following superpotential

\[
W_{\text{eff}} = \frac{a}{\Lambda^{2N-1}} \left( \det M - B_i M^i_j \tilde{B}^j \right),
\]

where \( i = 1, 2, \ldots, N+1 \) is a flavor index, \( 2N-1 \) is the one-loop \( \beta \)-function coefficient and \( a \), as usual, is for now an undetermined coefficient. The above superpotential has all correct symmetry properties, including the R-charge, which is indeed equal to 2. Notice, though, that since the rank of the meson matrix \( k \leq N \), then \( \det M = 0 \), classically. So the above equation should be really thought of as a quantum equation, valid off-shell, so to say.

Let us now add a mass \( m \) to the \( F \)-th flavor. This gives

\[
W_{\text{eff}} = \frac{a}{\Lambda^{2N-1}} \left( \det M - B_i M^i_j \tilde{B}^j \right) - m M^{N+1}_{N+1}.\]

The F-flatness conditions for \( M^{N+1}_{N+1}, M^i_{N+1}, B_i \) and \( \tilde{B}^i \) for \( i < N + 1 \) reduce the meson matrix and the baryons to

\[
M = \begin{pmatrix} \hat{M}_{ij} & 0 \\ 0 & t \end{pmatrix}, \quad B = \begin{pmatrix} 0_i \\ \hat{B} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0^t \\ \hat{\tilde{B}} \end{pmatrix},
\]

where now \( i, j = 1, \ldots, N \), and \( t = M_{N+1,N+1} \). The F-flatness condition for \( t \) reads

\[
\frac{a}{\Lambda^{2N-1}} \left( \det \hat{M} - \hat{B} \hat{\tilde{B}} \right) - m = 0
\]

34
which implies
\[
\det \tilde{M} - \tilde{B} \tilde{B} = \frac{1}{a} m \Lambda^{2N-1} = \frac{1}{a} \Lambda^N^L,
\] (10.97)
where in the last step we have used the relation (10.52). This shows that the ansatz (10.93) is correct, since upon holomorphic decoupling we get exactly the quantum constraint of \( F = N \) SQCD (and \( a \) gets fixed to one). From eq. (10.93), by differentiating with respect to \( M, B_i \) and \( \tilde{B}^i \) we get the moduli space equations (i.e. the classical, still quantum exact, constraints between baryons and mesons)
\[
\begin{align*}
M \cdot \tilde{B} &= B \cdot M = 0 \\
\det M \cdot (M^{-1})^j_i - B_i \tilde{B}^j &= 0
\end{align*}
\] (10.98)
where \( \det M \cdot (M^{-1})^j_i \equiv \text{minor} \{ M \}^j_i = (-1)^{i+j} \times \det \text{of the matrix obtained from} \ M \text{by omitting the} \ i \text{-th row and the} \ j \text{-th column} \) (recall that above equations are on-shell, and on-shell \( \det M \) itself vanishes).

As a non-trivial check of this whole picture one can verify, choosing any preferred point in the space of vacua, that 't Hooft anomalies match (and hence that our effective description holds).

Now that we know eq. (10.93) is correct, let us try to understand what does it tell us about the vacuum structure of SQCD with \( F = N + 1 \). First, the origin of field space, \( M = B = \tilde{B} = 0 \), is now part of the moduli space. In such vacuum chiral symmetry is unbroken. This is an instance of a theory displaying confinement (actually charge screening) without chiral symmetry breaking.

Classically, the singularities at the origin are interpreted as extra massless gluons (and matter fields), since the theory gets unhiggsed for vanishing values of matter field VEVs. At the quantum level, the physical interpretation is different, since because the theory is UV-free, the region around the origin is the more quantum one. Singularities are more naturally associated with additional massless mesons and baryons which pop-up since eqs. (10.98) are trivially realized at the origin, and do not provide any actual constraint between meson and baryon components. In other words, at the origin the number of mesonic and baryonic massless degrees of freedom is larger than the dimension of the moduli space.

So we see that similarly to the \( F = N \) case, also this theory exhibits complementarity in the sense that one can move smoothly from a confining phase (near the origin) to a Higgs phase (at large field VEVs) without any order parameter being able to distinguish between them.
Theories with charge screening and no chiral symmetry breaking, like SQCD with $F = N + 1$, are known as s-confining.

One could in principle try to go further, and apply the same logic to $F = N + 2$ (and on). On general ground one would expect $M$, $B^{ij}$, $\tilde{B}_{ij}$ (baryons have now two free flavor indices) to be the dynamical degrees of freedom in the IR, and could then try to construct an effective (off-shell) superpotential of the kind of (10.93). This, however, does not work. Looking at the charges of the various gauge invariant operators and dynamical scale $\Lambda$ one can easily see that an effective superpotential with R-charge equal to 2, correct physical dimensions and symmetries, cannot be constructed. Indeed, the only $SU(F)_L \times SU(F)_R$ invariant superpotential one could construct should be the obvious generalization of (10.93), that is

$$W_{\text{eff}} \sim \det M - B_{il} M^i_j M^l_m \tilde{B}_{jm}, \quad (10.99)$$

which does not have $R = 2$ (things get worse the larger the number of flavors). Even 't Hooft anomaly matching condition can be proven not to work. For instance, choosing (for simplicity) the origin of field space where meson and baryons are unconstrained, one can see that 't Hooft anomalies do not match.

In fact, things turn out to be rather different. As we will show, the correct degrees of freedom to describe the dynamics around SQCD vacua for $F = N + 2$ are those of an IR-free theory (!) described by $SU(2)$ SYM coupled to $F$ chiral superfields $q$ transforming in the fundamental of $SU(2)$, $F$ chiral superfields $\tilde{q}$ transforming in the anti-fundamental and $F^2$ singlet chiral superfields $\Phi$, plus a cubic tree level superpotential coupling $q, \tilde{q}$ and $\Phi$. What’s that?

Two pieces of information are needed in order to understand this apparently weird result and, more generally, to understand what is going on for $F \geq N + 2$. Both are due to Seiberg. In the following we will review them in turn.

10.4.5 Conformal window

A first proposal is that SQCD in the range $\frac{3}{2}N < F < 3N$ flows to an interacting IR fixed point (meaning it does not confine). In other words, even if the theory is UV-free and hence the gauge coupling $g$ increases through the IR, at low energy $g$ reaches a constant RG-fixed value. Let us try to see how such claim comes about. The SQCD $\beta$-function for the physical gauge coupling (which hence takes into account
wave-function renormalization effects) is

\[ \beta(g) = - \frac{g^3}{16\pi^2} \frac{3N - F[1 - \gamma(g^2)]}{1 - Ng^2/8\pi^2}, \quad (10.100) \]

where \( \gamma \) is the anomalous dimension of matter fields and can be computed in perturbation theory to be

\[ \gamma(g^2) = - \frac{g^2}{8\pi^2} \frac{N^2 - 1}{N} + O(g^4). \quad (10.101) \]

Expanding formula (10.100) in powers of \( g^2 \) we get

\[ \beta(g) = - \frac{g^3}{16\pi^2} \left[ 3N - F + \left( 3N^2 - 2FN + \frac{F}{N} \right) \frac{g^2}{8\pi^2} + O(g^4) \right]. \quad (10.102) \]

From the above expression it is clear that there can exist values of \( F \) and \( N \) such that the one-loop contribution is negative but the two-loops contribution is positive. This suggests that in principle there could be a non-trivial fixed point, a value of the gauge coupling \( g = g_\ast \), for which \( \beta(g_\ast) = 0 \).

Let us consider \( F \) slightly smaller than \( 3N \). Defining

\[ \epsilon = 3 - \frac{F}{N} << 1 \quad (10.103) \]

we can re-write the \( \beta \)-function as

\[ \beta(g) = - \frac{g^3}{16\pi^2} \left[ \epsilon N - [3(N^2 - 1) + O(\epsilon)] \frac{g^2}{8\pi^2} + O(g^4) \right]. \quad (10.104) \]

The first term inside the parenthesis is positive while the second is negative and hence we see we have a solution \( \beta(g) = 0 \) at

\[ g_\ast^2 = \frac{8\pi^2}{3} \frac{N}{N^2 - 1} \epsilon, \quad (10.105) \]

up to \( O(\epsilon^2) \) corrections. This is called Banks-Zaks (BZ) fixed point. Seiberg argued that an IR fixed point like the one above exists not only for \( F \) so near to \( 3N \) but actually for any \( F \) in the range \( \frac{2}{3}N < F < 3N \), the so-called conformal window. According to this proposal, the IR dynamics of SQCD in the conformal window is described by an interacting superconformal theory: quarks and gluons are not confined but appear as interacting massless particles, the Coulomb-like potential being

\[ V(r) \sim \frac{g_\ast^2}{r}. \quad (10.106) \]
Hence, according to this proposal, SQCD in the conformal window enjoys a non-abelian Coulomb phase.

Let us try to understand why the conformal window is bounded from below and from above. The possibility of making exact computations in a SCFT shows that for $F < \frac{3}{2}N$ the theory should be in a different phase. In a SCFT the dimension of a field satisfies the following relation

$$\Delta \geq \frac{3}{2} |R| ,$$

where $R$ is the field R-charge (recall that in a SCFT the generator of the R-symmetry enters the algebra, and hence an R-symmetry is always present). The equality holds for chiral (or anti-chiral) operators. This implies that

$$\Delta(M) = \frac{3}{2} R(M) = \frac{3}{2} R(Q\bar{Q}) = \frac{3}{2} \frac{F - N}{F} = 2 + \gamma_s ,$$

given that $M$ is a chiral operator. This means that the anomalous dimension of the meson matrix at the IR fixed point is $\gamma_s = 1 - 3N/F$.

The dimension of a scalar field must satisfy

$$\Delta \geq 1 .$$

Indeed, when $\Delta < 1$ the operator, which is in a unitary representation of the superconformal algebra, would include a negative norm state which cannot exist in a unitary theory. This implies that $F = \frac{3}{2}N$ is a lower bound since there $\Delta(M) = 1$ and lower values of $F$ make no sense (recall that the lowest component of the superfield $M$ is a scalar field): for $F < \frac{3}{2}N$ the theory should be in a different phase. A clue to what such phase could be is that at $F = \frac{3}{2}N$ the field $M$ becomes free. Indeed, for $F = \frac{3}{2}N$ we get that $\Delta(M) = 1$ which is possible only for free, non-interacting scalar fields. Perhaps it is the whole theory of mesons and baryons which becomes free, somehow. We will make this intuition more precise later.

As for the upper bound, let us notice that for $F \geq 3N$ SQCD is not asymptotically free anymore (the $\beta$-function changes sign). The spectrum at large distance consists of elementary quarks and gluons interacting through a potential

$$V \sim \frac{g^2}{r} \quad \text{with} \quad g^2 \sim \frac{1}{\log(rA)} ,$$

which implies that SQCD is in a non-abelian free phase. It is interesting to notice that for $F = 3N$ the anomalous dimension of $M$ is actually zero, consistent with
the fact that from that value on, the IR dimension of gauge invariant operators is not renormalized since the theory becomes IR-free. A summary of the IR behavior of SQCD for $F > \frac{3}{2}N$ is reported in Figure 10.12.

\[
\begin{array}{c}
3/2 \, N < F < 3N \\
g = g_* \quad \leftrightarrow \quad g = 0
\end{array}
\]

\[
\begin{array}{c}
F \geq 3N \\
g = 0 \quad \leftrightarrow \quad \text{any } g
\end{array}
\]

Figure 10.12: The IR behavior of SQCD in the window $\frac{3}{2}N < F < 3N$, where the theory flows to an IR fixed point with $g = g_*$, and for $F \geq 3N$, where $g_* = 0$ and the theory is in a non-abelian IR-free phase.

### 10.4.6 Electric-magnetic duality (aka Seiberg duality)

The second proposal put forward by Seiberg regards the existence of a electromagnetic-like duality. The IR physics of SQCD for $F > N + 1$ has an equivalent description in terms of another supersymmetric gauge theory, known as the magnetic dual theory. Such dual theory is IR-free for $N + 1 < F \leq \frac{3}{2}N$, and UV-free for $F > \frac{3}{2}N$. In the conformal window defined before, that is for $\frac{3}{2}N < F < 3N$, it has a IR fixed point (the same as the original SQCD theory!), while for $F \geq 3N$, where SQCD becomes IR-free, it enters into a confining phase. So these two theories are very different: as such, the equivalence is an IR equivalence. SQCD, sometime called electric theory in this context, and its magnetic dual are not equivalent in the UV neither along the RG-flow. They just provide two equivalent ways to describe the dynamics around the space of vacua (in fact, perturbing SQCD by suitable operators, e.g., by quartic operators, one can sometime promote this IR duality to a full duality, valid along the whole RG; however, discussing such instances is beyond our present scope).

In order to understand this claim (and its implications), and define such dual theory more precisely, we first need to do a step back. In trying to extend to higher values of $F$ the reasoning about SQCD with $F = N + 1$, one would like to consider the following gauge invariant operators

\[
M^i_j, \quad B_{i_1i_2...i_{F-N}}, \quad \tilde{B}^{i_1i_2...i_{F-N}}. \quad (10.111)
\]
The baryons have $\tilde{N} = F - N$ free indices so one might like to view them as bound states of $\tilde{N}$ components, some new quark fields $q$ and $\tilde{q}$ of some SYM theory with gauge group $SU(\tilde{N}) = SU(F - N)$ for which $q$ and $\tilde{q}$ transform in the $\tilde{N}$ and $\bar{N}$ representations, respectively. Then the SQCD baryons would have a dual description as

$$B_{i_1i_2...i_{\tilde{N}}} \sim \epsilon_{a_1a_2...a_{\tilde{N}}} q^{a_1} q^{a_2} \cdots q^{a_{\tilde{N}}}$$

(10.112)

and similarly for $\tilde{B}$. Recall that in terms of the original matter fields $Q$ and $\tilde{Q}$, the baryons are composite fields made out of $N$ components.

Seiberg made this naive idea concrete (and physical), putting forward the following proposal: SQCD with gauge group $SU(N)$ and $F > N + 1$ flavors, can be equivalently described, in the IR, by a SQCD theory with gauge group $SU(F - N)$ and $F$ flavors plus an additional chiral superfield $\Phi$ which is a gauge singlet and which transforms in the fundamental representation of $SU(F)_L$ and in the anti-fundamental representation of $SU(F)_R$, and which interacts with $q$ and $\tilde{q}$ via a cubic superpotential

$$W = q_i \Phi^j \tilde{q}^j.$$  

(10.113)

As bizarre this proposal may look like, let us try to understand it better. Let us first consider the Seiberg dual theory (which from now on we dub mSQCD, where 'm' stands for magnetic) without superpotential term, and let us focus on the SQCD conformal window, $\frac{3}{2}N < F < 3N$, first. For $W = 0$ the field $\Phi$ is completely decoupled and the theory is just SQCD with gauge group $SU(F - N)$ and $F$ flavors. Interestingly, the SQCD conformal window is a conformal window also for mSQCD! Hence mSQCD (without the singlet $\Phi$) flows to an IR fixed point for $\frac{3}{2}N < F < 3N$. At such fixed point the superpotential coupling, that we now switch-on, is relevant, since $\Delta(W) = \Delta(\Phi) + \Delta(q) + \Delta(\tilde{q}) = 1 + \frac{3}{2}N/F + \frac{3}{2}N/F < 3$. The claim is that the perturbation (10.113) drives the theory to some new fixed point which is actually the same fixed point of SQCD.

How does mSQCD look like for $F \leq \frac{3}{2}N$? The one-loop $\beta$-function coefficient of mSQCD is $b_1 = 2F - 3N$. Hence, for $F = \frac{3}{2}N$ the $\beta$-function vanishes and for lower values of $F$ it changes its sign and mSQCD becomes IR-free. Hence, the bound $F = \frac{3}{2}N$ has the same role that the bound $F = 3N$ has for SQCD. Not surprisingly, one can apply the BZ-fixed point argument to mSQCD for $F$ slightly larger than $\frac{3}{2}N$ and find the existence of a perturbative fixed point. This explains why, if Seiberg duality is correct, the IR dynamics of SQCD in the range $N + 1 < F \leq \frac{3}{2}N$ differs from the behavior in the conformal window - something we had some indications
of, when studying the lower bound in $F$ of the SQCD conformal window. Indeed, we now can make our former intuition precise: using a clever set of variables (i.e. the magnetic dual variables), one concludes that for $N + 1 < F \leq \frac{3}{2}N$ SQCD IR dynamics is described by a theory of freely interacting (combinations of) meson and baryon fields. These can be described in terms of free dual quarks interacting with a Coulomb-like potential

$$V_m \sim \frac{g_m^2}{r} \quad \text{with} \quad g_m^2 \sim \frac{1}{\log(r \Lambda)} \quad (10.114)$$

This phase of SQCD is dubbed free magnetic phase, a theory of freely interacting (dual) quarks. The fact that the IR dynamics of SQCD for $N + 1 < F \leq \frac{3}{2}N$, where the theory is confining, can be described this way is a rather powerful statement: since $m_{\text{SQCD}}$ is IR-free, in terms of magnetic dual variables the Kahler potential is canonical (up to subleading $1/\Lambda^2$ corrections), meaning we know the full effective IR Lagrangian of SQCD for $N + 1 < F \leq \frac{3}{2}N$, at low enough energies!

As for the conformal window, which variables to use depends on $F$. The larger $F$, the nearer to IR-freedom SQCD is, and the more UV-free $m_{\text{SQCD}}$ is. In other words, the conformal window IR-fixed point is at smaller and smaller value of the electric gauge coupling the nearer $F$ is to $3N$, and eventually becomes 0 for $F = 3N$. For $m_{\text{SQCD}}$ things are reversed. The IR-fixed point arises at weaker coupling the nearer $F$ is to $\frac{3}{2}N$, and for $F = \frac{3}{2}N$ we have that $g_e^m = 0$. Therefore, the magnetic description is the simplest to describe SQCD non-abelian Coulomb phase for $F$ near to $\frac{3}{2}N$; the electric description is instead the most appropriate one when $F$ is near to $3N$.

For $F \geq 3N$ the magnetic theory does not reach anymore an IR interacting fixed point. The value $F = 3N$ plays for $m_{\text{SQCD}}$ the same role the value $F = \frac{3}{2}N$ plays for SQCD. Indeed, the $m_{\text{SQCD}}$ meson matrix, $U = q\tilde{q}$ has $\Delta = 1$ for $F = 3N$, and becomes a free field, while for larger values of $F$ it would get dimension lower than one, which is not acceptable. For $F \geq 3N$ the theory should enter in a new phase. This is something we know already: in this region we are in the SQCD IR-free phase.

In the remainder of this section, we provide several consistency checks for the validity of this proposed duality.

First, we note that two basic necessary requirements for its validity are met: the two theories have the same global symmetry group as well as the same number of IR degrees of freedom. In order to see this, let us first make the duality map precise. The mapping between chiral operators of SQCD and $m_{\text{SQCD}}$ (at the IR fixed point)
is

\[ M \leftrightarrow \Phi : \quad \Phi^i_j = \frac{1}{\mu} M^i_j \quad (10.115) \]

\[ B \leftrightarrow b : \quad b_{i_1 j_2 \cdots j_N} = \epsilon^{i_1 j_2 \cdots i_{F-N} j_1 j_2 \cdots j_N} B_{i_1 j_2 \cdots i_{F-N}} \]

and similarly for \( \tilde{b} \) and \( \tilde{B} \). The scale \( \mu \) relating SQCD mesons with the mSQCD gauge singlet \( \Phi \) appears for the following reason. In SQCD mesons are composite fields and their dimension in the UV, where SQCD is free, is \( \Delta = 2 \). On the other hand, \( \Phi \) is an elementary field in mSQCD and its dimension in the UV is \( \Delta = 1 \). Hence the scale \( \mu \) needs to be introduced to match \( \Phi \) to \( M \) in the UV. Clearly, upon RG-flow both fields acquire an anomalous dimension and should flow to one and the same operator in the IR, if the duality is correct. Applying formula (10.107), which for chiral operators is an equality, one easily sees that this is indeed what should happen, since \( R(M) = R(\Phi) \).

From the map (10.115) it easily follows that the magnetic theory has a global symmetry group which is nothing but the one of SQCD, \( G_F = SU(F)_L \times SU(F)_R \times U(1)_B \times U(1)_R \), with the following charges for the elementary fields

<table>
<thead>
<tr>
<th>Field</th>
<th>( SU(F)_L )</th>
<th>( SU(F)_R )</th>
<th>( U(1)_B )</th>
<th>( U(1)_R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q^0_i )</td>
<td>( \overline{\mathcal{F}} )</td>
<td>( \bullet )</td>
<td>( \frac{N}{F-N} )</td>
<td>( \frac{N}{F} )</td>
</tr>
<tr>
<td>( \tilde{q}^j_b )</td>
<td>( \bullet )</td>
<td>( \mathcal{F} )</td>
<td>( \frac{-N}{F-N} )</td>
<td>( \frac{N}{F} )</td>
</tr>
<tr>
<td>( \Phi )</td>
<td>( \mathcal{F} )</td>
<td>( \overline{\mathcal{F}} )</td>
<td>( 0 )</td>
<td>( 2\frac{F-N}{F} )</td>
</tr>
<tr>
<td>( \tilde{\lambda} )</td>
<td>( \bullet )</td>
<td>( \bullet )</td>
<td>( 0 )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>

while the superpotential (10.113) has \( R = 2 \).

We can now use global symmetries to see that SQCD and mSQCD have the same number of IR degrees of freedom. Basically, there is a one-to-one map between gauge invariant operators, and these operators have the same global symmetries (which counts physically distinct degrees of freedom). Indeed, the meson matrix \( M \) enjoys the same symmetries as the mSQCD singlet \( \Phi \), and the SQCD baryons \( B, \tilde{B} \) the same as the mSQCD baryons \( b, \tilde{b} \) (the latter being gauge invariant operators constructed in terms of \( F - N \) dual quarks \( q, \tilde{q} \)). One might feel uncomfortable since the mesons of the magnetic dual theory, \( U^i_j = q_i \tilde{q}^j \) seem not to match with anything in the electric theory. This is where the superpotential (10.113) comes into play. Recall the supposed equivalence between SQCD and mSQCD is just a IR equivalence. The F-equations for \( \Phi \) fix the dual meson to vanish on the moduli space: \( F_{\Phi} = q \tilde{q} = U = 0 \). Hence, in the IR the two theories do have the same number of degrees of freedom!
In what follows, we are going to present further non-trivial checks for the validity of Seiberg’s proposal.

- The first very non-trivial check comes from ’t Hooft anomaly matching. The computation of ’t Hooft anomalies gives

\[
\begin{align*}
\text{SQCD} & \\
SU(F)_{L}^{2} U(1)_{B} & = \frac{1}{2} N (+1) = \frac{1}{2} N \\
U(1)_{B}^{3} U(1)_{R} & = 2 NF (+1) \left( -\frac{N}{F} \right) = -2 N^{2} \\
U(1)_{R}^{3} & = \left( -\frac{N}{F} \right)^{3} 2 NF + N^{2} - 1 = -2 \frac{N^{4}}{F} + N^{2} - 1 \\
\end{align*}
\]

which shows there is indeed matching between SQCD and its IR-equivalent mSQCD description.

Note that for the matching to work it turns out that the presence of dual gauginos is crucial (as well as that of the magnetic superpotential term). This explicitly shows that the description of SQCD baryons in terms of some sort of dual quarks is not just a mere group representation theory accident. There is a truly dynamical dual gauge group, under which dual quarks are charged, and dual vector superfields (which include dual gauginos) which interact with them.

- The duality relation is a duality, which means that acting twice with the duality map one recovers the original theory (as far as IR physics!). Let us start from SQCD with \( N \) colors and \( F \) flavors and act with the duality map twice

\[
\begin{align*}
\text{SQCD} & : SU(N) \ , \ F \ , \ W = 0 \\
\downarrow \text{duality} \\
\text{mSQCD} & : SU(F - N) \ , \ F \ , \ W = \frac{1}{\mu} q_{i} M_{j}^{i} \tilde{q}^{j} = q_{i} \Phi_{j}^{i} \tilde{q}^{j} \\
\downarrow \text{duality} \\
\text{mmSQCD} & : SU(N) \ , \ F \ , \ W = \frac{1}{\mu} q_{i} M_{j}^{i} \tilde{q}^{j} + \frac{1}{\mu} d^{i} U_{j}^{i} \tilde{d}^{j} = q_{i} \Phi_{j}^{i} \tilde{q}^{j} + d^{i} \Psi_{j}^{i} \tilde{d}^{j}
\end{align*}
\]

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where \( U^i_j = q_i \tilde{q}^j \) is the meson matrix of mSQCD, while \( \Psi^i_j \) is the gauge singlet chiral superfield dual to \( U \) and belonging to the magnetic dual of mSQCD. Choosing \( \tilde{\mu} = -\mu \), we can rewrite the superpotential of mmSQCD as

\[
W = \frac{1}{\mu} \text{Tr} \left[ UM - dU \bar{d} \right].
\]

Choosing \( \tilde{\mu} = \mu \), we can rewrite the superpotential of mmSQCD as

\[
W = \frac{1}{\mu} \text{Tr} \left[ UM - dU \bar{d} \right].
\]

The fields \( U \) and \( M \) are hence massive and can be integrated out (recall we claim the IR equivalence of Seiberg-dual theories, not the equivalence at all scales). This implies

\[
\frac{\partial W}{\partial U} = 0 \rightarrow M^i_j = d^i \bar{d}^j, \quad \frac{\partial W}{\partial M} = 0 \rightarrow U = 0
\]

showing that the dual of the dual quarks are nothing but the original quark superfields \( Q \) and \( \tilde{Q} \), and that \( U = 0 \) (hence \( W = 0 \)) in the IR. Summarizing, after integrating out heavy fields, we are left with SQCD with gauge group \( SU(N) \), \( F \) flavors and no superpotential, exactly the theory we have started with! In passing, let us note that in order to make the duality working we have to set \( \tilde{\mu} = -\mu \), a mass scale which is not fixed by the duality itself.

- The duality is preserved under mass perturbations, namely upon holomorphic decoupling. Let us again consider SQCD with gauge group \( SU(N) \) and \( F \) flavors and let us add a mass term to the \( F \)-th flavor, \( W = m M^r_F \). This corresponds to \( SU(N) \) SQCD with \( F - 1 \) massless flavors and one massive one. In the dual magnetic theory this gives a superpotential

\[
W = \frac{1}{\mu} q_i M^i_j \tilde{q}^j + m M^r_F.
\]

The F-flatness conditions for \( M^r_F \) and \( q^r_F \) and \( \tilde{q}^r_F \) are

\[
q^r_a \tilde{q}^r_a + \mu m = 0, \quad (M \cdot \tilde{q}_a)^r = (q^a \cdot M)_r = 0,
\]

where \( a \) is a \( SU(F - N) \) gauge index. The first equation induces a VEV for the dual quarks with flavor index \( F \), which breaks the gauge group down to \( SU(F - N - 1) \). The other two equations imply that the \( F \)-th row and column of the SQCD meson matrix \( M \) vanish. We hence end-up with \( SU(F - N - 1) \) SQCD with \( F - 1 \) flavors, a gauge singlet \( M \) which is a \( (F - 1) \times (F - 1) \) matrix, while the superpotential (10.118) reduces to eq. (10.113) where now \( i, j \) run from 1 to \( F - 1 \) only. This is the correct Seiberg dual mSQCD theory at low energy.
This analysis shows that a mass term in the electric theory corresponds to higgsing in the magnetic dual theory, according to the table below.

<table>
<thead>
<tr>
<th>SQCD</th>
<th>mSQCD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(N), F$</td>
<td>$SU(F - N), F$</td>
</tr>
<tr>
<td>$\downarrow$ mass</td>
<td>$\downarrow$ higgsing</td>
</tr>
<tr>
<td>$SU(N), F - 1$</td>
<td>$SU(F - N - 1), F - 1$</td>
</tr>
</tbody>
</table>

The converse is also true (though slightly harder to prove): a mass term in mSQCD corresponds to higgsing in SQCD.

- Let us use holomorphic decoupling to go from the last value of $F$ where we have the duality, $F = N + 2$, to $F = N + 1$. If Seiberg duality is correct, we should recover the description of SQCD with $F = N + 1$ flavors we discussed previously. Let us consider mSQCD when $F = N + 2$. The magnetic gauge group is $SU(2)$. Upon holomorphic decoupling, an analysis identical to the one we did above produces a cubic superpotential at low energy as

$$W \sim q_i M_j^i \tilde{q}^j$$

where $q_i$ are the baryons $B^i$ of SQCD with $F = N + 1$ and $\tilde{q}_j$ the baryons $\tilde{B}_j$. At the same time, the VEVs for $q_r$ and $\tilde{q}_r$ break the $SU(2)$ gauge symmetry completely. From mSQCD view point this is a situation similar to SQCD with $F = N - 1$ where the full breaking of gauge symmetry group allowed an exact instanton calculation providing the $\sim \det M$ contribution to the effective superpotential. The same happens here and the final answer one gets for the low energy effective superpotential is

$$W_{\text{eff}} \sim \left(q_i M_j^i \tilde{q}^j - \det M\right),$$

where $q_i M_j^i \tilde{q}^j$ is precisely the effective superpotential of SQCD with $F = N + 1$!

This also shows that by holomorphic decoupling we can actually connect the description of the IR dynamics of SQCD for any number of flavors, from $F = 0$ to any larger values of $F$, at fixed $N$.

Let us finally notice, in passing, that even for $F = N + 1$ we can sort of speak of a magnetic dual theory. Just it is trivial, since there is no magnetic dual gauge group.
• There is yet an important relation between the three a priori different mass scales entering the duality: the electric dynamical scale $\Lambda_{el}$, the magnetic scale $\Lambda_{m}$, and the matching scale $\mu$. This reads

$$\Lambda_{el}^{3N-F} \Lambda_{m}^{3(F-N)-F} = (-1)^{F-N} \mu^F .$$

(10.122)

That this relation is there, can be seen in different ways. First, one can check that the relation is duality invariant, as it should. Indeed, applying the duality map (recall that $\tilde{\mu} = -\mu$, while $\tilde{\mu}_{el}$ and $\tilde{\mu}_{m}$ get interchanged by the duality) one gets

$$\Lambda_{el}^{3(F-N)-F} \Lambda_{m}^{3N-F} = (-1)^N \tilde{\mu}^F = (-1)^{N-F} \mu^F ,$$

(10.123)

which is identical to (10.122). One can also verify the consistency of the relation (10.122) upon higgsing and/or holomorphic decoupling. The check is left to the reader.

Eq. (10.122) shows that as the electric theory becomes stronger (i.e. $\Lambda_{el}$ increases), the magnetic theory becomes weaker (i.e. $\Lambda_{m}$ decreases). By using the relation between dynamical scales and gauge couplings, this can be translated into a relation between gauge coupling constants, and gives an inverse relation between them

$$g_{el}^2 \sim g_{m}^{-2} ,$$

(10.124)

showing that large values of the electric gauge coupling $g_{el}$ correspond to small values of the magnetic one, and viceversa. This is why Seiberg duality is an electric-magnetic duality.

Depending on where in the $(F, N)$ space one sits, the meaning of the dynamical scales changes. In the conformal window both SQCD and mSQCD are UV-free. Both theories have a non-trivial RG-flow and, upon non-perturbative effects, driven by $\Lambda_{el}$ and $\Lambda_{m}$, reach an IR fixed point (which is one and the same, in fact). In the free-magnetic phase, mSQCD is IR-free and SQCD is UV-free. Therefore, in this regime $\Lambda_{m}$ should be better thought of as a UV-scale for the magnetic theory, which is an effective theory. In this regime SQCD can be thought of as the (or better, a possible) UV-completion of mSQCD (the electric free phase can be thought of in a similar way, with the role of SQCD and mSQCD reversed). Within this interpretation it is natural to tune the free parameter $\mu$ to make the two theories have one single non-perturbative scale, the scale at which non-perturbative SQCD effects come into play and
the scale below which the magnetic effective description takes over. From the relation (10.122) one sees that this is obtained by equating, up to an overall phase, the matching scale $\mu$ with $\Lambda_{el}$ and $\Lambda_m$

$$\mu = \Lambda_{el} = \Lambda_m.$$  \hspace{1cm}  (10.125)

It is left as an exercise to check that using the above relation for $F = N + 2$ and adding a mass term for the $F$-th flavor, upon holomorphic decoupling one gets the expression (10.121) including the correct power of $\Lambda$, that is

$$W_{\text{eff}} = \frac{1}{\Lambda^{2N-1}} (q_i M_j \tilde{q}^j - \text{det} M) .$$  \hspace{1cm}  (10.126)

Figure 10.13 contains a qualitative description of the three different regimes we have just discussed.

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10.5 The phase diagram of $N=1$ SQCD

After this long tour on quantum properties of SQCD, it is time to wrap-up and summarize its phase diagram.

For $F = 0$ SQCD (pure SYM in this case) enjoys strict confinement, displays $N$ isolated supersymmetric vacua and a mass gap. For $0 < F < N$ the theory doesn’t exist by its own. The classical moduli space is completely lifted and a runaway potential, with no absolute minima at finite distance in field space, is generated.
For $F = N, N + 1$ a moduli space persists at the quantum level and SQCD enjoys confinement with charge screening (the asymptotic states are gauge singlets but flux lines can break) and no mass gap. Asymptotic states are mesons and baryons. The theory displays complementarity, as any theory where there are scalars transforming in the fundamental representation of the gauge group: there is no invariant distinction between Higgs phase, which is the more appropriate description for large field VEVs, and confinement phase, which takes over near the origin of field space. The potential between static test charges goes to a constant asymptotically since in the Higgs phase gauge bosons are massive and there are no long-range forces. As already observed, this holds also in the confining description, since we have charge screening and no area-law for Wilson loops.

For $N + 2 \leq F \leq \frac{3}{2}N$ we are still in a confinement phase, but the theory is in the so-called free magnetic phase and can be described at large enough distance in terms of freely interacting dual quarks and gluons. What is amusing here is that while asymptotic massless states are composite of elementary electric degrees of freedom (i.e. mesons and baryons), they are magnetically charged with respect to a gauge group whose dynamics is not visible in the electric description and which is generated, non-perturbatively, by the theory itself.

For $F > \frac{3}{2}N$ SQCD does not confine anymore, not even in the weak sense: asymptotic states are quarks and gluons (and their superpartners). The potential between asymptotic states, though, differs if $F \geq 3N$ or $F < 3N$. In the former case the theory is IR-free and it is described by freely interacting particles. Hence the potential vanishes, at large enough distance. For $\frac{3}{2}N < F < 3N$, instead, the theory (which is still UV-free) is in a non-abelian Coulomb phase. Charged particles are not confined but actually belong to a SCFT, and interact by a $1/r$ potential with coupling $g = g_*$.

A diagram summarizing the gross features of the quantum dynamics of SQCD is reported below.

10.6 Exercises

1. Consider SQCD with $F = N$ with superpotential

$$W = A \left( \det M - B\tilde{B} - aA^{2N} \right) + mQ^N\tilde{Q}^N$$  \hspace{1cm} (10.127)

By integrating out the massive flavor, show that one recovers the ADS superpotential for $F = N - 1$ SQCD if and only if $a = 1$.  

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2. Check ’t Hooft anomaly matching for SQCD with $F = N$ along the baryonic branch, $M = 0, B = -\tilde{B} = \Lambda^N$.

3. Check ’t Hooft anomaly matching for SQCD with $F = N + 1$ at the origin of the moduli space.

4. Consider mSQCD for $F = 3(F - N) - \epsilon(F - N)$ with $\epsilon << 1$, and find the BZ perturbative fixed point (i.e. the value of $g$ such that the $\beta$-function vanishes).

References


