5 Supersymmetric actions: minimal supersymmetry

In the previous lecture we have introduced the basic superfields one needs to construct $N = 1$ supersymmetric theories, if one is not interested in describing gravitational interactions. We are now ready to look for supersymmetric actions describing the dynamics of these superfields. We will first concentrate on matter actions and construct the most general supersymmetric action describing the interaction of a set of chiral superfields. Then we will introduce SuperYang-Mills theory which is nothing but the supersymmetric version of Yang-Mills. Finally, we will couple the two sectors with the final goal of deriving the most general $N = 1$ supersymmetric action describing the interaction of radiation with matter. In all these cases, we will consider both renormalizable as well as non-renormalizable theories, the latter being relevant to describe effective low energy theories.

Note: in what follows we will deal with gauge theories, and hence gauge groups, like $U(N)$ and alike. In order to avoid confusion, in the rest of these lectures we will use calligraphic $\mathbb{N}$ when referring to the number of supersymmetry, $\mathbb{N} = 1, 2$ or 4.

5.1 $\mathbb{N} = 1$ Matter actions

Following the general strategy outlined in §4.3 we want to construct a supersymmetric invariant action describing the interaction of a (set of) chiral superfield(s). Let us first notice that a product of chiral superfields is still a chiral superfield and a product of anti-chiral superfields is an anti-chiral superfield. Conversely, the product of a chiral superfield with its hermitian conjugate (which is anti-chiral) is a (very special, in fact) real superfield.

Let us start analyzing the theory of a single chiral superfield $\Phi$. Consider the following integral

$$\int d^2 \theta \ d^2 \bar{\theta} \ \Phi \Phi \ .$$

This integral satisfies all necessary conditions to be a supersymmetric Lagrangian. First, it is supersymmetric invariant (up to total space-time derivative) since it is the integral in superspace of a superfield. Second, it is real and a scalar object. Indeed, the first component of $\Phi \Phi$ is $\bar{\phi} \phi$ which is real and a scalar. Now, the $\theta^2 \bar{\theta}^2$ component of a superfield, which is the only term contributing to the above integral, has the same tensorial structure as its first component since $\theta^2 \bar{\theta}^2$ does not have any free space-time indices and is real, that is $(\theta^2 \bar{\theta}^2)^\dagger = \theta^2 \bar{\theta}^2$. Finally, the above integral has
also the right physical dimensions for being a Lagrangian, i.e. \([M]^4\). Indeed, from the expansion of a chiral superfield, one can see that \(\theta\) and \(\bar{\theta}\) have both dimension \([M]^{-1/2}\) (compare the first two components of a chiral superfield, \(\phi(x)\) and \(\theta\psi(x)\), and recall that a spinor in four dimensions has physical dimension \([M]^{3/2}\)). This means that the \(\theta^2\bar{\theta}^2\) component of a superfield \(Y\) has dimension \([Y] + 2\) if \([Y]\) is the dimension of the superfield (which is that of its first component). Since the dimension of \(\Phi\Phi\) is 2, it follows that its \(\theta^2\bar{\theta}^2\) component has dimension 4 (notice that \(\int d^2\theta d^2\bar{\theta} \theta^2\bar{\theta}^2\) is dimensionless since \(d\theta (d\bar{\theta})\) has opposite dimensions with respect to \(\theta\) \((\bar{\theta})\), given that the differential is equivalent to a derivative, for Grassman variables). Summarizing, eq. (5.1) is an object of dimension 4, is real, and transforms as a total space-time derivative under SuperPoincaré transformations.

To perform the integration in superspace one can start from the expression of \(\Phi\) and \(\Phi\) in the \(y\) (resp. \(\bar{y}\)) coordinate system, take the product of \(\Phi(\bar{y}, \bar{\theta})\Phi(y, \theta)\), expand the result in the \((x, \theta, \bar{\theta})\) space, and finally pick up the \(\theta^2\bar{\theta}^2\) component, only. The computation is left to the reader. The end result is

\[
L_{\text{kin}} = \int d^2\theta d^2\bar{\theta} \Phi \Phi = \partial_\mu \phi \partial^\mu \phi + \frac{i}{2} \left( \partial_\mu \psi \sigma^\mu \psi - \psi \sigma^\mu \partial_\mu \psi \right) + F + \text{total der. (5.2)}
\]

What we get is precisely the kinetic term describing the degrees of freedom of a free chiral superfield! In doing so we also see that, as anticipated, the \(F\) field is an auxiliary field, namely a non-propagating degrees of freedom. Integrating it out (which is trivial in this case since its equation of motion is simply \(F = 0\)) one gets a (supersymmetric) Lagrangian describing physical degrees of freedom, only. Notice that after integrating \(F\) out, supersymmetry is realized on-shell, only, as it can be easily checked.

The equations of motion for \(\phi, \psi\) and \(F\) following from the Lagrangian (5.2) can be easily derived using superfield formalism readily from the expression in superspace. This might not look obvious at a first sight since varying the action (5.1) with respect to \(\Phi\) we would get \(\Phi = 0\), which does not provide the equations of motion we would expect, as it can be easily inferred expanding it in components. The point is that the integral in eq. (5.1) is a constrained one, since \(\Phi\) is a chiral superfield and hence subject to the constraint \(\bar{D}_\alpha \Phi = 0\). One can rewrite the above integral as an unconstrained one noticing that

\[
\int d^2\theta d^2\bar{\theta} \Phi \Phi = \frac{1}{4} \int d^2\bar{\theta} \Phi D^2\Phi . \quad (5.3)
\]

In getting the right hand side we have used the fact that \(\int d\theta_\alpha = D_\alpha\), up to total space-time derivative, and that \(\Phi\) is a chiral superfield (hence \(D_\alpha \Phi = 0\). Now,
varying with respect to $\Phi$ we get

$$D^2\Phi = 0,$$

(5.4)

which, upon expansion in $(x, \theta, \bar{\theta})$, does correspond to the equations of motion for $\phi, \psi$ and $F$ one would obtain from the Lagrangian (5.2). The check is left to the reader.

A natural question arises. Can we do more? Can we have a more general Lagrangian than just the one above? Let us try to consider a more generic function of $\Phi$ and $\bar{\Phi}$, call it $K(\Phi, \bar{\Phi})$, and consider the integral

$$\int d^2\theta d^2\bar{\theta} K(\Phi, \bar{\Phi}).$$

(5.5)

Again, in order for the integral (5.5) to be a promising object to describe a supersymmetric Lagrangian, the function $K$ should satisfy a number of properties. First, it should be a superfield. This ensures supersymmetric invariance. Second, it should be a real and scalar function. As before, this is needed since a Lagrangian should have these properties and the $\theta^2 \bar{\theta}^2$ component of $K$, which is the only one contributing to the above integral, is a real scalar object, if so is the superfield $K$. Third, $[K] = 2$, since then its $\theta^2 \bar{\theta}^2$ component will have dimension 4, as a Lagrangian should have. Finally, $K$ should be a function of $\Phi$ and $\bar{\Phi}$ but not of $D_\alpha \Phi$ and $\bar{D}_\alpha \bar{\Phi}$. The reason is that, as it can be easily checked, covariant derivatives would provide $\theta \theta \bar{\theta} \bar{\theta}$-term contributions giving a higher derivative theory (third order and higher), which cannot be accepted for a local field theory. It is not difficult to get convinced that the most general expression for $K$ which is compatible with all these properties is

$$K(\Phi, \bar{\Phi}) = \sum_{m,n=1}^{\infty} c_{mn} \Phi^m \Phi^n$$

where $c_{mn} = c_{nm}^*$. (5.6)

where the reality condition on $K$ is ensured by the relation between $c_{mn}$ and $c_{nm}$. All coefficients $c_{mn}$ with either $m$ or $n$ greater than one have negative mass dimension, while $c_{11}$ is dimensionless. This means that, in general, a contribution as that in eq. (5.5) will describe a supersymmetric but non-renormalizable theory, typically defined below some cut-off scale $\Lambda$. Indeed, generically, the coefficients $c_{mn}$ will be of the form

$$c_{mn} \sim \Lambda^{2-(m+n)}$$

(5.7)

with the constant of proportionality being a pure number. The function $K$ is called Kähler potential. The reason for such fancy name will become clear later (see §5.1.1).
If renormalizability is an issue, the lowest component of $K$ should not contain operators of dimensionality bigger than 2, given that the $\theta^2\bar{\theta}^2$ component has dimensionality $[K] + 2$. In this case all $c_{mn}$ but $c_{11}$ should vanish and the Kähler potential would just be equal to $\Phi\Phi$, the object we already considered before and which leads to the renormalizable (free) Lagrangian (5.2).

In passing, notice that the combination $\Phi + \bar{\Phi}$ respects all the physical requirements discussed above. However, a term like that would not give any contribution since its $\theta^2\bar{\theta}^2$ component is a total derivative. This means that two Kähler potentials $K$ and $K'$ related as

$$K(\Phi, \bar{\Phi})' = K(\Phi, \bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) ,$$

(5.8)

where $\Lambda$ is a chiral superfield (obtained out of $\Phi$) and $\bar{\Lambda}$ is the corresponding antichiral superfield (obtained out of $\bar{\Phi}$), are different, but their integrals in full superspace, which is all what matters for us, are the same

$$\int d^4x \, d^2\theta \, d^2\bar{\theta} \, K(\Phi, \bar{\Phi})' = \int d^4x \, d^2\theta \, d^2\bar{\theta} \, K(\Phi, \bar{\Phi}) .$$

(5.9)

This is the reason why we did not consider $m, n = 0$ in the expansion (5.6).

Thus far, we have not been able to describe any renormalizable interaction, like non-derivative scalar interactions and Yukawa interactions, which should certainly be there in a supersymmetric theory. How to describe them? As we have just seen, the simplest possible integral in superspace full-filling the minimal and necessary physical requirements, $\Phi\Phi$, already gives two-derivative contributions, see eq. (5.2). Not to mention the more general expression (5.6). What can we do, then?

When dealing with chiral superfields, there is yet another possibility to construct supersymmetric invariant superspace integrals. Let us consider a generic chiral superfield $\Sigma$ (which can be obtained from products of $\Phi$’s, in our case). Integrating it in full superspace would give

$$\int d^4x \, d^2\theta \, d^2\bar{\theta} \, \Sigma = 0 ,$$

(5.10)

since its $\theta^2\bar{\theta}^2$ component is a total derivative. Consider instead integrating $\Sigma$ in half superspace

$$\int d^4x \, d^2\theta \, \Sigma .$$

(5.11)

Differently from the previous one, this integral does not vanish, since now it is the $\theta^2$ component which contributes, and this is not a total derivative for a chiral superfield.
Note that while $S = S(y, \theta)$, in computing (5.11) one can safely take $y^\mu = x^\mu$ since the terms one is missing would just provide total space-time derivatives, which do not contribute to $\int d^4x$. Another way to see it, is to notice that in $(x^\mu, \theta_\alpha, \bar{\theta}_\dot{\alpha})$ coordinate, the chiral superfield $S$ reads $S(x, \theta, \bar{\theta}) = \exp(i\theta \sigma^\mu \bar{\theta} \partial_\mu) S(x, \theta)$. Besides being non-vanishing, (5.11) is also supersymmetric invariant, since the $\theta^2$ component of a chiral superfield transforms as a total derivative under supersymmetry transformations, as can be seen from eq. (4.59)!

An integral like (5.11) is more general than an integral like (5.5). The reason is the following. Any integral in full superspace can be re-written as an integral in half superspace. Indeed, for any superfield $Y$

$$\int d^4 x \, d^2 \theta \, d^2 \bar{\theta} \, Y = \frac{1}{4} \int d^4 x \, d^2 \theta \, \bar{D}^2 Y,$$

(in passing, let us notice that for any arbitrary $Y$, $\bar{D}^2 Y$ is manifestly chiral, since $\bar{D}^3 = 0$ identically). This is because when going from $d\bar{\theta}$ to $\bar{D}$ the difference is just a total space-time derivative, which does not contribute to the above integral. On the other hand, the converse is not true in general. Consider a term like

$$\int d^4 x \, d^2 \theta \, \Phi^\alpha,$$

where $\Phi$ is a chiral superfield. This integral cannot be converted into an integral in full superspace, essentially because there are no covariant derivatives to play with. Integrals like (5.13), which cannot be converted into integral in full superspace, are called F-terms. All others, like (5.12), are called D-terms.

Coming back to our problem, it is clear that since the simplest non-vanishing integral in full superspace, eq. (5.1), already contains field derivatives, we must turn to F-terms. First notice that any holomorphic function of $\Phi$, namely a function $W(\Phi)$ such that $\partial W/\partial \bar{\Phi} = 0$, is a chiral superfield, if so is $\Phi$. Indeed

$$\bar{D}_a W(\Phi) = \frac{\partial W}{\partial \Phi} \bar{D}_a \Phi + \frac{\partial W}{\partial \bar{\Phi}} \bar{D}_\dot{\alpha} \bar{\Phi} = 0.$$  (5.14)

The proposed term for describing interactions in a theory of a chiral superfield is

$$L_{int} = \int d^2 \theta W(\Phi) + \int d^2 \bar{\theta} \, \overline{W(\Phi)},$$  (5.15)

where the hermitian conjugate has been added to make the whole thing real. The function $W$ is called superpotential. Which properties should the (otherwise arbitrary) function $W$ satisfy? First, as already noticed, $W$ should be a holomorphic
function of $\Phi$. This ensures it to be a chiral superfield and the integral (5.15) to be supersymmetric invariant. Second, $W$ should not contain covariant derivatives since $D_\alpha \Phi$ is not a chiral superfield, given that $D_\alpha$ and $\bar{D}_\alpha$ do not (anti)commute. Finally, $[W] = 3$, to make the expression (5.15) have dimension 4. The upshot is that the superpotential should have an expression like

$$W(\Phi) = \sum_{n=1}^{\infty} a_n \Phi^n$$  \hspace{1cm} (5.16)

If renormalizability is an issue, the lowest component of $W$ should not contain operators of dimensionality bigger than 3, given that the $\theta^2$ component has dimensionality $[W] + 1$. Since $\Phi$ has dimension one, it follows that to avoid non-renormalizable operators the highest power in the expansion (5.16) should be $n = 3$, so that the $\theta^2$ component will have operators of dimension 4, at most. In other words, a renormalizable superpotential should be at most cubic.

The superpotential is also constrained by R-symmetry. Given a chiral superfield $\Phi$, if the R-charge of its lowest component $\phi$ is $r$, then that of $\psi$ is $r - 1$ and that of $F$ is $r - 2$. This follows from the commutation relations (2.72) and the variations (4.59). Therefore, we have

$$R[\theta] = 1 \ , \ R[\bar{\theta}] = -1 \ , \ R[d\theta] = -1 \ , \ R[d\bar{\theta}] = 1$$  \hspace{1cm} (5.17)

(recall that $d\theta = \partial/\partial \theta$, and similarly for $\bar{\theta}$). In theories where R-symmetry is a (classical) symmetry, it follows that the superpotential should have R-charge equal to 2

$$R[W] = 2 \ ,$$  \hspace{1cm} (5.18)

in order for the Lagrangian (5.15) to have R-charge 0 and hence be R-symmetry invariant. As far as the Kähler potential is concerned, first notice that the integral measure in full superspace has R-charge 0, because of eqs. (5.17). This implies that for theories with a R-symmetry, the Kähler potential should also have R-charge 0. This is trivially the case for a canonical Kähler potential, since $\Phi \Phi$ has R-charge 0. If one allows for non-canonical Kähler potential, that is non-renormalizable interactions, then besides the reality condition, one should also impose that $c_{nm} = 0$ whenever $n \neq m$, see eq. (5.6).

The integration in superspace of the Lagrangian (5.15) is easily done recalling the expansion of the superpotential in powers of $\theta$. We have

$$W(\Phi) = W(\phi) + \sqrt{2} \frac{\partial W}{\partial \phi} \theta \psi - \theta \theta \left( \frac{\partial W}{\partial \phi} F + \frac{1}{2} \frac{\partial^2 W}{\partial \phi \partial \phi} \bar{\psi} \psi \right),$$  \hspace{1cm} (5.19)
where $\frac{\partial W}{\partial \phi}$ and $\frac{\partial^2 W}{\partial \phi^2}$ are evaluated at $\Phi = \phi$. So we have, modulo total space-time derivatives
\[
L_{\text{int}} = \int d^2 \theta W(\Phi) + \int d^2 \bar{\theta} \bar{W}(\Phi) = -\frac{\partial W}{\partial \phi} F - \frac{1}{2} \frac{\partial^2 W}{\partial \phi^2} \psi \bar{\psi} + \text{h.c. ,}
\] (5.20)
where, again, the r.h.s. is already evaluated at $x^\mu$.

We can now assemble all what we have found. The most generic supersymmetric matter Lagrangian has the following form
\[
L_{\text{matter}} = \int d^2 \theta d^2 \bar{\theta} K(\Phi, \bar{\Phi}) + \int d^2 \theta W(\Phi) + \int d^2 \bar{\theta} \bar{W}(\Phi) .
\] (5.21)
For renormalizable theories the Kähler potential is just $\bar{\Phi} \Phi$ and the superpotential at most cubic. In this case we get
\[
\mathcal{L} = \int d^2 \theta d^2 \bar{\theta} \Phi \bar{\Phi} + \int d^2 \bar{\theta} W(\Phi) + \int d^2 \bar{\theta} \bar{W}(\Phi)
\] (5.22)
\[
= \partial_i \bar{\Phi} \partial^i \Phi + \frac{i}{2} \left( \partial_i \psi \sigma^i \bar{\psi} - \psi \sigma_i \partial_i \bar{\psi} \right) + \bar{F} F - \frac{\partial W}{\partial \phi} F - \frac{1}{2} \frac{\partial^2 W}{\partial \phi^2} \psi \bar{\psi} + \text{h.c. ,}
\] where
\[
W(\Phi) = \sum_{1}^{3} a_n \Phi^n .
\] (5.23)

We can now integrate the auxiliary fields $F$ and $\bar{F}$ out by substituting in the Lagrangian their equations of motion which read
\[
\bar{F} = \frac{\partial W}{\partial \phi} , \quad F = \frac{\partial \bar{W}}{\partial \bar{\phi}} .
\] (5.24)
Doing so, we get the on-shell Lagrangian
\[
\mathcal{L}_{\text{on-shell}} = \partial_i \bar{\Phi} \partial^i \Phi + \frac{i}{2} \left( \partial_i \psi \sigma^i \bar{\psi} - \psi \sigma_i \partial_i \bar{\psi} \right) - \left| \frac{\partial W}{\partial \phi} \right|^2 - \frac{1}{2} \frac{\partial^2 W}{\partial \phi^2} \psi \bar{\psi} - \frac{1}{2} \frac{\partial^2 \bar{W}}{\partial \bar{\phi}^2} \bar{\psi} \psi
\] (5.25)

From the on-shell Lagrangian we can now read the scalar potential which is
\[
V(\phi, \bar{\phi}) = \left| \frac{\partial W}{\partial \phi} \right|^2 = \bar{F} F ,
\] (5.26)
where the last equality holds on-shell, namely upon use of eqs. (5.24).

All what we said so far can be easily generalized to a set of chiral superfields $\Phi^i$ where $i = 1, 2, \ldots, n$. In this case the most general Lagrangian reads
\[
\mathcal{L} = \int d^2 \theta d^2 \bar{\theta} K(\Phi^i, \bar{\Phi}^i) + \int d^2 \theta W(\Phi^i) + \int d^2 \bar{\theta} \bar{W}(\Phi^i) .
\] (5.27)
For renormalizable theories we have
\[ K(\Phi^i, \bar{\Phi}_i) = \bar{\Phi}_i \Phi^i \quad \text{and} \quad W(\Phi^i) = a_i \Phi^i + \frac{1}{2} m_{ij} \Phi^i \Phi^j + \frac{1}{3} g_{ijk} \Phi^i \Phi^j \Phi^k, \quad (5.28) \]
where summation over dummy indices is understood (notice that a quadratic Kähler potential can always be brought to such diagonal form by means of a $GL(n, \mathbb{C})$ transformation on the most general term $K^i_j \Phi_i \Phi^j$, where $K^i_j$ is a constant hermitian matrix). In this case the scalar potential reads
\[ V(\phi^i, \bar{\phi}_i) = \sum_{i=1}^n | \frac{\partial W}{\partial \phi^i} |^2 = \bar{F}_i F^i, \quad (5.29) \]
where
\[ \bar{F}_i = \frac{\partial W}{\partial \phi^i} , \quad F^i = \frac{\partial W}{\partial \bar{\phi}_i} . \quad (5.30) \]

5.1.1 Non-linear sigma model I

The possibility to deal with non-renormalizable supersymmetric field theories we alluded to previously, is not just academic. In fact, one often has to deal with effective field theories at low energy. The Standard Model itself, though renormalizable, is best thought of as an effective field theory, valid up to a scale of order the TeV scale. Not to mention other effective field theories which are relevant beyond the realm of particle physics. In this section we would like to say something more about the Lagrangian (5.27), allowing for the most general Kähler potential and superpotential, and showing that what one ends-up with is nothing but a supersymmetric non-linear $\sigma$-model. Though a bit heavy notation-wise, the effort we are going to do here will be very instructive as it will show the deep relation between supersymmetry and geometry.

Since we do not care about renormalizability here, the superpotential is no more restricted to be cubic and the Kähler potential is no more restricted to be quadratic (though it must still be real and with no covariant derivatives acting on the chiral superfields $\Phi^i$). For later purposes it is convenient to define the following quantities
\[ K_i = \frac{\partial}{\partial \phi^i} K(\phi, \bar{\phi}) , \quad K^i = \frac{\partial}{\partial \bar{\phi}_i} K(\phi, \bar{\phi}) , \quad K^i_j = \frac{\partial^2}{\partial \bar{\phi}^i \partial \phi^j} K(\phi, \bar{\phi}) \]
\[ W_i = \frac{\partial}{\partial \phi^i} W(\phi) , \quad W^i = \bar{W}_i , \quad W_{ij} = \frac{\partial^2}{\partial \phi^i \partial \phi^j} W(\phi) , \quad W^{ij} = \bar{W}_{ij} , \]
where in the above formulæ both the Kähler potential and the superpotential are meant as their restriction to the scalar component of the chiral superfields, while $\phi$
stands for the full $n$-dimensional vector made out of the $n$ scalar fields $\phi^i$ (similarly for $\bar{\phi}$).

Extracting the F-term contribution in terms of the above quantities is pretty simple. The superpotential can be written as

$$W(\Phi) = W(\phi) + W_i \Delta^i + \frac{1}{2} W_{ij} \Delta^i \Delta^j,$$  \hspace{1cm} (5.31)

where we have defined

$$\Delta^i(y) = \Phi^i(y) - \phi^i(y) = \sqrt{2} \theta \psi^i(y) - \theta F^i(y),$$  \hspace{1cm} (5.32)

and we get for the F-term

$$\int d^2 \theta W(\Phi^i) + \int d^2 \bar{\theta} W(\bar{\Phi}_i) = \left( -W_i F^i - \frac{1}{2} W_{ij} \psi^i \psi^j \right) + h.c.,$$  \hspace{1cm} (5.33)

where, see the comment after eq. (5.11), all quantities on the r.h.s. are evaluated in $x^n$.

Extracting the D-term contribution is more tricky (but much more instructive). Let us first define

$$\Delta^i(x) = \Phi^i(x) - \phi^i(x), \quad \bar{\Delta}_i(x) = \bar{\Phi}_i - \bar{\phi}_i(x)$$  \hspace{1cm} (5.34)

which read

$$\Delta^i(x) = \sqrt{2} \theta \psi^i(x) + i \theta \sigma^\mu \partial_\mu \phi^i(x) - \theta F^i(x) - \frac{i}{\sqrt{2}} \theta \bar{\partial}_\mu \phi^i(x) \sigma^\mu \bar{\theta} - \frac{i}{4} \theta \bar{\theta} \bar{\theta} \square \phi^i(x)$$

$$\bar{\Delta}_j(x) = \sqrt{2} \bar{\psi}_j(x) - i \theta \sigma^\mu \partial_\mu \phi_j(x) - \bar{\theta} \bar{\bar{F}}_j(x) + \frac{i}{\sqrt{2}} \bar{\theta} \bar{\theta} \sigma^\mu \partial_\mu \bar{\phi}_j(x) - \frac{i}{4} \theta \bar{\theta} \bar{\theta} \square \bar{\phi}_j(x).$$

Note that $\Delta^i \Delta^j \Delta^k = \bar{\Delta}_i \bar{\Delta}_j \bar{\Delta}_k = 0$. With these definitions the Kähler potential can be written as follows

$$K(\Phi, \bar{\Phi}) = K(\phi, \bar{\phi}) + K_i \Delta^i + K^i \bar{\Delta}_i + \frac{1}{2} K_{ij} \Delta^i \Delta^j + \frac{1}{2} K^{ij} \bar{\Delta}_i \bar{\Delta}_j + K^i \Delta^i \bar{\Delta}_i + \frac{1}{2} K_k^k \Delta^i \bar{\Delta}_k + \frac{1}{2} K^{ij} \Delta^i \bar{\Delta}_j \Delta^k + \frac{1}{4} K_{ij}^{kl} \Delta^i \Delta^j \Delta^k \bar{\Delta}_l.$$  \hspace{1cm} (5.35)

We can now compute the D-term contribution to the Lagrangian. We get

$$\int d^2 \theta d^2 \bar{\theta} \; K(\Phi, \bar{\Phi}) = -\frac{1}{4} K_i \Box \phi^i - \frac{1}{4} K^i \Box \bar{\phi}_i - \frac{1}{4} K_{ij} \partial_\mu \phi^i \partial^\mu \phi^j - \frac{1}{4} K^{ij} \partial_\mu \bar{\phi}_i \partial^\mu \bar{\phi}_j +$$

$$+ \ K^i_j \left( F^i \bar{F}_j + \frac{1}{2} \partial_\mu \phi^i \partial^\mu \bar{\phi}_j - \frac{i}{2} \bar{\psi}^i \sigma^\mu \partial_\mu \bar{\psi}_j + \frac{i}{2} \partial_\mu \psi^i \sigma^\mu \bar{\psi}_j \right)$$

$$+ \ K^k_i \left( \psi^i \sigma^\mu \bar{\psi}_k \partial_\mu \phi^i + \bar{\psi}^i \sigma^\mu \bar{\psi}_k \partial_\mu \phi^i - 2i \psi^i \psi^j \bar{F}_k \right) - \frac{i}{4} K^i_k (h.c.) +$$

$$+ \frac{1}{4} K^{kl} \psi^i \psi^j \bar{\psi}_l \psi^k \psi^l \bigg).$$  \hspace{1cm} (5.36)
up to total derivatives. Notice now that
\[
\Box K(\phi, \bar{\phi}) = K_i \Box \phi^i + K^i_j \Box \bar{\phi}_j + 2K^i_j \partial_\mu \bar{\phi}_j \partial^\mu \phi^i + K_i \partial_\mu \phi^i \partial^\mu \phi^i + K^i_j \partial_\mu \bar{\phi}_j \partial^\mu \bar{\phi}_j .
\]
(5.37)

Using this identity we can eliminate \(K_{ij}\) and \(K^{ij}\), and rewrite eq. (5.36) as
\[
\int d^2 \theta d^2 \bar{\theta} K(\Phi, \bar{\Phi}) = K^j_i \left( F^i_j + \partial_\mu \phi^i \partial^\mu \bar{\phi}_j - \frac{i}{2} \psi^i \sigma^\mu \bar{\psi}_{\bar{j}} \partial_\mu \bar{\psi}_j + \frac{i}{2} \partial_\mu \psi^i \sigma^\mu \bar{\psi}_j \right) +
\]
\[
+ \frac{i}{4} K^k_i \left( \psi^i \sigma^\mu \bar{\psi}_{k} \partial_\mu \phi^i \psi^k + \psi^j \sigma^\mu \bar{\psi}_{k} \partial_\mu \phi^j - 2i \psi^i \psi^j \bar{F}_k \right) - \frac{i}{4} K^{ij} (h.c.) +
\]
\[
+ \frac{1}{4} K^{kl} \psi^i \psi^j \bar{\psi}_{k} \psi_{l} ,
\]
(5.38)
again up to total derivatives.

A few important comments are in order. As just emphasized, independently whether the fully holomorphic and fully anti-holomorphic Kähler potential components, \(K_{ij}\) and \(K^{ij}\) respectively, are or are not vanishing, they do not enter the final result (5.38). In other words, from a practical view point it is as if they are not there. The only two-derivative contribution entering the effective Lagrangian is hence \(K^j_i\). This means that the transformation
\[
K(\phi, \bar{\phi}) \to K(\phi, \bar{\phi}) + \Lambda(\phi) + \overline{\Lambda}(\bar{\phi}) ,
\]
known as Kähler transformation, is a symmetry of the theory (in fact, such symmetry applies to the full Kähler potential, as we have already observed). This is important for our second comment.

The function \(K^j_i\) which normalizes the kinetic term of all fields in eq. (5.38), is hermitian, i.e. \(K^j_i = K^i_j\), since \(K(\phi, \bar{\phi})\) is a real function. Moreover, it is positive definite and non-singular, because of the correct sign for the kinetic terms of all non-auxiliary fields. That is to say \(K^j_i\) has all necessary properties to be interpreted as a metric of a manifold \(\mathcal{M}\) of complex dimension \(n\) whose coordinates are the scalar fields \(\phi^i\) themselves. This is the (supersymmetric) \(\sigma\)-model. The metric \(K^j_i\) is in fact the second derivative of a (real) scalar function \(K\), since
\[
K^j_i = \frac{\partial^2}{\partial \phi^i \partial \bar{\phi}_j} K(\phi, \bar{\phi}) .
\]
(5.40)
In this case we speak of a Kähler metric and the manifold \(\mathcal{M}\) is what mathematically is known as Kähler manifold. The scalar fields are maps from space-time to this Riemannian manifold, which supersymmetry dictates to be a Kähler manifold.
Actually, in order to prove that the Lagrangian is a $\sigma$-model, with target space the Kähler manifold $\mathcal{M}$, we should prove that not only the kinetic term but any other term in the Lagrangian can be written in terms of geometric quantities defined on $\mathcal{M}$: the affine connection, the curvature tensor, etc... With a bit of an effort one can compute the affine connection $\Gamma$ and the curvature tensor $R$ out of the Kähler metric $K_{i}^{j}$ and, using the auxiliary field equations of motion

$$F^{i} = (K^{-1})_{k}^{j}K_{l}^{k}\psi^{l}\psi^{m}$$

(remark: the above equation shows that when the Kähler potential is not canonical, the F-fields can depend also on fermion fields!), get for the Lagrangian

$$\mathcal{L} = K_{i}^{j} \left( \partial_{\mu}\phi^{i}\partial^{\mu}\tilde{\phi}^{j} + \frac{i}{2} D_{\mu}\psi^{j}\sigma^{\mu}\tilde{\psi}^{j} + \frac{i}{2} \psi^{j}\sigma^{\mu}D_{\mu}\tilde{\psi}^{j} \right) - (K^{-1})_{j}^{i}W_{i}W^{j}$$

where

$$V(\phi, \tilde{\phi}) = (K^{-1})_{j}^{i}W_{i}W^{j}$$

is the scalar potential, and the covariant derivatives for the fermions are defined as

$$D_{\mu}\psi^{i} = \partial_{\mu}\psi^{i} + \Gamma_{\mu}^{j}\partial_{\mu}\phi^{i}\psi^{k}$$

$$D_{\mu}\tilde{\psi}_{i} = \partial_{\mu}\tilde{\psi}_{i} + \Gamma_{k}^{j}\partial_{\mu}\phi_{k}\tilde{\phi}_{j}.$$

With our conventions on indices, $\Gamma_{jk}^{i} = (K^{-1})_{m}^{i}K_{jk}^{m}$ while $\Gamma_{k}^{j} = (K^{-1})_{i}^{l}K_{k}^{l}$ and $R_{ijkl}^{k} = K_{k}^{ij} - K_{ij}^{m}(K^{-1})_{m}^{n}K_{nk}^{l}$. As anticipated, a complicated component field Lagrangian is uniquely characterized by the geometry of the target space. Once a Kähler potential $K$ is specified, anything in the Lagrangian (masses and couplings) depends geometrically on this potential (and on $W$). This shows the strong connection between supersymmetry and geometry. There are of course infinitely many Kähler metrics and therefore infinitely many $N = 1$ supersymmetric $\sigma$-models. The normalizable case, $K_{i}^{j} = \delta_{i}^{j}$, is just the simplest such instances.

The more supersymmetry the more constraints, hence one could imagine that there should be more restrictions on the geometric structure of the $\sigma$-model for theories with extended supersymmetry. This is indeed the case, as we will see explicitly when discussing the $N = 2$ version of the supersymmetric $\sigma$-model. In this case, the scalar manifold is further restricted to be a special class of Kähler manifolds, known as special-Kähler manifolds. For $N = 4$ constraints are even
sharper. In fact, in this case the Lagrangian turns out to be unique, the only possible scalar manifold being the trivial one, \( \mathcal{M} = \mathbb{R}^n \), if \( n \) is the number of \( \mathcal{N} = 4 \) vector multiplets (recall that a \( \mathcal{N} = 4 \) vector multiplet contains six scalars). So, for \( \mathcal{N} = 4 \) supersymmetry the only allowed Kähler potential is the canonical one! As we will discuss later, this has immediate (and drastic) consequences on the quantum behavior of \( \mathcal{N} = 4 \) theories.

5.2 \( \mathcal{N} = 1 \) SuperYang-Mills

We would like now to find a supersymmetric invariant action describing the dynamics of vector superfields. In other words, we want to write down the supersymmetric version of Yang-Mills theory. Let us start considering an abelian theory, with gauge group \( G = U(1) \). The basic object we should play with is the vector superfield \( V \), which is the supersymmetric extension of a spin one field. Notice, however, that the vector \( v^\mu \) appears explicitly in \( V \) so the first thing to do is to find a suitable supersymmetric generalization of the field strength, which is the gauge invariant object which should enter the action. Let us define the following superfield

\[
W_a = -\frac{1}{4} \bar{D} \bar{D} D_a V, \quad \bar{W}_\dot{a} = -\frac{1}{4} D \bar{D} \dot{D}_a V, \quad (5.44)
\]

and see if this can do the job. First, \( W_a \) is obviously a superfield, since \( V \) is a superfield and both \( \bar{D}_a \) and \( D_a \) commute with supersymmetry transformations. In fact, \( W_a \) is a chiral superfield, since \( \bar{D}^3 = 0 \) identically. The chiral superfield \( W_a \) is invariant under the gauge transformation (4.62). Indeed

\[
W_a \rightarrow W_a - \frac{1}{4} \bar{D} \bar{D} D_a (\Phi + \bar{\Phi}) = W_a + \frac{1}{4} \bar{D}^3 \bar{D}_a D \Phi \\
= W_a + \frac{1}{4} \bar{D}^3 \{ \bar{D}_a, D \} \Phi = W_a + \frac{i}{2} \sigma_{a\dot{a}} \partial_\mu \bar{D}_a^\mu \Phi = W_a. \quad (5.45)
\]

This also means that, as anticipated, as far as we deal with \( W_a \), we can stick to the WZ-gauge without bothering about compensating gauge transformations or anything.

In order to find the component expression for \( W_a \) it is useful to use the \((y, \theta, \bar{\theta})\) coordinate system, momentarily. In the WZ gauge the vector superfield reads

\[
V_{WZ} = \theta \sigma^\mu \theta \bar{v}_\mu (y) + i \theta \theta \bar{\theta} \lambda (y) - i \bar{\theta} \theta \theta \lambda (y) + \frac{1}{2} \theta \theta \bar{\theta} \bar{\theta} (D(y) - i \partial_\mu \nu^\mu (y)) . \quad (5.46)
\]

It is a simple exercise we leave to the reader to prove that expanding in \((x, \theta, \bar{\theta})\) coordinate system, the above expression reduces to eq. (4.65). Acting with \( D_a \) we
get
\[ D_\alpha V_{WZ} = \sigma^\mu_{\alpha\beta} \partial_\mu \nu + 2i \theta_\alpha \bar{\theta} \lambda - i \bar{\theta} \theta \lambda_\alpha + \theta_\alpha \bar{\theta} \theta D + 2i (\sigma^{\mu\nu})_{\alpha\beta} \theta_\beta \bar{\theta} \theta \partial_\mu \nu + \theta \bar{\theta} \theta \sigma^\mu_{\alpha\beta} \partial_\mu \lambda_{\beta} \]

(5.47)

(where we used the identity \( \sigma^\mu \bar{\sigma}^\nu - \eta^{\mu\nu} = 2 \sigma^{\mu\nu} \) and \( y \)-dependence is understood), and finally
\[ W_\alpha = -i \lambda_\alpha + \theta_\alpha D + i (\sigma^{\mu\nu})_{\alpha} F_{\mu\nu} + \theta \theta (\sigma^\mu \partial_\mu \bar{\lambda})_{\alpha}, \]

(5.48)

where \( F_{\mu\nu} = \partial_\mu \nu - \partial_\nu \nu \) is the usual gauge field strength. So it seems this is the right superfield we were searching for! Indeed, \( W_\alpha \) is the so-called supersymmetric field strength. Note that \( W_\alpha \) is an instance of a chiral superfield whose lowest component is not a scalar field, as we have been used to, but in fact a Weyl fermion, \( \lambda_\alpha \), the gaugino. For this reason, \( W_\alpha \) is also called the gaugino superfield.

Given that \( W_\alpha \) is a chiral superfield, a putative supersymmetric Lagrangian could be constructed out of the following integral in chiral superspace (notice: correctly, it has dimension four)
\[ \int d^2 \theta W^\alpha W_\alpha. \]

(5.49)

Plugging eq. (5.48) into the expression above and computing the superspace integral one gets after some simple algebra
\[ \int d^2 \theta W^\alpha W_\alpha = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - 2i \lambda \sigma^\mu \partial_\mu \bar{\lambda} + D^2 + \frac{i}{4} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \]

(5.50)

One can get a real object by adding the hermitian conjugate to (5.50), having finally
\[ \mathcal{L}_{\text{gauge}} = \int d^2 \theta W^\alpha W_\alpha + \int d^2 \bar{\theta} \bar{W}_a W_a = -F_{\mu\nu} F^{\mu\nu} - 4i \lambda \sigma^\mu \partial_\mu \bar{\lambda} + 2D^2. \]

(5.51)

This is the supersymmetric version of the abelian gauge Lagrangian (up to an overall normalization to be fixed later). As anticipated, \( D \) is an (real) auxiliary field.

The Lagrangian (5.51) has been written as an integral over chiral superspace, so one might be tempted to say it is a F-term. This is wrong since (5.51) is not a true F-term. Indeed, it can be re-written as an integral in full superspace (while F-terms cannot)
\[ \int d^2 \theta W^\alpha W_\alpha = \int d^2 \theta d^2 \bar{\theta} D^\alpha V \cdot W_\alpha, \]

(5.52)

and so it is in fact a D-term. As we will see later, this fact has important consequences at the quantum level, when discussing renormalization properties of supersymmetric Lagrangians.
All what we said, so far, has to do with abelian interactions. What changes if we consider a non-abelian gauge group? First we have to promote the vector superfield to

\[ V = V_a T^a \quad a = 1, \ldots, \text{dim } G, \]  

(5.53)

where \( T^a \) are hermitian generators and \( V_a \) are \( \text{dim } G \) vector superfields. Second, we have to define the finite version of the gauge transformation (4.62) which can be written as

\[ e^V \rightarrow e^{i\Lambda} e^V e^{-i\Lambda}. \]  

(5.54)

One can easily check that at leading order in \( \Lambda \) this indeed reduces to (4.62), upon the identification \( \Phi = -i\Lambda \). Again, it is straightforward to set the WZ gauge for which

\[ e^V = 1 + V + \frac{1}{2} V^2. \]  

(5.55)

In what follows this gauge choice is always understood. The gaugino superfield is generalized as follows

\[ W_\alpha = -\frac{1}{4} \bar{D} \bar{D} \left( e^{-V} D_\alpha e^V \right), \quad \bar{W}_\dot{\alpha} = -\frac{1}{4} D \bar{D} \left( e^V \bar{D}_\dot{\alpha} e^{-V} \right) \]  

(5.56)

which again reduces to the expression (5.44) to first order in \( V \). Let us look at eq. (5.56) more closely. Under the gauge transformation (5.54) \( W_\alpha \) transforms as

\[
\begin{align*}
W_\alpha & \rightarrow -\frac{1}{4} \bar{D} \bar{D} \left[ e^{i\Lambda} e^{-V} e^{-i\bar{D}_\alpha} \left( e^{i\bar{\Lambda}} e^V e^{-i\bar{\Lambda}} \right) \right] \\
& = -\frac{1}{4} \bar{D} \bar{D} \left[ e^{i\Lambda} e^{-V} \left( (D_\alpha e^V) e^{-i\bar{\Lambda}} + e^V D_\alpha e^{-i\bar{\Lambda}} \right) \right] \\
& = -\frac{1}{4} e^{i\Lambda} \bar{D} \bar{D} \left( e^{-V} D_\alpha e^V \right) e^{-i\bar{\Lambda}} = e^{i\Lambda} W_\alpha e^{-i\bar{\Lambda}},
\end{align*}
\]

(5.57)

where we used the fact that, given that \( \Lambda \) (and products thereof) is a chiral superfield, \( \bar{D}_\alpha e^{-i\bar{\Lambda}} = 0 \), \( D_\alpha e^{i\bar{\Lambda}} = 0 \) and also \( \bar{D} \bar{D} D_\alpha e^{-i\bar{\Lambda}} = 0 \). The end result is that \( W_\alpha \) transforms covariantly under a finite gauge transformation, as it should. Similarly, one can prove that

\[ \bar{W}_\dot{\alpha} \rightarrow e^{i\bar{\Lambda}} \bar{W}_\dot{\alpha} e^{-i\bar{\Lambda}}. \]  

(5.58)

Let us now expand \( W_\alpha \) in component fields. We would expect the non-abelian
generalization of eq. (5.48). We have

\[
W_\alpha = -\frac{1}{4} \bar{D} \bar{D} \left[ \left( 1 - V + \frac{1}{2} V^2 \right) D_\alpha \left( 1 + V + \frac{1}{2} V^2 \right) \right] \\
= -\frac{1}{4} \bar{D} \bar{D} D_\alpha V - \frac{1}{8} \bar{D} \bar{D} D_\alpha V^2 + \frac{1}{4} \bar{D} \bar{D} V D_\alpha V \\
= -\frac{1}{4} \bar{D} \bar{D} D_\alpha V - \frac{1}{8} \bar{D} \bar{D} V D_\alpha V - \frac{1}{8} \bar{D} \bar{D} D_\alpha V \cdot V + \frac{1}{4} \bar{D} \bar{D} V D_\alpha V \\
= -\frac{1}{4} \bar{D} \bar{D} D_\alpha V + \frac{1}{8} \bar{D} \bar{D} \left[ V, D_\alpha V \right].
\]

The first term is the same as the one we already computed in the abelian case. As for the second term we get

\[
\frac{1}{8} \bar{D} \bar{D} \left[ V, D_\alpha V \right] = \frac{1}{2} (\sigma^{\mu\nu})_\alpha [v_\mu, v_\nu] - \frac{i}{2} \theta \theta \sigma_{\alpha\beta} [v_\mu, \bar{\lambda}^\beta].
\] (5.59)

Adding everything up simply amounts to turn ordinary derivatives into covariant ones, finally obtaining

\[
W_\alpha = -i \lambda_\alpha (y) + \theta_\alpha D(y) + i (\sigma^{\mu\nu})_\alpha F_{\mu\nu} + \theta \left( \sigma^\mu D_\mu \bar{\lambda}(y) \right)_\alpha
\] (5.60)

with

\[
F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - \frac{i}{2} [v_\mu, v_\nu], \quad D_\mu = \partial_\mu - \frac{i}{2} [v_\mu, ],
\] (5.61)

which provides the correct non-abelian generalization for the field strength and the (covariant) derivatives.

In view of coupling the pure SYM Lagrangian with matter, it is convenient to introduce the coupling constant \( g \) explicitly, making the redefinition

\[
V \to 2gV \leftrightarrow v_\mu \to 2gv_\mu, \quad \lambda \to 2g\lambda, \quad D \to 2gD,
\] (5.62)

which implies the following changes in the final Lagrangian. First, we have now

\[
F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - ig [v_\mu, v_\nu], \quad D_\mu = \partial_\mu - ig [v_\mu, ],
\] (5.63)

Moreover, the gaugino superfield (5.60) should be multiplied by \( 2g \) and the (non-abelian version of the) Lagrangian (5.51) by \( 1/4g^2 \). The end result for the SuperYang-Mills Lagrangian (SYM) reads

\[
\mathcal{L}_{SYM} = \frac{1}{32\pi} \text{Im} \left( \tau \int d^2 \theta \text{Tr} W^\alpha W_\alpha \right)
= \text{Tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i \lambda \sigma^\mu D_\mu \bar{\lambda} + \frac{1}{2} D^2 \right] + \frac{\theta_{YM}}{32\pi^2} g^2 \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu},
\] (5.64)
where we have introduced the complexified gauge coupling
\[
\tau = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2} \quad (5.65)
\]
and the dual field strength
\[
\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad (5.66)
\]
while gauge group generators are normalized as \( \text{Tr} T^a T^b = \delta^{ab} \).

### 5.3 \( \mathcal{N} = 1 \) Gauge-matter actions

We want to couple radiation with matter in a supersymmetric consistent way. To this end, let us consider a chiral superfield \( \Phi \) transforming in some representation \( R \) of the gauge group \( G \), \( T^a \to (T^a_R)_i^j \) where \( i, j = 1, 2, \ldots, \text{dim} R \). Under a gauge transformation we expect
\[
\Phi \to \Phi' = e^{i\Lambda} \Phi, \quad \Lambda = \Lambda_a T^a_R, \quad (5.67)
\]
where \( \Lambda \) must be a chiral superfield, otherwise the transformed \( \Phi \) would not be a chiral superfield anymore. Notice, however, that in this way the chiral superfield kinetic action we have derived previously would not be gauge invariant since
\[
\bar{\Phi} \Phi \to \bar{\Phi} e^{-i\Lambda} e^{i\Lambda} \Phi \neq \bar{\Phi} \Phi. \quad (5.68)
\]
So we have to change the kinetic action. The correct expression is in fact
\[
\bar{\Phi} e^V \Phi, \quad (5.69)
\]
which is indeed gauge invariant and also supersymmetric invariant (modulo total space-time derivatives), when integrated in superspace. Therefore, the complete Lagrangian for charged matter reads
\[
\mathcal{L}_{\text{matter}} = \int d^2\theta d^2\bar{\theta} \, \bar{\Phi} e^V \Phi + \int d^2\theta \, W(\Phi) + \int d^2\bar{\theta} \, \bar{W}(\bar{\Phi}). \quad (5.70)
\]
Obviously the superpotential should be compatible with the gauge symmetry, \textit{i.e.} it should be gauge invariant itself. This means that a term like
\[
a_{a_1 a_2 \ldots a_n} \Phi^{i_1} \Phi^{i_2} \ldots \Phi^{i_n} \quad (5.71)
\]
is allowed only if \( a_{a_1 a_2 \ldots a_n} \) is an invariant tensor of the gauge group and if \( R \times R \times \cdots \times R \) \( n \) times contains the singlet representation of the gauge group \( G \).
As an explicit example, take the gauge group of strong interactions, \( G = SU(3) \), and consider quarks as matter field. In this case \( R \) is the fundamental representation, \( R = 3 \). Since \( 3 \times 3 \times 3 = 1 + \ldots \) and \( \epsilon_{ijk} \) is an invariant tensor of \( SU(3) \), while \( 1 \not\subset 3 \times 3 \) it follows that a supersymmetric and gauge invariant cubic term is allowed, but a mass term is not. In order to have mass terms for quarks, one needs \( R = 3 + \bar{3} \) corresponding to a chiral superfield \( \Phi \) in the \( 3 \) (quark) and a chiral superfield \( \bar{\Phi} \) in the \( \bar{3} \) (anti-quark). In this case \( \bar{\Phi} \Phi \) is gauge invariant and does correspond to a mass term. Notice this is consistent with the fact that a chiral superfield contains a Weyl fermion only and quarks are described by Dirac fermions. The lesson is that to construct supersymmetric actions with colour charged matter, one needs to introduce two sets of chiral superfields which transform in conjugate representations of the gauge group. This is just the supersymmetric version of what happens in ordinary QCD or in any non-abelian gauge theory with fermions transforming in complex representations (\( G = SU(2) \) is an exception because \( 2 \not\cong \bar{2} \)).

Let us now compute the D-term of the Lagrangian (5.70). We have (as usual we work in the WZ gauge)

\[
\bar{\Phi}e^\Phi = \bar{\Phi} \Phi + \Phi V \Phi + \frac{1}{2} \Phi V^2 \Phi .
\] (5.72)

The first term is the one we have already calculated, so let us focus on the D-term contribution of the other two. After some algebra we get

\[
\bar{\Phi}V \Phi |_{00\bar{0}} = \frac{i}{2} \bar{\phi} v^\mu \partial_\mu \phi - \frac{i}{2} \partial_\mu \bar{\phi} v^\mu \phi - \frac{1}{2} \bar{\psi} \sigma^\mu v_\mu \psi + \frac{i}{\sqrt{2}} \bar{\phi} \lambda \phi - \frac{i}{\sqrt{2}} \bar{\psi} \lambda \phi + \frac{1}{2} \bar{\phi} D \phi
\]

\[
\bar{\Phi}V^2 \Phi |_{00\bar{0}} = \frac{1}{2} \bar{\phi} v^\mu v_\mu \phi .
\]

Putting everything together we finally get (up to total derivatives)

\[
\bar{\Phi}e^\Phi V |_{00\bar{0}} = (D_\mu \bar{\phi}) D^\mu \phi - i \bar{\psi} \sigma^\mu D_\mu \psi + \bar{F} F + i \frac{1}{\sqrt{2}} \bar{\phi} \lambda \phi - i \frac{1}{\sqrt{2}} \bar{\psi} \lambda \phi + \frac{1}{2} \bar{\phi} D \phi ,
\] (5.73)

where \( D_\mu = \partial_\mu - \frac{i}{2} v_\mu T^a_R \).

Performing the rescaling \( V \rightarrow 2g V \) and rewriting \( \bar{\psi} \sigma^\mu D_\mu \psi = \psi \sigma^\mu D_\mu \bar{\psi} \) (recall the spinor identity \( \chi \sigma^\mu \bar{\psi} = -\bar{\psi} \sigma^\mu \chi \)) we get finally

\[
\bar{\Phi}e^{2gV} \Phi |_{00\bar{0}} = (D_\mu \bar{\phi}) D^\mu \phi - i \psi \sigma^\mu D_\mu \bar{\psi} + \bar{F} F + i \sqrt{2} g \bar{\phi} \lambda \phi - i \sqrt{2} g \bar{\psi} \lambda \phi + g \bar{\phi} D \phi ,
\] (5.74)

where now \( D_\mu = \partial_\mu - ig v_\mu T^a_R \). So we see that the D-term in the Lagrangian (5.70) not only provides matter kinetic terms but also interaction terms between matter
fields $\phi, \psi$ and gauginos $\lambda$, where it is understood that
\[ \bar{\phi}\lambda\psi = \bar{\phi}_i (T_R^a)^i_j \lambda^j, \] (5.75)
and similarly for the other couplings.

To get the most general action there is one term still missing: the so called Fayet-Iliopulos term. Suppose that the gauge group is not semi-simple, i.e. it contains $U(1)$ factors. Let $V^A$ be the vector superfields corresponding to the abelian factors, $A = 1, 2, \ldots, n$, where $n$ is the number of abelian factors. The D-term of $V^A$ transforms as a total derivative under super-gauge transformations, since
\[ V^A \rightarrow V^A - i\Lambda + i\bar{\Lambda} : \quad D^A \rightarrow D^A + \partial_\mu \partial^\mu \ldots \]. (5.76)
Therefore a Lagrangian of this type
\[ \mathcal{L}_{FI} = \sum_A \xi_A \int d^2\theta d^2\bar{\theta} V^A = \frac{1}{2} \sum_A \xi_A D^A \] (5.77)
is supersymmetric invariant (since $V^A$ are superfields) and gauge invariant, modulo total space-time derivatives.

We can now assemble all ingredients and write the most general $\mathcal{N} = 1$ supersymmetric Lagrangian (with canonical Kähler potential, hence renormalizable, if the superpotential is at most cubic)
\[ \mathcal{L} = \mathcal{L}_{SYM} + \mathcal{L}_{matter} + \mathcal{L}_{FI} = \]
\[ \mathcal{L} = \frac{1}{32\pi} \text{Im} \left[ \tau \int d^2\theta \text{Tr} W^\alpha W_\alpha \right] + 2g \sum_A \xi_A \int d^2\theta d^2\bar{\theta} V^A + \]
\[ + \int d^2\theta d^2\bar{\theta} \Phi e^{2g \Phi} + \int d^2\theta W (\Phi) + \int d^2\bar{\theta} \bar{W}(\bar{\Phi}) \]
\[ = \text{Tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\lambda \sigma^\mu D_\mu \bar{\lambda} + \frac{1}{2} D_\mu \bar{\lambda} \right] + \frac{\theta_{YM}}{32\pi^2} g^2 \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \]
\[ + g \sum_A \xi_A D^A + (D_{\mu\phi}) D^\mu \phi - i\psi \sigma^\mu D_\mu \bar{\psi} + \bar{F} F + i\sqrt{2} g \bar{\phi} \lambda \psi \]
\[ - i\sqrt{2} \bar{\psi} \bar{\phi} \lambda \phi + \bar{\phi} D^i \phi - \frac{\partial W}{\partial \phi^i} F^i - \frac{\partial\bar{W}}{\partial \bar{\phi}^i} \bar{F}_i - \frac{1}{2} \frac{\partial^2 W}{\partial \phi^i \partial \phi^j} \psi^i \psi^j - \frac{1}{2} \frac{\partial^2 \bar{W}}{\partial \bar{\phi}^i \partial \bar{\phi}^j} \bar{\psi}_i \bar{\psi}_j . \]
Notice that both $D^a$ and $F^i$ are auxiliary fields and can be integrated out. The equations of motion of the auxiliary fields read
\[ \bar{F}_i = \frac{\partial W}{\partial \phi^i}, \quad D^a = -g \bar{\psi} T^a \phi - g \xi^a \quad (\xi^a = 0 \text{ if } a \neq A) . \] (5.79)
These can plugged back into (5.78) leading to the following on-shell Lagrangian

\[ \mathcal{L} = \text{Tr} \left[ -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - i \lambda \sigma^\mu D_\mu \lambda \right] + \frac{\theta_{YM}}{32 \pi^2} g^2 \text{Tr} F_{\mu \nu} \tilde{F}^{\mu \nu} + \partial_\mu \phi \partial^\mu \phi - i \dot{\psi} \sigma^\mu D_\mu \bar{\psi} + i \sqrt{2} g \phi \dot{\phi} \psi - i \sqrt{2} g \bar{\phi} \bar{\psi} - \frac{1}{2} \partial^2 W \psi^i \psi^j - \frac{1}{2} \partial^2 \bar{W} \bar{\psi}^i \bar{\psi}^j - V(\phi, \bar{\phi}) , \tag{5.80} \]

where the scalar potential \( V(\phi, \bar{\phi}) \) is

\[ V(\phi, \bar{\phi}) = \frac{\partial W}{\partial \phi^i} \frac{\partial \bar{W}}{\partial \bar{\phi}^i} + \frac{g^2}{2} \sum_a |\bar{\phi}_a(T^a)^i_j \phi^j + \xi^a|^2 \tag{5.81} \]

\[ = \bar{F} F + \frac{1}{2} D^2 \text{on the solution } \geq 0 . \]

We see that the potential is a semi-positive definite quantity in supersymmetric theories. \textit{Supersymmetric vacua}, if any, are described by its zero’s which are described by sets of scalar field VEVs which solve simultaneously the so-called D-term and F-term equations

\[ \bar{F}_i(\phi) = 0 \ , \ D^a(\phi, \bar{\phi}) = 0 . \tag{5.82} \]

That supersymmetric vacua correspond to the zeros of the potential can be easily seen as follows. First recall that a vacuum is a Lorentz invariant state configuration. This means that all field derivatives and all fields but scalar ones should vanish in a vacuum state. Hence, the only non trivial thing of the Hamiltonian which can be different from 0 is the non-derivative scalar part, which, by definition, is indeed the scalar potential. Therefore, the vacua of a theory, which are the minimal energy states, are in one-to-one correspondence with the (global or local) minima of the scalar potential. Now, as we have already seen, in a supersymmetric theory the energy of any state is semi-positive definite. This holds also for vacua (which are the minimal energy states). Hence for a vacuum \( \Omega \) we have

\[ \langle \Omega | P^0 | \Omega \rangle \sim \sum_a (||Q_a|\Omega||^2 + ||Q^a_\dagger|\Omega||^2) \geq 0 . \tag{5.83} \]

This means that the vacuum energy is 0 if and only if it is a supersymmetric state, that is \( Q_a|\Omega\rangle = 0, Q^a_\dagger|\Omega\rangle = 0 \ \forall a \). Conversely, supersymmetry is broken (in the perturbative theory based on this vacuum) if and only if the vacuum energy is positive. Hence, supersymmetric vacua are indeed in one-to-one correspondence with the zero’s of the scalar potential.

To find such zero’s, one first looks for the space of scalar field VEVs such that

\[ D^a(\phi, \bar{\phi}) = 0 , \tag{5.84} \]
which is called the space of D-flat directions. If a superpotential is present, one should then consider the F-term equations, which will put further constraints on the subset of scalar field VEVs already satisfying the D-term equations (5.84). The subspace of the space of D-flat directions which is also F-flat, i.e. which also satisfies the equations

\[ \bar{F}^i(\phi) = 0 , \]

is called (classical) moduli space and represents the space of (classical) supersymmetric vacua. Clearly, in solving for the D-term equations, one should mod out by gauge transformations, since solutions which are related by gauge transformations are physically equivalent and describe the same vacuum state.

That the moduli space parametrizes the so-called flat directions is because it represents the space of fields the potential does not depend on, and each such flat direction has a massless particle associated to it, a modulus. The moduli represent the lightest degrees of freedom of the low energy effective theory (think about the supersymmetric \(\sigma\)-model we discussed in §5.1.1). As one moves along the moduli space one spans physically inequivalent (supersymmetric) vacua, since the mass spectrum of the theory changes from point to point, as generically particles masses will depend on scalar field VEVs.

In passing, let us notice that while in a non-supersymmetric theory (or in a supersymmetry breaking vacuum of a supersymmetric theory) the space of classical flat directions, if any, is generically lifted by radiative corrections (which can be computed at leading order by e.g. the Coleman-Weinberg potential), in supersymmetric theories this does not happen. If the ground state energy is zero at tree level, it remains so at all orders in perturbation theory. This is because perturbations around a supersymmetric vacuum are themselves described by a supersymmetric Lagrangian and quantum corrections are protected by cancellations between fermionic and bosonic loops. This means that the only way to lift a classical supersymmetric vacuum, namely to break supersymmetry, if not at tree level, are non-perturbative corrections. We will have much more to say about this issue in later lectures.

### 5.3.1 Classical moduli space: examples

To make concrete the previous discussion on moduli space, in what follows we would like to consider two examples explicitly. Before we do that, however, we would like to rephrase our definition of moduli space, presenting an alternative (but equivalent) way to describe it.
Suppose we are considering a theory without superpotential. For such a theory the space of D-flat directions coincides with the moduli space. The space of D-flat directions is defined as the set of scalar field VEVs satisfying the D-flat conditions

\[ \mathcal{M}_d = \{ \langle \phi_i \rangle \mid D^a = 0 \ \forall a \}/\text{gauge transformations} . \]  

(5.86)

Generically it is not at all easy to solve the above constraints and find a simple parametrization of the \( \mathcal{M}_d \). An equivalent, though less transparent definition of the space of D-flat directions can help in this respect. It turns out that the same space can be defined as the space spanned by all (single trace) gauge invariant operator VEVs made out of scalar fields, modulo classical relations between them

\[ \mathcal{M}_d = \{ \langle \text{Gauge invariant operators } \equiv X_r(\phi) \rangle \}/\text{classical relations} . \]  

(5.87)

The latter parametrization is very convenient since, up to classical relations, the construction of the moduli space is unconstrained. In other words, the gauge invariant operators provide a direct parametrization of the space of scalar field VEVs satisfying the D-flat conditions (5.84).

Notice that if a superpotential is present, this is not the end of the story: F-equations will put extra constraints on the \( X_r(\phi) \)'s and may lift part of (or even all) the moduli space of supersymmetric vacua. In later lectures we will discuss some such instances in detail. Below, in order clarify the equivalence between definitions (5.86) and (5.87), we will consider two concrete models with no superpotential term, instead.

**SQED.** The first example we want to consider is SQED, the supersymmetric version of quantum electrodynamics. This is a SYM theory with gauge group \( U(1) \) and \( F \) (couples of) chiral superfields \( (Q_i, \tilde{Q}_i) \) having opposite charge with respect to the gauge group (we will set for definiteness the charges to be \( \pm 1 \)) and no superpotential, \( W = 0 \). The vanishing of the superpotential implies that for this system the space of D-flat directions coincides with the moduli space of supersymmetric vacua. The Lagrangian is an instance of the general one we derived before and reads

\[ \mathcal{L}_{SQED} = \frac{1}{32\pi} \text{Im} \left( \tau \int d^2\theta \ W^\alpha W_\alpha \right) + \int d^2\theta d^2\bar{\theta} \left( Q_i^\dagger e^{2V} Q^i + \tilde{Q}_i^\dagger e^{-2V} \tilde{Q}^i \right) \]  

(5.88)

(in order to ease the notation, we have come back to the most common notation for indicating the hermitian conjugate of a field).
The (only one) D-equation reads

\[ D = Q_i^\dagger Q^i - \hat{Q}_i^\dagger \hat{Q}^i = 0 \tag{5.89} \]

where here and in the following a \( \langle \rangle \) is understood whenever \( Q \) or \( \hat{Q} \) appear.

What is the moduli space? Let us first use the definition (5.86). The number of putative complex scalar fields parametrizing the moduli space is \( 2F \). We have one D-term equation only, which provides one real condition, plus gauge invariance

\[ Q^i \to e^{i\alpha} Q^i, \quad \hat{Q}^i \to e^{-i\alpha} \hat{Q}^i, \tag{5.90} \]

which provides another real condition. Therefore, the dimension of the moduli space is

\[ \dim \mathcal{M}_d = 2F - \frac{1}{2} - \frac{1}{2} = 2F - 1. \tag{5.91} \]

At a generic point of the moduli space the gauge group \( U(1) \) is broken. Indeed, the \(-1\) above corresponds to the complex field which, together with its fermionic superpartner, gets eaten by the vector superfield to give a massive vector superfield (recall the content of a massive vector multiplet). One component of the complex scalar field provides the third polarization to the otherwise massless photon; the other real component provides the real physical scalar field a massive vector superfield has; finally, the Weyl fermion provides the extra degrees of freedom to let the photino become massive. This is nothing but the supersymmetric version of the Higgs mechanism. As anticipated, the vacua are physically inequivalent, generically, since e.g. the mass of the photon depends on the VEV of the scalar fields.

Let us now repeat the above analysis using the definition (5.87). The only gauge invariants we can construct are

\[ M^i_j = Q^i \hat{Q}^j, \tag{5.92} \]

the mesons. As for the classical relation, this can be found as follows. The meson matrix \( M \) is a \( F \times F \) matrix with rank one since so is the rank of \( Q \) and \( \hat{Q} \) (\( Q \) and \( \hat{Q} \) are vectors of length \( F \), since the gauge group is abelian). This implies that the meson matrix has only one non-vanishing eigenvalue which means

\[ \det (M - \lambda \mathbf{1}) = \lambda^{F-1}(\lambda - \lambda_0)(-1)^F. \tag{5.93} \]

Recalling that for a matrix \( A \)

\[ \epsilon_{i_1i_2\ldots i_F} A_{j_1}^{i_1} A_{j_2}^{i_2} \cdots A_{j_F}^{i_F} = \det A \epsilon_{j_1j_2\ldots j_F} \tag{5.94} \]
with \( \epsilon_{i_1i_2\ldots i_F} \) the fully antisymmetric tensor with \( F \) indices, we have

\[
\epsilon_{i_1i_2\ldots i_F} (M_{j_1}^{i_1} - \lambda \delta_{j_1}^{i_1}) \ldots (M_{j_F}^{i_F} - \lambda \delta_{j_F}^{i_F}) = \lambda^{F-1}(\lambda - \lambda_0)(-1)^F \epsilon_{j_1j_2\ldots j_F} \tag{5.95}
\]

which means that from the left hand side only the coefficients of the terms \( \lambda^F \) and \( \lambda^{F-1} \) survive. The next contribution, proportional to \( \lambda^{F-2} \), should vanish, that is

\[
\epsilon_{i_1i_2\ldots i_F} M_{j_1}^{i_1} M_{j_2}^{i_2} \epsilon_{j_1j_2\ldots j_F} = 0 . \tag{5.96}
\]

These are the classical relations: \((F-1)^2\) complex conditions the meson matrix \( M \) should satisfy. Since the meson matrix is a complex \( F \times F \) matrix, we finally get that

\[
\text{dim}_C \mathcal{M}_{cl} = F^2 - (F - 1)^2 = 2F - 1 . \tag{5.97}
\]

which coincides with what we have found before!

The parametrization in terms of (single trace) gauge invariant operators is very useful if one wants to find the low energy effective theory around the supersymmetric vacua. Indeed, up to classical relations, these gauge invariant operators (in fact, their fluctuations) directly parametrize the massless degrees of freedom of the perturbation theory constructed upon these same vacua. Using equation (5.89) and the fact that the meson matrix has rank one, one can easily show that on the moduli space

\[
\text{Tr} Q^\dagger Q = \text{Tr} \tilde{Q}^\dagger \tilde{Q} = \text{Tr} M^\dagger M . \tag{5.98}
\]

Therefore, the Kähler potential, which is canonical in terms of the microscopic UV degrees of freedom \( Q \) and \( \tilde{Q} \), once projected on the moduli space reads

\[
K = \text{Tr} \left[ Q^\dagger Q + \tilde{Q}^\dagger \tilde{Q} \right] = 2\text{Tr} \sqrt{M^\dagger M} . \tag{5.99}
\]

The Kähler metric of the non-linear \( \sigma \)-model hence reads

\[
ds^2 = K_{MM^\dagger} dM dM^\dagger = \frac{1}{2} \frac{1}{\sqrt{M^\dagger M}} dM dM^\dagger \tag{5.100}
\]

which is manifestly non-canonical. Notice that the (scalar) kinetic term

\[
\frac{1}{2} \int d^4x \frac{1}{\sqrt{M^\dagger M}} \partial_\mu M \partial^\mu M^\dagger \tag{5.101}
\]

is singular at the origin, since the Kähler metric diverges there. This has a clear physical interpretation: at the origin the theory is un-higgsed, the photon becomes massless and the correct low energy effective theory should include it in the description. This is a generic feature in all this business: singularities showing-up at
specific points of the moduli space are a signal of extra-massless degrees of freedom that, for a reason or another, show up at those specific points

\[ \text{Singularities} \leftrightarrow \text{New massless d.o.f.} \]  

The correct low energy, singularity-free, effective description of the theory should include them. The singular behavior of \( K_{\mu\nu} \) at the origin is simply telling us that.

**SQCD.** Let us now consider the non-abelian version of the previous theory. We have now a gauge group \( SU(N) \) and \( F \) flavors. The quarks superfields \( Q \) and \( \tilde{Q} \) are \( F \times N \) matrices, and again there is no superpotential, \( W = 0 \). Looking at the Lagrangian, which is the obvious generalization of (5.88), we see there are two independent flavor symmetries, one associated to \( Q \) and one to \( \tilde{Q} \), \( SU(F)_L \) and \( SU(F)_R \) respectively

\[
\begin{array}{ccc}
SU(N) & SU(F)_L & SU(F)_R \\
Q^i_a & N & F & 1 \\
\tilde{Q}^b_j & \bar{N} & 1 & F \\
\end{array}
\]  

where \( i, j = 1, 2, \ldots, F \) and \( a, b = 1, 2, \ldots, N \). The convention for gauge indices is that lower indices are for an object transforming in the fundamental representation and upper indices for an object transforming in the anti-fundamental. The convention for flavor indices is chosen to be the opposite one. Given these conventions, the D-term equations read

\[
D^A = Q^i_j (T^A)^b_i Q^i_c + \tilde{Q}^i_j (T^A)^b_i \tilde{Q}^i_c = Q^i_j (T^A)^b_i Q^i_c - \tilde{Q}^i_j (T^A)^b_i \tilde{Q}^i_c = 0 ,
\]

where \( A = 1, 2, \ldots, N^2 - 1 \), and we used the fact that \( (T^A)^a_b = -(T^A)^a_b \equiv (T^A)^a_b \).

Let us first focus on the case \( F < N \). Using the two \( SU(F) \) flavor symmetries and the (global part of the) gauge symmetry \( SU(N) \), one can show that on the moduli space (5.104) the matrices \( Q \) and \( \tilde{Q} \) can be put, at most, in the following form (recall that the maximal rank of \( Q \) and \( \tilde{Q} \) is \( F \) in this case, since \( F < N \))

\[
Q = \begin{pmatrix}
v_1 & 0 & \ldots & 0 & \ldots \\
0 & v_2 & \ldots & 0 & \ldots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
0 & 0 & \ldots & v_F & \ldots \\
\end{pmatrix} = \tilde{Q}^T
\]  

(5.105)
This means that at a generic point of the moduli space the gauge group is broken to \( SU(N - F) \). So, the complex dimension of the classical moduli space is

\[
\dim_{\mathbb{C}} \mathcal{M}_c = 2FN - \{ N^2 - 1 - [(N - F)^2 - 1] \} = F^2 .
\] (5.106)

Let us now use the parametrization in terms of gauge invariant single trace operators (5.87). In this case we have

\[
M^i_j = Q^i_a \tilde{Q}^a_j
\] (5.107)

(notice the contraction on the \( N \) gauge indices). The meson matrix has now maximal rank, since \( F < N \), so there are no classical constraints it has to satisfy: its \( F^2 \) entries are all independent. In terms of the meson matrix the classical moduli space dimension is then (trivially) \( F^2 \), in agreement with eq. (5.106). Again, playing with global symmetries, \( M \) can be diagonalized in terms of \( F \) complex eigenvalues \( V_i \), which, obviously, are nothing but the square of the ones in (5.105), \( V_i = v_i^2 \).

A similar reasoning as the one working for SQED would hold about possible singularities in the moduli space. On the moduli space we have \( Q^i_a Q^i_b = \tilde{Q}^a_i \tilde{Q}^i_b \). Using this identity we get

\[
( M^\dagger M )^i_j = \tilde{Q}^i_a Q^i_k \tilde{Q}^k_j = \tilde{Q}^i_a \tilde{Q}^a_k \tilde{Q}^i_j
\] (5.108)

which implies \( \tilde{Q}^i \tilde{Q} = \sqrt{M^\dagger M} \) as a matrix equation. So the Kähler potential is

\[
K = 2 \text{Tr} \sqrt{M^\dagger M} .
\] (5.109)

The Kähler metric is singular whenever the meson matrix \( M \) is not invertible. This does not only happen at the origin of field space as for SQED, but actually at subspaces where some of the \( N^2 - 1 - [(N - F)^2 - 1] = (2N - F)F \) massive gauge bosons parametrizing the coset \( SU(N)/SU(N - F) \) become massless, and they need to be included in the low energy effective description.

Let us now consider the case \( F \geq N \). Following a similar procedure as the one before, the matrices \( Q \) and \( \tilde{Q} \) can be brought to the following form on the moduli space

\[
Q = \begin{pmatrix}
    v_1 & 0 & \ldots & 0 \\
    0 & v_2 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & v_N \\
    0 & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & 0
\end{pmatrix}, \quad \tilde{Q}^T = \begin{pmatrix}
    \tilde{v}_1 & 0 & \ldots & 0 \\
    0 & \tilde{v}_2 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & \tilde{v}_N \\
    0 & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & 0
\end{pmatrix}
\] (5.110)
where $|v_i|^2 - |\tilde{v}_i|^2 = a$, with $a$ a $i$-independent number. Since $F \geq N$, at a generic point in the moduli space the gauge group is now completely higgsed. Therefore, the dimension of the classical moduli space is now

$$\dim \mathcal{M}_{cl} = 2NF - (N^2 - 1). \quad (5.111)$$

The parametrization in terms of gauge invariant operators is slightly more involved, in this case. The mesons are still there, and defined as in eq. (5.107). However, there are non-trivial classical constraints one should take into account, since the rank of the meson matrix, which is at most $N$, is now smaller than its dimension, $F$. Moreover, besides the mesons, there are now new gauge invariant single trace operators one can build, the baryons, which are operators made out of $N$ fields $Q$ respectively $N$ fields $\tilde{Q}$, with fully anti-symmetrized indices.

As an explicit example of this richer structure, let us apply the above rationale to the case $N = F$. According to eq. (5.111), in this case $\dim \mathcal{M}_{cl} = F^2 + 1$. The gauge invariant operators are the meson matrix plus two baryons, $B$ and $\tilde{B}$, defined as

$$B = \epsilon^{a_1a_2...a_N} Q^1_{a_1} Q^2_{a_2} ... Q^N_{a_N}$$
$$\tilde{B} = \epsilon_{a_1a_2...a_N} \tilde{Q}^1_{a_1} \tilde{Q}^2_{a_2} ... \tilde{Q}^N_{a_N}.$$ 

Since $F = N$ the anti-symmetrization on the flavor indices is automatically taken care of, once anti-symmetrization on the gauge indices is imposed. All in all, we have apparently $F^2 + 2$ complex moduli space directions. There is however one classical constraints between them which reads

$$\det M - B\tilde{B} = 0, \quad (5.112)$$

as can be easily checked from the definition of the meson matrix (5.107) and that of the baryons above. Hence, the actual dimension of the moduli space is $F^2 + 2 - 1 = F^2 + 1$, as expected. As for the case $F < N$, there is a subspace in the moduli space, which includes the origin, in which some fields become massless and the low energy effective analysis should be modified to include them.

In fact, all we said, so far, is true classically. As we will later see, (non-perturbative) quantum corrections sensibly change this picture and the final structure of SQCD moduli space differs in many respects from the one above. For instance, for $F = N$ SQCD, it turns out that the classical constraint (5.112) is modified at the quantum level; this has the effect of excising the origin of field space,
\( Q_{ia} = \tilde{Q}_{jb} = 0 \) from the actual quantum moduli space, removing, in turn, all singularities and corresponding new massless degrees of freedom, which are then just an artifact of the classical analysis, in this case.

We will have much more to say about SQCD and its classical and quantum properties at some later stage.

### 5.3.2 The SuperHiggs mechanism

We have alluded several times to a supersymmetric version of the Higgs mechanism. In the following, by considering a concrete example, we would like to show how the superfield degrees of freedom rearrangement explicitly works upon higgsing. As we already noticed, it is expected that upon supersymmetric higgsing a full vector superfield becomes massive, eating up a chiral superfield.

Let us consider once again SQCD with gauge group \( SU(N) \) and \( F \) flavors and focus on a point of the moduli space where all scalar field VEVs \( v_i \) are 0 but \( v_1 \). At such point of the moduli space the gauge group is broken to \( SU(N-1) \) and flavor symmetries are broken to \( SU(F-1)_L \times SU(F-1)_R \). The number of broken generators is

\[
N^2 - 1 - [(N-1)^2 - 1] = 2(N-1) + 1 \tag{5.113}
\]

which just corresponds to the statement that if we decompose the adjoint representation of \( SU(N) \) into \( SU(N-1) \) representations we get

\[
\text{Adj}_N = 1 \oplus \bigoplus_{N-1} \oplus \bigoplus_{N-1} \oplus \text{Adj}_{N-1} \rightarrow G^A = X^0, X^a_1, X^a_2, T^a \tag{5.114}
\]

where \( G^A \) are the generators of \( SU(N) \), \( T^a \) those of \( SU(N-1) \) and the \( X^a \)'s the generators of the coset \( SU(N)/SU(N-1) \) \( (A = 1, 2, \ldots, N^2 - 1, a = 1, 2, \ldots, (N-1)^2 - 1 \) and \( \alpha = 1, 2, \ldots, N-1 \).

Upon this decomposition, the matter fields matrices can be re-written schematically as

\[
Q = \begin{pmatrix} \omega^0 & \psi \\ \omega & Q' \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} \tilde{\omega}^0 & \tilde{\psi} \\ \tilde{\omega} & \tilde{Q}' \end{pmatrix} \tag{5.115}
\]

where, with respect to the surviving gauge and flavor symmetries, \( \omega^0 \) and \( \tilde{\omega}^0 \) are singlets, \( \omega \) and \( \tilde{\omega} \) are flavor singlets but carries the fundamental (resp anti-fundamental) representation of \( SU(N-1) \), \( \psi \) (resp \( \tilde{\psi} \)) are gauge singlets and transform in the fundamental representation of \( SU(F-1)_L \) (resp \( SU(F-1)_R \)), and finally \( Q' \) (resp...
are in the fundamental (resp. anti-fundamental) representation of $SU(N-1)$ and in the fundamental representation of $SU(F-1)_L$ (resp. $SU(F-1)_R$).

By expanding the scalar fields around their VEVs (which are all vanishing but $v_1$) and plugging them back into the SQCD Lagrangian, after some tedious algebra one finds all $v_1$-dependent fermion and scalar masses, together with massive gauge bosons and corresponding massive gauginos. On top of this, there remains a set of massless fields, belonging to the massless vector superfields spanning $SU(N-1)$ and the massless chiral superfields $Q'$ and $\tilde{Q}'$. We refrain to perform this calculation explicitly and just want to show that the number of bosonic and fermionic degrees of freedom, though arranged differently in the supersymmetry algebra representations, are the same before and after Higgsing.

We just focus on bosonic degrees of freedom, since, due to supersymmetry, the same result holds for the fermionic ones. For $v_1 = 0$ we have a fully massless spectrum. As far as bosonic degrees of freedom are concerned we have $2(N^2 - 1)$ of them coming from the gauge bosons and $4NF$ coming from the complex scalars. All in all there are

$$2(N^2 - 1) + 4NF$$

bosonic degrees of freedom.

For $v_1 \neq 0$ things are more complicated. As for the vector superfield degrees of freedom, we have $(N-1)^2 - 1$ massless ones, which correspond to $2[(N-1)^2 - 1]$ bosonic degrees of freedom, and $1 + 2(N-1)$ massive ones which correspond to $4 + 8(N-1)$ bosonic degrees of freedom. As for the matter fields (5.115), the massive ones are not there since they have been eaten by the (by now massive) vectors. These are $\omega$ and $\tilde{\omega}$, which are eaten by the vector multiplets associated to the generators $X_1^a$ and $X_2^a$, and the combination $\omega^0 - \tilde{\omega}^0$ which is eaten by the vector multiplet associated to the generator $X^0$. We have already taken them into account, then. The massless chiral superfields $Q'$ and $\tilde{Q}'$ provide $2(N-1)(F-1)$ bosonic degrees of freedom each, the symmetric combination $S = \omega^0 + \tilde{\omega}^0$ another 2 bosonic degrees of freedom, and finally the massless chiral superfields $\psi$ and $\bar{\psi}$ 2$(F-1)$ each. All in all we get

$$2[(N-1)^2 - 1] + 4 + 8(N-1) + 4(N-1)(F-1) + 2 + 4(F-1) = 2(N^2 - 1) + 4NF,$$

which are exactly the same as those of the un-higgsed phase (5.116).
5.3.3 Non-linear sigma model II

In section 5.1.1 we discussed the supersymmetric non-linear $\sigma$-model for matter fields, which is relevant to describe supersymmetric low energy effective theories. Though it is not often the case, it may happen to face effective theories with some left over propagating gauge degrees of freedom at low energy. Therefore, in this section we will generalize the $\sigma$-model of section 5.1.1 to such a situation: a supersymmetric but non-renormalizable effective theory coupled to gauge fields. Note that the choice of the gauge group cannot be arbitrary here. In order to preserve the structure of the non-linear $\sigma$-model one can gauge only a subgroup $G$ of the isometry group of the scalar manifold.

Following the previous strategy, one gets easily convinced that the pure SYM part changes simply by promoting the (complexified) gauge coupling $\tau$ to a holomorphic function of the chiral superfields, getting

$$
\tau \int d^2 \theta \, \text{Tr} W^a W_a \longrightarrow \int d^2 \theta \, F_{ab}(\Phi) W^{a} W^{b},
$$

(5.118)

where the chiral superfield $F_{ab}(\Phi)$ should transform in the $\text{Adj} \otimes \text{Adj}$ of the gauge group $G$ in order for the whole action to be $G$-invariant. Notice that for $F_{ab} = \tau \text{Tr} T_a T_b$ one gets back the usual result (recall that we have normalized the gauge group generators as $\text{Tr} T_a T_b = \delta_{ab}$). For this reason the function $F_{ab}$ (actually its restriction to the scalar fields) is dubbed generalized complex gauge coupling.

As for the matter Lagrangian, given what we have already seen, namely that whenever one has to deal with charged matter fields the gauge invariant combination is $(\Phi e^{2gV}), \Phi^i$, one should simply observe that the same holds for any real $G$-invariant function of $\Phi$ and $\bar{\Phi}$. In other words, the $\sigma$-model Lagrangian for charged chiral superfields is obtained from the one we derived in section 5.1.1 upon the substitution

$$
K(\Phi^i, \bar{\Phi}_i) \longrightarrow K(\Phi^i, (\Phi e^{2gV})_i).
$$

(5.119)

The end result is then

$$
\mathcal{L} = \frac{1}{32\pi} \text{Im} \left[ \int d^2 \theta \, F_{ab}(\Phi) W^{a} W^{b} \right] + 
$$

$$
+ \int d^2 \theta d^2 \bar{\theta} K(\Phi^i, (\Phi e^{2gV})_i) + \int d^2 \theta W(\Phi^i) + \int d^2 \bar{\theta} W(\bar{\Phi}_i).
$$

(5.120)

By expanding and integrating in superspace one gets the final result. The derivation is a bit lengthy and we omit it here. Let us just mention some important differences
with respect to our previous results. The gauge part has the imaginary part of $F_{ab}$ multiplying the kinetic term (the generalized gauge coupling) and the real part multiplying the instanton term (generalized $\theta$-angle). Moreover, there are higher order couplings between fields belonging to vector and scalar multiplets which are proportional to derivatives of $F_{ab}$ with respect to the scalar fields, and which are obviously absent for the renormalizable Lagrangian (5.78). As for the matter part, one important difference with respect to the $\sigma$-model Lagrangian (5.42) is that all derivatives are (also) gauge covariantized. More precisely we have

$$
\bar{D}_\mu \psi^i = \partial_\mu \psi^i - igv_a \partial_\mu \phi^a \psi^k \\
\bar{D}_\mu \bar{\psi}_j = \partial_\mu \bar{\psi}_j - igv_a \partial_\mu \bar{\phi}_a \bar{\psi}_i,
$$

which are covariant both with respect to the $\sigma$-model metric and the gauge connection. As compared to the Lagrangian (5.78) the Yukawa couplings have the Kähler metric inserted, that is

$$
\bar{\phi} \lambda \psi \rightarrow K^i_j \bar{\phi}_i \lambda \psi^j = K^i_j (\bar{\phi}_i) M (T^a_R) N \lambda_a (\psi^j)^N,
$$

where $M, N$ are gauge indices. Moreover, the term $g\bar{\phi} D \phi$ is also modified into $g\bar{\phi}_i D K^i$, where as usual $K^i = \partial_{\bar{\phi}_i} K$.

All these changes are important to keep in mind. However, it is worth noticing that in $\mathcal{N} = 1$ supersymmetry vectors belong to different multiplets with respect to those where scalars sit. Hence, any geometric operation on the scalar manifold $\mathcal{M}$ will not have much effect on the vectors, and vice versa. In other words, the structure of the $\mathcal{N} = 1$ non-linear $\sigma$-model is essentially unchanged by gauging some of the isometries of the scalar manifold. This is very different from what happens in models with extended supersymmetry, as we will see in the next lecture.

After solving for the auxiliary fields which read (with obvious notation)

$$
F^i = (K^{-1})^i_j W^j - \frac{1}{2} \Gamma^i_{jk} \psi^j \psi^k - i \frac{g^2}{16\pi} (K^{-1})^i_j (\mathcal{F}_{a,b})^j \bar{\lambda}^a \lambda^b
$$

$$
D^a = -4\pi \frac{g^2}{g^2} (\text{Im} \mathcal{F})^{-1}_{ab} \left( g\bar{\phi}_i T^b K^i + g^2 \frac{1}{8\pi \sqrt{2}} [(\mathcal{F}_{a,b}) \psi^j \lambda^c + h.c.] \right)
$$

one finds for the potential

$$
V(\phi, \bar{\phi}) = (K^{-1})^i_j W_i W^j + 2\pi (\text{Im} \mathcal{F})^{-1}_{ab} (\bar{\phi}_i T^a K^i) (\bar{\phi}_j T^b K^j),
$$

which is the $\sigma$-model version of the potential (5.81).
As far as the potential, we cannot resist making a comment which will actually be relevant later, when we will discuss supersymmetry breaking. Whenever the effective theory one is dealing with does not have any propagating gauge degrees of freedom (due to Higgs mechanism, confinement or alike) the scalar potential (5.123) gets contributions from the first term, only. In this case the zero’s of the potential, which correspond to the supersymmetric vacua of the theory, are described just by

$$W_i = 0 ,$$  \hspace{5em} (5.124)

as in cases where the Kähler potential is canonical, since $K$ is a positive definite matrix (provided the integrating out procedure has been done correctly along the whole moduli space; cf. our previous discussion). This means that it is possible to see whether supersymmetry is broken/unbroken independently of any knowledge of the Kähler potential! Still, other important features, as field VEVs, the exact value of the vacuum energy (if not zero), the mass and the interactions of the lightest excitations, etc... do depend on $K$. With an abuse of notation, eqs. (5.124) are usually referred to as F-term equations, even though for a theory with non-canonical Kähler potential the contribution to the scalar potential by the F-terms is really given by the first term of eq. (5.123).

5.4 Exercises

1. Consider a chiral superfield $\Phi$ with components $\phi, \psi$ and $F$. Compute the D-term of the real superfield $\bar{\Phi} \Phi$ (up to total derivatives). Using eqs. (4.59), show that the resulting expression transform as a total space-time derivative under supersymmetry transformations.

2. Compute $D^2\Phi = 0$ and show that the different components provide the equations of motion for a free massless WZ multiplet.

3. Consider a theory of a chiral superfield $\Phi$ with canonical Kähler potential, $K = \bar{\Phi} \Phi$ and superpotential $W(\Phi) = \frac{1}{2} m\Phi^2 + \frac{1}{3} g\Phi^3$. This is the renowned Wess-Zumino (WZ) model. Compute the off-shell and on-shell Lagrangians, which is obtained from the latter integrating out auxiliary fields, and the scalar potential. Finally, show that supersymmetry closes only on-shell, namely that the algebra closes on the on-shell representation only upon use of (some of) the equations of motion.
4. Consider the theory of a single chiral superfield $\Phi$ and Kähler potential $K = \ln(1 + \Phi \Phi)$. Compute the off-shell and on-shell Lagrangians and study the geometry of the one-dimensional supersymmetric non-linear $\sigma$ model.

5. Using all possible available symmetries, show that in $SU(N)$ SQCD with $F < N$ flavors, the complex scalar field matrices parametrizing the moduli space can be put in the form (5.105). Using the same procedure, show that the structure (5.110) holds for $F \geq N$.

6. Consider the following matter theories

1. $K = \bar{Q}Q$, $W = \frac{1}{2}mQ^2$
2. $K = \bar{X}X + \bar{Y}Y$, $W = (X - m)Y^2$
3. $K = \bar{X}X + \bar{Y}Y + \bar{Z}Z$, $W = gXYZ$
4. $K = \Lambda\sqrt{XX}$, $W = \lambda X$.

Determine whether there are supersymmetric vacua and, if they exist, compute the mass spectrum of the theory around them.

References


