COMPACT OPERATORS

S:defi

1. Definitions

An operator $\mathbf{M} : X \mapsto Y$, X, Y Banach, is *compact* if $\mathbf{M}(B_X(0,1))$ is relatively compact, i.e. it has compact closure. We denote

E:kk

$$\mathcal{K}(X,Y) = \Big\{ \mathbf{M} \in \mathcal{L}(X,Y), \mathbf{M} \text{ compact} \Big\} \tag{1.1}$$

the set of compact operators from X into Y Banach spaces.

P:closu

Proposition 1.1. *The set $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{L}(x,y)$.*

Proof. Clearly $\mathcal{K}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$.

Let $\mathbf{M}_n \to \mathbf{M}$ in the operator norm, where $\mathbf{M}_n$ is compact. Fixed $\epsilon > 0$, let n such that

$$\|\mathbf{M} - \mathbf{M}_n\|_{\mathcal{L}(X,Y)} \leq \frac{\epsilon}{2}.$$

Since $\mathbf{M}_n(B(0,1))$ is relatively compact, then it can be covered by a finite number of balls

$$B_Y(y_i, \epsilon/2)$$

of radius $\epsilon/2$. Then $\mathbf{M}(B_X(0,1))$ is covered by

$$\bigcup_i B_Y(y_i, \epsilon).$$

□

As for degenerate maps, $\mathbf{M} \circ \mathbf{L}$ is compact if one is compact and the other continuous: thus $\mathcal{K}(X) = \mathcal{K}(X,X)$ is an ideal w.r.t. map composition.

We recall that a linear operator $\mathbf{M}$ is *degenerate* if it has *finite rank*:

E:finite

$$\dim(R_{\mathbf{M}}) < \infty. \tag{1.2}$$

Clearly such an operator is continuous if X is Banach, and thus it is compact. We thus have that

if $\mathbf{M}$ is the limit of a sequence of finite rank operators $\mathbf{M}_n$, then it is compact.

In Hilbert spaces the converse is true:

L:hilbecomp

Lemma 1.2. *If Y is a Hilbert space, then every compact operator is the limit of a sequence of finite rank operators.*

Proof. Consider a converging of $\overline{\mathbf{M}(B(0,1))}$ with balls of radius $\epsilon > 0$,

$$K = \bigcup_i B(y_i, \epsilon).$$

Let $S = \mathrm{span}\{y_i\}_i$, and consider the projector $\mathbf{P}_S$. This projection exists because Y is Hilbert.

Define the finite rank operator

$$\mathbf{M}_\epsilon = \mathbf{P}_S \circ \mathbf{M}.$$

By construction, if $x \in B_X(0,1)$, then there is y_i such that

$$\|\mathbf{M}x - y_i\| < \epsilon,$$

so that, since the operator norm of a projection in Hilbert spaces is 1 and $\mathbf{P}_S y_i = y_i$, we have

$$\|(\mathbf{P}_S \circ \mathbf{M})x - y_i\| < \epsilon,$$

It follows that

$$\big\|\mathbf{M}x - \mathbf{M}_\epsilon x\big\| = \big\|\mathbf{M}x - (\mathbf{P}_S \circ \mathbf{M})x\big\| < 2\epsilon,$$

□

S:adjoint

2. Transpose of a linear operator

Let X, Y be Banach spaces, with duals X^*, Y^*, respectively. Let $\mathbf{M} : X \mapsto Y$ be a bounded linear map. Define the *transpose* $\mathbf{M}^* : Y^* \mapsto X^*$ by

E:tranp

$$(\mathbf{M}^*\xi)x = \xi(\mathbf{M}x). \tag{2.1}$$

Because of the estimate

$$\big|\xi(\mathbf{M}x)\big| \leq \|\xi\|_{Y^*}\|\mathbf{M}\|_{\mathcal{L}(X,Y)}\|x\|,$$

the right hand side is a linear functional over X, which we denote by $\mathbf{M}^*\xi$. Thus $\mathbf{M}^* : Y^* \mapsto X^*$ is well defined. It is clearly linear and by the above estimate

$$\|\mathbf{M}^*\|_{\mathcal{L}(Y^*,X^*)} \leq \|\mathbf{M}\|_{\mathcal{L}(X,Y)}.$$

P:adjprop

Proposition 2.1. *If $\mathbf{M} \in \mathcal{L}(X,Y)$, then*

E:norm13

$$\|\mathbf{M}^*\|_{\mathcal{L}(Y^*,X^*)} = \|\mathbf{M}\|_{\mathcal{L}(X,Y)}. \tag{2.2}$$

Moreover,

1. $N_{\mathbf{M}^*} = R_{\mathbf{M}}^{\perp}$;
2. $N_{\mathbf{M}} = R_{\mathbf{M}^*}^{\perp}$;
3. $(\mathbf{M}+\mathbf{N})^* = \mathbf{M}^* + \mathbf{N}^*$.

Proof. The equality (2.2) is an application of Hahn Banach theorem in the space Y.

The other relations follow easily from (2.1). □

We now prove that if $\mathbf{M}$ is compact, then also its transpose is compact.

T:comptrap

Theorem 2.2 (Schauder)**.** *The operator $\mathbf{M} \in \mathcal{K}(X,Y)$ if and only if $\mathbf{M}^* \in \mathcal{K}(Y^*,X^*)$.*

Proof. Let ξ_n be a sequence in $B_{Y^*}(0,1)$, and $K = \overline{\mathbf{M}(B_X(0,1))}$. Consider the functions

$$\phi_n(y) = \xi_n y \in C(K), \quad \xi_n \in B_{Y^*}(0,1).$$

Clearly these functions are equicontinuous (they are Lipschitz continuous with modulus 1) and K is compact, so that there is a converging subsequence, which we denote again by ϕ_n.

Since ϕ_n is Cauchy, we have

$$\big|\xi_n(\mathbf{M}u) - \xi_m(\mathbf{M}u)\big| = \big|(\mathbf{M}^*\xi_n)u - (\mathbf{M}^*\xi_m)u\big| < \epsilon, \quad \forall u \in B_X(0,1),\ n,m \gg 1.$$

Hence $\mathbf{M}^*\xi_n$ is a Cauchy sequence in $\mathbf{M}^*(B_{Y^*}(0,1))$.

Conversely, if $\mathbf{M}^*$ is compact, then $\mathbf{M}^{**}$ is compact because of the first part of the proof. It is easy to see that if $\mathbf{J}_X : X \mapsto X^{**}$, $\mathbf{J}_Y : Y \mapsto Y^{**}$ are the canonical immersions, then

$$\mathbf{M}^{**}(\mathbf{J}_X x) = \mathbf{J}_Y(\mathbf{M}x).$$

Since $\mathbf{J}_X(B_X(0,1)) \subset B_{X^{**}}(0,1)$, then $\mathbf{M}^{**}(\mathbf{J}_X(B_X(0,1))) = \mathbf{J}_Y(\mathbf{M}(B_X(0,1)))$ is relatively compact in Y^{**}. Since the canonical immersion $\mathbf{J}$ is an isometry, then $\mathbf{M}(B_X(0,1))$ is relatively compact. □

S:fredh

3. Fredholm's alternative

This section is devoted to the proof of *Fredholm's alternative*:

If $\mathbf{M} : X \mapsto X$, X Banach, is compact, then

- *either the equation $u - \mathbf{M}u = v$ has a unique solution,*
- *or $u - \mathbf{M}u = 0$ has n linearly independent solutions, and $u - \mathbf{M}u = v$ has a solution if and only if v satisfies the linear conditions*

$$v \in (R_{\mathbf{M}}^{\perp})^{\perp} = \Big\{\ell v = 0, \forall v \in R_{\mathbf{M}}^{\perp}\Big\}. \tag{3.1}$$

From $\mathbf{M}$ compact it follows that $R_{\mathbf{M}}^{\perp}$ is finite dimensional.

We prove in fact the following theorem:

T:fredh

Theorem 3.1. *If $\mathbf{M} : X \mapsto X$, X Banach, is compact, then*

1. *$N_{\mathbf{I}-\mathbf{M}}$ has finite dimension;*
2. *$R_{\mathbf{I}-\mathbf{M}}$ is closed and $R_{\mathbf{I}-\mathbf{M}} = N_{\mathbf{I}-\mathbf{M}^*}^{\perp}$;*

(3) $N_{\mathbf{I}-\mathbf{M}} = 0$ *is equivalent to* $R_{\mathbf{I}-\mathbf{M}} = X$*;*
(4) *the dimension of* $N_{\mathbf{I}-\mathbf{M}}$ *is equal to the dimension of* $N_{I-\mathbf{M}^*}$.

In particular, the index of the operator $\mathbf{I}-\mathbf{M}$, $\mathbf{M}$ compact, is 0:

$$\mathrm{ind}(\mathbf{I}-\mathbf{M}) = \dim N_{\mathbf{I}-\mathbf{M}} - \dim R_{\mathbf{I}-\mathbf{M}} = 0.$$

Proof. Point (1) follows from the observation that

$$N_{\mathbf{I}-\mathbf{M}} = \mathbf{M}(N_{\mathbf{I}-\mathbf{M}}),$$

and since $\mathbf{M}$ is compact, the space $N_{\mathbf{I}-\mathbf{M}}$ is locally compact, hence finite dimensional.

Let $u_n - \mathbf{M}u_n = y_n \to y$. Since $N_{\mathbf{I}-\mathbf{M}}$ has finite dimension, there is $v_n \in N_{\mathbf{I}-\mathbf{M}}$ which minimize

$$\|u_n - v_n\| = \inf_{v \in N_{\mathbf{I}-\mathbf{M}}} \|u_n - v\|.$$

and $(u_n - v_n) - \mathbf{M}(u_n - v_n) = y_n$. By dividing the above equation by $\|u_n - v_n\|$, one sees that if $\|u_n - v_n\| \to \infty$, then the sequence $w_n = (u_n - v_n)/\|u_n - v_n\|$ satisfies

$$w_n + \mathbf{M}w_n \to 0, \quad \|w_n\| = 1$$

Since $\mathbf{M}$ is compact, then we can extract a subsequence $\mathbf{M}w_n \to w$, so that $w + \mathbf{M}w = 0$, but $\|w\| = 1$, This is a contradiction, because $w \notin N_{\mathbf{I}-\mathbf{M}}$.

It follows that $\|u_n - v_n\|$ remains bounded. Thus up to subsequences we have that $u_n - v_n$ converges. This prove that $R_{\mathbf{I}-\mathbf{M}}$ is closed.

Since for a closed subspace Y, the Hahn Banach theorem implies $(Y^\perp)^\perp = Y$, then (2) follows.

To prove (3), assume that $N_{\mathbf{I}-\mathbf{M}} = \{0\}$, and $R_{\mathbf{I}-\mathbf{M}} = X_1 \neq X$, then for $v \in R_{\mathbf{I}-\mathbf{M}}$

$$\mathbf{M}v = \mathbf{M}\big(\mathbf{I}-\mathbf{M}\big)x = \big(\mathbf{I}-\mathbf{M}\big)\mathbf{M}x \subset X_1, \quad X_1 \text{ closed.}$$

The operator $\mathbf{M} \in \mathcal{K}(X_1)$, so that we can consider again $X_2 = (\mathbf{I}-\mathbf{M})(X_1) = (\mathbf{I}-\mathbf{M})^2(X) \subsetneq X_1$, because $\mathbf{I}-\mathbf{M}$ is injective and $X_1 = (\mathbf{I}-\mathbf{M})(X)$.

Proceding in this way we find a sequence of subspaces $X_n = (\mathbf{I}-\mathbf{M})^n(X)$, and thus we can find points $x_n \in X_{n-1}$ such that

$$\|x_n - y\| \geq \frac{1}{2}, \quad y \in X_n.$$

We have for $n \leq m$

$$\mathbf{M}(x_n - x_m) = u_n - u_m + (\mathbf{I}-\mathbf{M})u_m - (\mathbf{I}-\mathbf{M})u_n = u_n - y_{nm}, \quad y_{nm} \in X_n.$$

Hence $\|\mathbf{M}(x_n - x_m)\| \geq 1/2$, but this contradicts the assumption $\mathbf{M}$ compact.

Conversely, if $R_{\mathbf{I}-\mathbf{M}} = X$, then we have $N_{\mathbf{I}-\mathbf{M}^*} = \{0\}$, and thus using the first part $R_{\mathbf{I}-\mathbf{M}^*} = X^*$. Using again Proposition 2.1, we conclude $N_{\mathbf{I}-\mathbf{M}} = \{0\}$. This concludes (3).

Since $\mathbf{M} \in \mathcal{K}(X)$, then $\mathbf{M}^* \in \mathcal{K}(X^*)$, so that both kernels have finite dimension. Assume that $d = \dim N_{\mathbf{I}-\mathbf{M}} < d^* = \dim N_{\mathbf{I}-\mathbf{M}^*}$. Since $N_{\mathbf{I}-\mathbf{M}}$ is finite dimensional, then there is a continuous projector from X into $N_{\mathbf{I}-\mathbf{M}}$.

Using the fact that $R_{\mathbf{I}-\mathbf{M}}$ has finite codimension, there is a continuous projector on a linear complement E of $R_{\mathbf{I}-\mathbf{M}}$. By assumption, there is $\mathbf{\Lambda} : N_{\mathbf{I}-\mathbf{M}} \mapsto E$ which is injective but not surjective. Define

$$\mathbf{S} = \mathbf{M} + \mathbf{\Lambda} \circ P_{N_{\mathbf{I}-\mathbf{M}}}.$$

Then $\mathbf{S} \in \mathcal{K}(X)$, because $\mathbf{\Lambda}$ has finite rank. Moreover $N_{\mathbf{I}-\mathbf{S}} = \{0\}$. From point (3) it follows that $R_{\mathbf{I}-\mathbf{S}} = X$, but this contradict the fact that $\mathbf{S}$ is not surjective. We have thus proved that $d^* \leq d$.

Using the above result, it follows that for $\mathbf{M}^*$

$$\dim N_{\mathbf{I}-\mathbf{M}^{**}} \leq \dim N_{\mathbf{I}-\mathbf{M}^*} \leq \dim N_{\mathbf{I}-\mathbf{M}}.$$

Since $N_{\mathbf{I}-\mathbf{M}^{**}} \supset \mathbf{J}(N_{\mathbf{I}-\mathbf{M}})$, we have proved (4). □

S:spect

4. Spectral analysis

If $\mathbf{M} \in \mathcal{L}(X)$, then the *resolvent set* of $\mathbf{M}$ is

E:resolv
$$\rho(\mathbf{M}) = \Big\{ \lambda \in \mathbb{C} : (\lambda \mathbf{I} - \mathbf{M})^{-1} \in \mathcal{L}(X) \Big\}. \tag{4.1}$$

The *spectrum* of $\mathbf{M}$ is

E:spectr
$$\sigma(\mathbf{M}) = \mathbb{C} \setminus \rho(\mathbf{M}) = \left\{ \lambda \in \mathbb{C} : \begin{cases} \lambda \mathbf{I} - \mathbf{M} & \text{not injective} \\ \lambda \mathbf{I} - \mathbf{M} & \text{injective but not surjective} \\ \lambda \mathbf{I} - \mathbf{M} & \text{injective, surjective but with not continuous inverse} \end{cases} \right\} \tag{4.2}$$

For bounded operators the last case cannot occur, because of the open mapping theorem.

The values λ such that the first case holds are the *eigenvalues* of $\mathbf{M}$. The space $N_{\lambda \mathbf{I} - \mathbf{M}} \neq \{0\}$ is the *eigenspace* associated to λ, and its elements are the eigenvectors of $\mathbf{M}$.

E:solve

Proposition 4.1. *If* $\mathbf{M} \in \mathcal{L}(X)$*, then*

E:contr
$$\sigma(\mathbf{M}) \subset \big\{ |\lambda| \leq \|\mathbf{M}\|_{\mathcal{L}(X)} \big\}. \tag{4.3}$$

Proof. The proof follows from the fact that the series

$$\sum_{n=0}^{+\infty} \frac{1}{\lambda^{n+1}} \mathbf{M}^n$$

converges strongly and it is the inverse of $\lambda \mathbf{I} - \mathbf{M}$. □

For compact operators the spectrum has a precise form.

Theorem 4.2. *Let* $\mathbf{M} \in \mathcal{K}(X)$*, with* X *infinite dimensional Banach. Then*

- $0 \in \sigma(\mathbf{M})$;
- $\lambda \in \sigma(\mathbf{M}) \setminus \{0\}$ *is an eigenvalue;*
- $\sigma(\mathbf{M}) \setminus \{0\}$ *is either empty, or finite, or it is a sequence of eigenvalues converging to* 0.

Proof. The first point follows because $\mathbf{M}^{-1}$ cannot exists, otherwise $\mathbf{M}^{-1} \circ \mathbf{M}(X) = X$ is compact.

To prove point (2), we just use the (3) implication of Theorem 3.1, which gives a contradiction if $N_{\lambda \mathbf{I} - \mathbf{M}} = \{0\}$.

To prove the last point, we consider a sequence $\lambda_n \in \sigma(\mathbf{M}) \setminus \{0\}$. converging to some λ. For all eigenvalues λ_n, let $e_n \in N_{\lambda_n \mathbf{I} - \mathbf{M}}$ with norm 1. It is easy to verify that $N_{\lambda_n \mathbf{I} - \mathbf{M}} \cap N_{\lambda_m \mathbf{I} - \mathbf{M}} = \{0\}$ if $n \neq m$, so that all e_n are different.

Define

$$E_n = \operatorname{span}\Big\{ e_1, e_2, \dots, e_n \Big\},$$

and consider $u_n \in E_n$ such that $\|u_n\| = 1$, $\|u_n - y\| \geq 1/2$ for $y \in E_{n-1}$. We have for $m < n$

$$\left\| \frac{1}{\lambda_n} \mathbf{M} u_n - \frac{1}{\lambda_m} \mathbf{M} u_m \right\| = \left\| u_n - u_m + \frac{1}{\lambda_n} (\lambda u_n \mathbf{I} - \mathbf{M}) u_n - \frac{1}{\lambda_m} (\lambda_m \mathbf{I} - \mathbf{M}) u_n \right\| \geq \frac{1}{2},$$

since

$$\frac{1}{\lambda_n} (\lambda u_n \mathbf{I} - \mathbf{M}) u_n \in E_{n-1}.$$

Since $\mathbf{M} u_n$ has a converging subsequence, then $\lambda_n \to 0$.

This shows that the set $\sigma(\mathbf{M}) \cap \{|\lambda| \geq 1/n\}$ has at most a finite number of eigenvalues. □

S:sefladj1

5. Spectral decomposition of compact self adjoint operators

We say that $\mathbf{M} \in \mathcal{L}(H)$, H Hilbert space is *self adjoint* if

E:selfadj
$$(\mathbf{M}x, y) = (x, \mathbf{M}y), \quad \forall x, y \in H. \tag{5.1}$$

Proposition 5.1. *Let* $\mathbf{M} \in \mathcal{L}(H)$ *be self adjoint, and define*

E:minmax
$$m = \inf_{\|u\|=1} (\mathbf{M}u, u), \quad M = \sup_{\|u\|=1} (\mathbf{M}u, u). \tag{5.2}$$

Then $\sigma(\mathbf{M}) \subset [m, M]$, $m, M \in \sigma(\mathbf{M})$.

Proof. If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ or $\lambda > M$, then

$$\big(\lambda \mathbf{I} - \mathbf{M})u, u\big) = \lambda \|u\|^2 - (\mathbf{M}u, u) \neq 0,$$

since $(\mathbf{M}u, u) \in \mathbb{R}$. Moreover, $(\lambda \mathbf{I} - \mathbf{M})(H)$ is a subspace of H, which is closed because from the above relation

$$\big(|\lambda - M| + \Im\lambda\big)\|u\| \leq \|(\lambda \mathbf{I} - \mathbf{M})u\|.$$

The same argument implies that $(\lambda \mathbf{I} - \mathbf{M})(H) = H$. This proves that $\sigma(\mathbf{M}) \subset (-\infty, M)$.

For $\lambda = M$, then we have as in the proof of Schwartz inequality that

$$|(Mu - \mathbf{M}u, v)| \leq |(Mu - \mathbf{M}u, u)|^{1/2}|(Mv - \mathbf{M}v, v)|^{1/2}$$

If u_n, $\|u_n\| = 1$, is a maximizing sequence $(\mathbf{M}u, u) \to M$, it follows that $(M\mathbf{I} - \mathbf{M})u_n$ converges to 0. If now $M \in \rho(\mathbf{M})$, then

$$u_n = (M\mathbf{I} - \mathbf{M})^{-1}(M\mathbf{I} - \mathbf{M})u_n \to 0,$$

which contradicts $\|u_n\| = 1$.

Replacing $\mathbf{M}$ with $-\mathbf{M}$, we obtain the other part of (1). □

In particular, if $\sigma(\mathbf{M}) = \{0\}$, then $(\mathbf{M}u, u) = 0$, and $\mathbf{M}u = 0$.

T:deco

Theorem 5.2. *If $\mathbf{M} \in \mathcal{K}(H)$, H Hilbert, is self adjoint, then there exists an Hilbert base generated by eigenvector of $\mathbf{M}$.*

Proof. The result follows if we can prove that

$$H = \overline{N_{\mathbf{M}} \cup \bigcup_{\lambda_n \neq 0} N_{\lambda_n \mathbf{I} - \mathbf{M}}}.$$

In fact, the orthonormal base is just the union of the orthonormal bases of each eigenspace. Moreover, as in the finite dimensional space, one sees that the eigenvalue of $\mathbf{M}$ are real, and the spaces $N_{\lambda_n \mathbf{I} - \mathbf{M}}$ are orthogonals each other.

To prove that the vector space Y generated by $N_{\mathbf{M}}$ and $\{N_{\lambda_n \mathbf{I} - \mathbf{M}}\}_n$ is dense in H, we first observe that Y is invariant for $\mathbf{M}$, so that $\mathbf{M}(Y^\perp) \subset Y^\perp$, because $\mathbf{M}$ is self adjoint.

The operator $\mathbf{M}|_{Y^\perp}$ is self adjoint and compact, and by construction $\sigma(\mathbf{M}|_{Y^\perp}) = \{0\}$. It follows $\mathbf{M}|_{Y^\perp} = 0$ and $Y^\perp \subset N_{\mathbf{M}}$. □

S:exerc1

6. Exercises

1. Let $\mathbf{M} : X \mapsto Y$, X Banach, Y reflexive. Show that if $x_n \rightharpoonup x$, then $\mathbf{M}x_n \rightharpoonup \mathbf{M}x$.
2. Define the *adjoint* of $\mathbf{M} : H \mapsto H$, H Hilbert space, by
$$(x, \mathbf{M}^* y) = (\mathbf{M}x, y).$$
Prove that Proposition 2.1 holds for the adjoint operator.
3. Prove that if $Y \subset X$, X Banach, is a subspace, then $\bar{Y} = (Y^\perp)^\perp$.
4. On ℓ^∞, consider the linear operator
$$\mathbf{S}u(n) = u(n+1).$$
Compute the spectrum of $\mathbf{S}$ (consider the functions λ^n).
5. Consider the Hilbert space ℓ^2 and a sequence of real numbers $x_n \to 0$. Define
$$\mathbf{M}u(n) = x_n u(n).$$
Show that T is compact and find its spectrum.
6. Find the eigenvalues and eigenvectors of the orthogonal projection $\mathbf{P}_M$ on M subspace of H Hilbert. Is $\mathbf{P}_M$ compact?
7. Fixed $g(t, s) \in C^1([0, 1]^2, \mathbb{C})$, consider the linear operator
$$\mathbf{M} : C([0, 1], \mathbb{C}) \mapsto C([0, 1], \mathbb{C}), \quad \mathbf{M}u(t) = \int_0^1 g(t, s)u(s)ds.$$
Discuss its spectrum.
8. Let H be a separable Hilbert space, $K \subset \mathbb{C}$ a compact set in $\mathbb{C}$, $\{\lambda_n\}_n$ a countable dense sequence in K.

- Show that there is a unique bounded linear operator $\mathbf{M} \in \mathcal{L}(H)$ such that

$$\mathbf{M}e_n = \lambda_n e_n.$$

- Show that $\sigma(\mathbf{M} = K$, but the eigenvalues of $\mathbf{M}$ are $\{\lambda_n\}_n$.
- Prove that for $\lambda \in K \setminus \{\lambda_n\}_n$, then $R_{\boldsymbol{\lambda}\mathbf{I}-\mathbf{M}}$ is dense in H.