An introduction to Hamilton-Jacobi equations

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Outline

The Hamilton-Jacobi equation (HJ equation) is a special fully nonlinear scalar first order PDE.

It arises in many different context:

1. Hamiltonian dynamics
2. Classical limits of Schrödinger equation
3. Calculus of variation
4. Control theory
5. Optimal mass transportation problems
6. Conservation laws in one space dimension
7. etc...

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Even if it is fully nonlinear, there is a satisfactory theory of existence and regularity of solutions.

Hamilton's principal function

The function $S = S(q, P, t)$ defining a canonical transformation of coordinates $(p, q) \mapsto (Q, P)$

$$p = \nabla_q S, \quad Q = \nabla_P S,$$

yields a canonical transformation with the new Hamiltonian $H' = 0$ if

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The above equation is the **Hamilton-Jacobi equation**: the function H is called the *Hamiltonian*, and depending on the context the solution can be called *minimizer*, *value function*, *potential*, or in this case *Hamilton principal function*.

Schrödinger equation

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$$-\partial_t S = \frac{1}{2m}|\nabla S|^2 + U,$$

which is the Hamilton-Jacobi equation for the Hamiltonian

$$H = \frac{p^2}{2m} + U.$$

Calculus of variation

Consider the minimization problem in $\Omega \subset \mathbb{R}^d$

$$\min \left\{ \int \big(\mathbb{1}_{|p| \leq 1}(\nabla u(x)) + u(x) \big) dx : u|_{\partial\Omega} = u_0 \right\}.$$

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The Euler-Lagrange equation reads as

$$\mathrm{div}(\rho d) = 1,$$

with d the direction of the optimal ray (see later).

Control theory

Consider the ODE

$$\dot{x} = f(x, u), \quad u \text{ control},$$

and the problem is to minimize the functional

$$A(t) = \min_u \left\{ \int_t^T L(x, u) dt + F(x(T)) \right\}.$$

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Defining

$$H(t, x, p) := \min_u \left\{ p \cdot f(x, u) + L(x, u) \right\},$$

the function $A(t)$ satisfies the Hamilton-Jacobi-Bellman equation

$$\partial_t A + H(t, x, \nabla A) = 0, \quad A(T) = F(x).$$

Optimal mass transportation

Let $\|\cdot\| : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a norm, $\mu, \nu \in \mathcal{P}([0,1])$ and

$$\Pi(\mu,\nu) := \Big\{\pi \in \mathcal{P}([0,1]^2) : (P_1)_\sharp \pi = \mu, (P_2)_\sharp \pi = \nu\Big\}.$$

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$$\int \|x-y\|\pi(dxdy), \quad \pi \in \Pi(\mu,\nu).$$

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$$\int \|x-y\| \pi(dxdy), \quad \pi \in \Pi(\mu,\nu).$$

By duality, this is equivalent to maximize

$$\int \phi(x)(\mu-\nu)(dx), \quad \big|\phi(x)-\phi(y)\big| \leq \|x-y\|,$$

and one can show that ϕ is the solution to the Hamilton-Jacoby equation

$$1 - \|\nabla \phi\| = 0.$$

Conservation laws

Consider the scalar conservation laws

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By the change of variable

$$U(x) = \int^x u(y)dy,$$

we can transform the PDE into

$$U_t + f(U_x) = 0,$$

which is a Hamilton-Jacobi equation with Hamiltonian $H(p) = f(p)$.

Outline

What we can expect

The natural space of functions where the solutions lives is Lipschitz.

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Example. The model Hamiltonian is $\frac{p^2}{2}$, and the function

$$u(t,x) = -\int_0^x \min\left\{1, -\frac{y}{t}\right\} dy$$

is a regular solution for $t < 1$ to

$$u_t + \frac{|u_x|^2}{2} = 0.$$

At $t = 1$ the solution becomes only 1-Lipschitz.

A *solution* to Hamilton-Jacobi can be defined as a Lipschitz function $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ such that

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Example. The function

$$u(t, x) = \min \left\{ |x| - \frac{t}{2}, 0 \right\},$$

satisfies

$$u_t + \frac{|u_x|^2}{2} = 0, \quad u(0, x) = 0.$$

Clearly the expected solution is $u(t, x) = 0$.

Maximum principle

For scalar equation, a natural requirement is

$$u(0,x) \le v(0,x) \quad \Rightarrow \quad u(t,x) \le v(t,x).$$

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If ϕ is a regular function such that $u^\epsilon - \phi$ has a local minimum in $(\bar{t}, \bar{x})$, then it follows

$$\Delta(u^\epsilon - \phi) \geq 0,$$

Since $\nabla u^\epsilon = \nabla \phi$, $u^\epsilon_t = \phi_t$, we recover

$$\phi_t + H(\bar{t}, \bar{x}, \nabla \phi) - \epsilon \Delta \phi \geq 0,$$

and in the limit

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Under mild assumptions on H and u_0,

Theorem (Crandall-Lions)

The viscosity solution exists and is unique.

Lax formula

If u is a viscosity solution and H convex in p, then it can be obtained by the formula

$$u(t,x) = \min \bigg\{ u(0,y) + \int_0^t L\big(s,\gamma(s),\dot{\gamma}(s)\big)\,ds, \\ \gamma : [0,t] \to \mathbb{R}^d, \gamma(0) = y, \gamma(t) = x \bigg\},$$

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where the *Lagrangian* L is given by the Legendre transform of H

$$L(t,x,a) = \sup_p \big\{ a \cdot p - H(t,x,p) \big\}.$$

The curve γ for which the minimum

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By Euler-Lagrange equation, it is a solution to the ODE system

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In the special case where $H = H(p)$, this curve is a straight line, and the min-formula reads as

$$u(t,x) = \inf_y \left\{ u(0,y) + tL\left(\frac{x-y}{t}\right)\right\}.$$

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Since the minimum of semiconcave functions is semiconcave, it follows that the solution u to the HJ equation

$$\partial_t u + \frac{|\nabla_x u|^2}{2} = 0$$

is semiconcave.

A regularity result

The following result can be proved: if the Hamiltonian is uniformly convex in p and the initial data is sufficiently regular then there exists piecewise smooth hypersurfaces $\{S_k\}_k$ of codimension 1 such that ∇u is regular outside $\cup_k S_k$ (Cannarsa-Sinestrari).

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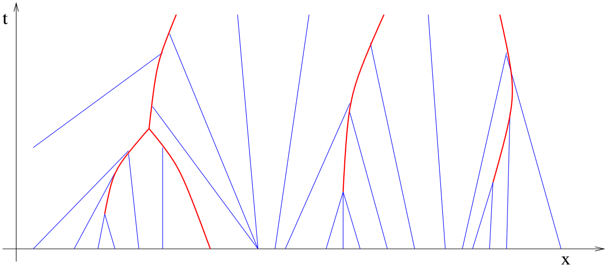
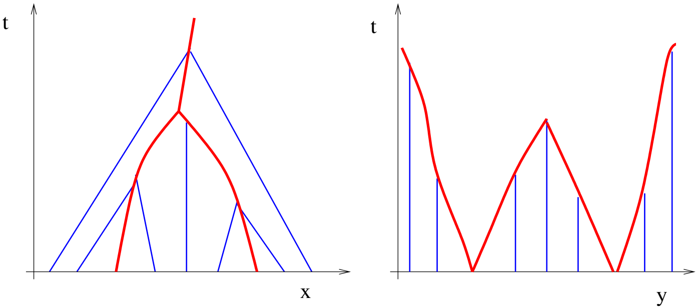


Due to the regularity outside the jumps of $\nabla_x u$, it is clear that the change of variable

$$\begin{cases} t = \tau, \\ x = \gamma(t, y), \end{cases} \qquad y \text{ initial point of the characteristic } \gamma,$$

is regular.



Solutions to hyperbolic conservation laws

For strictly hyperbolic system of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \; u \in \mathbb{R}^m,$$

one expects a similar structure: countably many shock curves and regularity of the solution in the remaining set.

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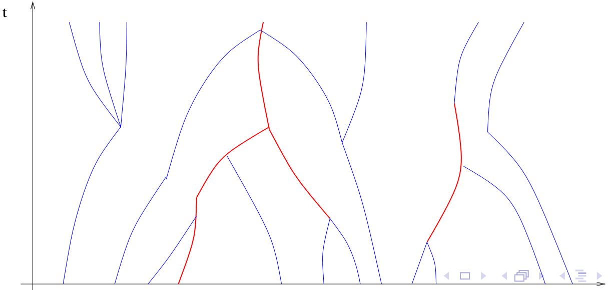

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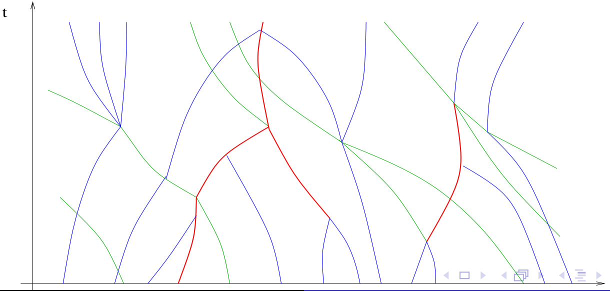
one expects a similar structure. However the presence of the other characteritic families generates a complicated structure.



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1. the structure of solutions if the initial data is only Lipschitz and H convex
2. the regularity of the decomposition of $\mathbb{R}^+ \times \mathbb{R}^d$ given by the characteristics
3. the applications/extension of these results to conservation laws and optimal transport on manifolds