An introduction to Glimm functional

Stefano Bianchini, SISSA-ISAS Trieste

http://www.sissa.it/~bianchin

June 8, 2005

Systems of Conservation Laws

$$u_t + f(u)_x = 0, \quad u \in \mathbb{R}^n,\ f : \mathbb{R}^n \mapsto \mathbb{R}^n. \tag{1}$$

Systems of Conservation Laws

$$u_t + f(u)_x = 0, \quad u \in \mathbb{R}^n, \ f : \mathbb{R}^n \mapsto \mathbb{R}^n. \tag{1}$$

Strictly hyperbolic if the Jacobian matrix $Df(u)$ has n distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u).$$

Systems of Conservation Laws

$$u_t + f(u)_x = 0, \quad u \in \mathbb{R}^n,\ f : \mathbb{R}^n \mapsto \mathbb{R}^n. \tag{1}$$

Strictly hyperbolic if the Jacobian matrix $Df(u)$ has n distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u).$$

Existence of solutions with small BV data: Glimm [1965]
Stability in L^1: Bressan [1995]

Systems of Conservation Laws

$$u_t + f(u)_x = 0, \quad u \in \mathbb{R}^n,\ f : \mathbb{R}^n \mapsto \mathbb{R}^n. \tag{1}$$

Strictly hyperbolic if the Jacobian matrix $Df(u)$ has n distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u).$$

Existence of solutions with small BV data: Glimm [1965]
Stability in L^1: Bressan [1995]

Technical difficulties:

Systems of Conservation Laws

$$u_t + f(u)_x = 0, \quad u \in \mathbb{R}^n, \; f : \mathbb{R}^n \mapsto \mathbb{R}^n. \tag{1}$$

Strictly hyperbolic if the Jacobian matrix $Df(u)$ has n distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u).$$

Existence of solutions with small BV data: Glimm [1965]
Stability in L^1: Bressan [1995]

Technical difficulties:

- The solution develops discontinuities in finite time

Systems of Conservation Laws

$$u_t + f(u)_x = 0, \quad u \in \mathbb{R}^n,\ f : \mathbb{R}^n \mapsto \mathbb{R}^n. \tag{1}$$

Strictly hyperbolic if the Jacobian matrix $Df(u)$ has n distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u).$$

Existence of solutions with small BV data: Glimm [1965]
Stability in L^1: Bressan [1995]

Technical difficulties:

- The solution develops discontinuities in finite time
- No monotonicity

Existence and stability are proved by means of a decreasing functional $Q(u)$.

Existence and stability are proved by means of a decreasing functional $Q(u)$.

This functional is a potential: it measures all the possible future interactions of non linear waves in (1).

Existence and stability are proved by means of a decreasing functional $Q(u)$.

This functional is a potential: it measures all the possible future interactions of non linear waves in (1).

The key estimate is that

$$TV(u) + CQ(u), \quad C \gg 1, \tag{2}$$

is decreasing in time for "entropic" solutions of (1).

Existence and stability are proved by means of a decreasing functional $Q(u)$.

This functional is a potential: it measures all the possible future interactions of non linear waves in (1).

The key estimate is that

$$TV(u) + CQ(u), \quad C \gg 1, \tag{2}$$

is decreasing in time for "entropic" solutions of (1).

Remark. This functional is different from the entropy. It is related to the growth of entropy dissipation.

Plan of the talk:

Plan of the talk:

- The linear part of the Glimm functional

Plan of the talk:

- The linear part of the Glimm functional
- Glimm functional for scalar conservation laws

$$u_t + f(u)_x = 0, \quad u \in R$$

Plan of the talk:

- The linear part of the Glimm functional
- Glimm functional for scalar conservation laws

$$u_t + f(u)_x = 0, \quad u \in R$$

- Glimm functional for vanishing viscosity and semidiscrete schemes

$$u_t + f(u)_x = \epsilon u_{xx}, \quad u_t^n + \frac{1}{\epsilon}(f(u^n) - f(u^{n-1})) = 0$$

Plan of the talk:

- The linear part of the Glimm functional
- Glimm functional for scalar conservation laws

$$u_t + f(u)_x = 0, \quad u \in R$$

- Glimm functional for vanishing viscosity and semidiscrete schemes

$$u_t + f(u)_x = \epsilon u_{xx}, \quad u_t^n + \frac{1}{\epsilon}(f(u^n) - f(u^{n-1})) = 0$$

- Glimm functional for kinetic models

$$\begin{cases} u_t + v_x &= 0 \\ v_t + u_x &= \frac{1}{\epsilon}(f(u) - v) \end{cases}$$

1. The linear part of the Glimm functional

1. The linear part of the Glimm functional

We say *linear part* because it decreases also for linear systems.

1. The linear part of the Glimm functional

We say *linear part* because it decreases also for linear systems.

Consider for example the linear 2×2 system

$$\begin{cases} u_{1,t} - u_{1,x} = 0 \\ u_{2,t} + u_{2,x} = 0 \end{cases} \tag{3}$$

1. The linear part of the Glimm functional

We say *linear part* because it decreases also for linear systems.

Consider for example the linear 2×2 system

$$\begin{cases} u_{1,t} - u_{1,x} = 0 \\ u_{2,t} + u_{2,x} = 0 \end{cases} \tag{3}$$

The component u_1 of the solution u travels with speed -1, while the component u_2 travels with speed 1:

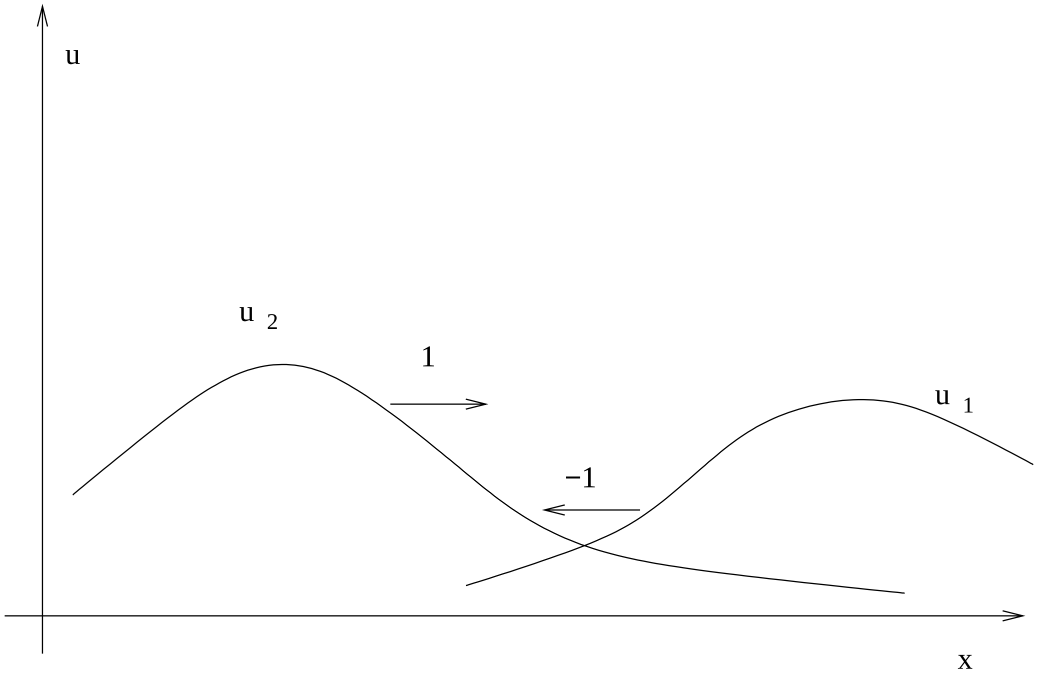
1. The linear part of the Glimm functional

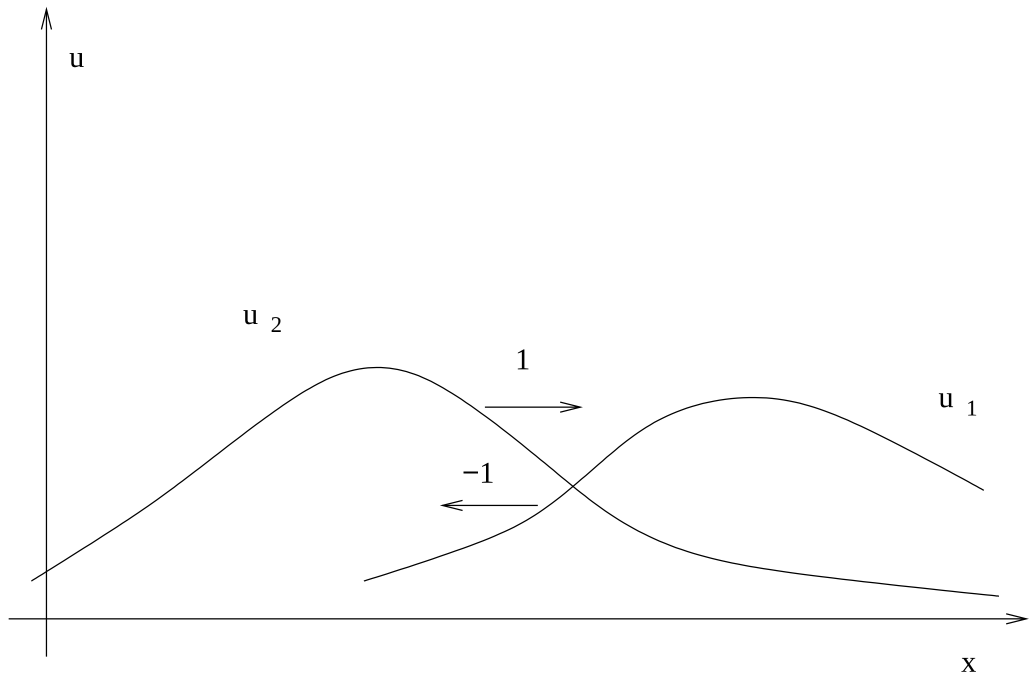
We say *linear part* because it decreases also for linear systems.

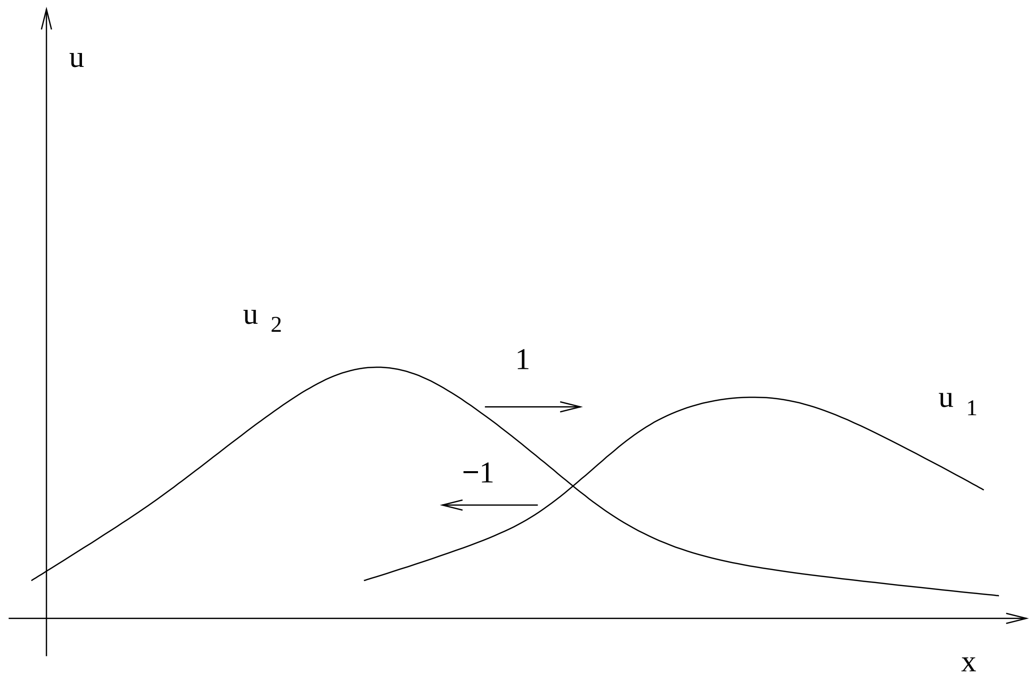
Consider for example the linear 2×2 system

$$\begin{cases} u_{1,t} - u_{1,x} &= 0 \\ u_{2,t} + u_{2,x} &= 0 \end{cases} \tag{3}$$

The component u_1 of the solution u travels with speed -1, while the component u_2 travels with speed 1:
the oscillations of u_1 belong to *first family of waves* of (3), corresponding to the eigenvalue -1, while u_2 is the *second family*, corresponding to the eigenvalue 1.


u
u 2
1
u 1
−1
x


u
u 2
1
u 1
−1
x



The two components u_1, and u_2 cross because have different speeds -1, and 1. Denote

$$P(t,x,y) = u_{1,x}(t,y)u_{2,x}(t,x), \quad P_t + \mathrm{div}_x((1,-1)P) = 0.$$

It follows that

$$Q(u) = \iint_{x<y} |u_{1,x}(t,y)||u_{2,x}(t,x)|dxdy = \|P\|_{L^1(x<y)}$$

is decreasing and

It follows that

$$Q(u) = \iint_{x<y} |u_{1,x}(t,y)||u_{2,x}(t,x)|dxdy = \|P\|_{L^1(x<y)}$$

is decreasing and

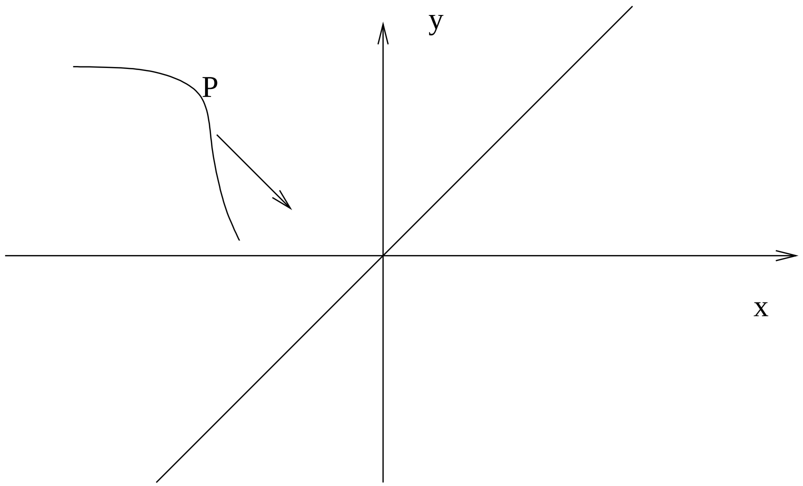
$$\frac{dQ}{dt} = -2\int_{\mathbb{R}} |u_{1,x}(t,x)||u_{2,x}(t,x)|dx = -\int_{\mathbb{R}} |P(t,x,x)|dx.$$

It follows that

$$Q(u) = \iint_{x<y} |u_{1,x}(t,y)||u_{2,x}(t,x)|dxdy = \|P\|_{L^1(x<y)}$$

is decreasing and

$$\frac{dQ}{dt} = -2\int_{\mathbb{R}} |u_{1,x}(t,x)||u_{2,x}(t,x)|dx = -\int_{\mathbb{R}} |P(t,x,x)|dx.$$

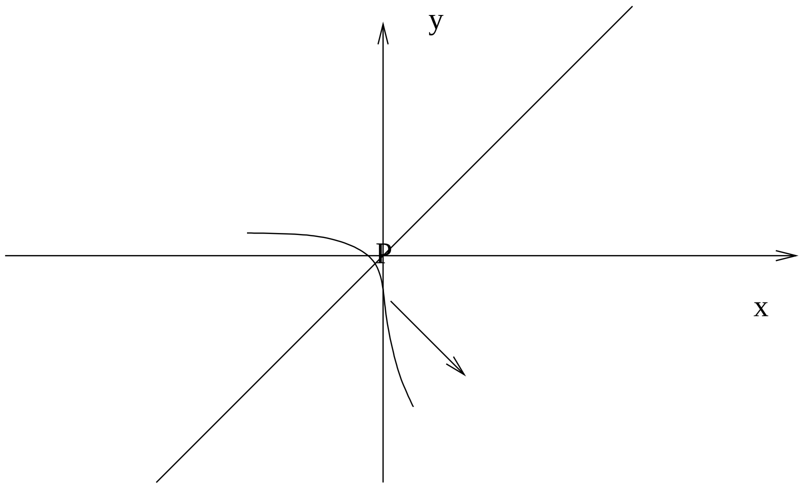


It follows that

$$Q(u) = \iint_{x<y} |u_{1,x}(t,y)||u_{2,x}(t,x)|dxdy = \|P\|_{L^1(x<y)}$$

is decreasing and

$$\frac{dQ}{dt} = -2\int_{\mathbb{R}} |u_{1,x}(t,x)||u_{2,x}(t,x)|dx = -\int_{\mathbb{R}} |P(t,x,x)|dx.$$



The extension of this linear part of the Glimm functional to vanishing viscosity, semidiscrete schemes and relaxation can be done by means of Fourier-Laplace transform, and some analytic tools.

The extension of this linear part of the Glimm functional to vanishing viscosity, semidiscrete schemes and relaxation can be done by means of Fourier-Laplace transform, and some analytic tools.

Remark. For a scalar conservation laws

$$u_t + f(u)_x = 0, \quad u \in R,$$

this part does not exists, and if $f(u) = \lambda u$, then there is no decreasing functional (the solution translates).

The extension of this linear part of the Glimm functional to vanishing viscosity, semidiscrete schemes and relaxation can be done by means of Fourier-Laplace transform, and some analytic tools.

Remark. For a scalar conservation laws

$$u_t + f(u)_x = 0, \quad u \in R,$$

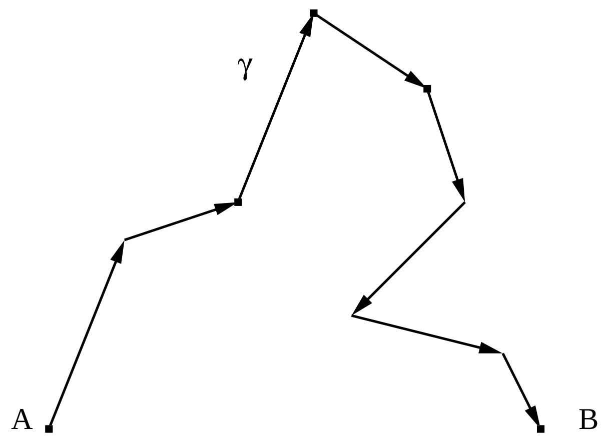
this part does not exists, and if $f(u) = \lambda u$, then there is no decreasing functional (the solution translates).

We will thus look for the part of the Glimm functional related to the nonlinearity of f.

Motion by in the direction of curvature

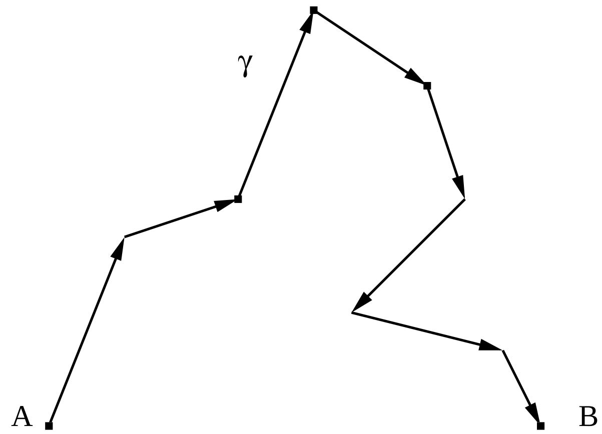
Motion by in the direction of curvature

Fix two points A, B in the plane $\mathbb{R}^2$ and consider a polygonal line joining A with B.



Motion by in the direction of curvature

Fix two points A, B in the plane $\mathbb{R}^2$ and consider a polygonal line joining A with B.

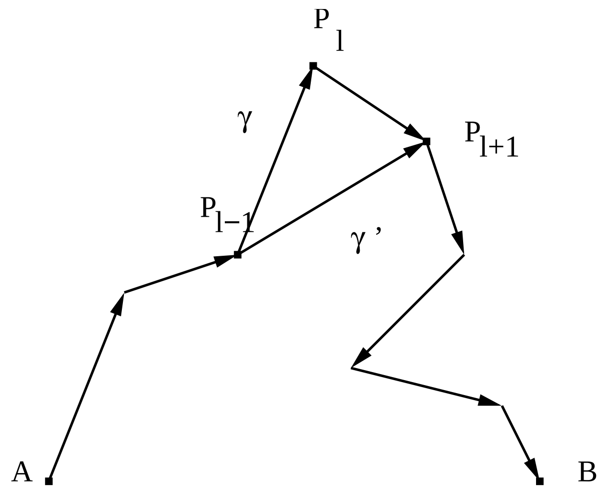


Define the functional ($\wedge$ is the external product in $\mathbb{R}^2$)

$$Q(\gamma) = \frac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^{n} |v_i \wedge v_j|.$$

Motion by in the direction of curvature

Fix two points A, B in the plane $\mathbb{R}^2$ and consider a polygonal line joining A with B.

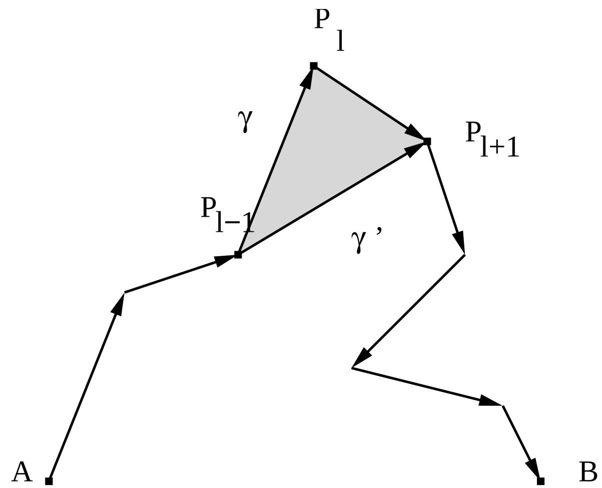


Define the functional ($\wedge$ is the external product in $\mathbb{R}^2$)

$$Q(\gamma) = \frac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^{n} |v_i \wedge v_j|.$$

Motion by in the direction of curvature

Fix two points A, B in the plane $\mathbb{R}^2$ and consider a polygonal line joining A with B.

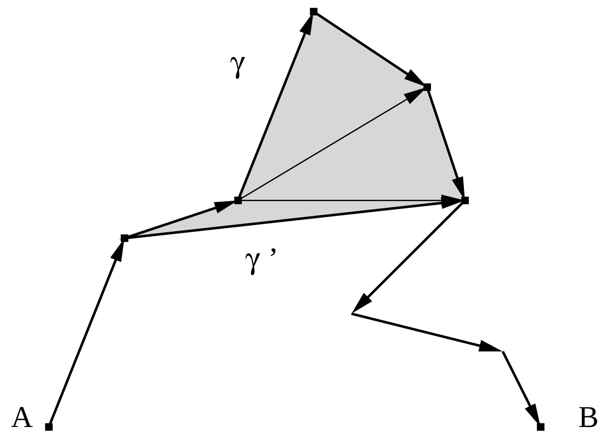


Define the functional ($\wedge$ is the external product in $\mathbb{R}^2$)

$$Q(\gamma) = \frac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^{n} |v_i \wedge v_j|.$$

Motion by in the direction of curvature

Fix two points A, B in the plane $\mathbb{R}^2$ and consider a polygonal line joining A with B.



Define the functional ($\wedge$ is the external product in $\mathbb{R}^2$)

$$Q(\gamma) = \frac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^{n} |v_i \wedge v_j|. \tag{4}$$

Let γ' be obtained from γ by replacing the two segments $P_{\ell-1}P_\ell$ and $P_\ell P_{\ell+1}$ by one single segment $P_{\ell-1}P_{\ell+1}$ (a *cut*). The area of the triangle with vertices $P_{\ell-1}$, P_ℓ, $P_{\ell+1}$ satisfies

$$\text{Area}(P_{\ell-1}P_\ell P_{\ell+1}) = \frac{1}{2}|v_{\ell+1} \wedge v_\ell| \leq Q(\gamma) - Q(\gamma'). \tag{5}$$

Let γ' be obtained from γ by replacing the two segments $P_{\ell-1}P_\ell$ and $P_\ell P_{\ell+1}$ by one single segment $P_{\ell-1}P_{\ell+1}$ (a *cut*). The area of the triangle with vertices $P_{\ell-1}$, P_ℓ, $P_{\ell+1}$ satisfies

$$\text{Area}(P_{\ell-1}P_\ell P_{\ell+1}) = \frac{1}{2}|v_{\ell+1} \wedge v_\ell| \leq Q(\gamma) - Q(\gamma'). \tag{5}$$

More generally, for absolutely continuous curves,

$$Q(\gamma) = \frac{1}{2}\int_0^1 \int_x^1 \Big|\gamma_x(x) \wedge \gamma_x(y)\Big| dydx, \tag{6}$$

Let γ' be obtained from γ by replacing the two segments $P_{\ell-1}P_\ell$ and $P_\ell P_{\ell+1}$ by one single segment $P_{\ell-1}P_{\ell+1}$ (a *cut*). The area of the triangle with vertices $P_{\ell-1}$, P_ℓ, $P_{\ell+1}$ satisfies

$$\text{Area}(P_{\ell-1}P_\ell P_{\ell+1}) = \frac{1}{2}|v_{\ell+1} \wedge v_\ell| \le Q(\gamma) - Q(\gamma'). \tag{5}$$

More generally, for absolutely continuous curves,

$$Q(\gamma) = \frac{1}{2}\int_0^1 \int_x^1 \Big|\gamma_x(x) \wedge \gamma_x(y)\Big| dydx, \tag{6}$$

(Area of the zonoid of the measure $d\mu(x) = d\gamma(x)$.)

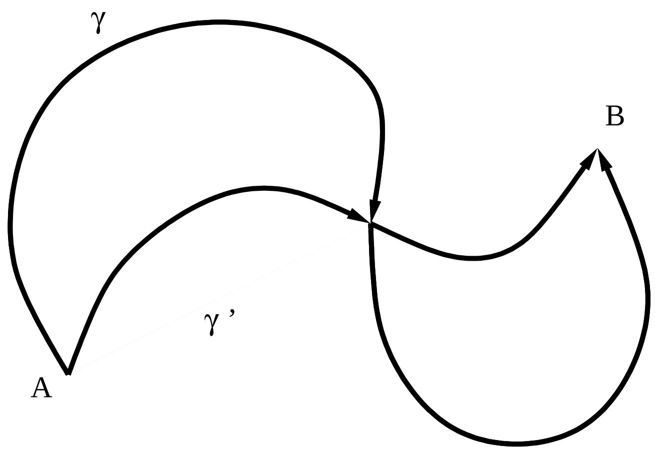
Let γ' be obtained from γ by replacing the two segments $P_{\ell-1}P_\ell$ and $P_\ell P_{\ell+1}$ by one single segment $P_{\ell-1}P_{\ell+1}$ (a *cut*). The area of the triangle with vertices $P_{\ell-1}$, P_ℓ, $P_{\ell+1}$ satisfies

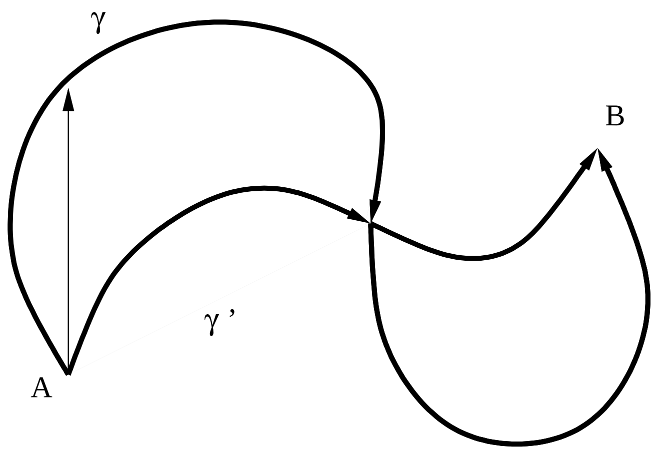
$$\text{Area}(P_{\ell-1}P_\ell P_{\ell+1}) = \frac{1}{2}|v_{\ell+1} \wedge v_\ell| \leq Q(\gamma) - Q(\gamma'). \tag{5}$$

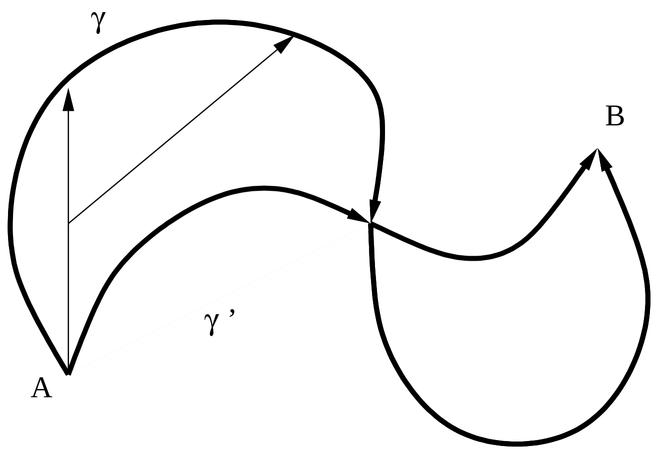
More generally, for absolutely continuous curves,

$$Q(\gamma) = \frac{1}{2}\int_0^1 \int_x^1 \Big|\gamma_x(x) \wedge \gamma_x(y)\Big| dy dx, \tag{6}$$

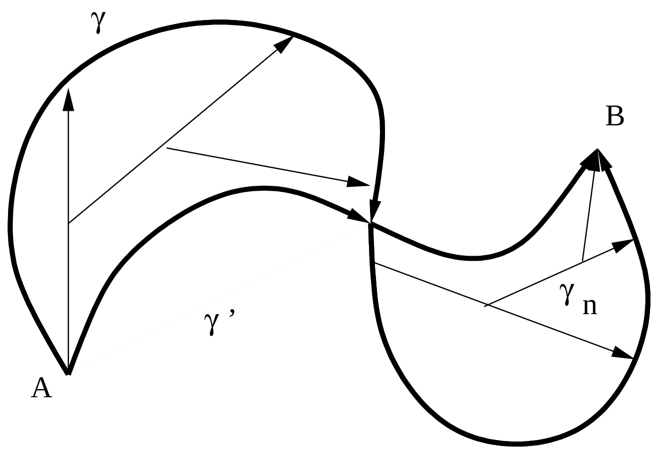
(Area of the zonoid of the measure $d\mu(x) = d\gamma(x)$.)
We say that γ *moves in the direction of curvature* if $\gamma(t)$ is obtained from $\gamma(s)$ by a sequence of cuts, for all $s < t$, $s, t \in [t_1, t_2]$.

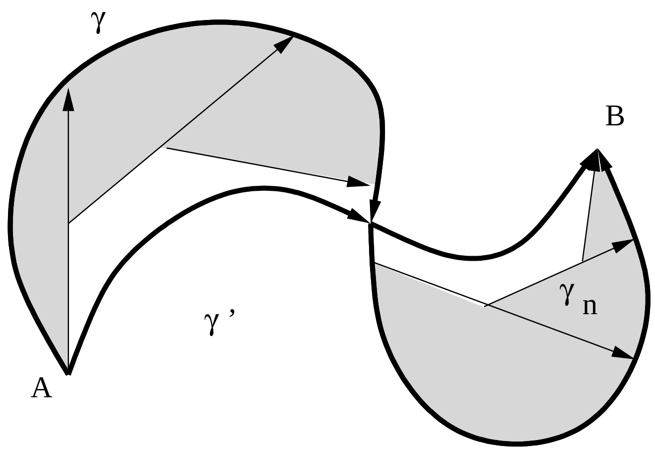

γ
B
γ ’
A

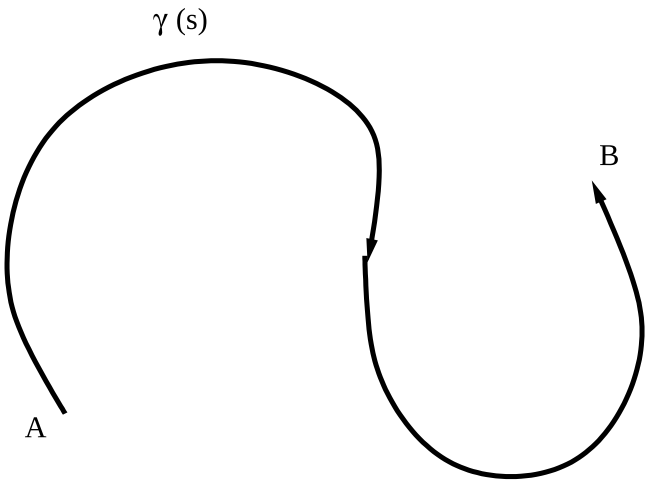

γ
B
γ ’
A

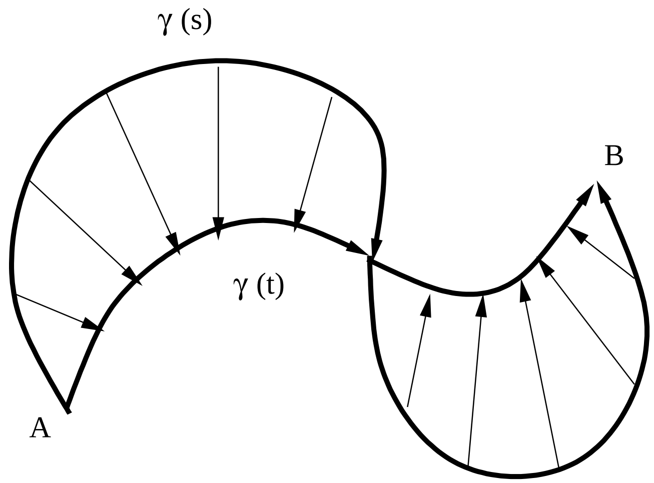


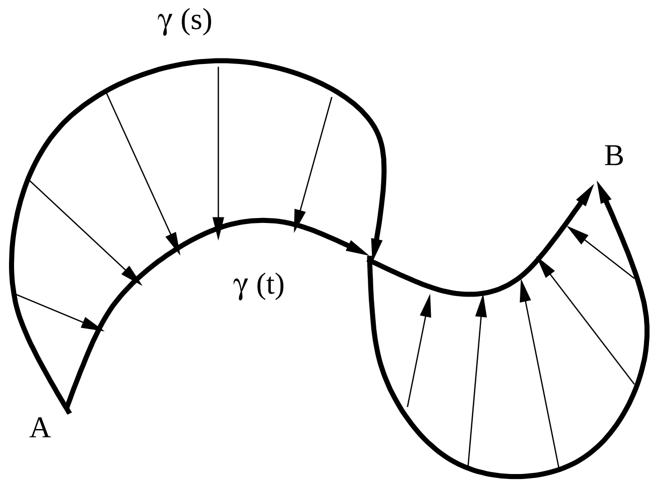
γ
B
γ ’
A


γ
B
γ n
γ '
A

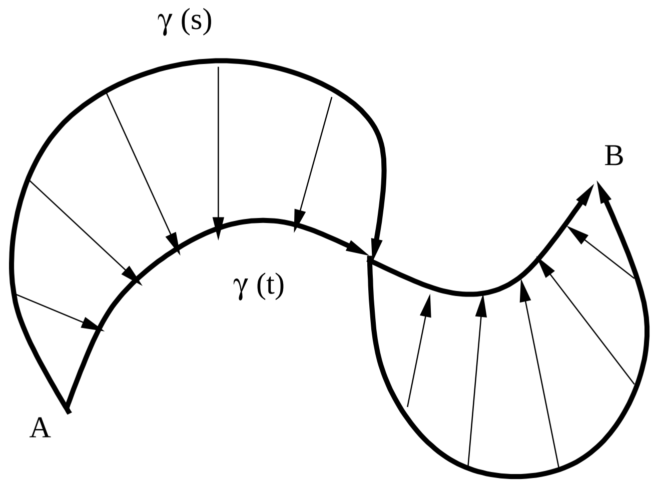

γ
B
γ n
γ ’
A


γ (s)
B
A


γ (s)
B
γ (t)
A



Let $\text{Area}\Big(\gamma;\ [t_1, t_2]\Big)$ be the area swept by γ in $[t_1, t_2]$.



Let Area$\Big(\gamma;\ [t_1, t_2]\Big)$ be the area swept by γ in $[t_1, t_2]$.

Theorem *Let $t \mapsto \gamma(t) \in \mathcal{F}$ denote a curve in the plane, moving in the direction of the curvature. Then, for every $t_1 < t_2$ one has*

$$\text{Area}\Big(\gamma;\ [t_1, t_2]\Big) \leq Q\Big(\gamma(t_1)\Big) - Q\Big(\gamma(t_2)\Big). \tag{7}$$

1.1. A curve moving in the direction of curvature for scalar conservation laws

1.1. A curve moving in the direction of curvature for scalar conservation laws

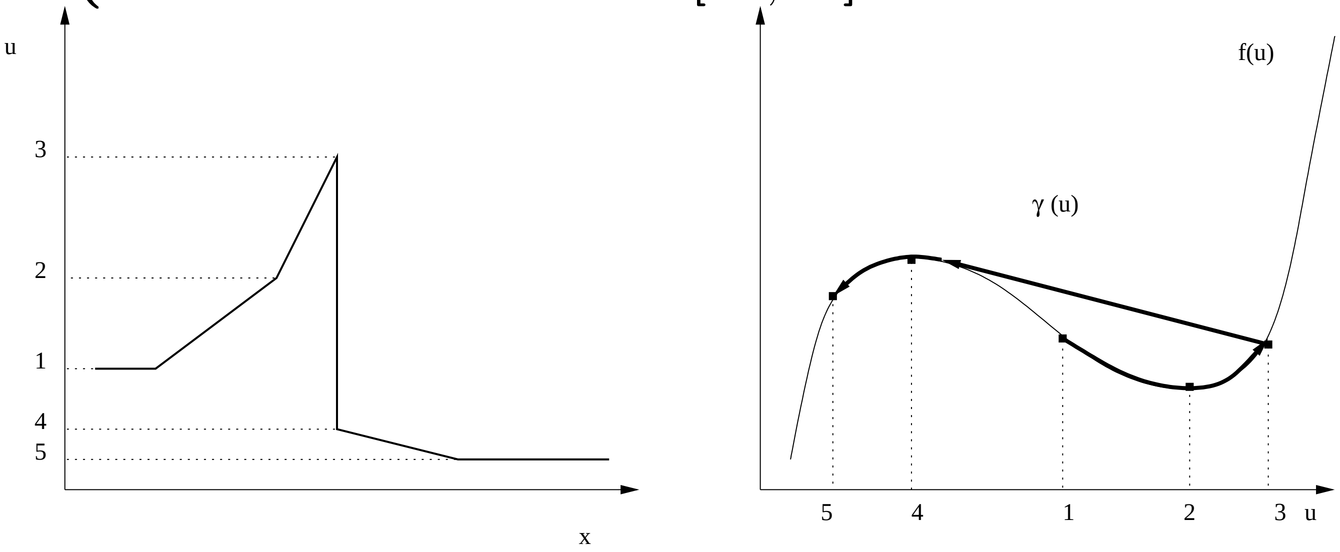
Given a map $u : \mathbb{R} \mapsto \mathbb{R}$ with bounded variation, define $\gamma(u)$ as

$$\gamma(u;x) = \begin{cases} (u(x), f(u(x))) & u \text{ is continuous at } x \\ \text{concave envelope of } f|_{[u^+,u^-]} & u \text{ has a jump in } x,\ u^- > u^+ \\ \text{convex envelope of } f|_{[u^-,u^+]} & u \text{ has a jump in } x,\ u^- < u^+ \end{cases}$$

1.1. A curve moving in the direction of curvature for scalar conservation laws

Given a map $u : \mathbb{R} \mapsto \mathbb{R}$ with bounded variation, define $\gamma(u)$ as

$$\gamma(u;x) = \begin{cases} (u(x), f(u(x)) & u \text{ is continuous at } x \\ \text{concave envelope of } f|_{[u^+,u^-]} & u \text{ has a jump in } x,\ u^- > u^+ \\ \text{convex envelope of } f|_{[u^-,u^+]} & u \text{ has a jump in } x,\ u^- < u^+ \end{cases}$$



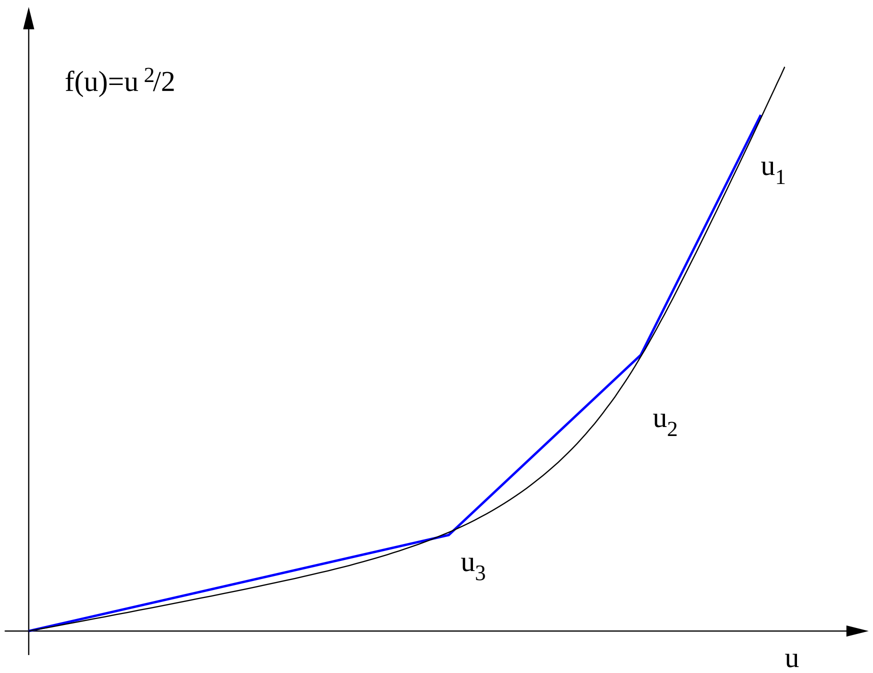
The fact that the curve γ corresponding to Kruzhkov entropy solution moves in the direction of curvature can be proved by considering wavefront tracking approximation.

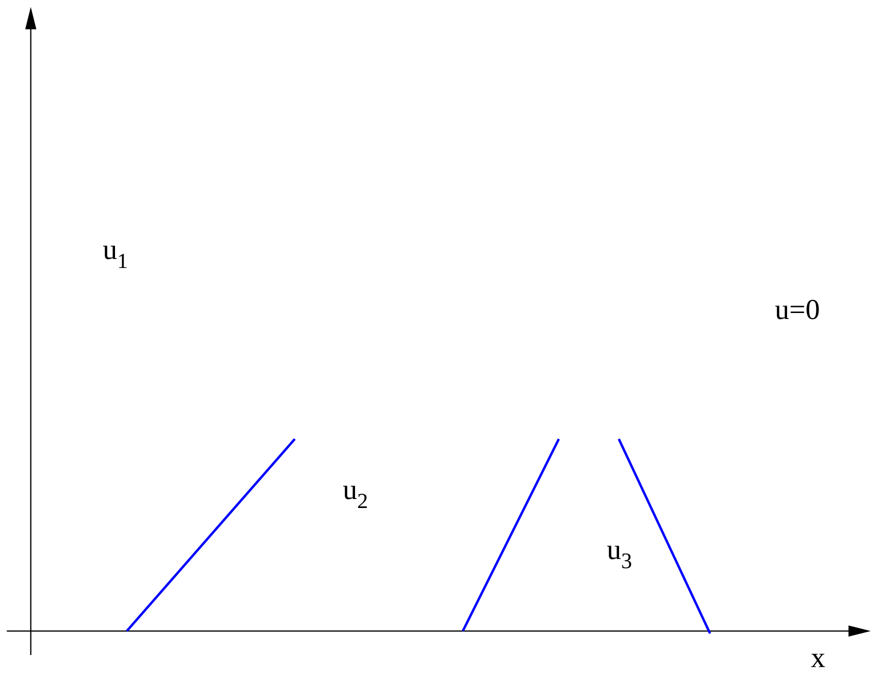
The fact that the curve γ corresponding to Kruzhkov entropy solution moves in the direction of curvature can be proved by considering wavefront tracking approximation.

Example: $u_t + (u^2/2)_x = 0$.

The fact that the curve γ corresponding to Kruzhkov entropy solution moves in the direction of curvature can be proved by considering wavefront tracking approximation.

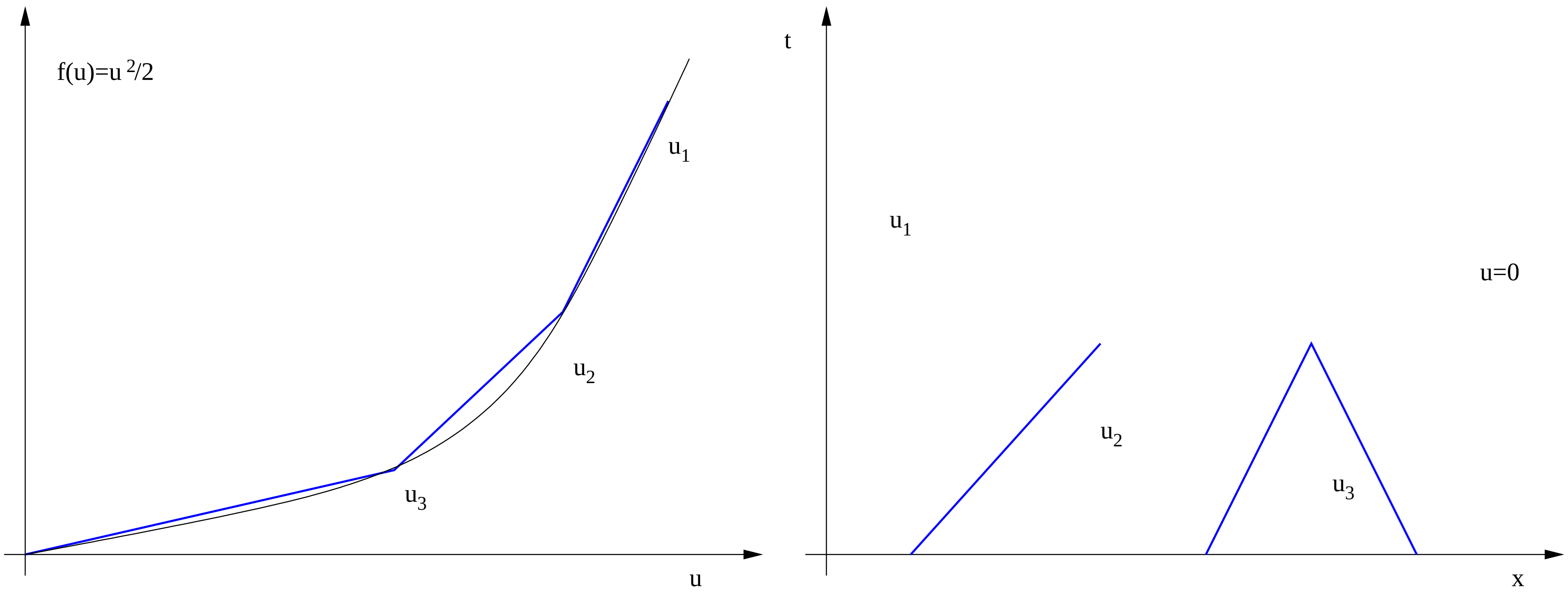
Example: $u_t + (u^2/2)_x = 0$.





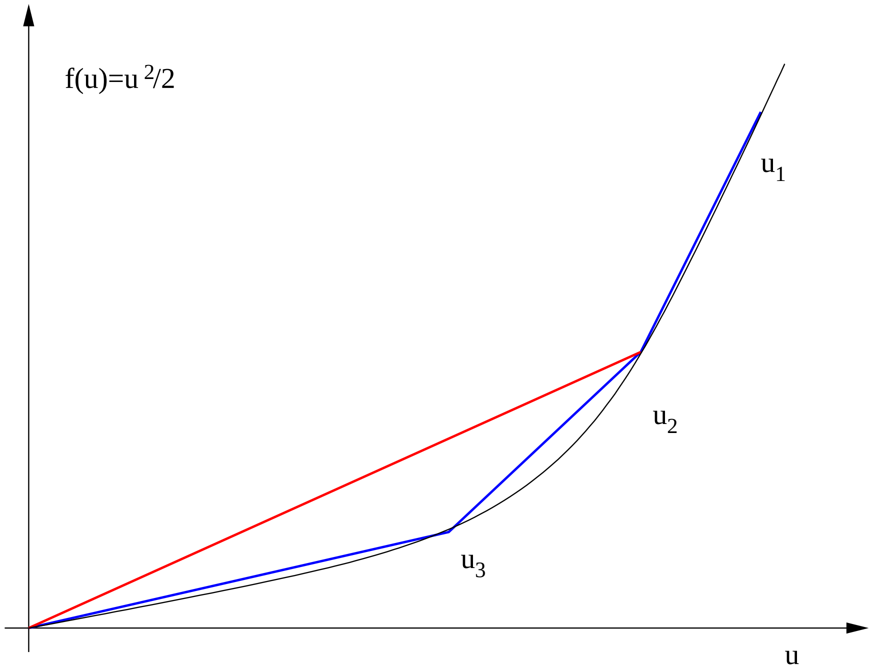
The fact that the curve γ corresponding to Kruzhkov entropy solution moves in the direction of curvature can be proved by considering wavefront tracking approximation.

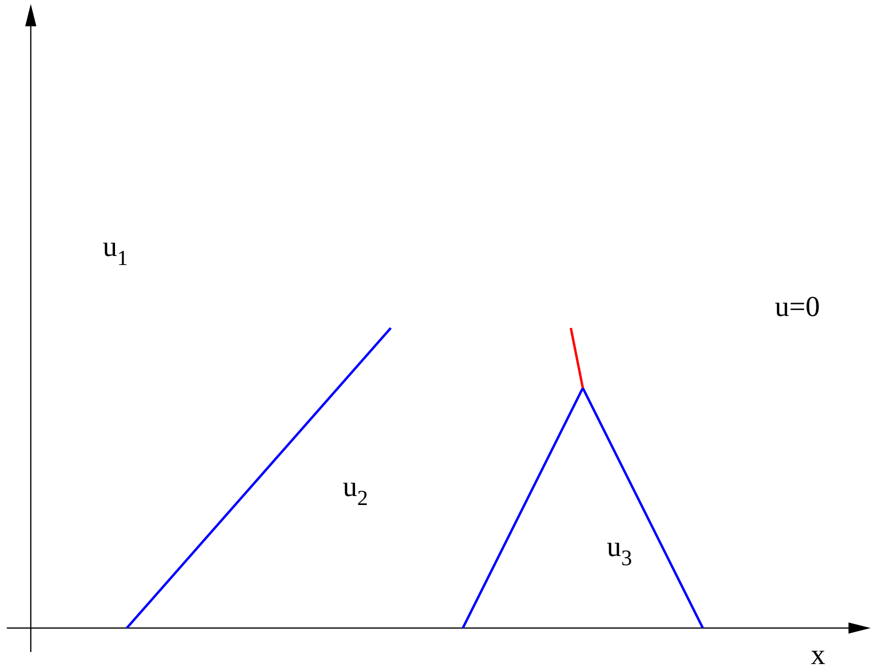
Example: $u_t + (u^2/2)_x = 0$.



The fact that the curve γ corresponding to Kruzhkov entropy solution moves in the direction of curvature can be proved by considering wavefront tracking approximation.

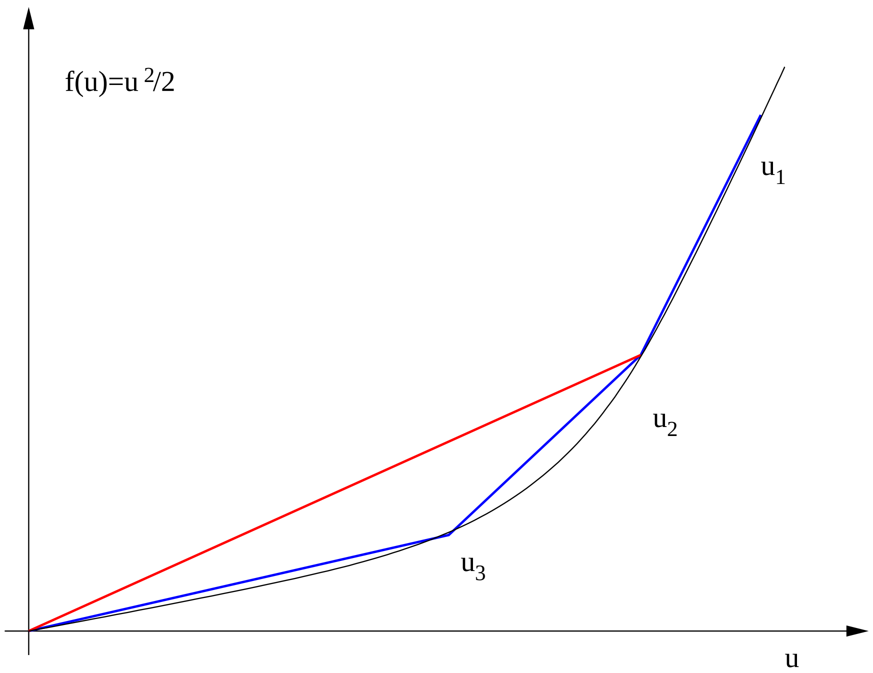
Example: $u_t + (u^2/2)_x = 0$.

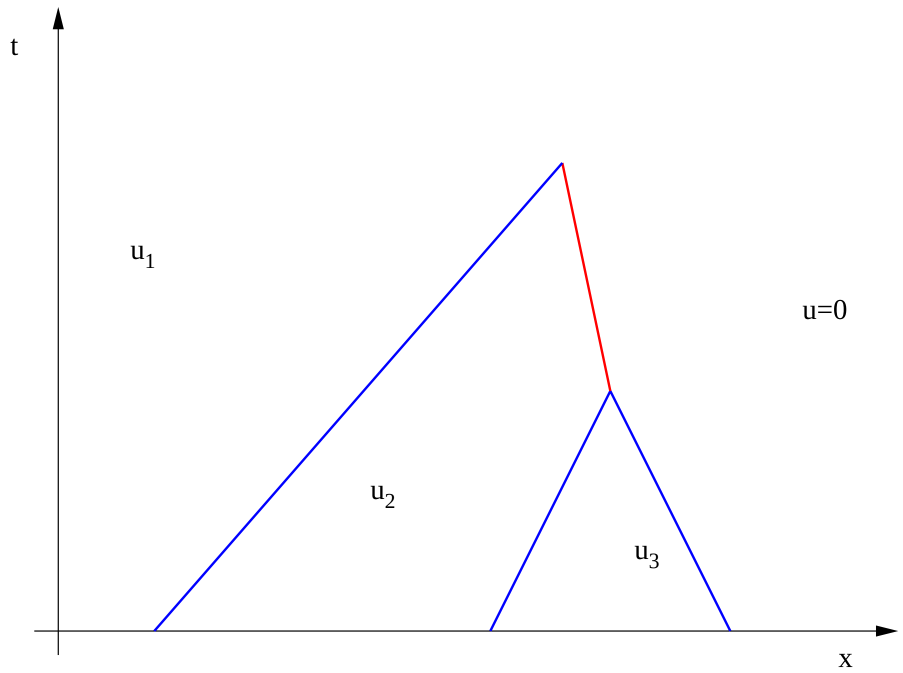




The fact that the curve γ corresponding to Kruzhkov entropy solution moves in the direction of curvature can be proved by considering wavefront tracking approximation.

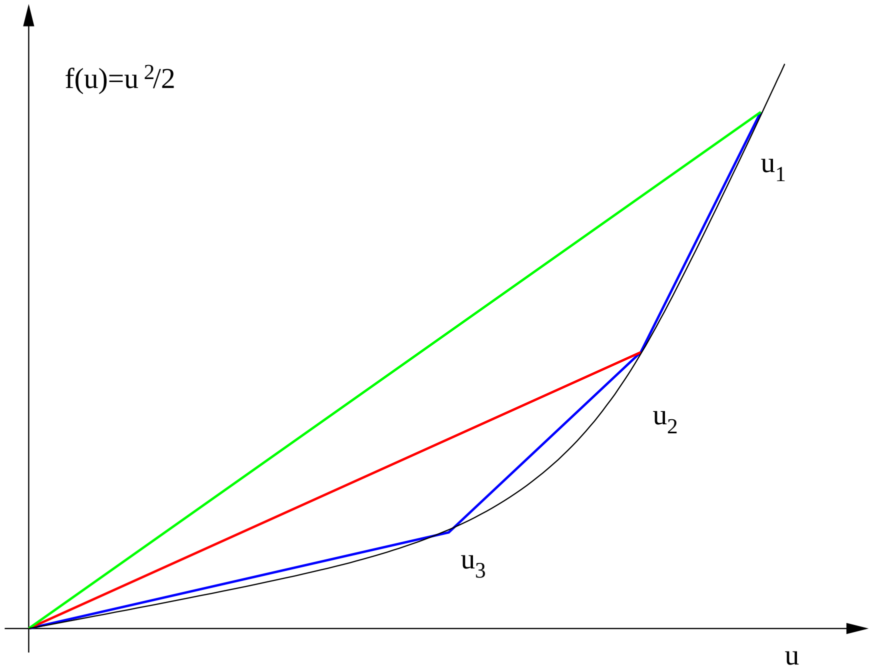
Example: $u_t + (u^2/2)_x = 0$.

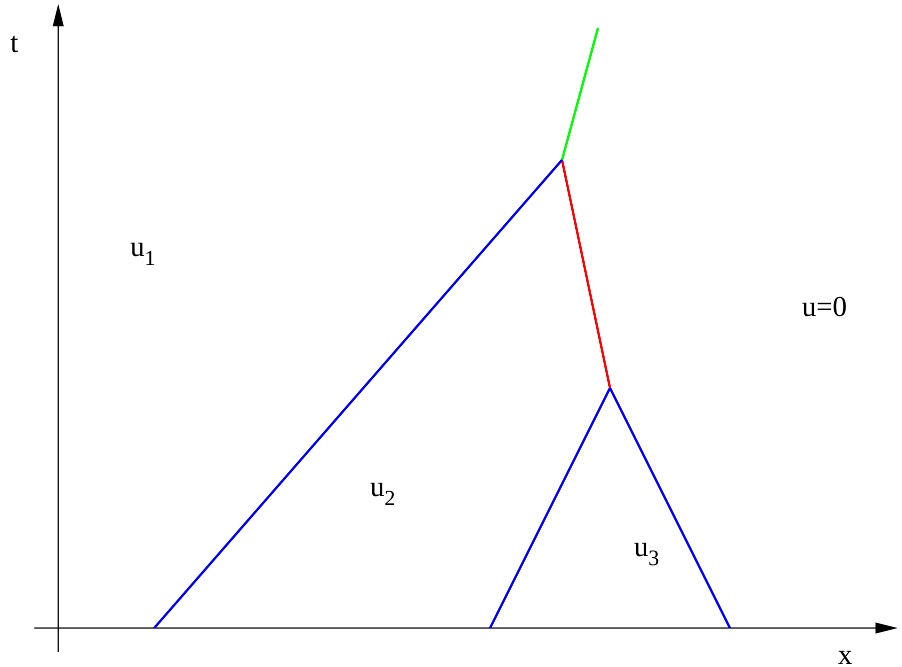




The fact that the curve γ corresponding to Kruzhkov entropy solution moves in the direction of curvature can be proved by considering wavefront tracking approximation.

Example: $u_t + (u^2/2)_x = 0$.





Viscous approximations

Viscous approximations

We can construct the interaction functional also for the viscous approximations

$$u_t + f(u)_x - u_{xx} = 0. \tag{8}$$

Viscous approximations

We can construct the interaction functional also for the viscous approximations

$$u_t + f(u)_x - u_{xx} = 0. \tag{8}$$

In fact, the curve in $\mathbb{R}^2$

$$\gamma(t,x) \doteq \begin{pmatrix} u \\ f(u) - u_x \end{pmatrix} \tag{9}$$

satisfies the parabolic system

$$\gamma_t + \lambda(t,x)\gamma_x - \gamma_{xx} = 0, \quad \lambda(t,x) = \lambda(u) = f'(u). \tag{10}$$

Viscous approximations

We can construct the interaction functional also for the viscous approximations

$$u_t + f(u)_x - u_{xx} = 0. \tag{8}$$

In fact, the curve in $\mathbb{R}^2$

$$\gamma(t,x) \doteq \begin{pmatrix} u \\ f(u) - u_x \end{pmatrix} \tag{9}$$

satisfies the parabolic system

$$\gamma_t + \lambda(t,x)\gamma_x - \gamma_{xx} = 0, \qquad \lambda(t,x) = \lambda(u) = f'(u). \tag{10}$$

γ *moves in the direction of curvature*

The functional

$$Q(u) = \frac{1}{2}\iint_{x<y}\Big|\gamma_x(x)\wedge\gamma_x(y)\Big|dydx$$
$$= \frac{1}{2}\iint_{x<y}\Big|u_x(t,x)u_t(t,y)-u_t(t,x)u_x(t,y)\Big|dxdy, \quad (11)$$

is decreasing, and controls the interaction quantity (Area swept)

$$\int_{\mathbb{R}}\Big|\gamma_t(t,x)\wedge\gamma_x(t,x)\Big|dx = \int_{\mathbb{R}}\Big|u_xu_{tx}-u_{xx}u_t\Big|dx. \quad (12)$$

The functional

$$Q(u) = \frac{1}{2}\iint_{x<y}\Big|\gamma_x(x)\wedge\gamma_x(y)\Big|dydx$$
$$= \frac{1}{2}\iint_{x<y}\Big|u_x(t,x)u_t(t,y) - u_t(t,x)u_x(t,y)\Big|dxdy, \quad (11)$$

is decreasing, and controls the interaction quantity (Area swept)

$$\int_{\mathbb{R}}\Big|\gamma_t(t,x)\wedge\gamma_x(t,x)\Big|dx = \int_{\mathbb{R}}\Big|u_x u_{tx} - u_{xx}u_t\Big|dx. \quad (12)$$

Theorem.

$$\frac{d}{dt}Q + \int_{\mathbb{R}}\Big|u_x u_{tx} - u_{xx}u_t\Big|dx \le 0.$$

Semidiscrete schemes

Semidiscrete schemes

The simplest semidiscrete scheme (stable and diffusive for $f' > 0$) is the *upwind* scheme,

$$u_t(t,x) + f(u(t,x)) - f(u(t,x-1)) = 0. \tag{13}$$

Semidiscrete schemes

The simplest semidiscrete scheme (stable and diffusive for $f' > 0$) is the *upwind* scheme,

$$u_t(t,x) + f(u(t,x)) - f(u(t,x-1)) = 0. \tag{13}$$

One can rewrite the scheme as

$$\begin{aligned} u_t(t,x) + \frac{f(u(t,x)) - f(u(t,x-1))}{u(t,x) - u(t,x-1)}\Big(u(t,x) - u(t,x-1)\Big) = \\ u_t(t,x) + \lambda\Big(u(t,x), u(t,x-1)\Big)\Big(u(t,x) - u(t,x-1)\Big) = 0, \end{aligned}$$

with $\lambda > 0$.

Semidiscrete schemes

The simplest semidiscrete scheme (stable and diffusive for $f' > 0$) is the *upwind* scheme,

$$u_t(t,x) + f(u(t,x)) - f(u(t,x-1)) = 0. \tag{13}$$

One can rewrite the scheme as

$$\begin{aligned} u_t(t,x) + \frac{f(u(t,x)) - f(u(t,x-1))}{u(t,x) - u(t,x-1)}\Big(u(t,x) - u(t,x-1)\Big) = \\ u_t(t,x) + \lambda\Big(u(t,x), u(t,x-1)\Big)\Big(u(t,x) - u(t,x-1)\Big) = 0, \end{aligned}$$

with $\lambda > 0$.
The curve γ solving

$$\gamma_t(t,x) + \lambda(t,x)\Big(\gamma(t,x) - \gamma(t,x-1)\Big) = 0 \tag{14}$$

moves in the direction of curvature for $\lambda > 0$.

Similarly for the discrete scheme,

$$u(t+1,x) - u(t,x) + f(u(t,x)) - f(u(t,x-1)) = 0,$$

Similarly for the discrete scheme,

$$u(t+1,x) - u(t,x) + f(u(t,x)) - f(u(t,x-1)) = 0,$$

and the curve γ solving $(\lambda(t,x) = \lambda(u(x), u(x-1)))$

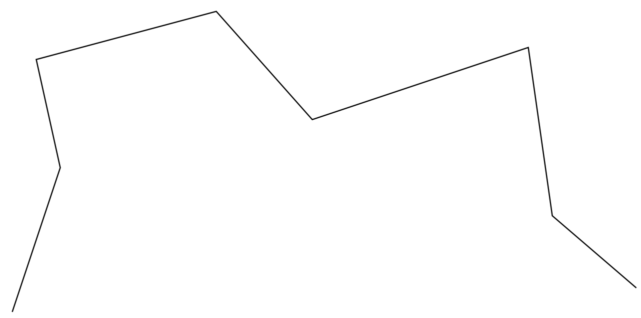
$$\gamma_t(t,x) = (1 - \lambda(t,x))\gamma(t,x) + \lambda(t,x)\gamma(t,x-1)\Big)$$

Similarly for the discrete scheme,

$$u(t+1,x) - u(t,x) + f(u(t,x)) - f(u(t,x-1)) = 0,$$

and the curve γ solving $(\lambda(t,x) = \lambda(u(x), u(x-1)))$

$$\gamma_t(t,x) = (1-\lambda(t,x))\gamma(t,x) + \lambda(t,x)\gamma(t,x-1)\Big)$$

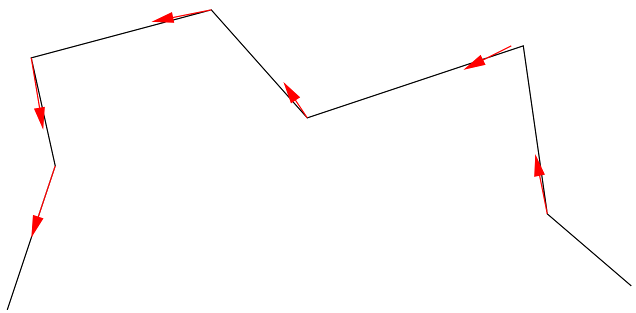


Similarly for the discrete scheme,

$$u(t+1,x) - u(t,x) + f(u(t,x)) - f(u(t,x-1)) = 0,$$

and the curve γ solving $(\lambda(t,x) = \lambda(u(x), u(x-1)))$

$$\gamma_t(t,x) = (1-\lambda(t,x))\gamma(t,x) + \lambda(t,x)\gamma(t,x-1)\Big)$$

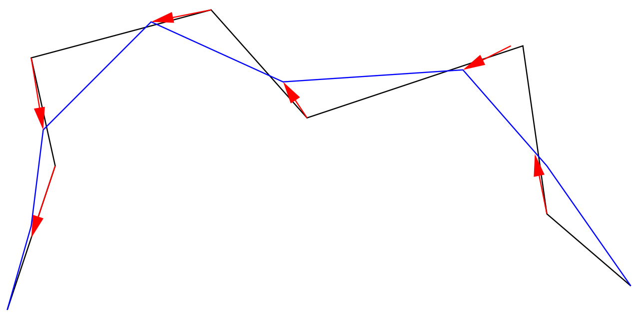


Similarly for the discrete scheme,

$$u(t+1,x) - u(t,x) + f(u(t,x)) - f(u(t,x-1)) = 0,$$

and the curve γ solving $(\lambda(t,x) = \lambda(u(x), u(x-1)))$

$$\gamma_t(t,x) = (1-\lambda(t,x))\gamma(t,x) + \lambda(t,x)\gamma(t,x-1)\Big)$$

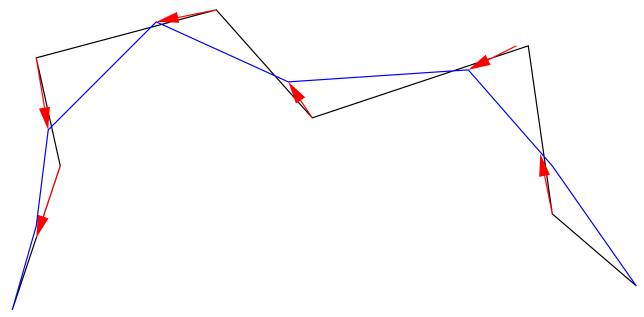


Similarly for the discrete scheme,

$$u(t+1,x) - u(t,x) + f(u(t,x)) - f(u(t,x-1)) = 0,$$

and the curve γ solving $(\lambda(t,x) = \lambda(u(x), u(x-1)))$

$$\gamma_t(t,x) = (1-\lambda(t,x))\gamma(t,x) + \lambda(t,x)\gamma(t,x-1)\Big)$$



Remark. The construction of γ as a function of u is nontrivial for the semidiscrete scheme, and open for the discrete.

Glimm functional and flux through the boundary

Glimm functional and flux through the boundary

Consider again the parabolic equation

$$u_t + f(u)_x - u_{xx} = 0,$$

Glimm functional and flux through the boundary

Consider again the parabolic equation

$$u_t + f(u)_x - u_{xx} = 0,$$

and construct the variable

$$P(t,x,y) \doteq u_t(t,x)u_x(t,y) - u_t(t,y)u_x(t,x). \qquad (15)$$

$$P_t + \mathrm{div}\Big(\big(f'(u(t,x)), f'(u(t,y))\big)P\Big) = \Delta P, \qquad x > y,\ \ P(t,x,x) = 0.$$

Glimm functional and flux through the boundary

Consider again the parabolic equation

$$u_t + f(u)_x - u_{xx} = 0,$$

and construct the variable

$$P(t,x,y) \doteq u_t(t,x)u_x(t,y) - u_t(t,y)u_x(t,x). \qquad (15)$$

$$P_t + \mathrm{div}\Big(\big(f'(u(t,x)), f'(u(t,y))\big)P\Big) = \Delta P, \quad x > y, \ P(t,x,x) = 0.$$

The interaction functional $Q(u)$ can be now interpreted as the L^1 norm of P in $\{x \geq y\}$,

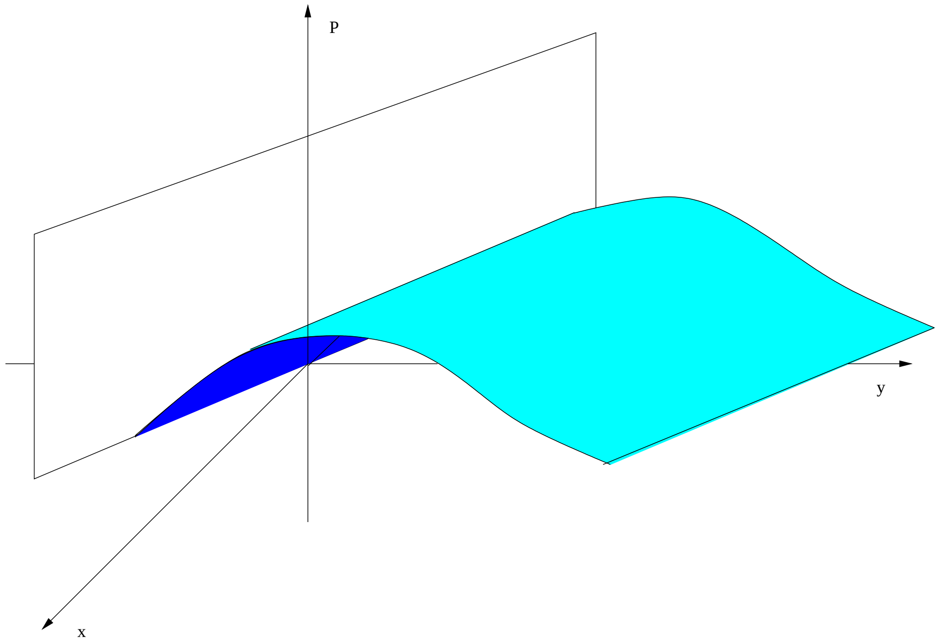
$$Q(P) = \iint_{x \geq y} |P(t,x,y)|dxdy, \qquad (16)$$

Its derivative controls the flux of P along the boundary $\{x = y\}$,

$$\frac{d}{dt}Q(P) \leq -\int_{x=y} \Big|\nabla P \cdot (1, -1)\Big| dx = -2 \int_{\mathbb{R}} \Big|u_{tx}u_x - u_t u_{xx}\Big| dx. \quad (17)$$

Its derivative controls the flux of P along the boundary $\{x = y\}$,

$$\frac{d}{dt}Q(P) \leq -\int_{x=y} \Big|\nabla P \cdot (1, -1)\Big| dx = -2 \int_{\mathbb{R}} \Big|u_{tx}u_x - u_t u_{xx}\Big| dx. \quad (17)$$



Kinetic models

Kinetic models

We consider two classes:

Kinetic models

We consider two classes:

BGK models: the simplest are

$$F_t^\alpha + \alpha F_x^\alpha = \frac{1}{\epsilon}(M^\alpha(u) - F^\alpha), \quad u = \sum_\alpha F^\alpha. \tag{18}$$

Kinetic models

We consider two classes:

BGK models: the simplest are

$$F_t^\alpha + \alpha F_x^\alpha = \frac{1}{\epsilon}(M^\alpha(u) - F^\alpha), \quad u = \sum_\alpha F^\alpha. \tag{18}$$

For stability and compatibility,

$$\sum_\alpha M^\alpha(u) = u, \quad 0 < DM^\alpha(u) < I.$$

Kinetic models

We consider two classes:

BGK models: the simplest are

$$F_t^\alpha + \alpha F_x^\alpha = \frac{1}{\epsilon}(M^\alpha(u) - F^\alpha), \quad u = \sum_\alpha F^\alpha. \tag{18}$$

For stability and compatibility,

$$\sum_\alpha M^\alpha(u) = u, \quad 0 < DM^\alpha(u) < I.$$

At equilibrium

$$f^\alpha = M^\alpha(u), \quad u_t + \left(\sum_\alpha M^\alpha(u)\right) = 0.$$

Broadwell model:

$$\begin{cases} F_t^- - F_x^- &= \frac{1}{\epsilon}((F^0)^2 - F^-F^+) \\ F_t^0 &= \frac{1}{\epsilon}(F^-F^+ - (F^0)^2) \\ F_t^+ - F_x^+ &= \frac{1}{\epsilon}((F^0)^2 - F^-F^+) \end{cases}$$

Broadwell model:

$$\begin{cases} F_t^- - F_x^- &=& \frac{1}{\epsilon}((F^0)^2 - F^-F^+) \\ F_t^0 &=& \frac{1}{\epsilon}(F^-F^+ - (F^0)^2) \\ F_t^+ - F_x^+ &=& \frac{1}{\epsilon}((F^0)^2 - F^-F^+) \end{cases}$$

Define

$$u^1 = F^- + F^0 + F^+, \; u^2 = F^+ - F^-, \; v = F^- + F^+$$

$$\begin{cases} u_t^1 + u_x^2 &=& 0 \\ u_t^2 + v_x &=& 0 \\ v_t + u_x^2 &=& \frac{1}{\epsilon}((u^1)^2 + (u^2)^2 - 2u^1v) \end{cases}$$

Broadwell model:

$$\begin{cases} F_t^- - F_x^- &= \frac{1}{\epsilon}((F^0)^2 - F^-F^+) \\ F_t^0 &= \frac{1}{\epsilon}(F^-F^+ - (F^0)^2) \\ F_t^+ - F_x^+ &= \frac{1}{\epsilon}((F^0)^2 - F^-F^+) \end{cases}$$

Define

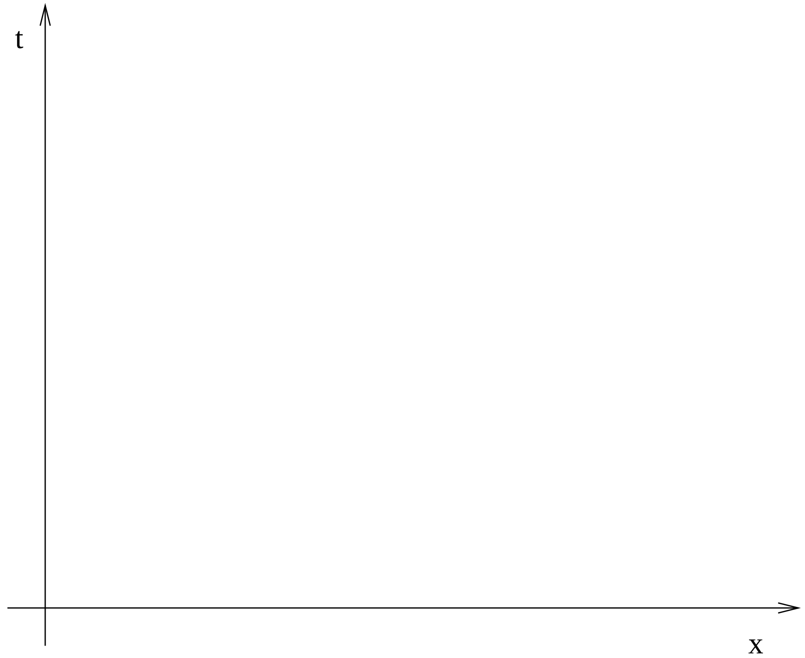
$$u^1 = F^- + F^0 + F^+,\ u^2 = F^+ - F^-,\ v = F^- + F^+$$

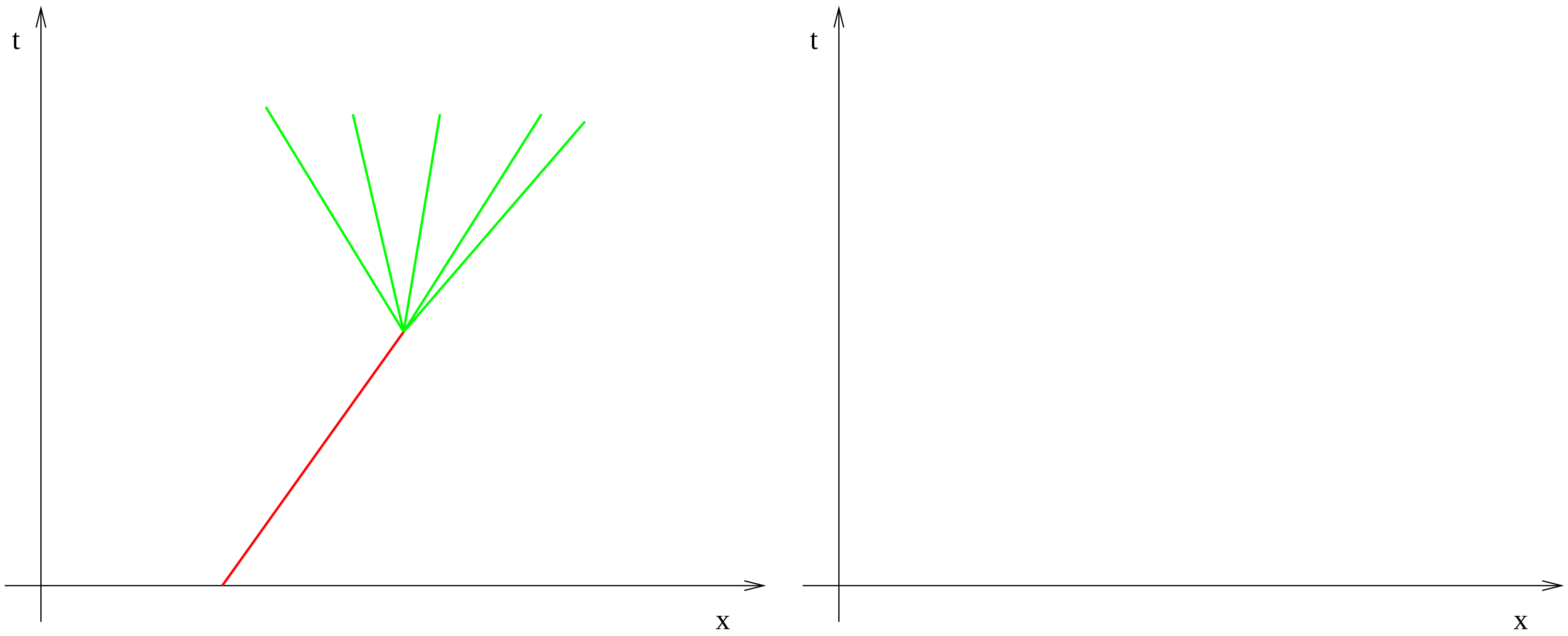
$$\begin{cases} u_t^1 + u_x^2 &= 0 \\ u_t^2 + v_x &= 0 \\ v_t + u_x^2 &= \frac{1}{\epsilon}((u^1)^2 + (u^2)^2 - 2u^1 v) \end{cases}$$

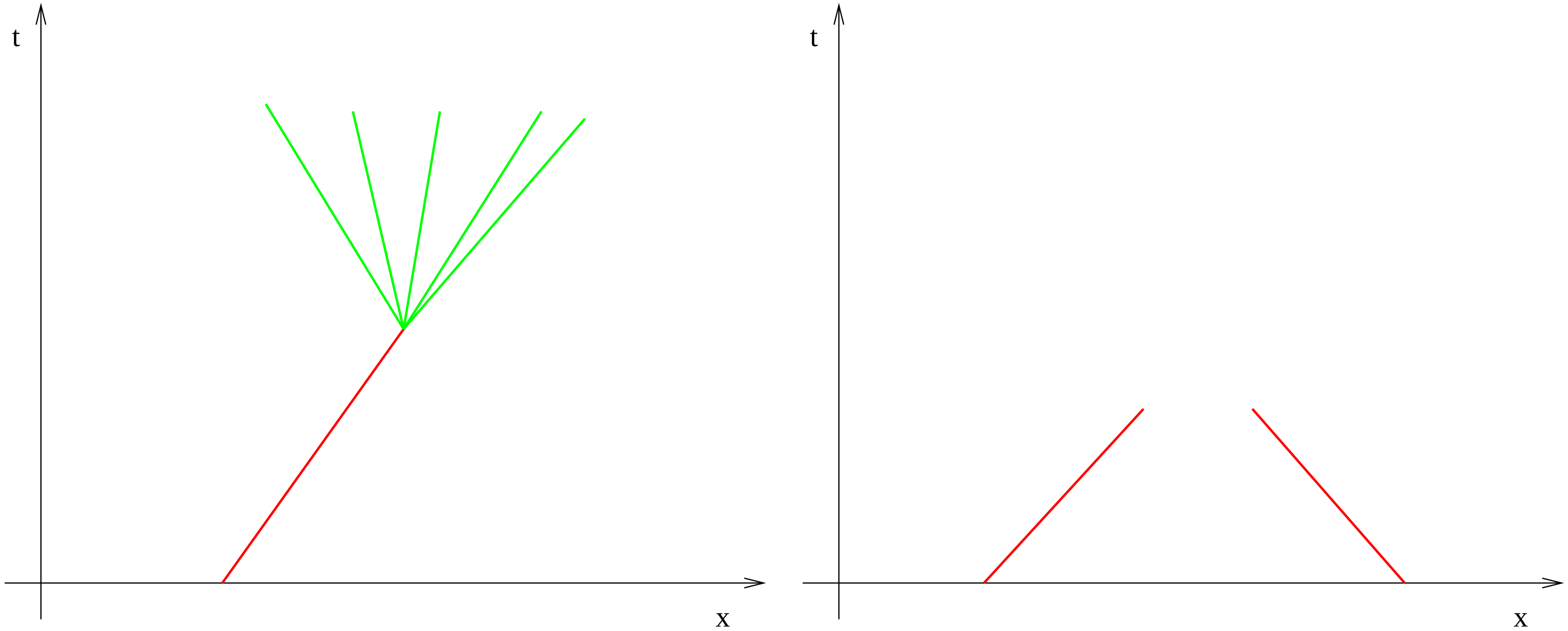
then its relaxation limit is

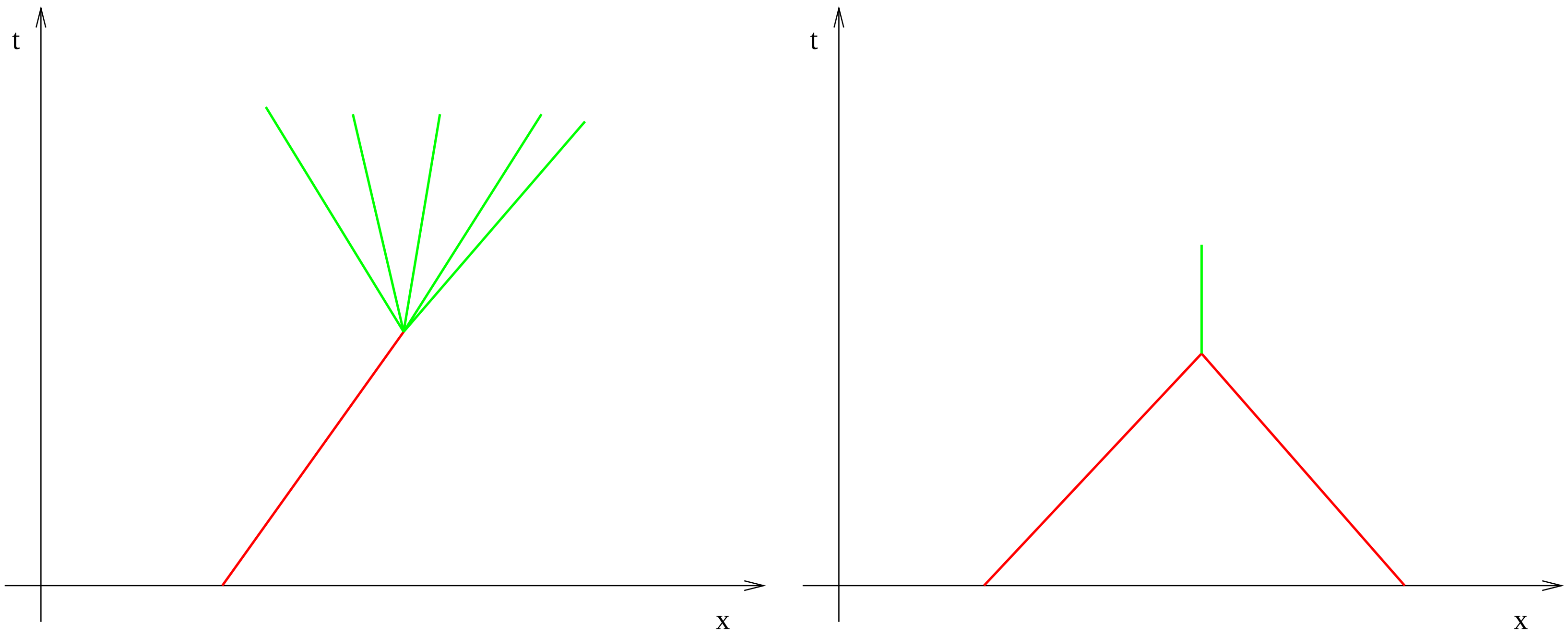
$$\begin{cases} u_t^1 + u_x^2 &= 0 \\ u_t^2 + \left(\frac{u^1}{2} + \frac{(u^2)^2}{2u^1}\right)_x &= 0 \end{cases}$$

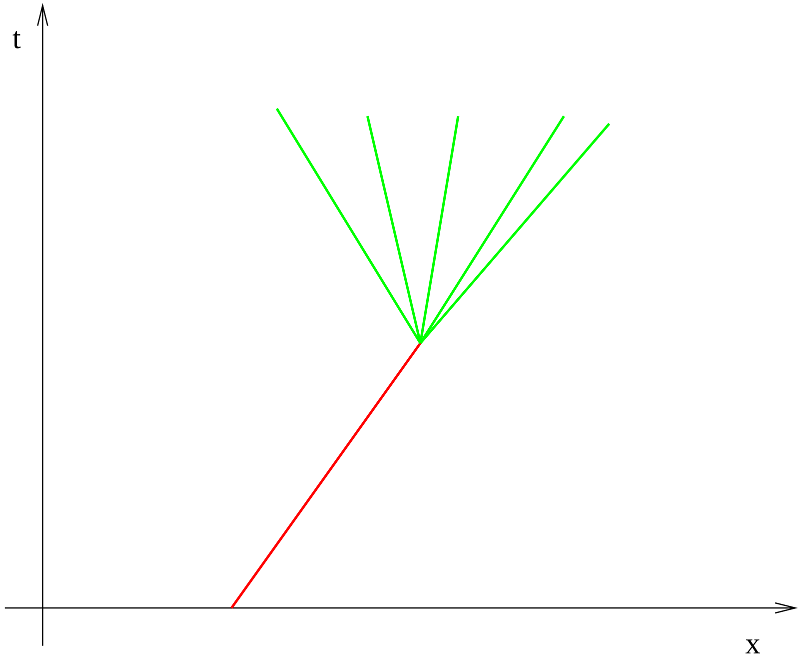

t
x

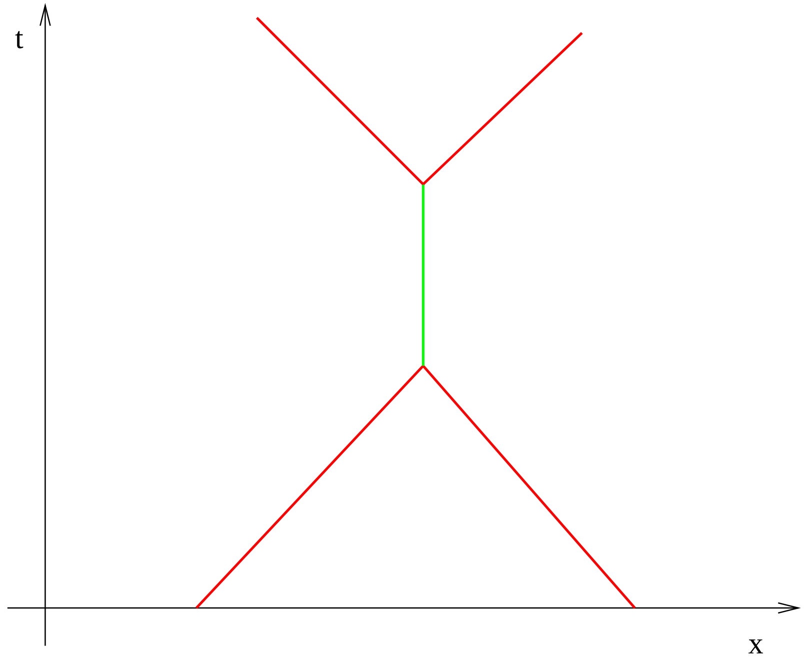

t
x

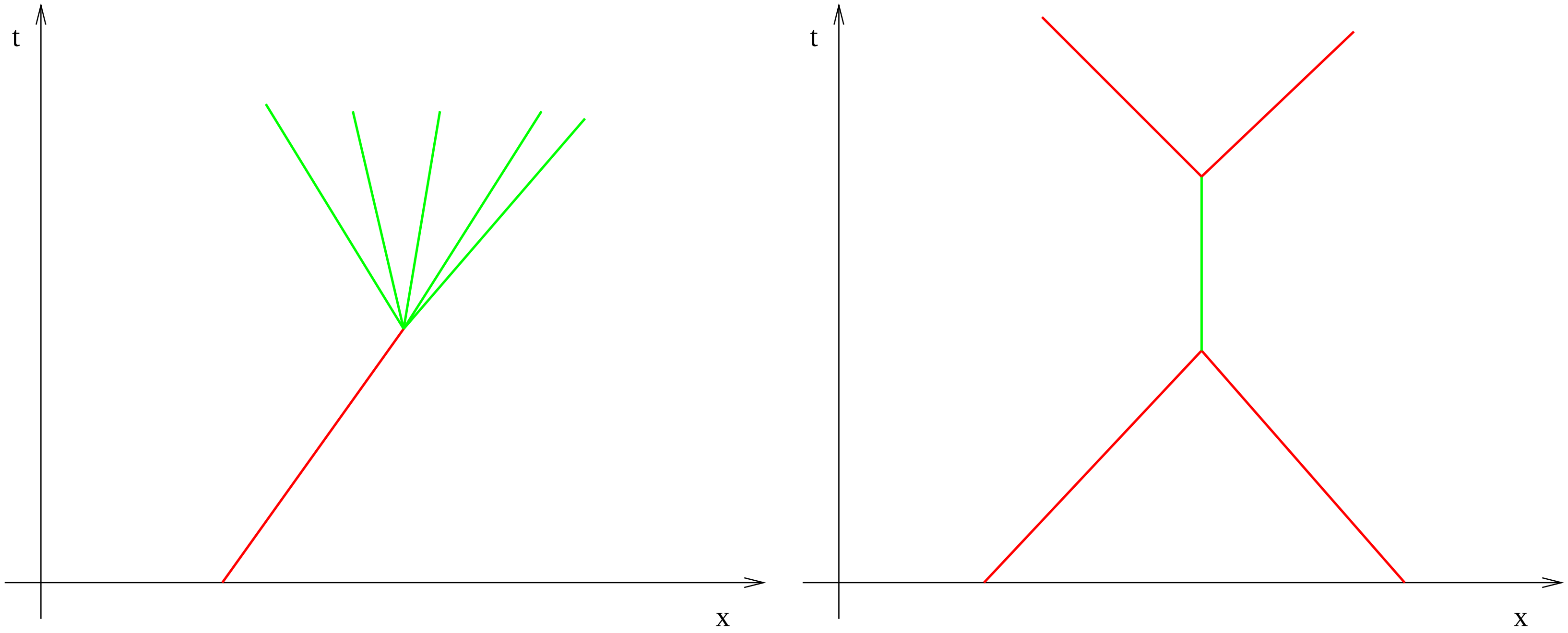

t
x
t
x


t
x
t
x


t
x
t
x


t
x


t
x



For BGK the probability of changing speed depends only on the state u, while for Broadwell depends on the density of the particles with different speeds.

An estimate for kinetic models

An estimate for kinetic models

Consider the simplest BGK model, i.e. linear with only two speeds,

$$\begin{cases} F_t^- - F_x^- &= -\frac{1}{2}F^- + \frac{1}{2}F^+ \\ F_t^+ + F_x^+ &= \frac{1}{2}F^- - \frac{1}{2}F^+ \end{cases} \tag{19}$$

An estimate for kinetic models

Consider the simplest BGK model, i.e. linear with only two speeds,

$$\left\{ \begin{array}{rcl} F_t^- - F_x^- & = & -\frac{1}{2}F^- + \frac{1}{2}F^+ \\ F_t^+ + F_x^+ & = & \frac{1}{2}F^- - \frac{1}{2}F^+ \end{array} \right. \tag{19}$$

The Dirichlet boundary conditions are given by

$$F^-(t,0) + F^+(t,0) = 0.$$

An estimate for kinetic models

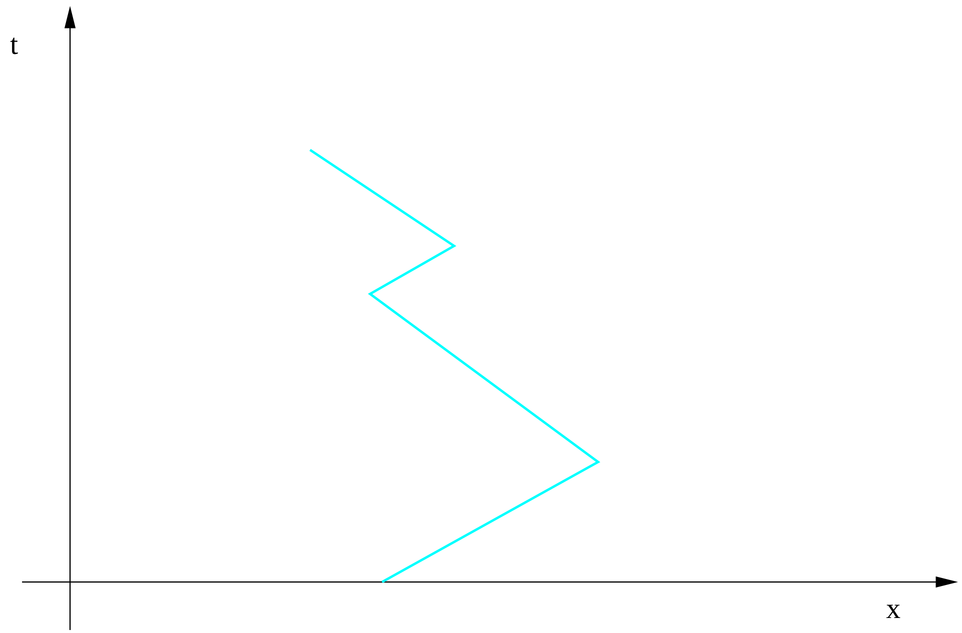
Consider the simplest BGK model, i.e. linear with only two speeds,

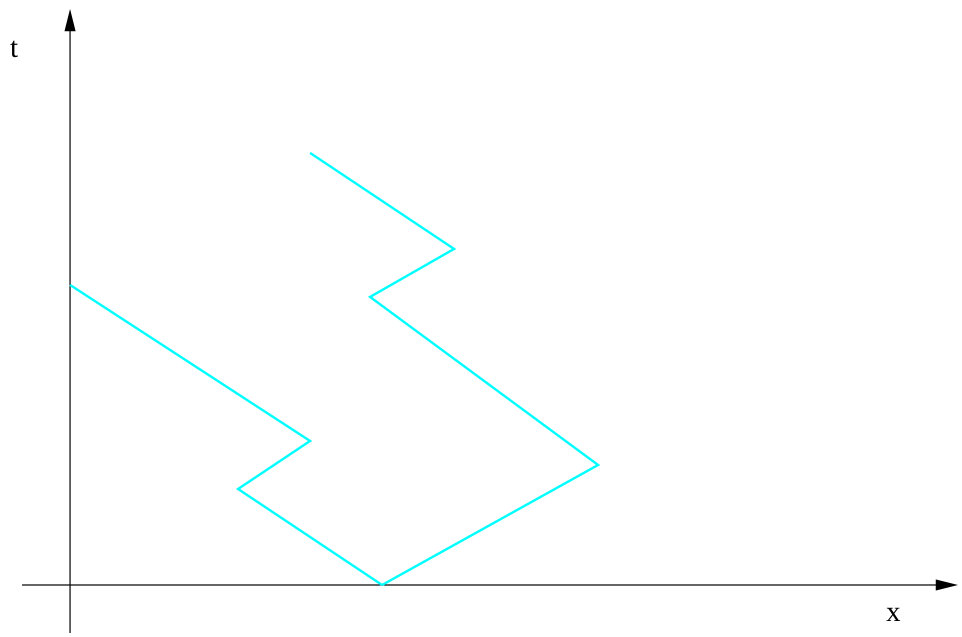
$$\begin{cases} F^-_t - F^-_x &= -\frac{1}{2}F^- + \frac{1}{2}F^+ \\ F^+_t + F^+_x &= \frac{1}{2}F^- - \frac{1}{2}F^+ \end{cases} \tag{19}$$

The Dirichlet boundary conditions are given by

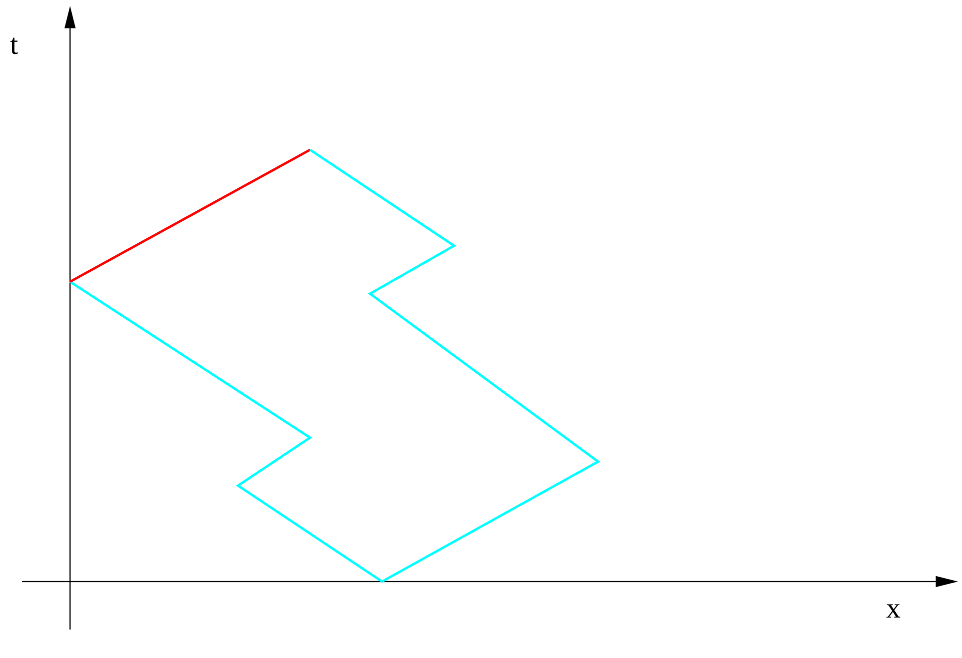
$$F^-(t,0) + F^+(t,0) = 0.$$

One can explain the above boundary condition by saying the when a particle hits the boundary $\{x = 0\}$ it changes sign.

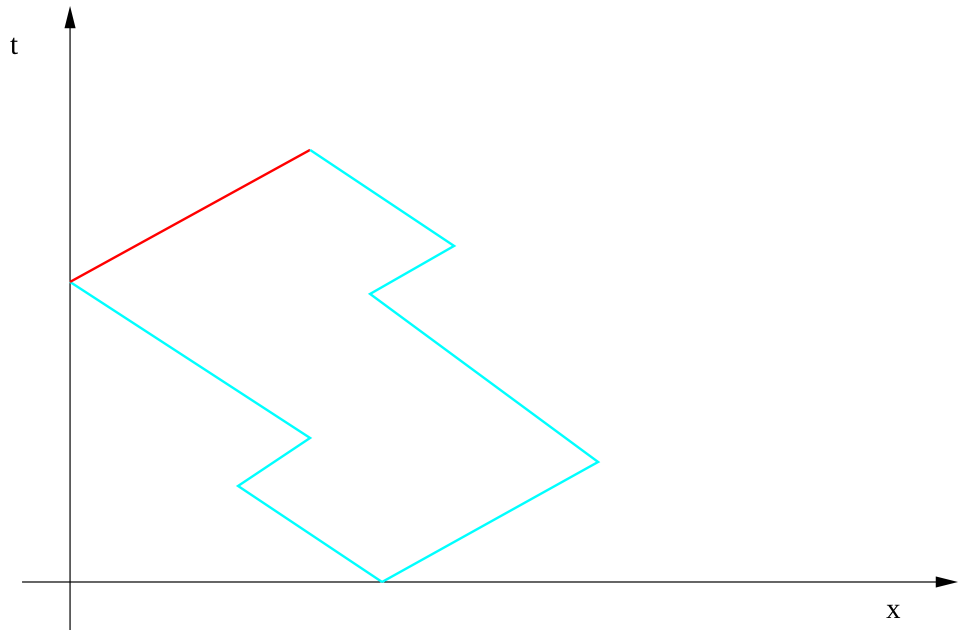

t
x



t
x



t
x



Due to diffusion, it is possible to verify that after some time, in each (t, x) the number of particle which have bounced at $x = 0$ an even number of times is very close to the number of particles which have bounced an odd number, more precisely

$$\int_0^{+\infty} |F^+(t,0)|dt \le 3 \int_{\mathbb{R}} \Big(|F^-(0,x)| + |F^+(0,x)|\Big) dx. \qquad (20)$$

Explicit computations

Explicit computations

We consider the slution as the sum of families of *generations of particles*,

$$(F^-, F^+) = \sum_{i=0}^{\infty} (F^{-,i}, F^{+,i}),$$

Explicit computations

We consider the slution as the sum of families of *generations of particles*,

$$(F^-, F^+) = \sum_{i=0}^{\infty} (F^{-,i}, F^{+,i}),$$

$$\begin{cases} F_t^{-,i+1} - F_x^{-,i+1} & = & \frac{F^{-,i}+F^{+,i}}{2} - F^{-,i+1} \\ F_t^{+,i+1} + F_x^{+,i+1} & = & \frac{F^{-,i}+F^{+,i}}{2} - F^{+,i+1} \end{cases}$$

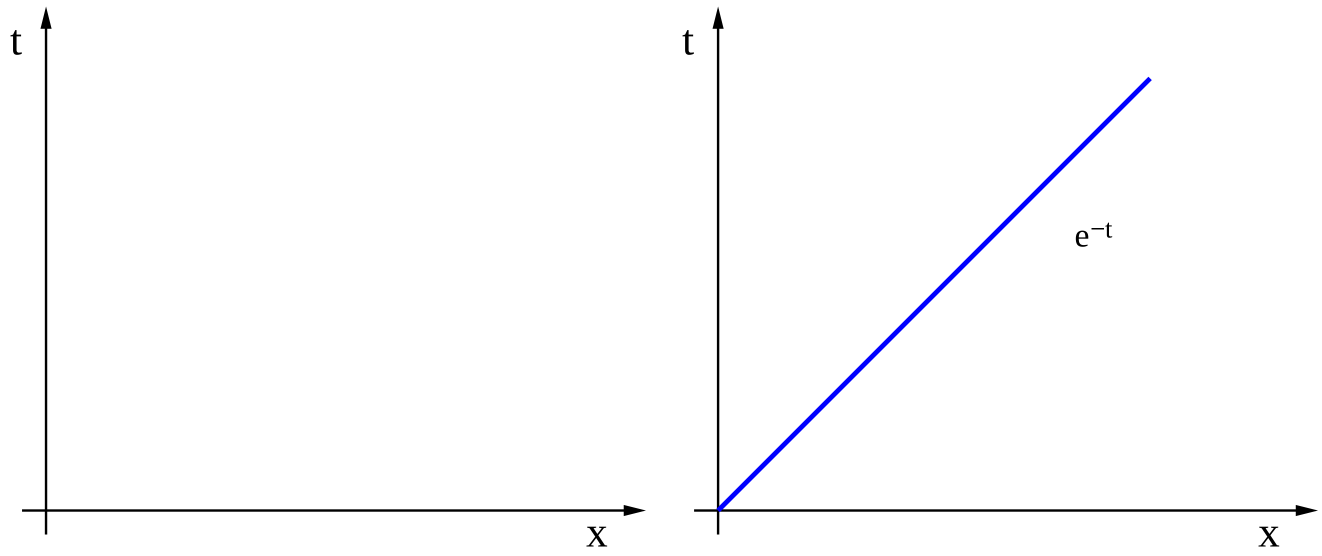
Explicit computations

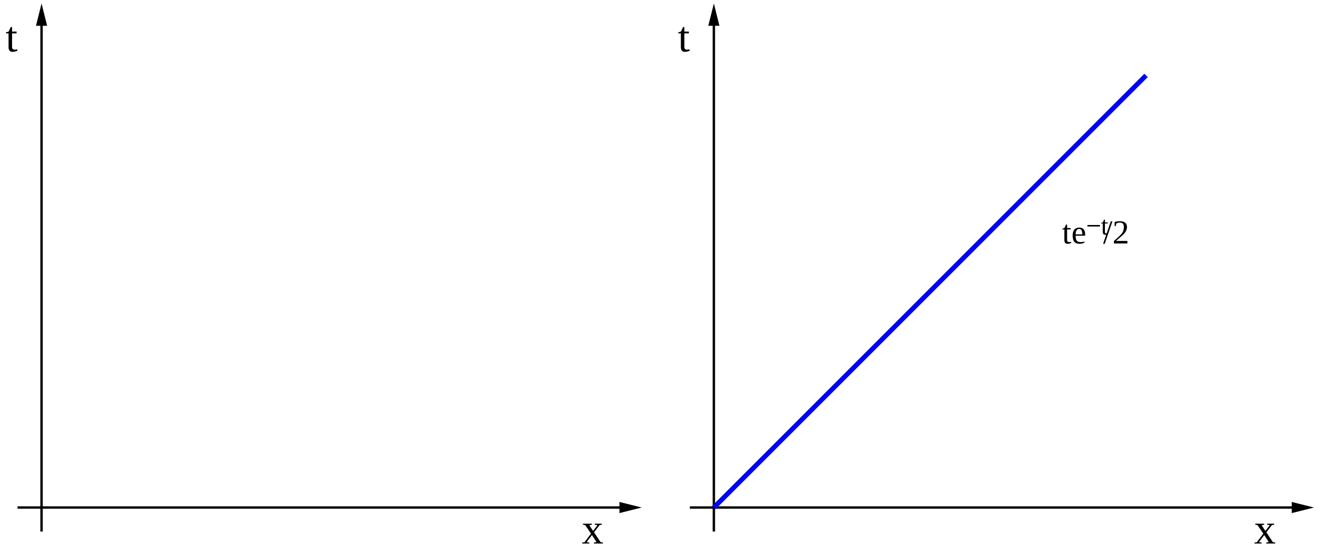
We consider the slution as the sum of families of *generations of particles*,

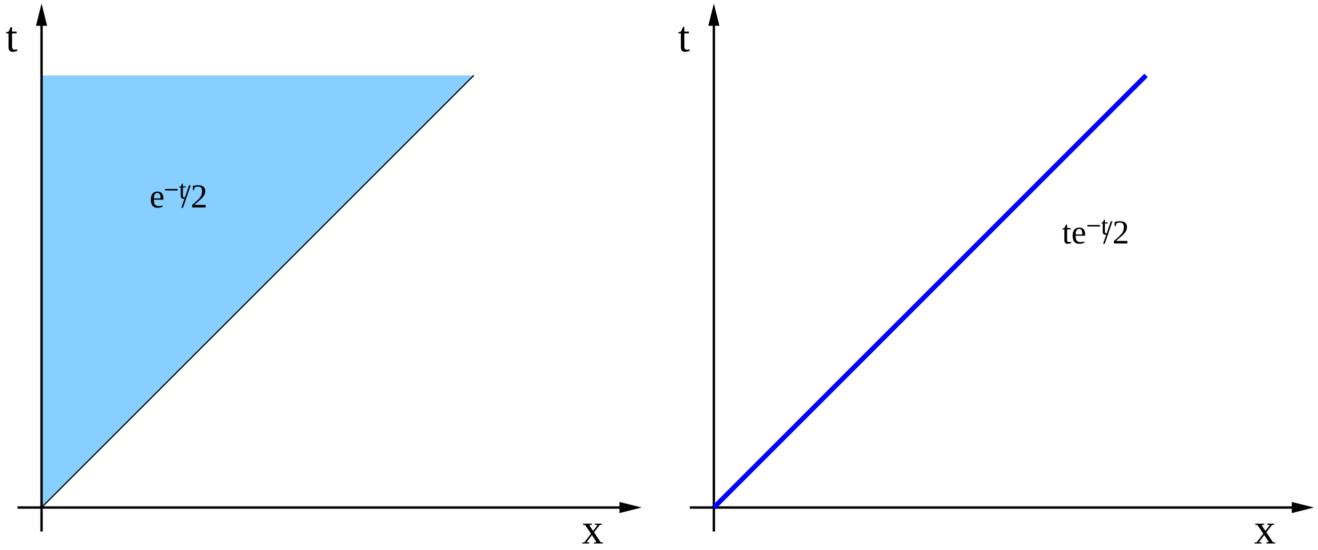
$$(F^-, F^+) = \sum_{i=0}^{\infty} (F^{-,i}, F^{+,i}),$$

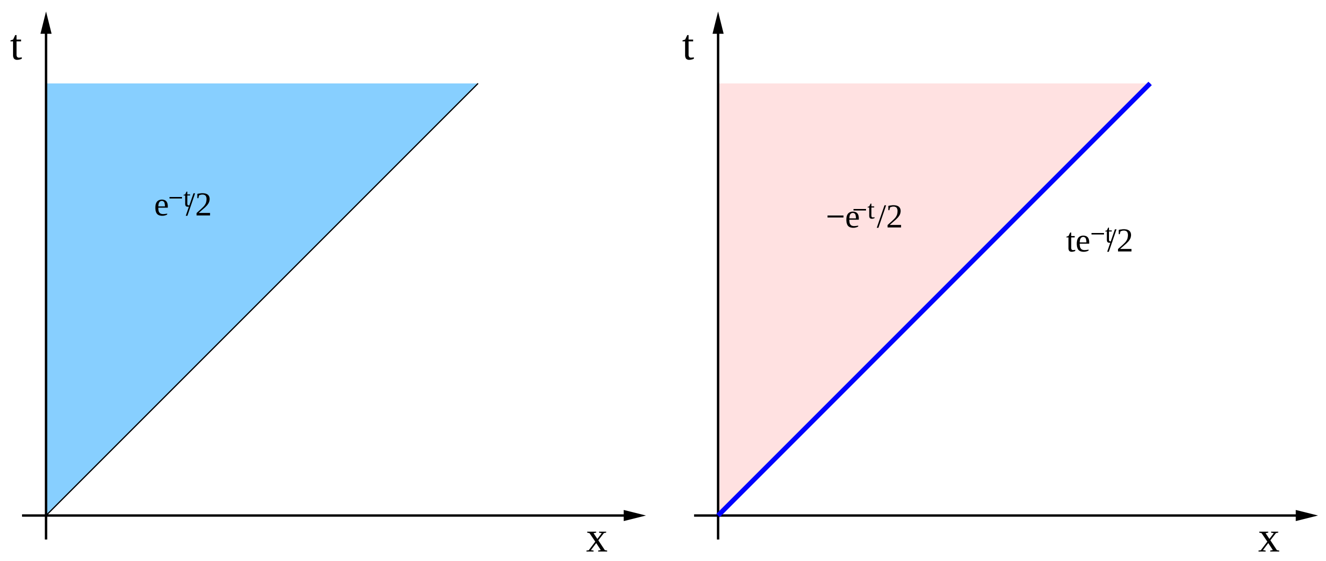
$$\begin{cases} F_t^{-,i+1} - F_x^{-,i+1} & = & \frac{F^{-,i}+F^{+,i}}{2} - F^{-,i+1} \\ F_t^{+,i+1} + F_x^{+,i+1} & = & \frac{F^{-,i}+F^{+,i}}{2} - F^{+,i+1} \end{cases}$$

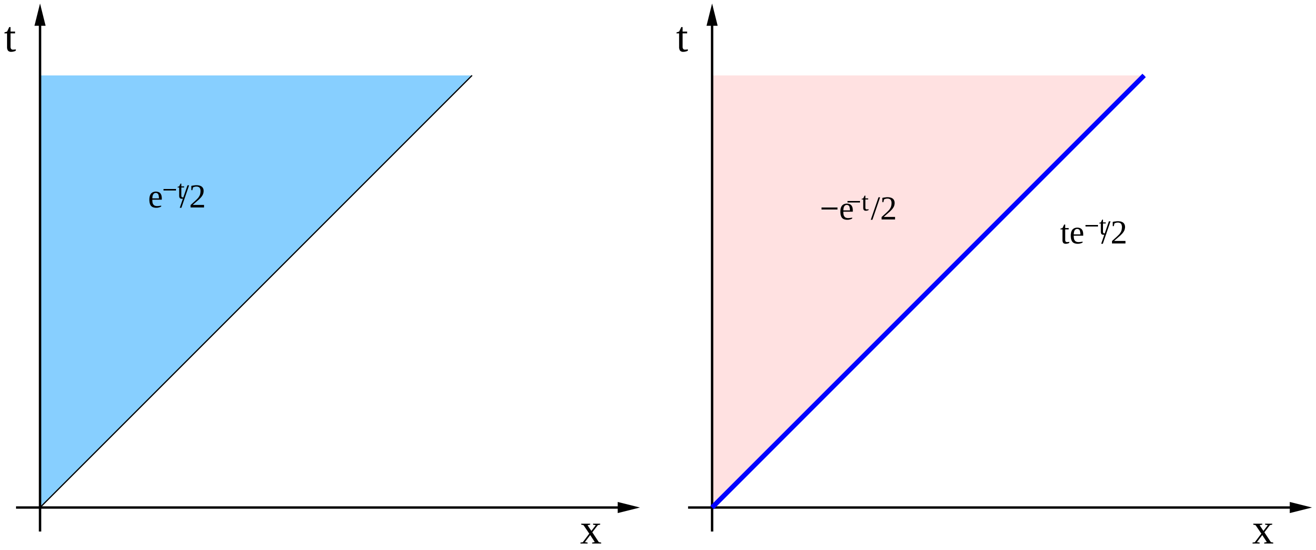
Each generation decays at a constant rate, and two particles of the next generation are created with opposite speeds.


t
x
t
e^{-t}
x

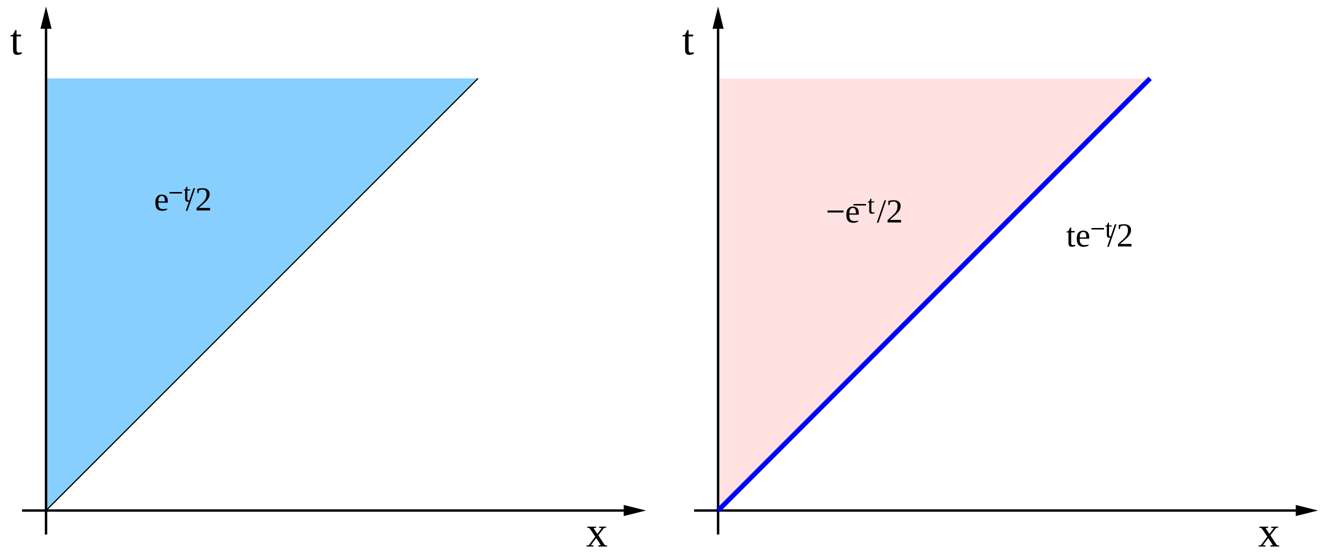

t
x
t
te−t/2
x


t
e−t/2
x
t
te−t/2
x


t
e−t/2
x
t
−e−t/2
te−t/2
x



$\frac{1}{2}$ of the initial number of particles annihilates.



$\frac{1}{2}$ of the initial number of particles annihilates.

The total amount of crossing is bounded by

$$\frac{\text{crossing of gen. } 1,2}{\text{mass disappearing}} = \frac{1+1/2}{1/2} = 3.$$

Estimate for BGK models

Estimate for BGK models

Consider the linearized BGK scheme

$$F_t^\alpha + \alpha F_x^\alpha = c^\alpha \left(\sum_\beta F^\beta \right) - F^\alpha, \quad c^\alpha > 0,\ \sum_\alpha c^\alpha = 1. \tag{21}$$

Estimate for BGK models

Consider the linearized BGK scheme

$$F_t^\alpha + \alpha F_x^\alpha = c^\alpha \left(\sum_\beta F^\beta \right) - F^\alpha, \quad c^\alpha > 0, \ \sum_\alpha c^\alpha = 1. \tag{21}$$

Define

$$g^\alpha = \frac{\partial F^\alpha}{\partial t}, \quad f^\alpha = \frac{\partial F^\alpha}{\partial x},$$

Estimate for BGK models

Consider the linearized BGK scheme

$$F_t^\alpha + \alpha F_x^\alpha = c^\alpha \left(\sum_\beta F^\beta \right) - F^\alpha, \quad c^\alpha > 0, \ \sum_\alpha c^\alpha = 1. \tag{21}$$

Define

$$g^\alpha = \frac{\partial F^\alpha}{\partial t}, \quad f^\alpha = \frac{\partial F^\alpha}{\partial x},$$

and introduce the functions

$$P^{\alpha\beta}(t,x,y) = f^\alpha(t,x) g^\beta(t,y) - f^\beta(t,y) g^\alpha(t,x). \tag{22}$$

A simple computation shows that

$$P_t^{\alpha\beta} + (\alpha, \beta) \cdot \nabla P^{\alpha\beta} = \sum_\gamma \Big(c^\beta P^{\alpha\gamma} + c^\alpha P^{\gamma\beta} \Big) - 2P^{\alpha\beta}.$$

A simple computation shows that

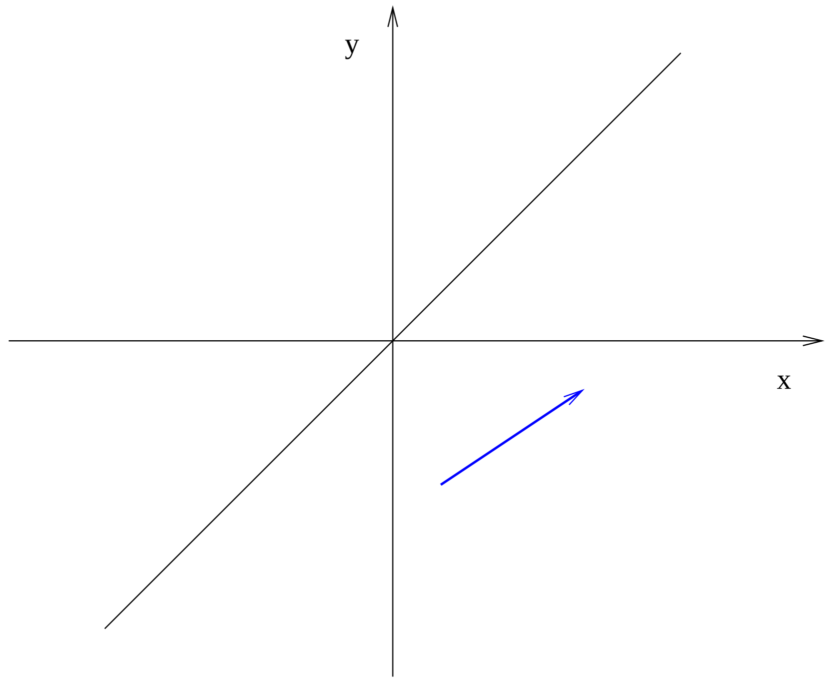
$$P_t^{\alpha\beta} + (\alpha, \beta) \cdot \nabla P^{\alpha\beta} = \sum_{\gamma} \Big(c^{\beta} P^{\alpha\gamma} + c^{\alpha} P^{\gamma\beta} \Big) - 2P^{\alpha\beta}.$$

Because of the symmetry of (22) one has $P^{\alpha\beta}(x, y) = -P^{\beta\alpha}(y, x)$.

A simple computation shows that

$$P_t^{\alpha\beta} + (\alpha,\beta)\cdot\nabla P^{\alpha\beta} = \sum_\gamma \Big(c^\beta P^{\alpha\gamma} + c^\alpha P^{\gamma\beta}\Big) - 2P^{\alpha\beta}.$$

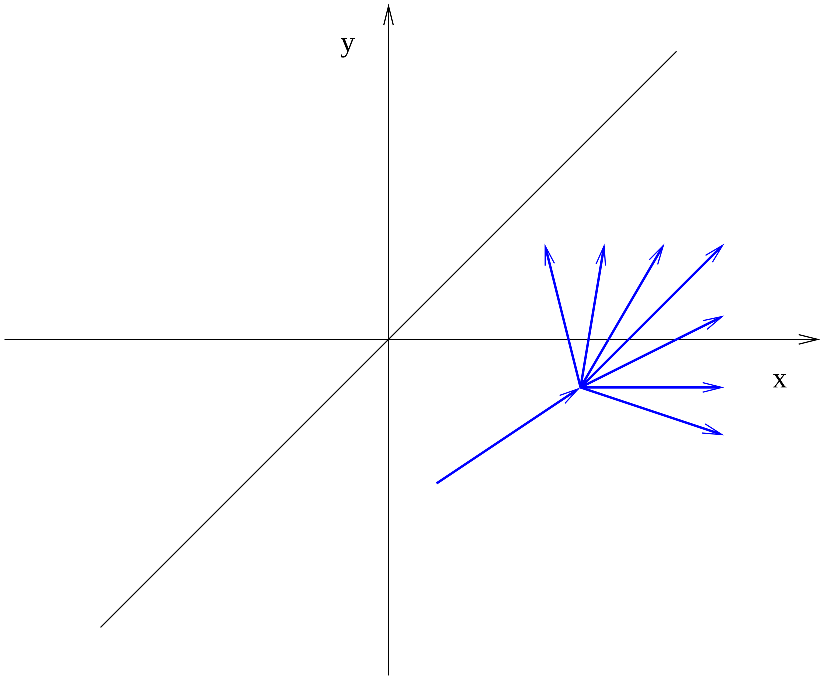
Because of the symmetry of (22) one has $P^{\alpha\beta}(x,y) = -P^{\beta\alpha}(y,x)$.



A simple computation shows that

$$P_t^{\alpha\beta} + (\alpha, \beta) \cdot \nabla P^{\alpha\beta} = \sum_{\gamma} \Big(c^{\beta} P^{\alpha\gamma} + c^{\alpha} P^{\gamma\beta} \Big) - 2P^{\alpha\beta}.$$

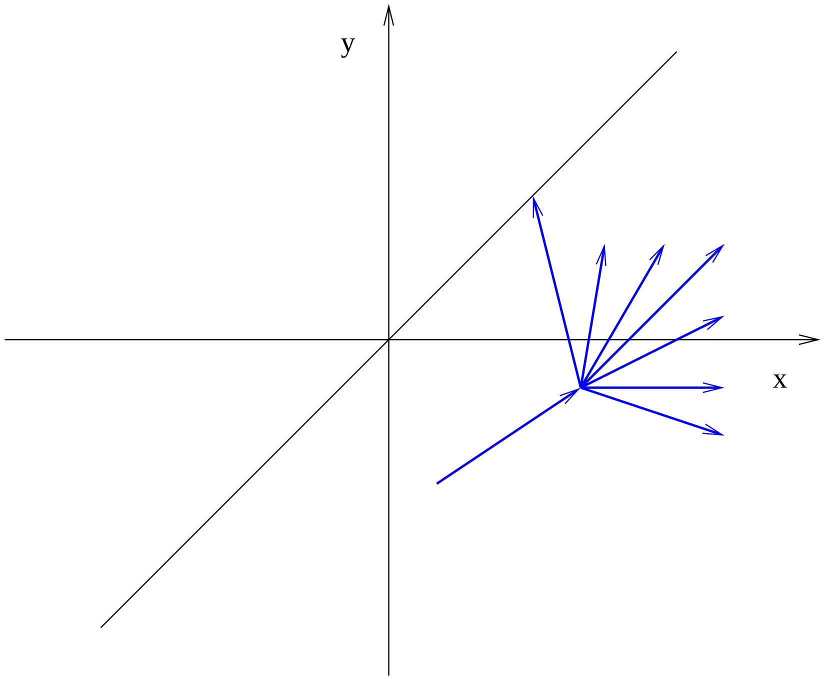
Because of the symmetry of (22) one has $P^{\alpha\beta}(x, y) = -P^{\beta\alpha}(y, x)$.



A simple computation shows that

$$P_t^{\alpha\beta} + (\alpha,\beta)\cdot\nabla P^{\alpha\beta} = \sum_{\gamma}\Big(c^{\beta}P^{\alpha\gamma} + c^{\alpha}P^{\gamma\beta}\Big) - 2P^{\alpha\beta}.$$

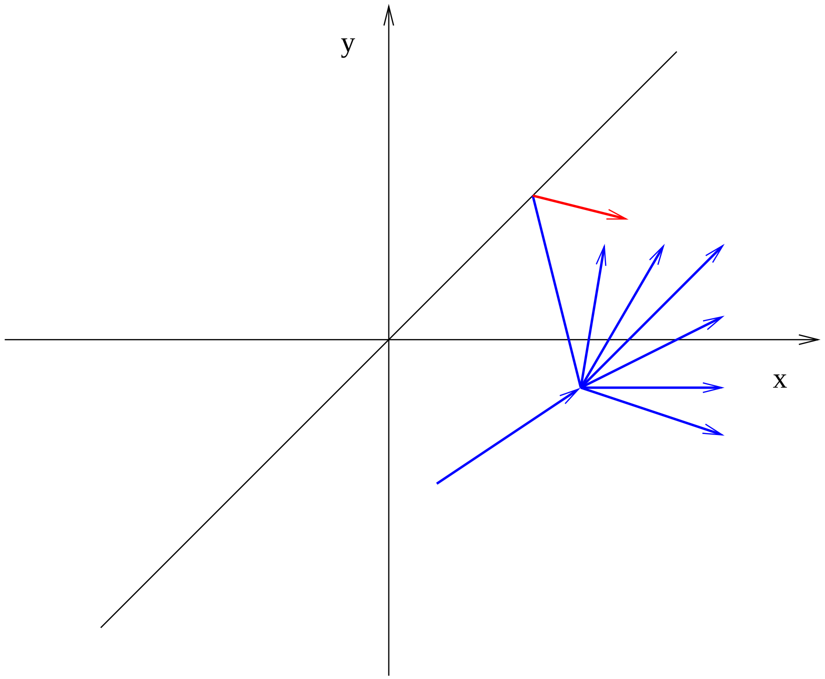
Because of the symmetry of (22) one has $P^{\alpha\beta}(x,y) = -P^{\beta\alpha}(y,x)$.



A simple computation shows that

$$P_t^{\alpha\beta} + (\alpha,\beta)\cdot\nabla P^{\alpha\beta} = \sum_\gamma \Big(c^\beta P^{\alpha\gamma} + c^\alpha P^{\gamma\beta}\Big) - 2P^{\alpha\beta}.$$

Because of the symmetry of (22) one has $P^{\alpha\beta}(x,y) = -P^{\beta\alpha}(y,x)$.



The Glimm Functional is

$$\begin{aligned}\mathcal{Q}(t) &= \sum_{\alpha\beta} \|P^{\alpha\beta}(t)\|_{L^1(x>y)} \\ &= \sum_{\alpha\beta} \iint_{\mathbb{R}^2} \Big| F_t^{\alpha}(t,x) F_x^{\beta}(t,x) - F_x^{\alpha}(t,x) F_t^{\beta}(t,x) \Big| dx dt, \quad (23)\end{aligned}$$

The Glimm Functional is

$$\begin{aligned}\mathcal{Q}(t) &= \sum_{\alpha\beta} \|P^{\alpha\beta}(t)\|_{L^1(x>y)} \\ &= \sum_{\alpha\beta} \iint_{\mathbb{R}^2} \Big| F_t^\alpha(t,x) F_x^\beta(t,x) - F_x^\alpha(t,x) F_t^\beta(t,x) \Big| dxdt, \quad (23)\end{aligned}$$

and the flux through the boundary $\{x = y\}$ is

$$\begin{aligned}\mathcal{I} &= \sum_{\alpha\beta} \int_0^{+\infty} \Big\| (1,-1) \cdot (\alpha,\beta) P^{\alpha\beta}(t) \Big\|_{L^1(x=y)} \\ &= \sum_{\alpha\beta} |\alpha - \beta| \int_0^{+\infty} \int_{\mathbb{R}} \Big| F_t^\alpha(t,x) F_x^\beta(t,x) - F_x^\alpha(t,x) F_t^\beta(t,x) \Big| dxdt.\end{aligned}$$

The solution to the BGK scheme can be written as

$$P^{\alpha\beta}(t,x,y) = \sum_{n=0}^{+\infty} P^{\alpha\beta,n}(t,x,y),$$

The solution to the BGK scheme can be written as

$$P^{\alpha\beta}(t,x,y) = \sum_{n=0}^{+\infty} P^{\alpha\beta,n}(t,x,y),$$

where each function $P^{\alpha\beta,n}$ satisfies

$$P_t^{\alpha\beta,n} + (\alpha,\beta)\cdot\nabla P^{\alpha\beta,n} = \frac{1}{2}\sum_{\gamma}\Big(c^{\beta}P^{\alpha\gamma,n-1} + c^{\alpha}P^{\gamma\beta,n-1}\Big) - P^{\alpha\beta,n}.$$

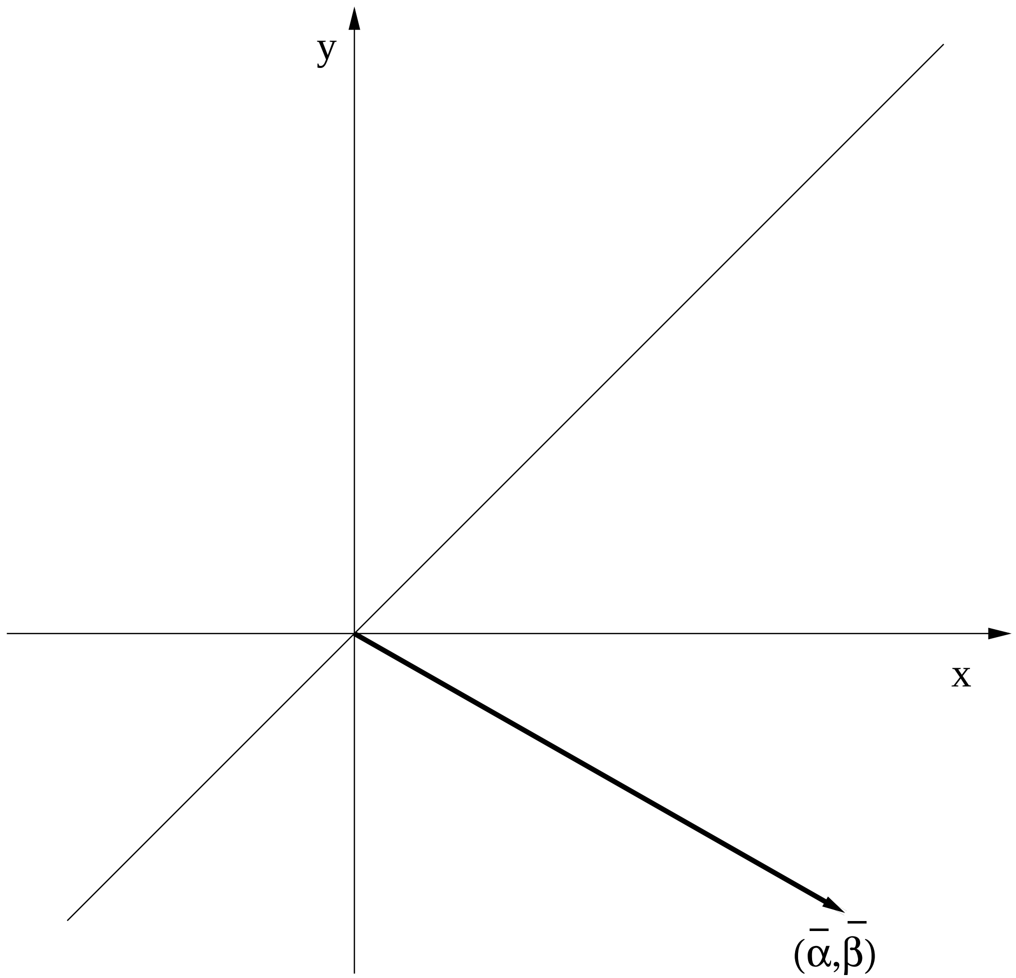
The solution to the BGK scheme can be written as

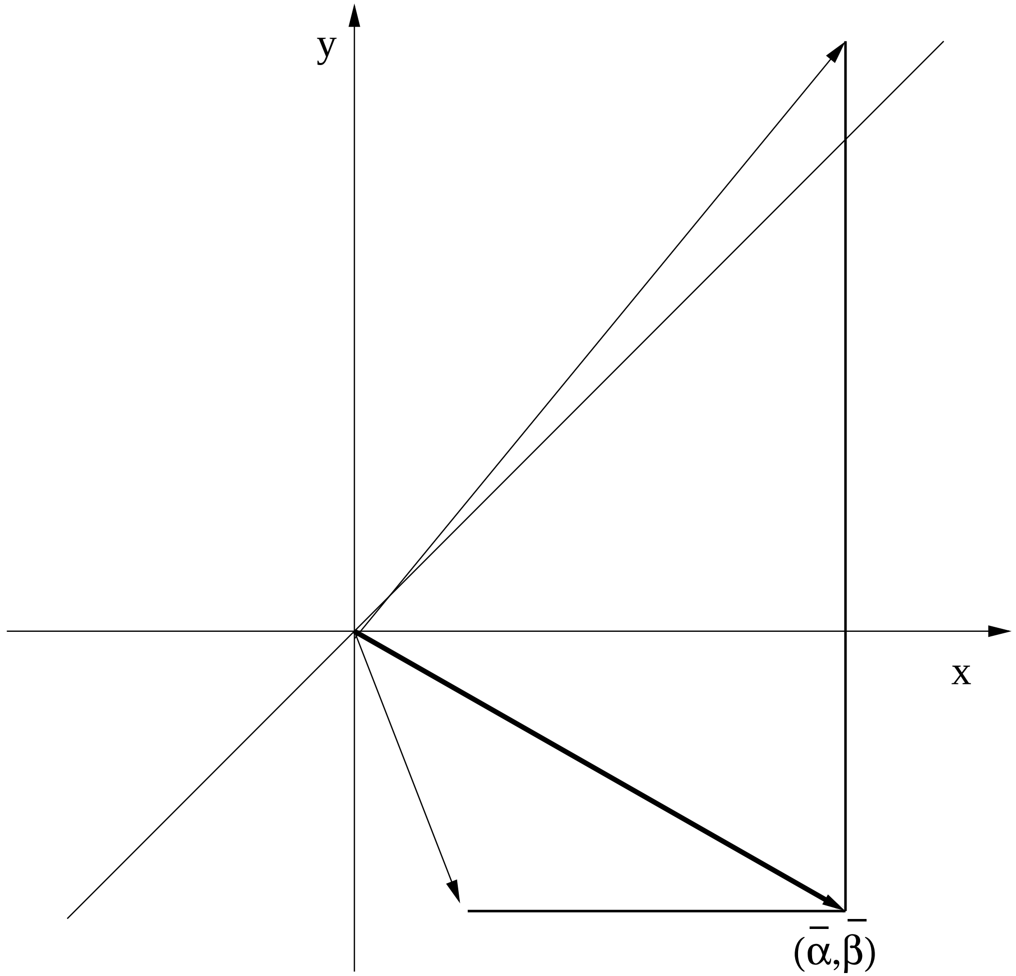
$$P^{\alpha\beta}(t,x,y) = \sum_{n=0}^{+\infty} P^{\alpha\beta,n}(t,x,y),$$

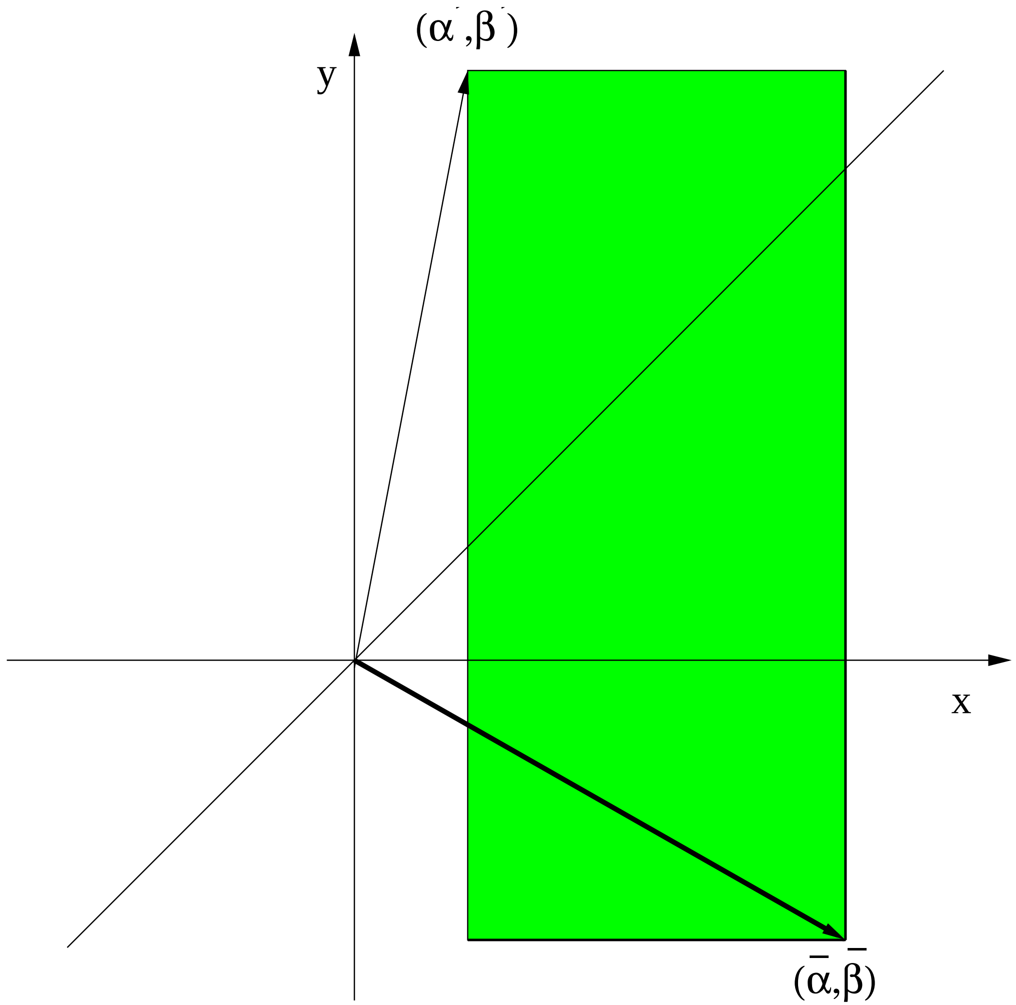
where each function $P^{\alpha\beta,n}$ satisfies

$$P_t^{\alpha\beta,n} + (\alpha,\beta)\cdot\nabla P^{\alpha\beta,n} = \frac{1}{2}\sum_{\gamma}\Big(c^{\beta}P^{\alpha\gamma,n-1} + c^{\alpha}P^{\gamma\beta,n-1}\Big) - P^{\alpha\beta,n}.$$

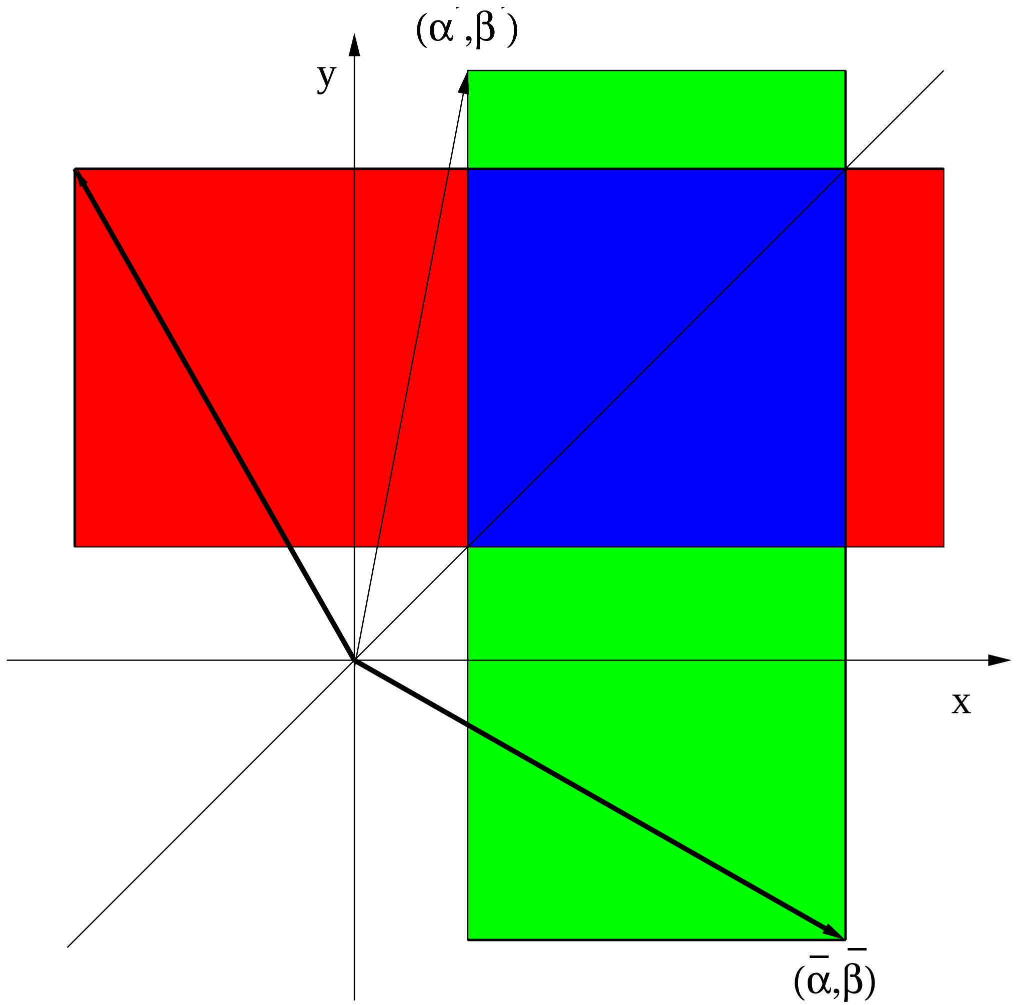
We will say that $P^{\alpha\beta,n}$ is the n-th generation of particle.


y
x
$(\bar{\alpha},\bar{\beta})$


y
x
$(\bar{\alpha},\bar{\beta})$



(α′,β′)
y
x
($\bar{\alpha}$,$\bar{\beta}$)



(α', β')
y
x
$(\bar{\alpha}, \bar{\beta})$

The cancellation is of the order

$$\sigma^{-8} = \Big(\frac{1}{2}\sum_{\alpha\beta}(\alpha-\beta)^2 c^\alpha c^\beta\Big)^{-4}.$$

The cancellation is of the order

$$\sigma^{-8} = \Big(\frac{1}{2}\sum_{\alpha\beta}(\alpha-\beta)^2 c^\alpha c^\beta\Big)^{-4}.$$

The quantity σ is the diffusion of the solutions F^α:

The cancellation is of the order

$$\sigma^{-8} = \Big(\frac{1}{2}\sum_{\alpha\beta}(\alpha-\beta)^2 c^\alpha c^\beta\Big)^{-4}.$$

The quantity σ is the diffusion of the solutions F^α:

as $t \to \infty$, $u = \sum_\alpha F^\alpha$ behaves like

$$u_t + \left(\sum_\alpha \alpha c^\alpha\right) u_x - \left(\frac{1}{2}\sum_{\alpha\beta}(\alpha-\beta)^2 c^\alpha c^\beta\right) u_{xx} =$$

$$u_t + \bar\lambda u_x - \sigma^2 u_{xx} = 0.$$

Decomposition in travelling profiles

Decomposition in travelling profiles

Writing Q as

$$Q(t) = \sum_{\alpha\beta} \iint_{\mathbb{R}^2} \Big| F_x^\alpha(t,x) F_x^\beta(t,x) \Big| \left| -\frac{F_t^\beta(t,x)}{F_x^\beta(t,x)} - \left(-\frac{F_t^\alpha(t,y)}{F_x^\alpha(t,y)} \right) \right| dx dt.$$

Decomposition in travelling profiles

Writing Q as

$$Q(t) = \sum_{\alpha\beta} \iint_{\mathbb{R}^2} \Big| F_x^\alpha(t,x) F_x^\beta(t,x) \Big| \left| -\frac{F_t^\beta(t,x)}{F_x^\beta(t,x)} - \left(-\frac{F_t^\alpha(t,y)}{F_x^\alpha(t,y)} \right) \right| dxdt.$$

and the flux as

$$\mathcal{I} = \sum_{\alpha\beta} |\alpha - \beta| \int_0^{+\infty} \int_{\mathbb{R}} \Big| F_x^\alpha(t,x) F_x^\beta(t,x) \Big| \left| -\frac{F_t^\beta(t,x)}{F_x^\beta(t,x)} - \left(-\frac{F_t^\alpha(t,x)}{F_x^\alpha(t,x)} \right) \right| dxdt.$$

Decomposition in travelling profiles

Writing Q as

$$Q(t) = \sum_{\alpha\beta} \iint_{\mathbb{R}^2} \left| F_x^\alpha(t,x) F_x^\beta(t,x) \right| \left| -\frac{F_t^\beta(t,x)}{F_x^\beta(t,x)} - \left(-\frac{F_t^\alpha(t,y)}{F_x^\alpha(t,y)} \right) \right| dxdt.$$

and the flux as

$$\mathcal{I} = \sum_{\alpha\beta} |\alpha - \beta| \int_0^{+\infty} \int_{\mathbb{R}} \left| F_x^\alpha(t,x) F_x^\beta(t,x) \right| \left| -\frac{F_t^\beta(t,x)}{F_x^\beta(t,x)} - \left(-\frac{F_t^\alpha(t,x)}{F_x^\alpha(t,x)} \right) \right| dxdt.$$

and noticing that $\sigma^\alpha = -\frac{F_t^\alpha}{F_x^\alpha}$ is the level set speed,

Decomposition in travelling profiles

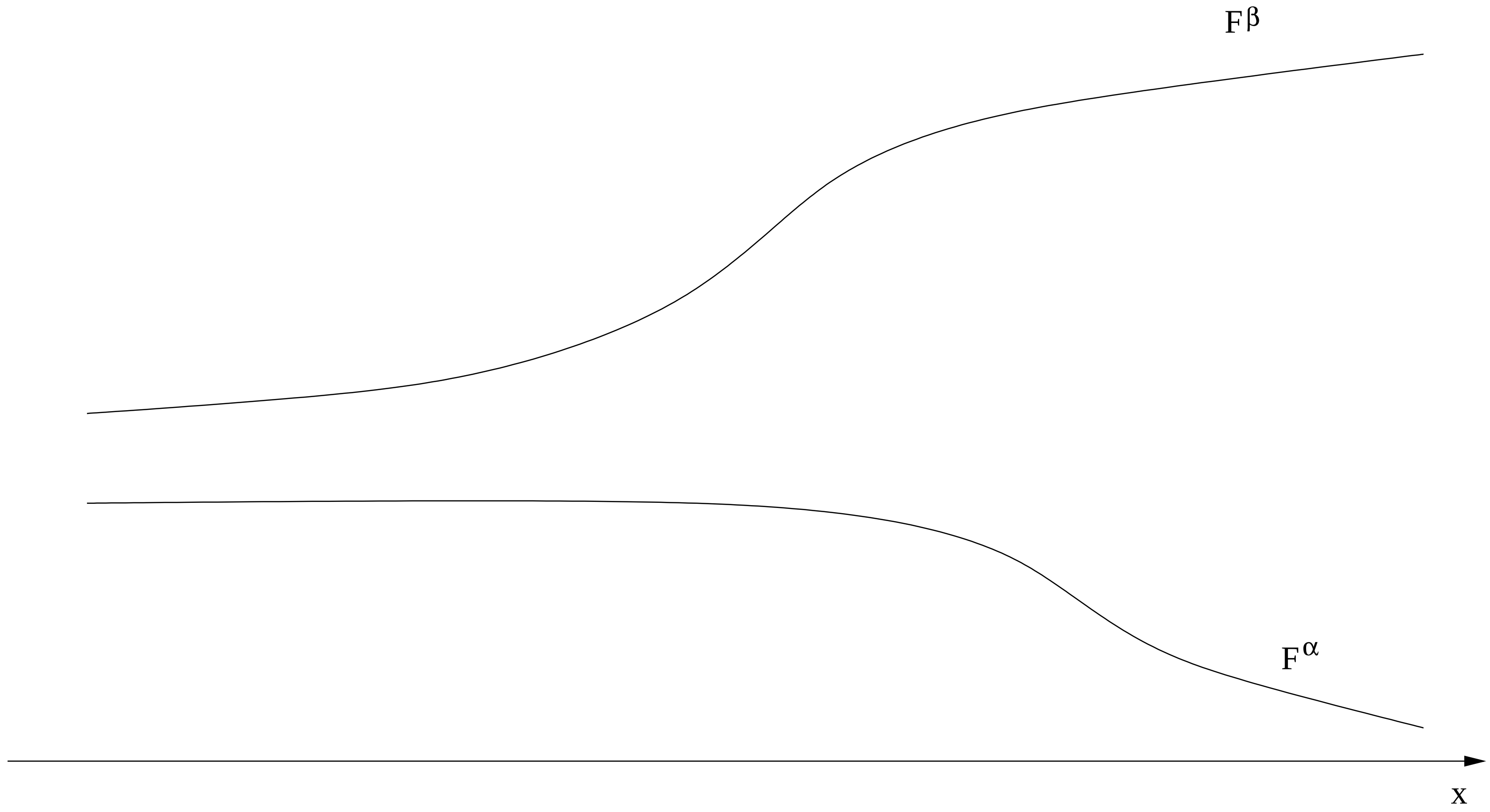
Writing Q as

$$Q(t) = \sum_{\alpha\beta} \iint_{\mathbb{R}^2} \left| F_x^\alpha(t,x) F_x^\beta(t,x) \right| \left| -\frac{F_t^\beta(t,x)}{F_x^\beta(t,x)} - \left(-\frac{F_t^\alpha(t,y)}{F_x^\alpha(t,y)} \right) \right| dxdt.$$

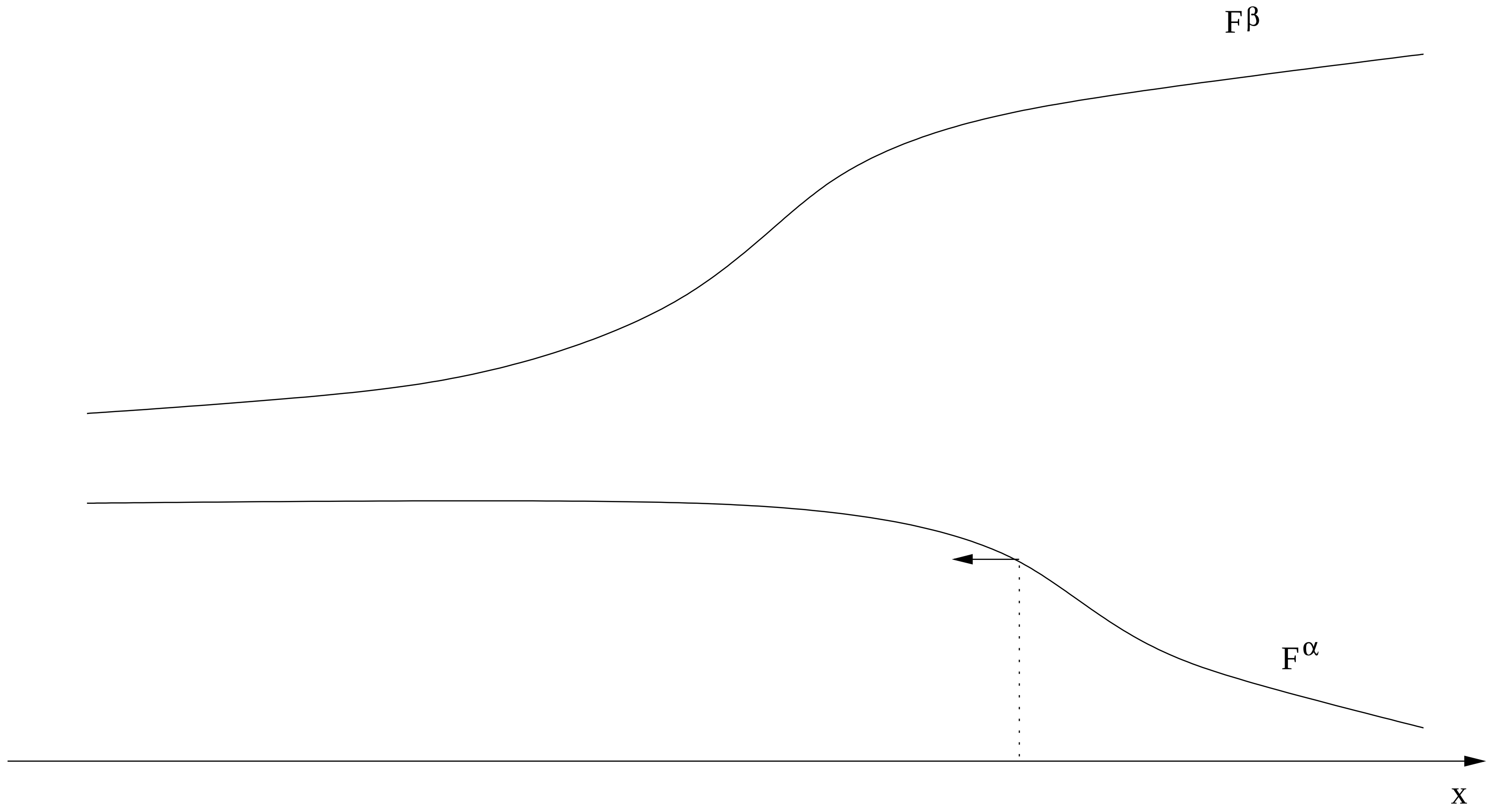
and the flux as

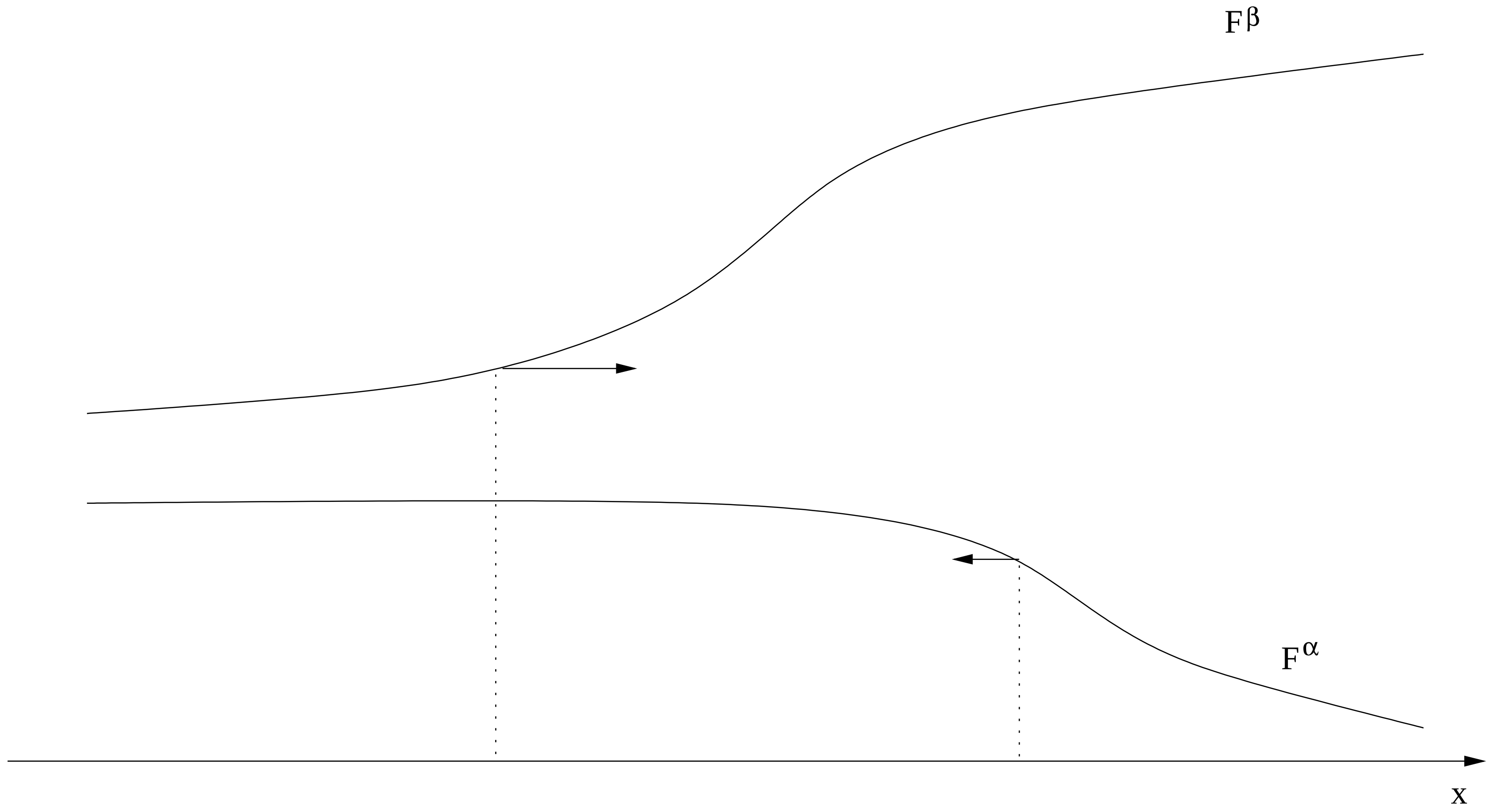
$$\mathcal{I} = \sum_{\alpha\beta} |\alpha-\beta| \int_0^{+\infty} \int_{\mathbb{R}} \left| F_x^\alpha(t,x) F_x^\beta(t,x) \right| \left| -\frac{F_t^\beta(t,x)}{F_x^\beta(t,x)} - \left(-\frac{F_t^\alpha(t,x)}{F_x^\alpha(t,x)} \right) \right| dxdt.$$

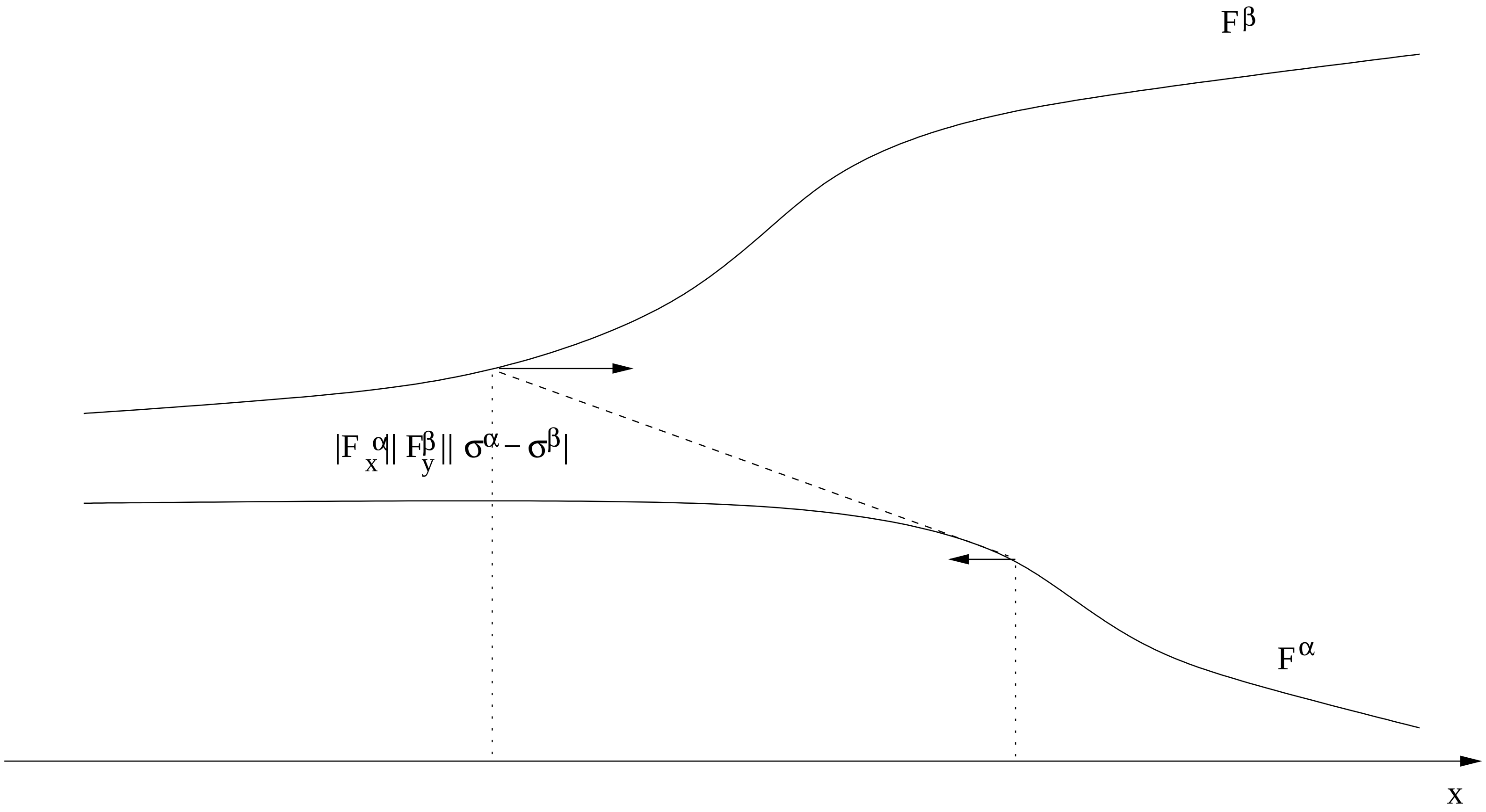
and noticing that $\sigma^\alpha = -\frac{F_t^\alpha}{F_x^\alpha}$ is the level set speed,
we obtain an interpretation in terms of wave interactions of the solution F^α.



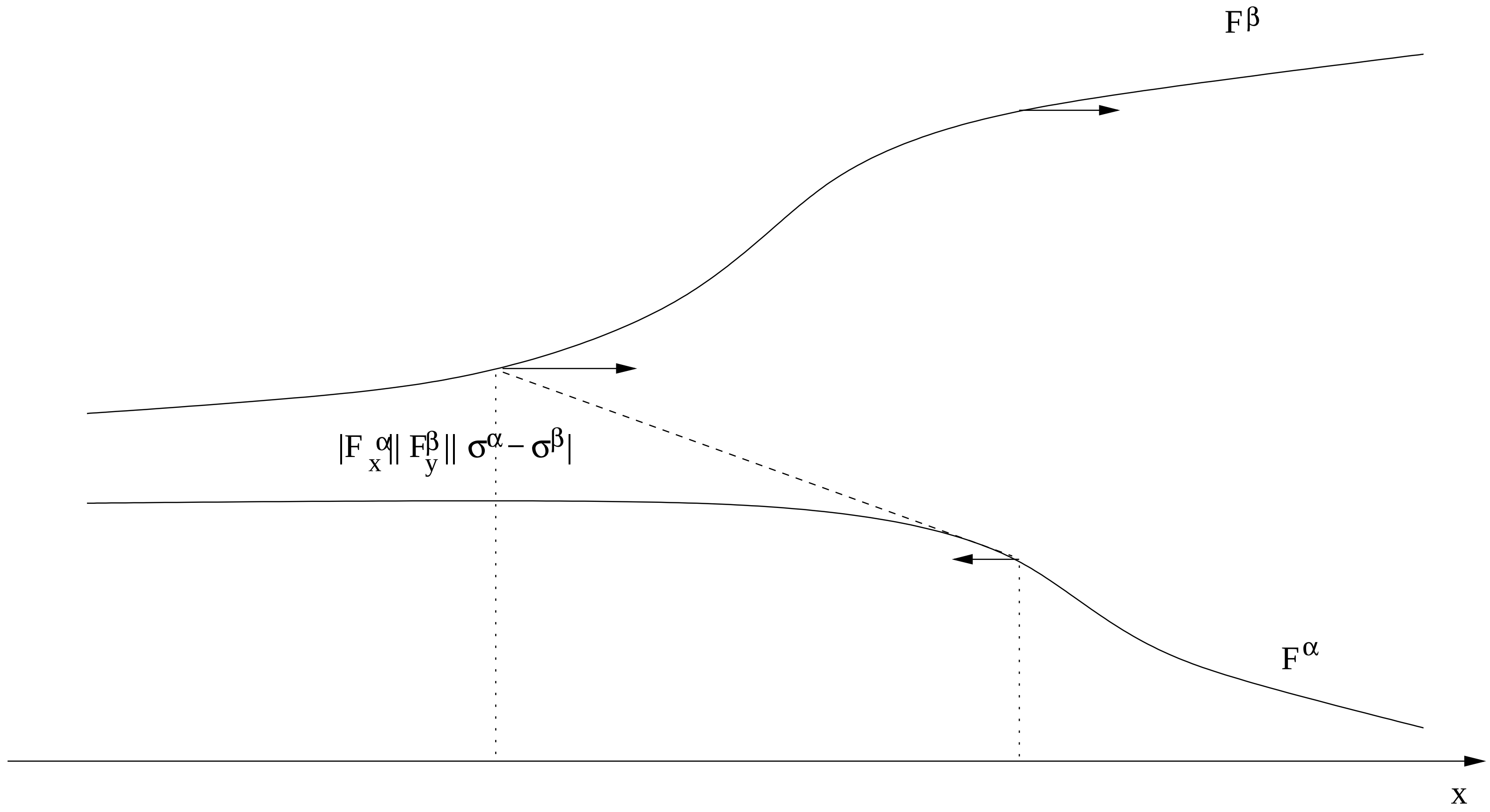
F^β
F^α
x


F^{β}
F^{α}
x

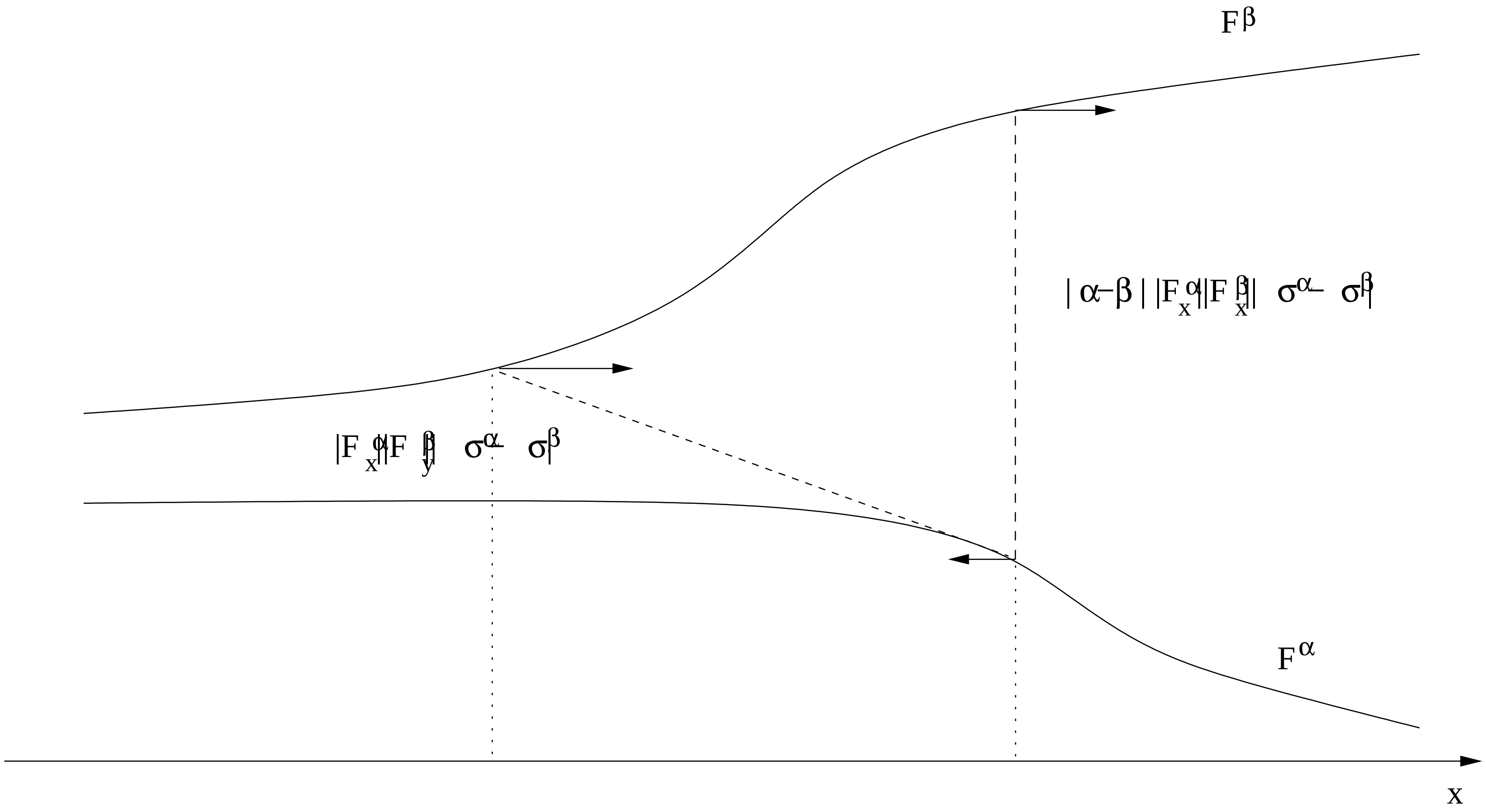

F^β
F^α
x

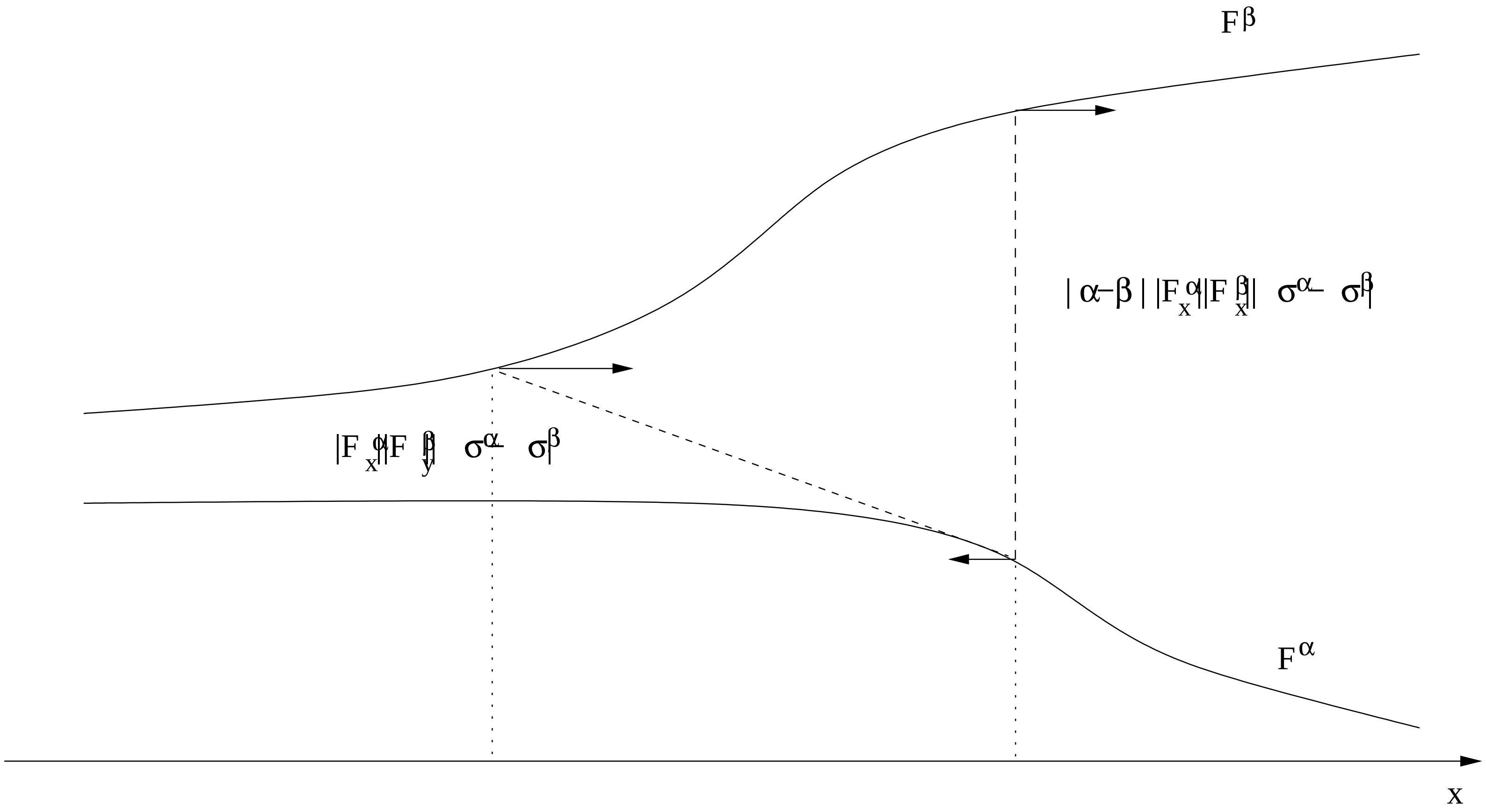


F^{β}
$|F_x^{\alpha}||F_y^{\beta}||\sigma^{\alpha}-\sigma^{\beta}|$
F^{α}
x



F^{β}
$|F_x^{\alpha}|\, F_y^{\beta}|\, |\sigma^{\alpha} - \sigma^{\beta}|$
F^{α}
x


F β
| α−β | |F x α|F x β| σα− σβ|
|F x α|F y β| σα− σβ|
F α
x



The interaction functional is the sum of the products of all waves in F^α, F^β multiplied by their speed.

References.

S. Bianchini. BV solutions to semidiscrete schemes. *Arch. Rat. Mech. Anal.*, 167(1):1–81, 2003.

S. Bianchini. Hyperbolic limit for the Jin-Xin relaxation model. preprint IAC-CNR, 2004.

S. Bianchini and A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. preprint SISSA, 2001.

S. Bianchini and A. Bressan. On a Lyapunov functional relating viscous conservation laws and shortening curves. *Nonlinear Analysis TMA*, 51(4):649–662, 2002.