Regularity of solutions to Hamilton-Jacobi and Hyperbolic Conservation Laws

L. Caravenna, C. De Lellis, M. Gloyer, R. Robyr, S. B.

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  What kind of regularity
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Structure of solutions to Hamilton-Jacobi

Consider the Hamilton-Jacobi equation

$$u_t + H(\nabla u) = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^m,$$

with uniformly convex Hamiltonian $H$ and Lipschitz initial data.
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with uniformly convex Hamiltonian $H$ and Lipschitz initial data. We expect a smooth function outside countably many regular hypersurfaces of codimension 1.
Structure of solutions to Hamilton-Jacobi

For strictly hyperbolic system of conservation laws in one space dimension

\[ u_t + f(u)_x = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \quad u \in \mathbb{R}^n, \]

one expects a similar structure: countably many shock curves and regularity of the solution in the remaining set.
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Precise notions of regularity

One correct form of these questions is:

1. $\mathcal{d}$-rectifiability of the jump set;
2. regularity of solutions to the linear transport PDE $\rho_t + \text{div} (\mathcal{d}\rho) = 0$, where $\mathcal{d}$ is the direction of the optimal ray for HJ or the $i$-eigenvalue for HCL.
3. SBV regularity of $\nabla u$ for HJ and $u$ for HCL.

The first question is an easy application of a well-known rectifiability criteria.
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Euler-Lagrange equation for singular variational problems

The Euler-Lagrange equation for the functional

$$\int_{\Omega} (\mathbf{1}_D (\nabla u) + u) \mathcal{L}^m, \quad u \in u_0 + W^{1,\infty}_0(\Omega),$$

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$$\text{div}(\rho d) - 1 = 0, \quad \rho \in W^{1,\infty}_0(\Omega),$$

where $d$ is the direction of the optimal ray for the viscosity solution

$$1 - (\mathbf{1}_D)^*(\nabla u) = 0, \quad u_{\mid \partial\Omega} = u_0.$$
Optimal transportation

In a geodesic space, under very general conditions the optimal transportation problem

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\min \left\{ \int d(x, y)\pi(dx dy), \quad \pi \in \Pi(\mu, \nu) \right\}
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$$\min \left\{ \int d(x, y)\pi(dx dy), \quad \pi \in \Pi(\mu, \nu) \right\}$$

can be written as transportation problems along a set of geodesic, and the dynamical interpretation of the transport correspond to solve the PDE (in the sense of currents)

$$\text{div}(d\rho) = \mu - \nu,$$

where $d$ is the “direction” of the geodesics.
Precise decay estimates

The derivative of a solution a gnl system of conservation laws can be decomposed in waves

\[ u_x = \sum_i v_i \tilde{r}_i, \quad u_t = \sum_i w_i \tilde{r}_i, \]

with

1. \( \tilde{r}_i \) direction of the \( i \)-th jumps or the \( i \)-th eigenvector;
2. \( w_i = -\tilde{\lambda}_i v_i \), with \( \tilde{\lambda}_i \) speed of the \( i \)-shock or the \( i \)-th eigenvalue;
3. the continuous part of \( v_i \) satisfies the equation \( (v_i)_t + (\lambda_i v_i)_x = J_i \), \( J_i \in M(R_+ \times R) \).
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\[ (v_i)_t + (\lambda_i v_i)_x = J_i, \quad J_i \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}). \]
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The solution to HJ equation

\[ u_t + H(\nabla u) = 0 \]

is given by

\[ u(t, x) = \min \left\{ u(0, y) + tL \left( \frac{x - y}{t} \right) \right\}, \quad L = H^*. \]
Area estimate

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\[ u(t, x) = \min \left\{ u(0, y) + tL\left(\frac{x-y}{t}\right) \right\}, \quad L = H^*. \]

In particular, it is uniformly approximated by the sequence of functions

\[ u_n(t, x) = \min \left\{ u(0, y) + tL\left(\frac{x-y}{t}\right), y \in \{y_1, \ldots, y_n\} \right\}. \quad (1) \]
These solutions have a very simple structure:
In particular, we have the estimates:

1. divergence is a measure

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2. the Jacobian \( c(t, x) \) of the flow \( x \mapsto x + td(x) \) satisfies

\[ c(t, s, x) = \left( \frac{t - s}{t} \right)^m \]
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\[ c(t, s, x) = \left( \frac{t - s}{t} \right)^m \]

By (1), one can show that this is the worst case, i.e.

\[
c(t, s, x) \begin{cases} 
\leq \left( \frac{t-s}{t} \right)^m & t \leq s \\
\geq \left( \frac{t-s}{t} \right)^m & t \geq s
\end{cases}
\]
Since along optimal rays we have the dual solution

\[ u(s, x) = \max \left\{ u(t, y) - (t - s)L\left(\frac{y - x}{t - s}\right) \right\}, \]

we obtain the bound on the Jacobian

\[ \min \left\{ \left(\frac{t - s}{t}\right)^m, \left(\frac{T - s}{T - t}\right)^m \right\} \leq c \leq \max \left\{ \left(\frac{t - s}{t}\right)^m, \left(\frac{T - s}{T - t}\right)^m \right\}, \]

where \([0, T]\) is the existence time of the ray.
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Disintegration of Lebesgue measure

The above estimate implies that the change of variable

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The correct form to state this is to write the disintegration of the Lebesgue measure along the rays:

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\mathcal{L}_{\mathbb{R}^+ \times \mathbb{R}^m}^{d+1} = \int c(t, y) dt m(dy),
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\[
\mathcal{L}^{d+1}_{\mathbb{R}^+ \times \mathbb{R}^m} = \int c(t, y) dt m(dy),
\]
i.e. \(\forall \phi \in C_c(\mathbb{R}^+ \times \mathbb{R}^m)\)
\[
\int \phi \mathcal{L}^{d+1} = \int \left( \int \phi(t, y + td(y))c(t, y) dt \right) m(dy).
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Reformulation of transport equations

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$$\text{div}d \in \mathcal{M} \implies \frac{dc}{dt} = (\text{div}d)_{a.c.c.}.$$
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1. Equation for the Jacobian $c$:

$$\text{div} d \in \mathcal{M} \implies \frac{dc}{dt} = (\text{div} d)_{a.c.} c.$$

2. Reformulation of transport equation as ODE:

$$\rho_t + \text{div}(d\rho) = f \implies \frac{d\rho}{dt} + (\text{div} d)_{a.c.} \rho = f.$$
Reformulation of transport equations

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Remark. The proof depends only on the convexity of $H$. 
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A formula for the divergence

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In particular, if there is a Cantor part (hence single rays), the area is strictly positive.
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SBV estimate for Hamilton-Jacobi

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Since $d(t, x) = D^2 H(\nabla u)D^2 u$, we obtain that $\text{tr}(D^2 u)$ has not Cantor parts, hence $D^2 u$ has not Cantor parts.
A measure for shock creation

If $v_i^s$ is the jump part of the $i$-th component $v_i$ of $u_x$, then we have the two equations:
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1. equation for $v_i$: if $Q$ is the interaction potential,

$$(v_i)_t + (\tilde{\lambda}_i v_i)_x = J_i, \quad |J_i|((s, t] \times \mathbb{R}) \leq C(Q(s) - Q(t));$$


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2. equation for $v_i^s$:

$$ (v_i^s)_t + (\tilde{\lambda}_i v_i^s)_x = J_i^s, $$

$$ |J_i^s|((s, t] \times \mathbb{R}) \leq \text{Tot.Var.}(v_v - v_i^s(s)) - \text{Tot.Var.}(v_v - v_i^s(t)) + C (Q(s) - Q(t)). $$
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1. it is easy to create shocks with negligible interactions (quadratic w.r.t. strength);
2. you need a interaction and cancellation of the order of the shock to cancel it.
The continuous part $v_i^c$ of $v_i$ thus satisfies

$$(v_i^c)_t + (\lambda_i v_i^c)_x = J_i^c, \quad J_i^c := J_i - J_i^s.$$
SBV regularity

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As argument similar to the estimate of the decay of positive waves yields now

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v_i^c(T, A) \geq -\frac{L_1(A)}{t-T} - |J_i^c| \text{ (Domain of influence of } A \text{).}
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