Vanishing Viscosity Solutions of Hyperbolic Systems with Boundary

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$$u_t + A(t,u)u_x = \epsilon u_{xx}, \qquad t, x > 0, \quad u \in \mathbb{R}^n, \tag{1}$$

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(1) the matrix $A(t,0)$ is smooth and strictly hyperbolic,

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(2) the map $t \mapsto A(t,u)$ is of uniform bounded variation,

$$\|A\| \doteq \sup_{|u|\leq\delta} \int_0^{+\infty} |A_t(s,u)|ds \leq C < +\infty. \tag{3}$$

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$$\|u_1(t) - u_2(s)\|_{L^1} \leq L\Big(|t-s| + \|u_{1,0} - u_{2,0}\|_{L^1} + \|u_{1,b} - u_{2,b}\|_{L^1(0,s)} + \textit{Tot.Var.}(u) \sup_u |A(u,\cdot) - B(u,\cdot)|_{L^1(0,s)}\Big), \quad (4)$$

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As $\epsilon \to 0$, $u^\epsilon(t)$ *converges in* L^1 *to a unique BV function* $u(t,x)$, *''vanishing viscosity solution'' to*

$$u_t + A(t,u)u_x = 0, \qquad u(0,x) = u_0(x), \; u(t,0) = u_b(t), \quad (5)$$

and satisfying again (4).

Example. Consider the system

$$u_t + A(u)u_x - \epsilon u_{xx} = 0, \qquad x \geq x_b(t),$$

which can be rewritten in form (1) by setting

$$y = x - x_b(t), \qquad A(t,u) = A(u) - \frac{dx_b}{dt} I.$$

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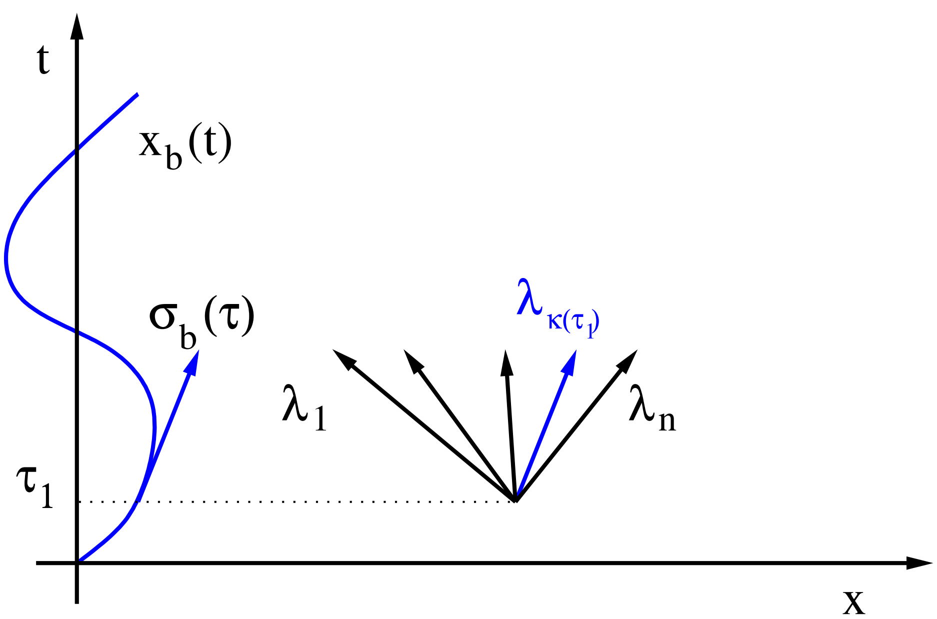

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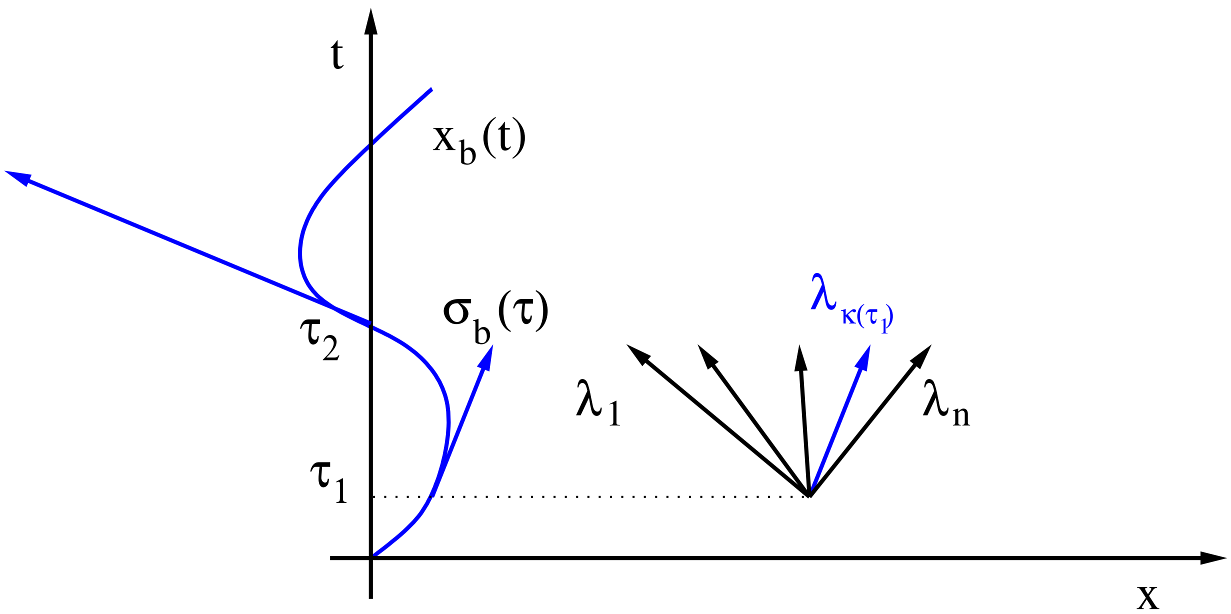

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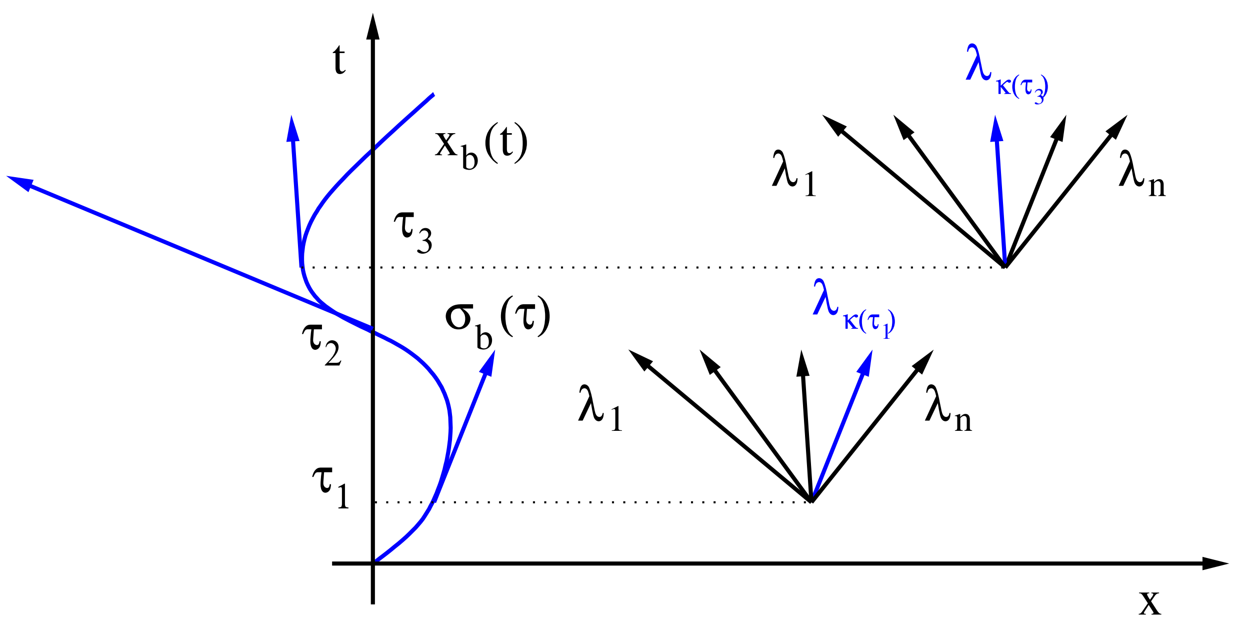

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• one has to study the interaction of travelling waves of (1) with the (non characteristic part of) boundary profiles;

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The equation for the boundary profile are

$$\begin{cases} u_x &= p \\ p_x &= A(\kappa, u)p \\ \kappa_x &= 0 \end{cases} \tag{8}$$

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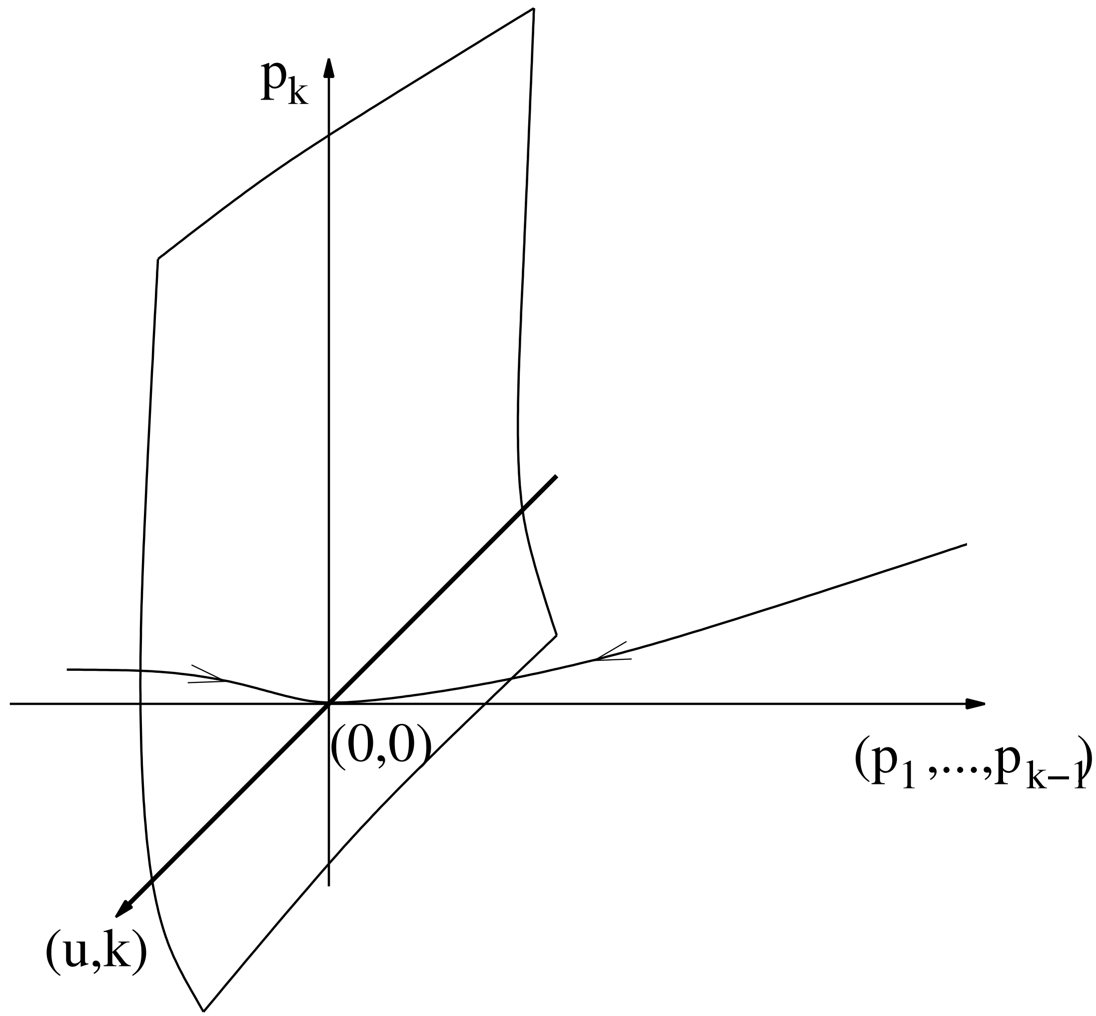
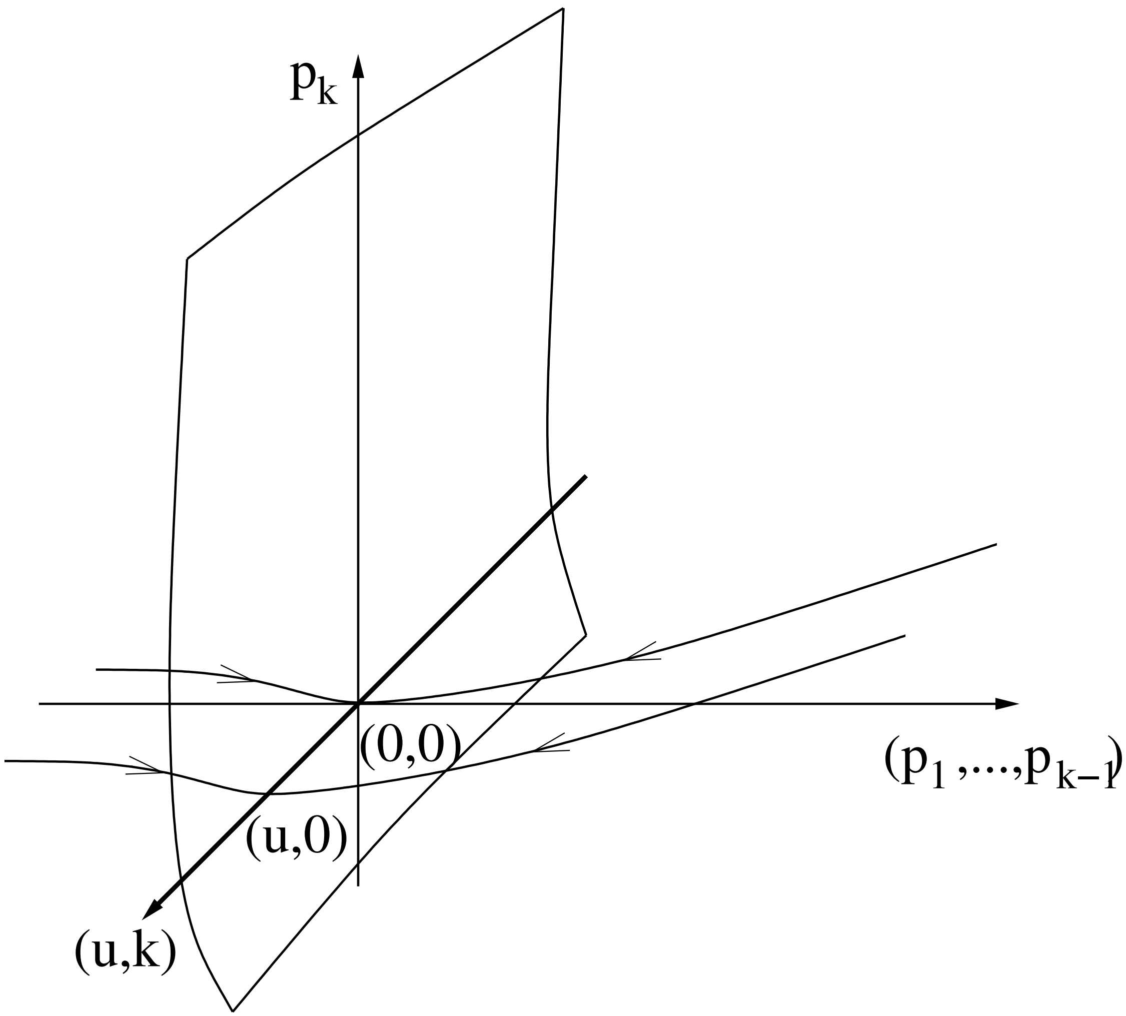
This manifold W^- *is identified uniquely by trajectories converging to* 0 *with speed* $\simeq \lambda$*.*

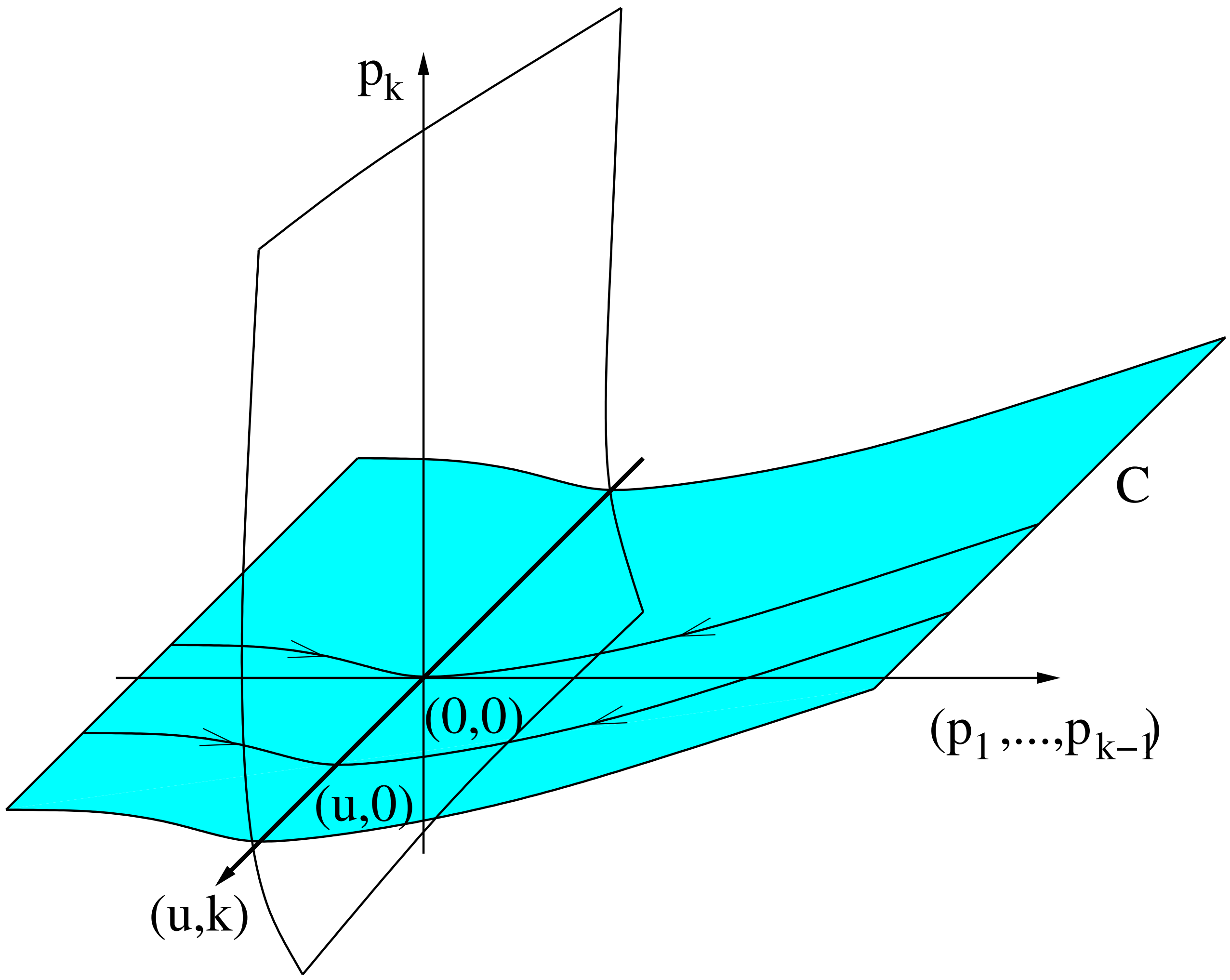
Center manifold and stable manifold near $(u, p) = (0, 0)$:



Applying the Hadamar-Perron theorem to the point $(u, 0)$



Manifold of all trajectories converging as $e^{-(\lambda_{k-1}-\epsilon)t}$ to $(u, 0)$



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Then:

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- $\hat{r}_k$ is ok for k-th travelling profiles or bdry profile ($\sigma_k = 0$).

Equation for the components v_b, v_i

By substituting into $u_t + A(t,x)u_x - u_{xx} = 0$

$$\left\{ \begin{array}{rcl} u_x & = & v_b \tilde{R}_b + v_k \hat{r}_k + \sum_{i \neq k} v_i \tilde{r}_i \\ u_t & = & w_b \tilde{R}_b + w_k \hat{r}_k + \sum_{i \neq k} w_i \tilde{r}_i \end{array} \right. \qquad \sigma_i = \theta_i(w_i/v_i), \qquad (12)$$

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after some computation one obtains (similarly for u_t)

$$\begin{aligned} &(\hat{R}_b + (\hat{R}_{b,v_b} \cdot) v_b + \hat{r}_{k,v_b} v_k)\Big[v_{b,t} + (\hat{A}_b v_b)_x - v_{b,xx}\Big] \\ &\quad + (\hat{R}_{b,v_k} v_b + \hat{r}_k + \hat{r}_{k,v_k} v_k + v_k \sigma_{k,v} \hat{r}_{k,\sigma})\Big[v_{k,t} + (\hat{\lambda}_k v_k)_x - v_{k,xx}\Big] \\ &\quad + \sum_{i \neq k} (\tilde{r}_i + v_i \tilde{r}_{i,v} + v_i \sigma_{i,v} \tilde{r}_{i,\sigma})\Big[v_{i,t} + (\tilde{\lambda}_i v_i)_x - v_{i,xx}\Big] \\ &\quad = \phi(\kappa, u, v, v_x, w, w_x) + \mathcal{O}(1)\left(|v_b| + \sum_{i=1}^{n} |v_i|\right) \sup_u \|A_t\|. \qquad (13) \end{aligned}$$

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There are $n + k - 1$ variables in n equations.

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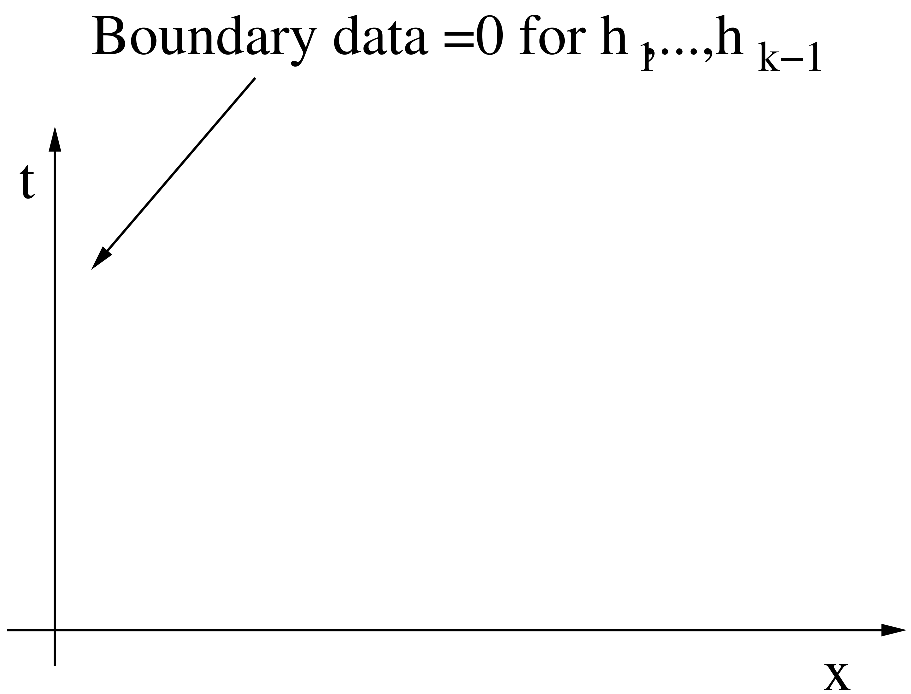

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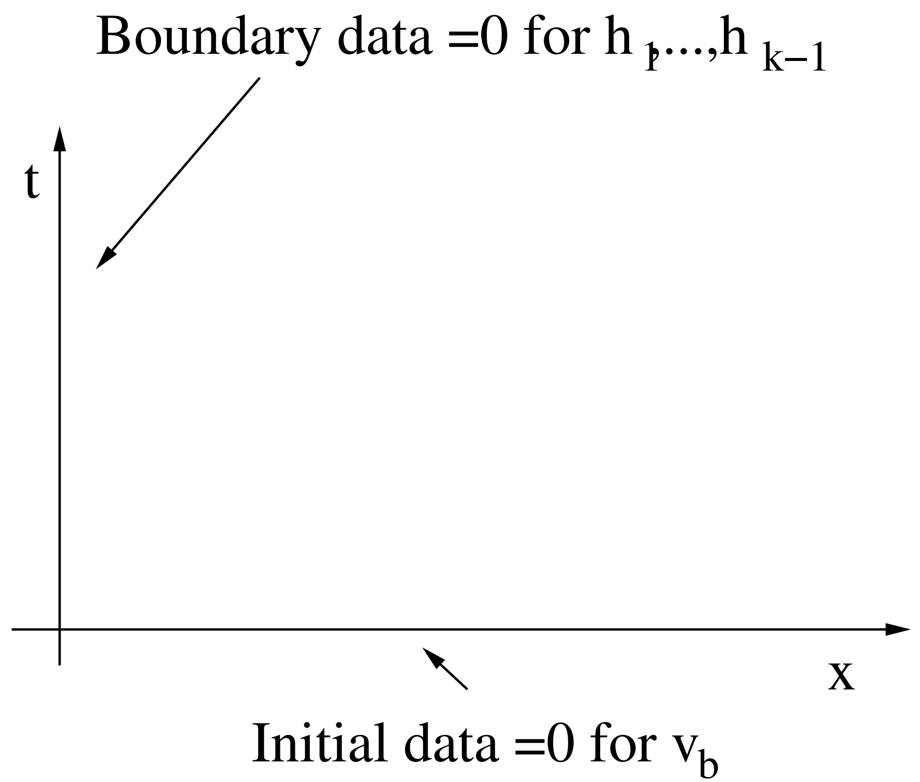

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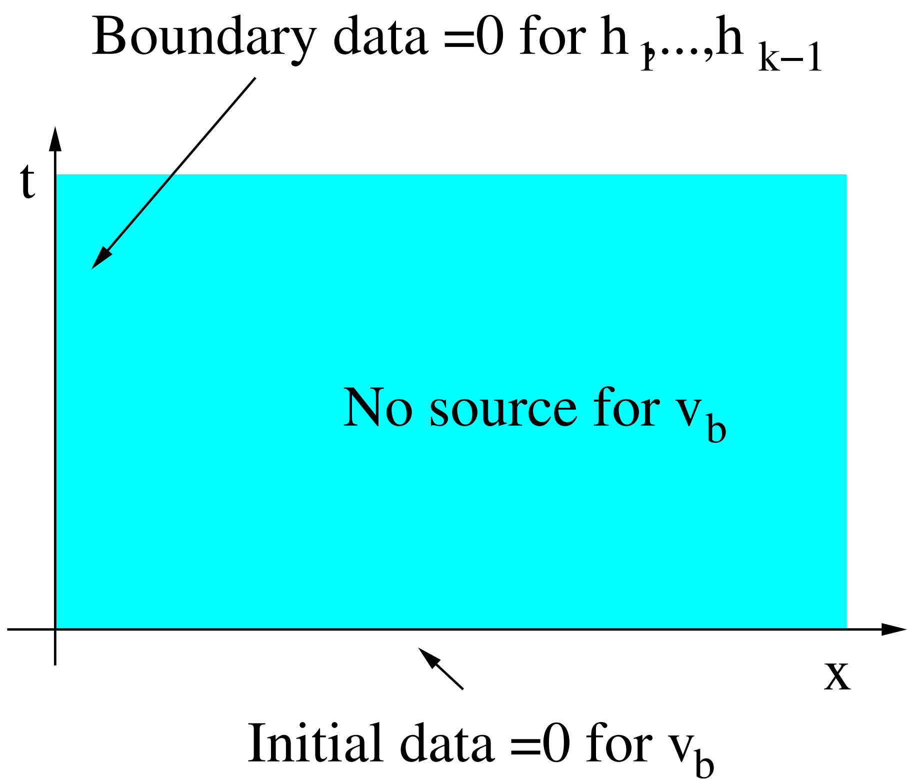

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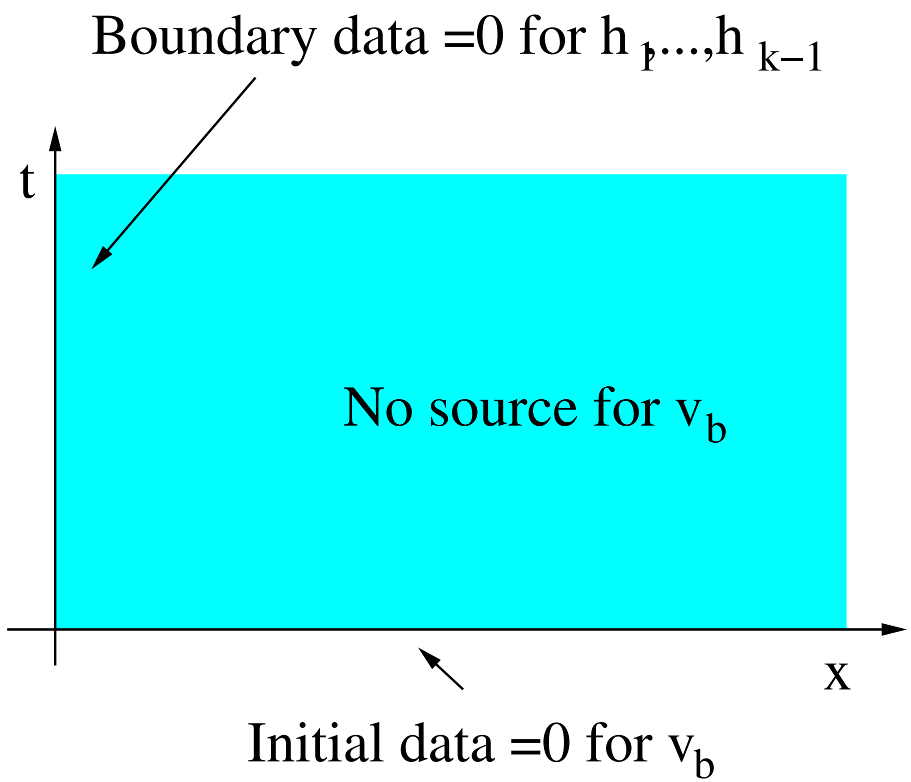

v_b, v_i determined by solving (13), not by the decomposition (12).

To understand the condition $v_i = 0$, $i = 1, \ldots, v_{k-1}$, consider the scalar equation

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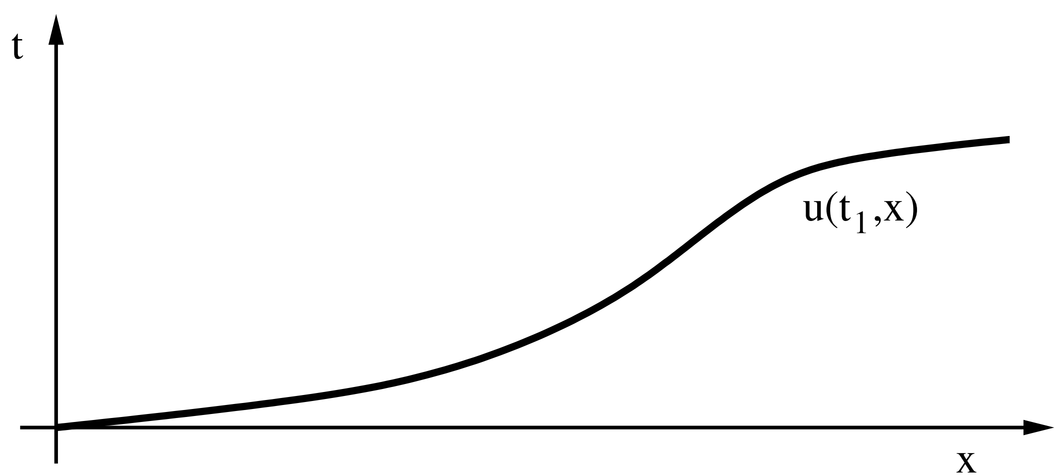

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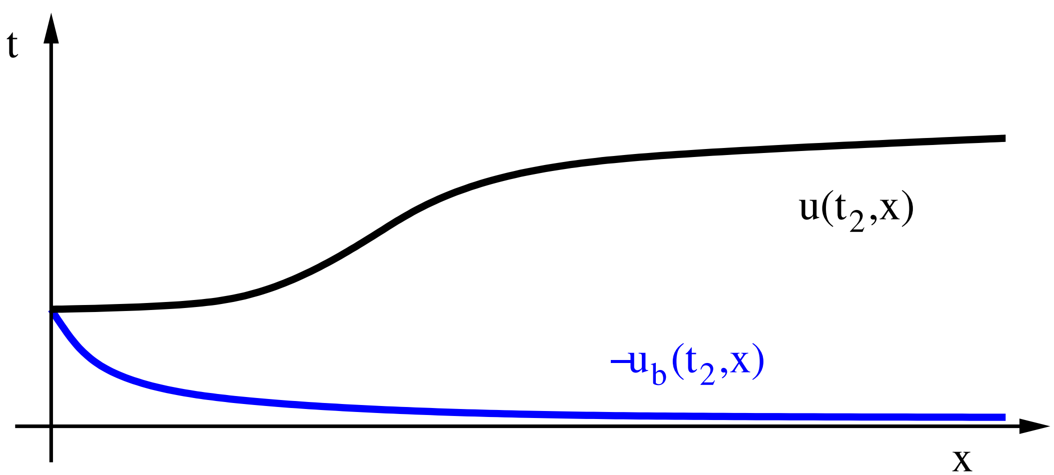

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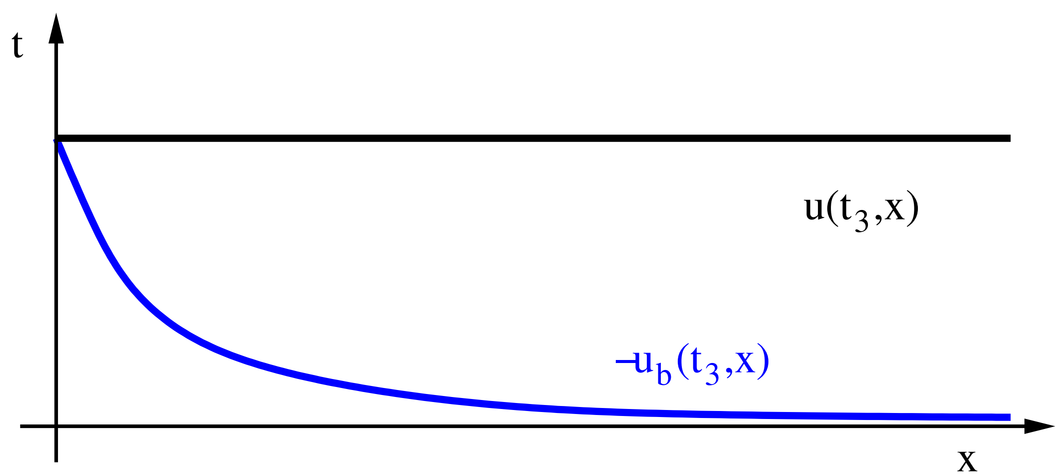

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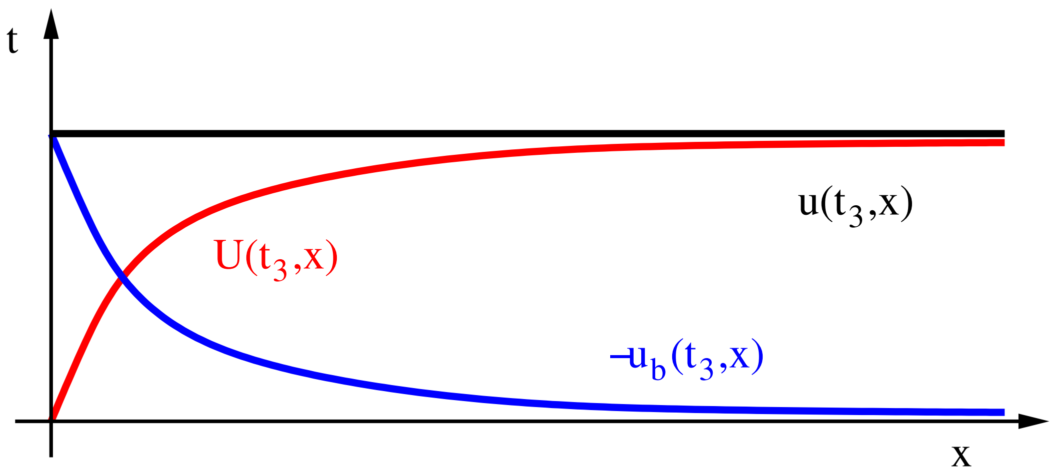

With the $k-1$ conditions on the initial-boundary data data and source terms, one arrives to the system

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Hence

$$\iint_{\mathbb{R}^+ \times \mathbb{R}^+} |v_b w_k| dx dt \le C \int_{\mathbb{R}^+} e^{(d-c)x} \int_{\mathbb{R}^+} |e^{-dy} w_k(t,y)| dt dx \le C.$$

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In $u(x/t)$ one sees only the first two points, the last two are in the jump at $x = 0$.

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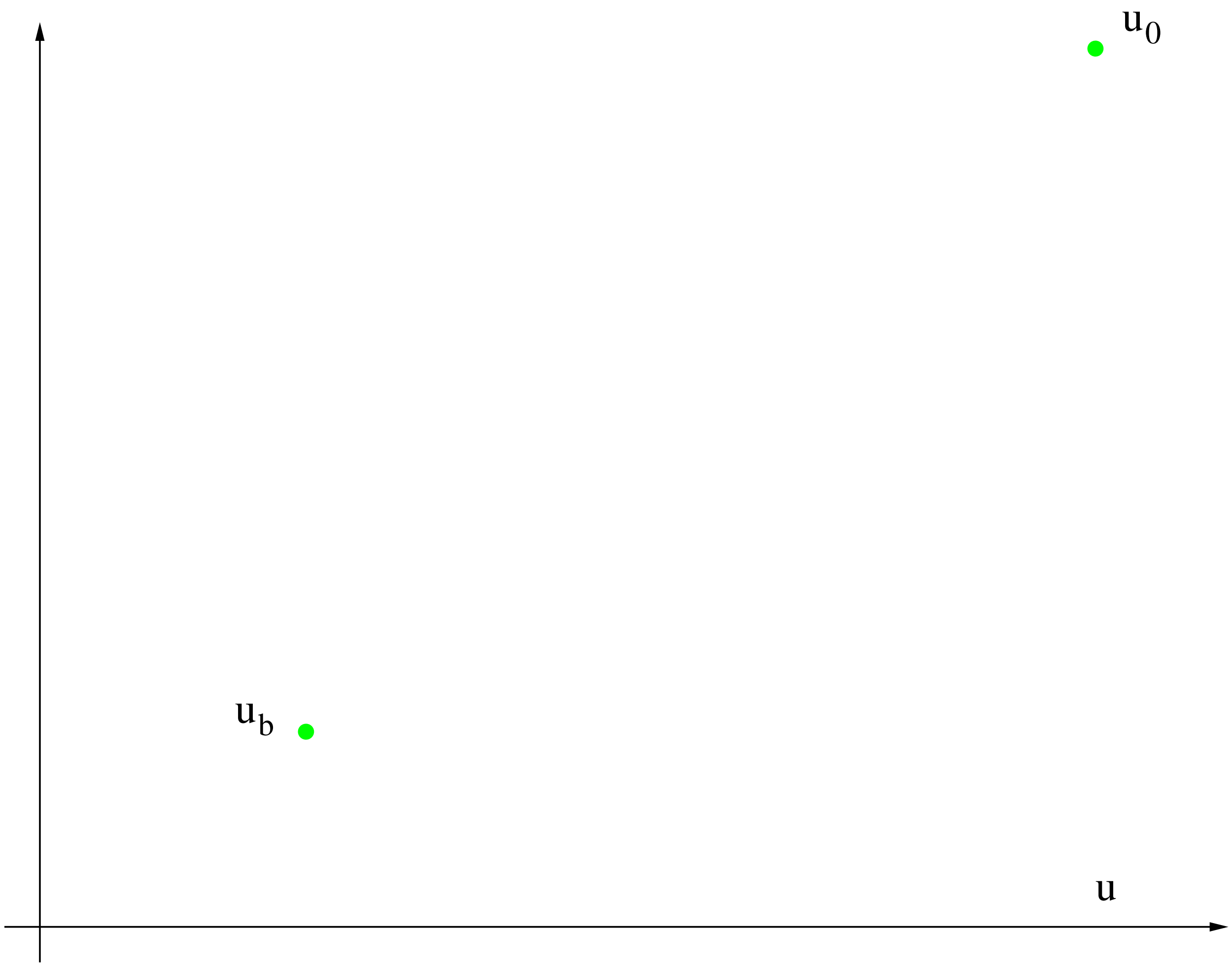
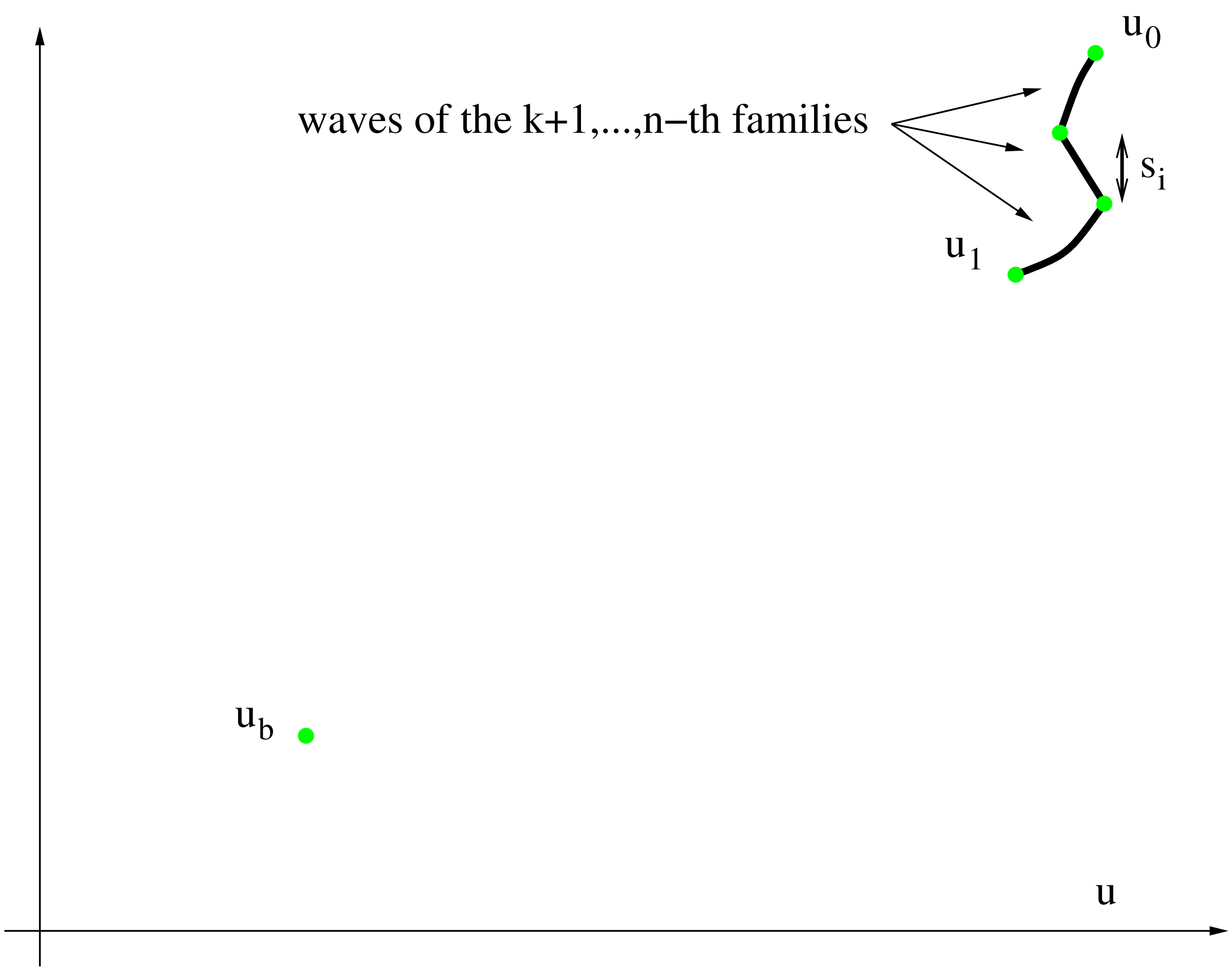
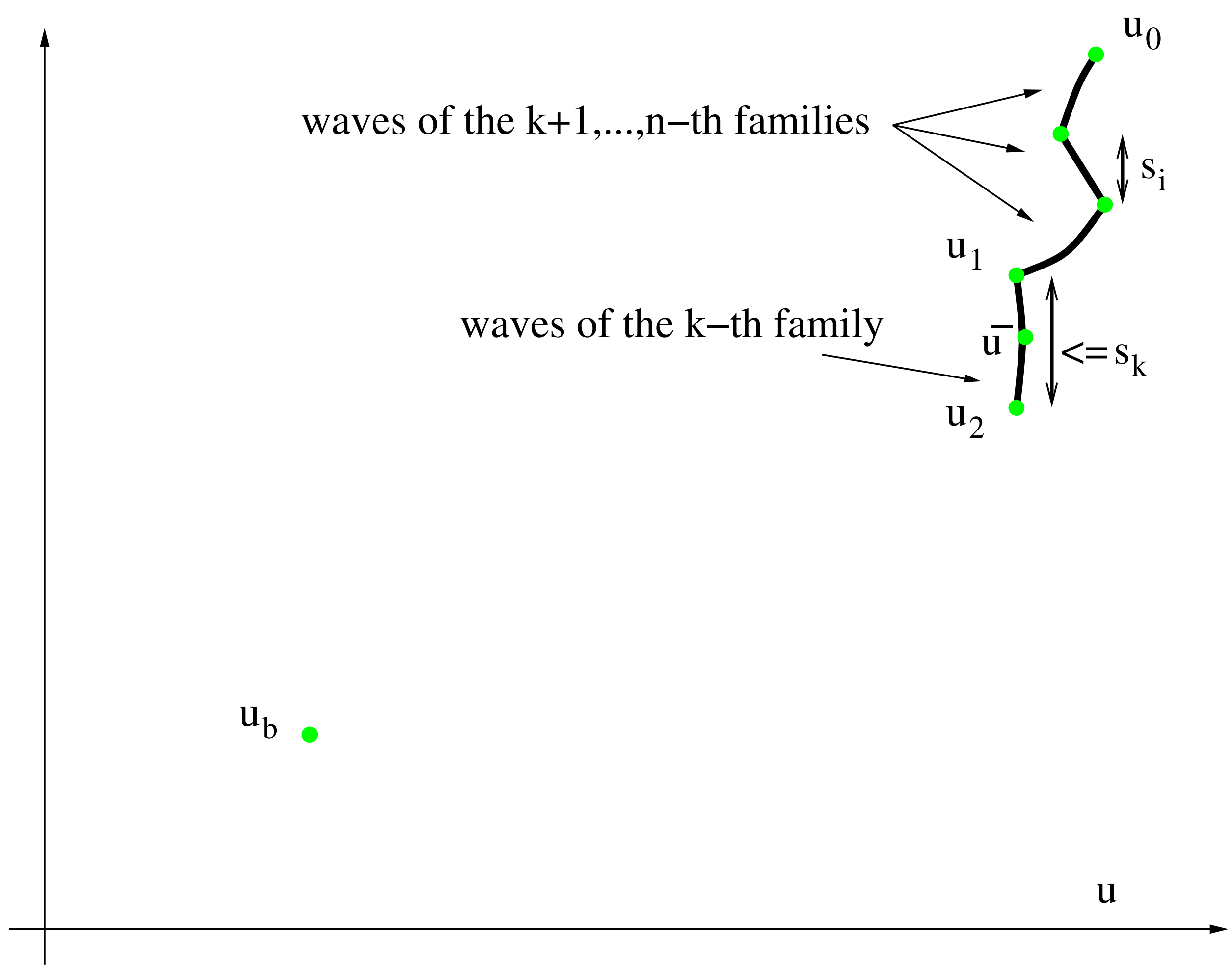


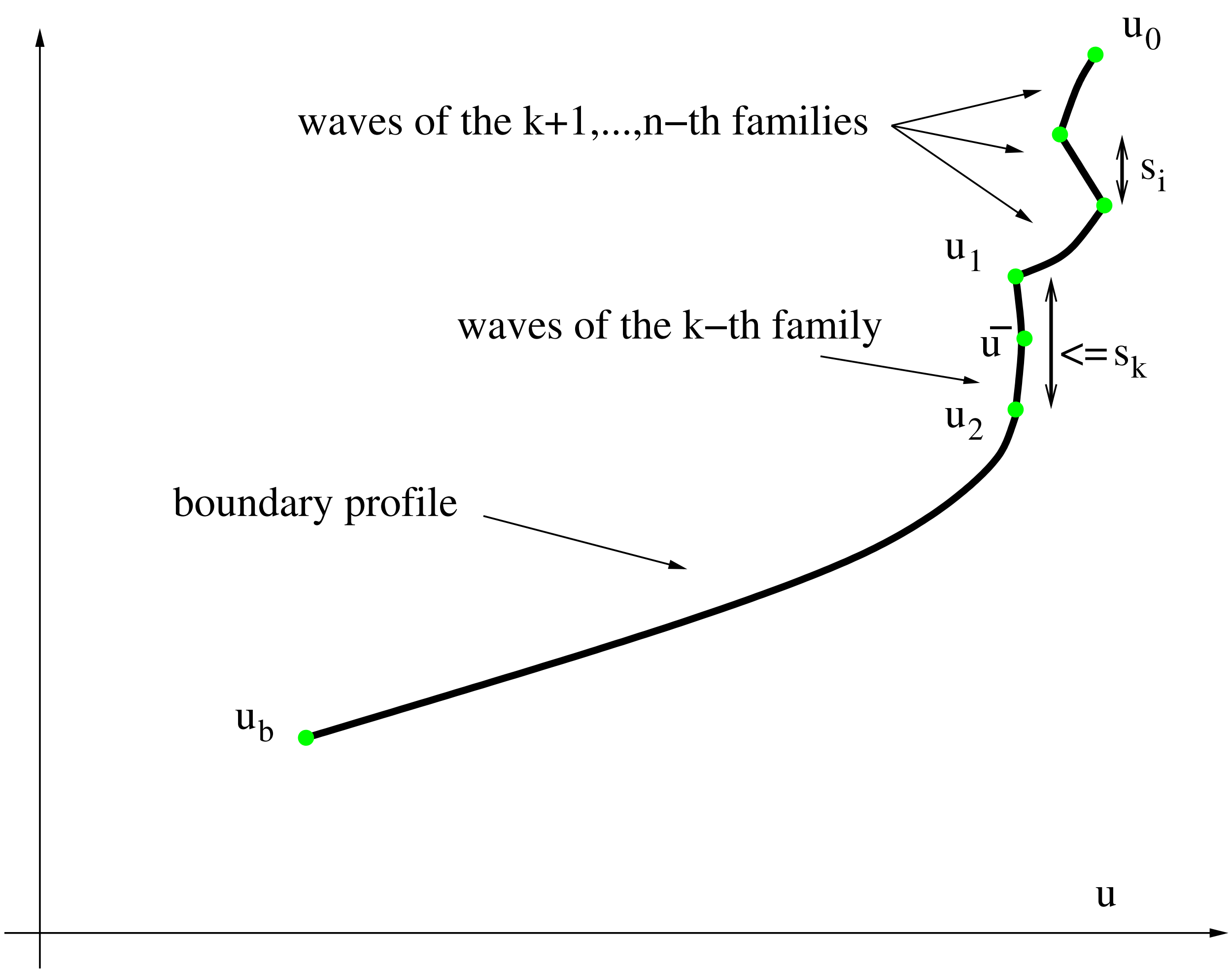
From u_0 to u_1, waves of the $i > k$ family,



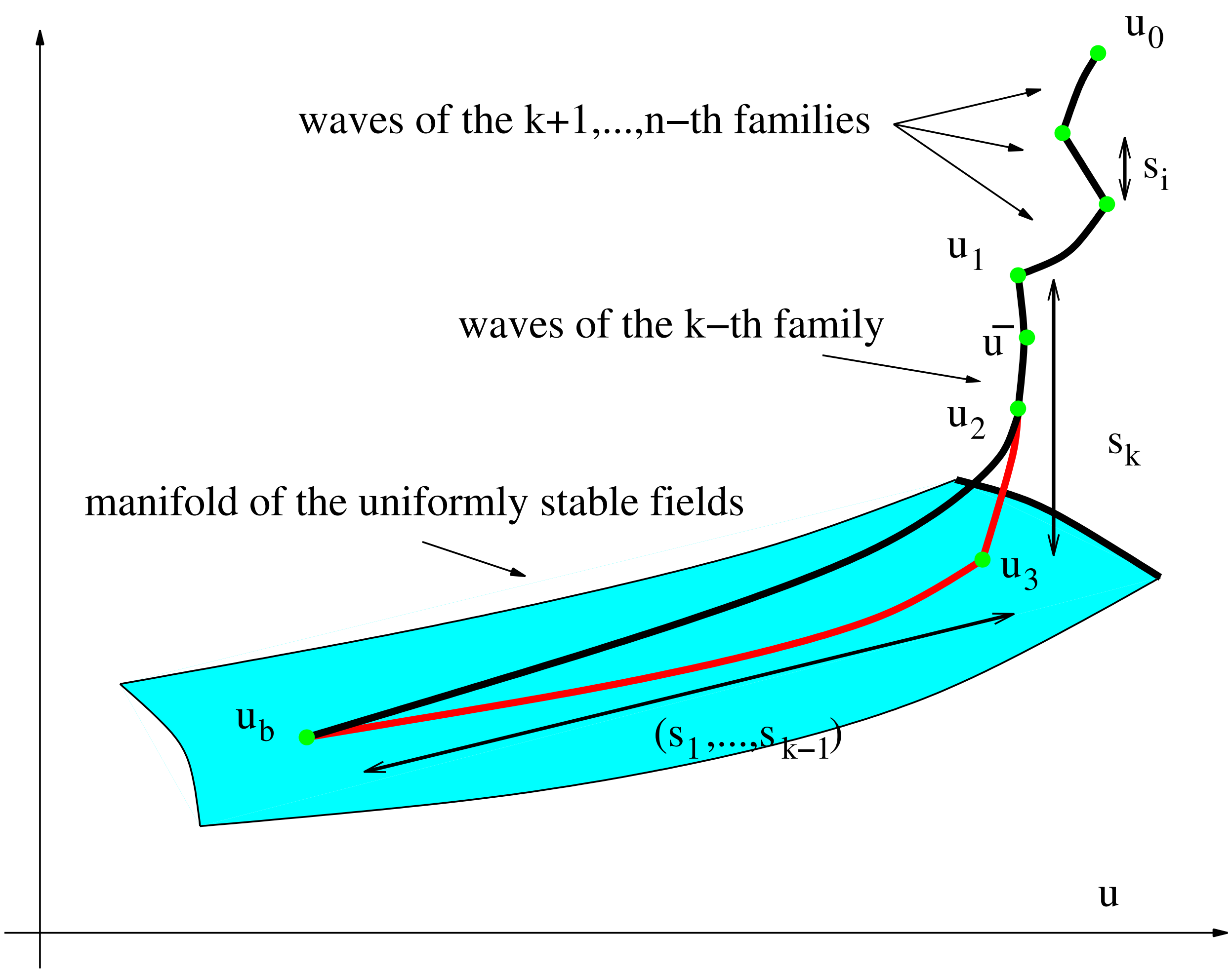
From u_1 to u_2, waves of the k-th family with $\sigma_k \geq 0$,



From u_2 to u_b there is a char. bdry profile,



By means of system (11), we decompose the bdry profile as



Exponentially decaying part of bdry profile

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This solves

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the manifold of solutions converging to 0 *as* $x \to \infty$ *is* $k-1$ *dimensional parameterized by* $(u_1(0), \ldots, u_{k-1}(0))$, *smoothly dependent on* u_k, p_k.

The characteristic part of bdry profile

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The system for u_k, p_k and σ_k is

$$\begin{cases} u_k(s) &= \quad u_1 + \int_0^s \widehat{r}_k(u_b + u_k, p_b, p_k, \sigma_k)d\tau \\ p_k(s) &= \quad \text{b-conc}_{[0,s_k]}\Big(\int_0^s \widehat{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k)d\tau\Big)(s) \\ \sigma_k &= \quad \frac{d}{ds}\text{b-conc}_{[0,s_k]}\Big(\int_0^s \widehat{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k)d\tau\Big)(s) \end{cases}$$

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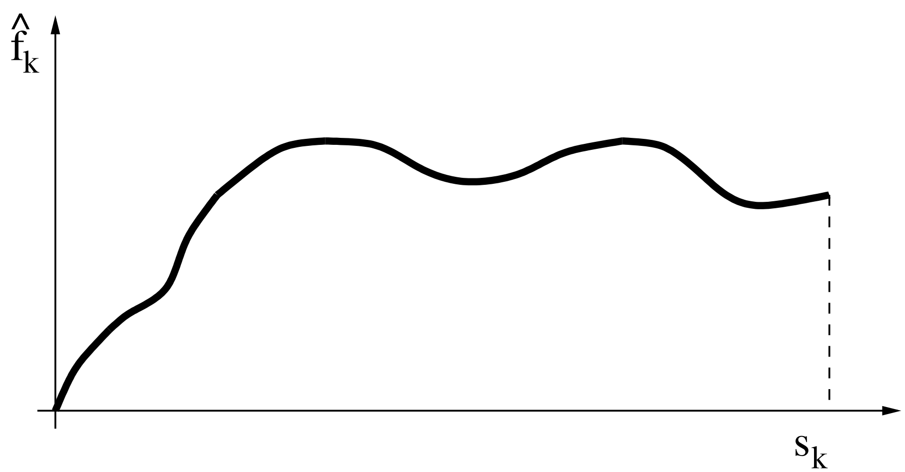

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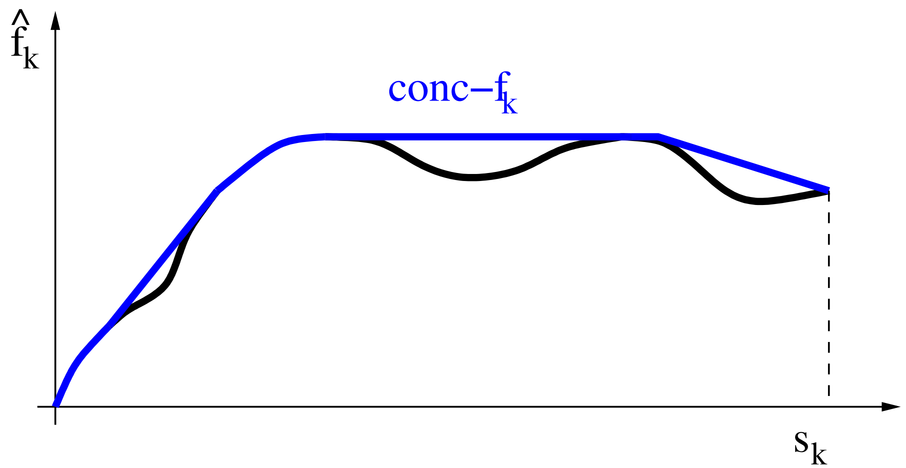
The concave hull for Riemann problem is



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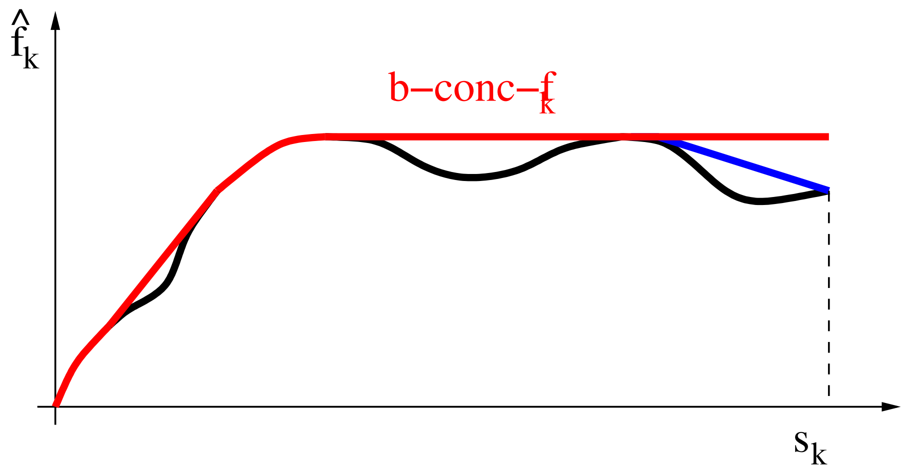
The boundary concave hull for Riemann problem is



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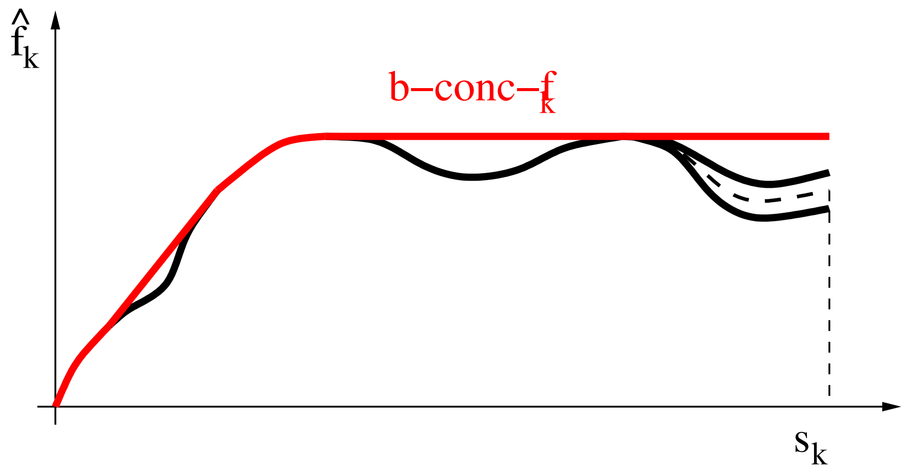
With the exponentially decaying (in space) perturbation u_b, p_b



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$$\Big|\widehat{f}_k(s; u_k = 0, p_k = 0) \; - \; \widehat{f}_k(s; u_k, p_k)\Big| \leq \frac{1}{2}\Big(\textit{b-conc}\widehat{f}_k - \widehat{f}_k\Big)(s; u_k = 0, p_k = 0).$$

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