Vanishing Viscosity Solutions of Hyperbolic Systems with Boundary

Fabio Ancona, CIRAM Bologna

Stefano Bianchini, IAC(CNR) Roma

http://www.iac.cnr.it/

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We consider the parabolic system

\[ u_t + A(t,u)u_x = \epsilon u_{xx}, \quad t, x > 0, \quad u \in \mathbb{R}^n, \quad (1) \]

with Dirichlet boundary conditions \( u_b(t) \) and initial data \( u_0(t) \).
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**Assumptions:**

(1) the matrix \( A(t, 0) \) is smooth and strictly hyperbolic,

\[ \inf_{t,u,v} \left\{ \lambda_{i+1}(t, u) - \lambda_i(t, v) \right\} \geq c > 0 \quad i = 1, \ldots, n - 1; \]  
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(2) the map \( t \mapsto A(t, u) \) is of uniform bounded variation,

\[ \| A \| = \sup_{|u| \leq \delta} \int_{0}^{+\infty} |A_t(s, u)| ds \leq C < +\infty. \quad (3) \]
Theorem. If

\[ |u_b(t)|, |u_0(x)|, \text{Tot.Var.}(u_b), \text{Tot.Var.}(u_0) < \min\left\{ K^{-1}, e^{-K\|A\|} \right\}, \]
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If \( u_1, u_2 \) are two different solution with matrices \( A, B \), for \( t \geq s \)

\[
\begin{aligned}
\|u_1(t) - u_2(s)\|_{L^1} &\leq L \left( |t - s| + \|u_1,0 - u_2,0\|_{L^1} + \|u_1,b - u_2,b\|_{L^1(0,s)} \\
&\quad + \text{Tot.Var.}(u) \sup_{\tilde{u}} |A(\tilde{u}, \cdot) - B(\tilde{u}, \cdot)|_{L^1(0,s)} \right),
\end{aligned}
\]

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\[ \|u_1(t) - u_2(s)\|_{L^1} \leq L\left(|t-s| + \|u_{1,0} - u_{2,0}\|_{L^1} + \|u_{1,b} - u_{2,b}\|_{L^1(0,s)} \right. \]
\[ \left. + \text{Tot.Var.}(u) \sup_u |A(u, \cdot) - B(u, \cdot)|_{L^1(0,s)} \right), \]
As \( \varepsilon \to 0 \), \( u^\varepsilon(t) \) converges in \( L^1 \) to a unique \( BV \) function \( u(t, x) \), "vanishing viscosity solution" to
\[ u_t + A(t, u)u_x = 0, \quad u(0, x) = u_0(x), \quad u(t, 0) = u_b(t), \quad (5) \]
and satisfying again (4).
Example. Consider the system

\[ u_t + A(u)u_x - \epsilon u_{xx} = 0, \quad x \geq x_b(t), \]

which can be rewritten in form (1) by setting

\[ y = x - x_b(t), \quad A(t, u) = A(u) - \frac{dx_b}{dt} I. \]
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![Diagram](image-url)
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- the boundary characteristic eigenvalue $\lambda_{\bar{k}}(t, 0)$ changes with time, i.e. $\bar{k} = \bar{k}(t)$;

- one has to study the interaction of travelling waves of (1) with the (non characteristic part of) boundary profiles;
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\[\text{non char. part boun. profile}\]  

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Decomposition of the boundary profile

The equation for the boundary profile are

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\begin{align*}
    u_x &= p \\
    p_x &= A(\kappa, u)p \\
    \kappa_x &= 0
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and we assume that the \( k \)-th eigenvalue of \( A(0,0) \) is 0.
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Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be $C^r$ diffeomorphism, with $r \geq 1$, such that 

$$Df(0) = (Ax, By), \quad \|A\| \leq \lambda, \quad \|B^{-1}\| \leq 1/\mu,$$

for $\lambda < \min\{1, \mu\}$, $(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. 
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This manifold \( W^- \) is identified uniquely by trajectories converging to 0 with speed \( \simeq \lambda \).
Center manifold and stable manifold near \((u, p) = (0, 0)\):
Applying the Hadamard-Perron theorem to the point \((u, 0)\)
Manifold of all trajectories converging as $e^{-(\lambda_{k-1}-\epsilon)t}$ to $(u,0)$
Write the center stable manifold of (7) as

\[ p = R_{cs}(\kappa, u, v_{cs})v_{cs}, \quad R_{cs} \in \mathbb{R}^{n \times k}; \]
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\[ v_{cs} = r_k(\kappa, u, v_k)v_k, \quad v_{cs} = R_s(\kappa, u, v_s)v_s, \]

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Then the vectors \( \tilde{r}_{k} \in \mathbb{R}^{n}, \ \tilde{R} \in \mathbb{R}^{n \times (k-1)} \) are given by

\[ \tilde{r}_{k}(\kappa, u, v_{b}, v_{k}) = R_{cs}(\kappa, u, R_{s}v_{b} + r_{k}v_{k})r_{k}(\kappa, u, v_{k}) \quad (9) \]
\[ \tilde{R}_{b}(\kappa, u, v_{b}, v_{k}) = R_{cs}(\kappa, u, R_{s}v_{b} + r_{k}v_{k})R_{s}(\kappa, u, v_{s}) \quad (10) \]
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\[ \tilde{R}_b(\kappa, u, v_b, v_k) = R_{cs}(\kappa, u, R_s v_b + r_k v_k)R_s(\kappa, u, v_s) \quad (10) \]
The dependence on \( \sigma \) can be added to \( \tilde{r}_k \) by replacing \( A(\kappa, u) \) with \( A(\kappa, u) - \sigma I \), with \( \sigma_x = 0 \).
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Then the vectors \( \hat{r}_{k} \in \mathbb{R}^{n}, \hat{R} \in \mathbb{R}^{n \times (k-1)} \) are given by
\begin{align*}
\hat{r}_{k}(\kappa, u, v_{b}, v_{k}) & = R_{cs}(\kappa, u, R_{s}v_{b} + r_{k}v_{k})r_{k}(\kappa, u, v_{k}) \quad (9) \\
\hat{R}_{b}(\kappa, u, v_{b}, v_{k}) & = R_{cs}(\kappa, u, R_{s}v_{b} + r_{k}v_{k})R_{s}(\kappa, u, v_{s}) \quad (10)
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The dependence on \( \sigma \) can be added to \( \hat{r}_{k} \) by replacing \( A(\kappa, u) \) with \( A(\kappa, u) - \sigma I \), with \( \sigma_{x} = 0 \).
Moreover the center manifold of (8) is \{ \[ p = v_{k}\hat{r}_{k}(\kappa, u, 0, v_{k}) \} \}. 
Write the center stable manifold of (7) as

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on this manifold, the center manifold and the manifold \( C \) as

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Then the vectors \( \hat{r}_k \in \mathbb{R}^n, \hat{R} \in \mathbb{R}^{n \times (k-1)} \) are given by

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\begin{align*}
\hat{r}_k(\kappa, u, v_b, v_k) & = R_{cs}(\kappa, u, R_sv_b + r_kv_k)r_k(\kappa, u, v_k) \\
\hat{R}_b(\kappa, u, v_b, v_k) & = R_{cs}(\kappa, u, R_sv_b + r_kv_k)R_s(\kappa, u, v_s)
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The dependence on \( \sigma \) can be added to \( \hat{r}_k \) by replacing \( A(\kappa, u) \) with \( A(\kappa, u) - \sigma I \), with \( \sigma_x = 0 \).

Moreover the center manifold of (8) is \( \{ p = v_k\hat{r}_k(\kappa, u, 0, v_k) \} \), and the stable manifold is \( \{ p = R_b(\kappa, u, v_b, 0)v_b \} \).
Diagonalization of system (8)
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By writing

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$$\begin{align*}
  u_x &= \tilde{R}_b(\kappa, u, u_x)v_b + \tilde{r}_k(\kappa, u, u_x)v_k \\
  v_{b,x} &= \tilde{A}_b(\kappa, u, u_x)v_b \\
  v_{k,x} &= \tilde{\lambda}_k(\kappa, u, u_x)v_k \\
  \kappa_x &= 0
\end{align*}$$

(11)
**Diagonalization of system (8)**

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    v_{k,x} &= \hat{\lambda}_k(\kappa, u, u_x)v_k \\
    \kappa_x &= 0
\end{align*}
\]

\[
\hat{A}_b(0, 0, 0) = \text{diag}(\lambda_1, \ldots, \lambda_{k-1}), \quad \hat{\lambda}_k(0, 0, 0) = \lambda_k.
\]
Diagonalization of system (8)

By writing

\[ u_x = \tilde{R}_b(\kappa, u, u_x) v_b + \tilde{r}_k(\kappa, u, u_x) v_k, \]

the equation (8) becomes

\[
\begin{cases}
  u_x &= \tilde{R}_b(\kappa, u, u_x) v_b + \tilde{r}_k(\kappa, u, u_x) v_k \\
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Then:

- \( v_b \) is exponentially decreasing (non characteristic part);
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**Diagonalization of system (8)**

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\]

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\tilde{A}_b(0,0,0) = \text{diag}(\lambda_1, \ldots, \lambda_{k-1}), \quad \tilde{\lambda}_k(0,0,0) = \lambda_k.
\]

Then:

- \( v_b \) is exponentially decreasing (non characteristic part);
- the eigenvalue \( \tilde{\lambda}_k \) determines the structure of boundary profile;
- \( \tilde{r}_k \) is ok for \( k \)-th travelling profiles or bdry profile \( (\sigma_k = 0) \).
Equation for the components $v_b, v_i$

By substituting into $u_t + A(t, x)u_x - u_{xx} = 0$

\[
\begin{align*}
    u_x &= v_b \tilde{R}_b + v_k \tilde{r}_k + \sum_{i \neq k} v_i \tilde{r}_i \\
    u_t &= w_b \tilde{R}_b + w_k \tilde{r}_k + \sum_{i \neq k} w_i \tilde{r}_i
\end{align*}
\]

\[\sigma_i = \theta_i(w_i/v_i), \quad (12)\]
Equation for the components $v_b, v_i$

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$$
\left\{
\begin{array}{l}
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u_t = w_b \tilde{R}_b + w_k \tilde{r}_k + \sum_{i \neq k} w_i \tilde{r}_i
\end{array}
\right.
\quad \sigma_i = \theta_i(w_i/v_i), \quad (12)
$$
after some computation one obtains (similarly for $u_t$)

$$
\begin{align*}
&\left(\tilde{R}_b + (\tilde{R}_b,v_b)\cdot v_b + \tilde{r}_k,v_b,v_k\right)[v_{b,t} + (\tilde{A}_b v_b)x - v_{b,xx}] \\
&\quad + \left(\tilde{R}_b,v_k\cdot v_b + \tilde{r}_k,v_k,v_k + v_k \sigma_k,v \tilde{r}_k,\sigma\right)[v_{k,t} + (\tilde{\lambda}_k v_k)x - v_{k,xx}] \\
&\quad + \sum_{i \neq k} (\tilde{r}_i + v_i \tilde{r}_i,v + v_i \sigma_i,v \tilde{r}_i,\sigma)[v_{i,t} + (\tilde{\lambda}_i v_i)x - v_{i,xx}] \\
&\quad = \phi(\kappa, u, v, v_x, w, w_x) + O(1)\left(|v_b| + \sum_{i=1}^n |v_i|\right) \sup_u \|A_t\|.
\end{align*}
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Equation for the components $v_b$, $v_i$

By substituting into $u_t + A(t, x)u_x - u_{xx} = 0$

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\begin{cases}
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\end{cases}
\]

\[\sigma_i = \theta_i(w_i/v_i), \quad (12)\]

after some computation one obtains (similarly for $u_t$)

\[
(\tilde{R}_b + (\tilde{R}_b, v_b) v_b + \tilde{r}_k, v_b v_k) [v_{b,t} + (\tilde{A}_b v_b)x - v_{b,xx}] \\
+ (\tilde{R}_b, v_k v_b + \tilde{r}_k, v_k v_k + v_k \sigma_k, v \tilde{r}_k, \sigma) [v_{k,t} + (\tilde{\lambda}_k v_k)x - v_{k,xx}] \\
+ \sum_{i \neq k} (\tilde{r}_i + v_i \tilde{r}_i, v + v_i \sigma_i, v \tilde{r}_i, \sigma) [v_{i,t} + (\tilde{\lambda}_i v_i)x - v_{i,xx}] \\
= \phi(\kappa, u, v, v_x, w, w_x) + O(1) \left( |v_b| + \sum_{i=1}^{n} |v_i| \right) \sup_u \|A_t\|. \quad (13)
\]

There are $n + k - 1$ variables in $n$ equations.
Ideas to recover one $k \times k$ system for $v_b$ and $n$ scalar equation with source for $v_i$: 
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Ideas to recover one \( k \times k \) system for \( v_b \) and \( n \) scalar equation with source for \( v_i \):

\[
\text{Boundary data } = 0 \text{ for } h_1, \ldots, h_{k-1}
\]
Ideas to recover one $k \times k$ system for $v_b$ and $n$ scalar equation with source for $v_i$:

- Initial data $= 0$ for $v_{b}$
- Boundary data $= 0$ for $h_1, ..., h_{k-1}$
Ideas to recover one $k \times k$ system for $v_b$ and $n$ scalar equation with source for $v_i$:

- Initial data $= 0$ for $v_b$
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Ideas to recover one $k \times k$ system for $v_b$ and $n$ scalar equation with source for $v_i$:

\[
\text{Boundary data} = 0 \text{ for } h_1, \ldots, h_{k-1} \\
\text{No source for } v_b \\
\text{Initial data} = 0 \text{ for } v_b
\]

$v_b, v_i$ determined by solving (13), not by the decomposition (12).
To understand the condition \( v_i = 0, \ i = 1, \ldots, v_{k-1} \), consider the scalar equation

\[
U_t - U_x = U_{xx}, \quad u(0, x) = u_0(x), \quad u(t, 0) = 0
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To understand the condition $v_i = 0$, $i = 1, \ldots, v_{k-1}$, consider the scalar equation

$$U_t - U_x = U_{xx}, \quad u(0, x) = u_0(x), \quad u(t, 0) = 0$$

which splits into $U = u + u_b$, with

$$\begin{cases} u_t - u_x = u_{xx} \\ u|_{t=0} = u_0(x), \\ u_x|_{x=0} = 0 \end{cases} \begin{cases} u_{b,t} - u_{b,x} = u_{b,xx} \\ u|_{x=0} = 0, \\ u|_{t=0} = -\int_0^t u_{xx}(s, 0) ds \end{cases}$$
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  u_{b,t} - u_{b,x} = u_{b,xx} \\
  u|_{x=0} = 0, \\
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\end{cases}$$
With the \( k - 1 \) conditions on the initial-boundary data data and source terms, one arrives to the system

\[
\begin{align*}
\frac{\partial v_b}{\partial t} + (\tilde{A}_b v_b)_x - v_{b,xx} &= 0 \\
\frac{\partial v_k}{\partial t} + (\tilde{\lambda}_k v_k)_x - v_{k,xx} &= s_k(t, x) \\
\frac{\partial v_i}{\partial t} + (\tilde{\lambda}_i v_i)_x - v_{i,xx} &= s_i(t, x)
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(14)

- **Interaction among** \( i \neq k \) **trav. waves and bdry profile**
With the $k - 1$ conditions on the initial-boundary data data and source terms, one arrives to the system

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  v_{b,t} + (\tilde{A}_b v_b)x - v_{b,xx} = 0 \\
  v_{k,t} + (\lambda_k v_k)x - v_{k,xx} = s_k(t, x) \\
  v_{i,t} + (\tilde{\lambda}_i v_i)x - v_{i,xx} = s_i(t, x)
\end{cases} \quad (14)$$

- **Interaction among $i \neq k$ trav. waves and bdry profile**

Since $\tilde{A}_b$ is strictly negative definite, one obtains that

$$|v_b(t, x)| \leq \text{Tot. Var.} (u)e^{-cx}, \quad c \text{ strict hyperbolicity.}$$
With the $k - 1$ conditions on the initial-boundary data data and source terms, one arrives to the system

$$
\begin{align*}
\frac{\partial v_b}{\partial t} + \left( \hat{A}_b v_b \right)_x - v_{b,xx} &= 0 \\
\frac{\partial v_k}{\partial t} + \left( \lambda_k v_k \right)_x - v_{k,xx} &= s_k(t, x) \\
\frac{\partial v_i}{\partial t} + \left( \tilde{\lambda}_i v_i \right)_x - v_{i,xx} &= s_i(t, x)
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Since $\lambda_i \neq 0$, $i \neq k$, then the following terms can be estimated

$$
\sum_{i \neq k} |v_i v_b|, \quad \sum_{i \neq k} |v_{i,x} v_b|,
$$
With the $k - 1$ conditions on the initial-boundary data and source terms, one arrives to the system

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    v_{i,t} + (\tilde{\lambda}_i v_i)x - v_{i,xx} &= s_i(t, x)
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\[
\sum_{i \neq k} |v_i v_b|, \quad \sum_{i \neq k} |v_i x v_b|,
\]

waves with speed $\neq 0$ cross an integrable function of $x$. 
• Interaction of $k$-th trav. waves and bdry profile
• *Interaction of k-th trav. waves and bdry profile*

Since for $\sigma_k = 0$ we have an exact boundary profile (11),
• Interaction of $k$-th trav. waves and bdry profile

Since for $\sigma_k = 0$ we have an exact boundary profile (11), the basic interaction term is

$$v_b v_k (\sigma_b - \sigma_k) = v_b w_k,$$
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with $w_k$ is $k$-th component of $u_t$. 
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with $w_k$ is $k$-th component of $u_t$.

Due to $\hat{\lambda}_k \simeq 0$ and the presence of boundary, it follows

$$\int_{\mathbb{R}^+} |e^{-dy} w_k(t, y)| dt \leq C \cdot \text{Tot.Var.}(u), \quad d \simeq \|\hat{\lambda}_k\|_{L^\infty},$$
• *Interaction of* $k$*-th trav. waves and bdry profile*

Since for $\sigma_k = 0$ we have an exact boundary profile (11), the basic interaction term is

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with $w_k$ is $k$-th component of $u_t$.

Due to $\lambda_k \approx 0$ and the presence of boundary, it follows

$$\int_{\mathbb{R}^+} |e^{-dy} w_k(t, y)| dt \leq C \cdot \text{Tot. Var.}(u), \quad d \simeq \|\lambda_k\|_{L^\infty},$$

Hence

$$\int \int_{\mathbb{R}^+ \times \mathbb{R}^+} |v_b w_k| dx \, dt \leq C \int_{\mathbb{R}^+} e^{(d-c)x} \int_{\mathbb{R}^+} |e^{-dy} w_k(t, y)| dt \, dx \leq C.$$
Solution of the Boundary Riemann problem
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To characterize the unique limit of $u^\varepsilon$ as $\varepsilon \to 0$, one has to study
Solution of the Boundary Riemann problem

To characterize the unique limit of $u^\epsilon$ as $\epsilon \to 0$, one has to study

$$u_t + A(\kappa, u)u_x = 0,$$

$$\begin{cases} u(0, x) = u_0 \\ u(t, 0) = u_b \end{cases} \quad (15)$$
Solution of the Boundary Riemann problem

To characterize the unique limit of \( u^\epsilon \) as \( \epsilon \to 0 \), one has to study

\[
\frac{\partial u}{\partial t} + A(\kappa, u) \frac{\partial u}{\partial x} = 0, \quad \begin{cases} 
    u(0, x) = u_0 \\
    u(t, 0) = u_b 
\end{cases} \quad (15)
\]

The solution \( u = u(x/t) \) will have the structure
Solution of the Boundary Riemann problem

To characterize the unique limit of $u^\epsilon$ as $\epsilon \to 0$, one has to study

$$u_t + A(\kappa, u)u_x = 0, \quad \begin{cases} u(0, x) = u_0 \\ u(t, 0) = u_b \end{cases}$$ (15)

The solution $u = u(x/t)$ will have the structure

- waves of the $i > k$ families entering the domain;
Solution of the Boundary Riemann problem

To characterize the unique limit of \( u^\varepsilon \) as \( \varepsilon \to 0 \), one has to study

\[
\frac{\partial u}{\partial t} + A(\kappa, u) \frac{\partial u}{\partial x} = 0, \quad \begin{cases} u(0, x) = u_0 \\ u(t, 0) = u_b \end{cases} \tag{15}
\]

The solution \( u = u(x/t) \) will have the structure

- waves of the \( i > k \) families entering the domain;
- waves of the \( k \)-th family entering the domain;
Solution of the Boundary Riemann problem

To characterize the unique limit of $u^\epsilon$ as $\epsilon \to 0$, one has to study

$$u_t + A(\kappa, u)u_x = 0,$$

with initial conditions

$$u(0,x) = u_0, \quad u(t,0) = u_b$$

(15)

The solution $u = u(x/t)$ will have the structure

- waves of the $i > k$ families entering the domain;
- waves of the $k$-th family entering the domain;
- waves of the $k$-th family with speed 0;
Solution of the Boundary Riemann problem

To characterize the unique limit of $u^\epsilon$ as $\epsilon \to 0$, one has to study

\[ u_t + A(\kappa, u)u_x = 0, \quad \begin{cases} u(0, x) = u_0 \\ u(t, 0) = u_b \end{cases} \quad (15) \]

The solution $u = u(x/t)$ will have the structure

- waves of the $i > k$ families entering the domain;
- waves of the $k$-th family entering the domain;
- waves of the $k$-th family with speed 0;
- a characteristic boundary profile.
Solution of the Boundary Riemann problem

To characterize the unique limit of $u^\epsilon$ as $\epsilon \to 0$, one has to study

$$u_t + A(\kappa, u)u_x = 0,$$

subject to

$$\begin{cases}
    u(0, x) = u_0 \\
    u(t, 0) = u_b
\end{cases} \quad (15)$$

The solution $u = u(x/t)$ will have the structure

- waves of the $i > k$ families entering the domain;
- waves of the $k$-th family entering the domain;
- waves of the $k$-th family with speed 0;
- a characteristic boundary profile.

In $u(x/t)$ one sees only the first two points, the last two are in the jump at $x = 0$. 

27
Starting from $u_0$, we construct the map $\Phi: (s_1, \ldots, s_n) \mapsto \mathbb{R}^n$. 
Starting from $u_0$, we construct the map $\Phi: (s_1, \ldots, s_n) \mapsto \mathbb{R}^n$
From $u_0$ to $u_1$, waves of the $i > k$ family,
From $u_1$ to $u_2$, waves of the $k$-th family with $\sigma_k \geq 0$, \\

waves of the $k+1,...,n$-th families \\

waves of the $k$-th family \\

$u_b$
From $u_2$ to $u_b$ there is a char. bdry profile,
By means of system (11), we decompose the bdry profile as
Exponentially decaying part of bdry profile
Exponentially decaying part of bdry profile

This solves

\[
\begin{align*}
    u_b, x &= \hat{R}_b(u_b + u_k(x), p_b, p_k(x))p_b \\
    p_b, x &= \hat{A}_b(u_b + u_b(x), p_b, p_k(x))p_b
\end{align*}
\]  \tag{16}
Exponentially decaying part of bdry profile

This solves

\[
\begin{align*}
    u_b, x &= \tilde{R}_b(u_b + u_k(x), p_b, p_k(x))p_b \\
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\end{align*}
\] (16)

Since $\tilde{A}_b$ strictly negative, then
Exponentially decaying part of bdry profile

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\begin{align*}
  u_{b,x} &= \tilde{R}_b(u_b + u_k(x), p_b, p_k(x))p_b \\
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\end{align*}
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(16)

Since \(\tilde{A}_b\) strictly negative, then

\[p_b(x) = \mathcal{O}(1)p_b(0)e^{-cx},\]
Exponentially decaying part of bdry profile

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\[
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\end{align*}
\] (16)

Since \( \tilde{A}_b \) strictly negative, then

\[
p_b(x) = \mathcal{O}(1)p_b(0)e^{-cx},
\]

\[
u_s(x) = u_s(0) + \int_0^x \tilde{R}_b(y; u_k, p_k)p_b(y; u_k, p_k)dy.
\]

By contraction principle (small data), we can verify that
Exponentially decaying part of bdry profile

This solves

\[
\begin{align*}
    u_{b,x} &= \tilde{R}_b(u_b + u_k(x), p_b, p_k(x)) p_b \\
    p_{b,x} &= \tilde{A}_b(u_b + u_b(x), p_b, p_k(x)) p_b
\end{align*}
\]

Since $\tilde{A}_b$ strictly negative, then

\[p_b(x) = O(1)p_b(0)e^{-cx},\]

\[u_s(x) = u_s(0) + \int_0^x \tilde{R}_b(y; u_k, p_k)p_b(y; u_k, p_k)dy.\]

By contraction principle (small data), we can verify that

the manifold of solutions converging to 0 as $x \to \infty$ is $k - 1$ dimensional parameterized by $(u_1(0), \ldots, u_{k-1}(0))$, smoothly dependent on $u_k, p_k$.\]
The characteristic part of bdry profile
The characteristic part of bdry profile

The system for \( u_k, p_k \) and \( \sigma_k \) is

\[
\begin{align*}
  u_k(s) &= u_1 + \int_0^s \tilde{r}_k(u_b + u_k, p_b, p_k, \sigma_k) d\tau \\
  p_k(s) &= \text{b-conc}_{[0,s_k]} \left( \int_0^s \tilde{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k) d\tau \right)(s) \\
  \sigma_k &= \frac{d}{ds} \text{b-conc}_{[0,s_k]} \left( \int_0^s \tilde{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k) d\tau \right)(s)
\end{align*}
\]
The characteristic part of bdry profile

The system for $u_k$, $p_k$, and $\sigma_k$ is

\[
\begin{align*}
u_k(s) &= u_1 + \int_0^s \hat{r}_k(u_b + u_k, p_b, p_k, \sigma_k) d\tau \\
p_k(s) &= b-\text{conc}_{[0,s_k]}\left(\int_0^s \hat{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k) d\tau\right)(s) \\
\sigma_k &= \frac{d}{ds}b-\text{conc}_{[0,s_k]}\left(\int_0^s \hat{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k) d\tau\right)(s)
\end{align*}
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The function $\hat{f}_k = \int \hat{\lambda}_k d\tau$ is
The characteristic part of bdry profile

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\end{align*}
$$

The concave hull for Riemann problem is
The characteristic part of bdry profile

The system for $u_k$, $p_k$ and $\sigma_k$ is

\[
\begin{align*}
    u_k(s) &= u_1 + \int_0^s \tilde{r}_k(u_b + u_k, p_b, p_k, \sigma_k) \, d\tau \\
    p_k(s) &= b \text{-conc}_{[0,s_k]} \left( \int_0^s \tilde{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k) \, d\tau \right)(s) \\
    \sigma_k &= \frac{d}{ds} b \text{-conc}_{[0,s_k]} \left( \int_0^s \tilde{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k) \, d\tau \right)(s)
\end{align*}
\]

The boundary concave hull for Riemann problem is
The characteristic part of bdry profile

The system for $u_k$, $p_k$ and $\sigma_k$ is

$$
\begin{align*}
    u_k(s) &= u_1 + \int_0^s \tilde{r}_k(u_b + u_k, p_b, p_k, \sigma_k) \, d\tau \\
    p_k(s) &= b\text{-conc}_{[0,s_k]}(\int_0^s \tilde{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k) \, d\tau)(s) \\
    \sigma_k &= \frac{d}{ds} b\text{-conc}_{[0,s_k]}(\int_0^s \tilde{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k) \, d\tau)(s)
\end{align*}
$$

With the exponentially decaying (in space) perturbation $u_b, p_b$
The characteristic part of bdry profile

The system for $u_k$, $p_k$ and $\sigma_k$ is

$$\begin{cases}
  u_k(s) = u_1 + \int_0^s \tilde{r}_k(u_b + u_k, p_b, p_k, \sigma_k) d\tau \\
  p_k(s) = b\text{-conc}_{[0,s_k]}(\int_0^s \tilde{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k) d\tau)(s) \\
  \sigma_k = \frac{d}{ds} b\text{-conc}_{[0,s_k]}(\int_0^s \tilde{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k) d\tau)(s)
\end{cases}$$

With the exponentially decaying (in space) perturbation $u_b$, $p_b$

the structure of $\tilde{f}_k$ remains essentially the same,
The characteristic part of bdry profile

The system for $u_k$, $p_k$ and $\sigma_k$ is

\[
\begin{aligned}
    u_k(s) &= u_1 + \int_0^s \hat{r}_k(u_b + u_k, p_b, p_k, \sigma_k) \, d\tau \\
    p_k(s) &= \text{b-conc}_{[0,s_k]} \left( \int_0^s \hat{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k) \, d\tau \right)(s) \\
    \sigma_k &= \frac{d}{ds} \text{b-conc}_{[0,s_k]} \left( \int_0^s \hat{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k) \, d\tau \right)(s)
\end{aligned}
\]

With the exponentially decaying (in space) perturbation $u_b$, $p_b$

the structure of $\hat{f}_k$ remains essentially the same,

because the uniform exponentially decaying estimate on $u_k$, $p_k$
yields
The characteristic part of bdry profile

The system for $u_k$, $p_k$ and $\sigma_k$ is

\[
\begin{aligned}
    u_k(s) &= u_1 + \int_0^s \tilde{r}_k(u_b + u_k, p_b, p_k, \sigma_k) d\tau \\
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    \sigma_k &= \frac{d}{ds}\text{b-conc}_{[0,s_k]}\left(\int_0^s \tilde{\lambda}_k(u_b + u_k, p_b, p_k, \sigma_k) d\tau\right)(s)
\end{aligned}
\]

With the exponentially decaying (in space) perturbation $u_b$, $p_b$ the structure of $\hat{f}_k$ remains essentially the same, because the uniform exponentially decaying estimate on $u_k$, $p_k$ yields

\[
\left| \hat{f}_k(s; u_k = 0, p_k = 0) - \hat{f}_k(s; u_k, p_k) \right| \leq \frac{1}{2}(b-\text{conc}\hat{f}_k - \hat{f}_k)(s; u_k = 0, p_k = 0).
\]
Final Remark. By studying the unperturbed $k$-th field we recover the structure of the boundary profile, hence the bdry RP.
**Final Remark.** By studying the unperturbed $k$-th field we recover the structure of the boundary profile, hence the bdry RP.