SBV REGULARITY OF GENUINELY NONLINEAR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS IN ONE SPACE DIMENSION

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Abstract. The problem of the presence of Cantor part in the derivative of a solution to a hyperbolic system of conservation laws is considered. An overview of the techniques involved in the proof is given, and a collection of related problems concludes the paper.

1. Introduction

Consider a strictly hyperbolic system of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R},\ u \in \mathbb{R}^n, \tag{1.1}$$

It is now a classical result that if the initial data

$$u(0,x) = u_0(x)$$

has a small BV norm, then the solution remains in BV for all $t > 0$. For a proof, one can use different methods: Glimm scheme [18, 3], wavefront tracking [2], vanishing viscosity [7] or other singular limits methods ([6, 5] for example).

For special systems, L^∞-solutions can be constructed, by means of uniform stability estimates [4], compensated compacness [17] or uniform decay estimates [19, 24].

All these results can be see as regularity properties of solutions, yielding some compacness in $L^\infty(\mathbb{R})$. It is important to notice that continuous solutions in general do not exists, as it is taught at every basic PDE course.

Other kinds of regularity can be considered. We here give a short list.

1.1. Decay of positive waves.

In the case $n = 1$, i.e. of a scalar conservation laws, Oleinik proved that the solution satisfies the one-sided Lipschitz bound

$$u(t,x+h) - u(t,x) \leq \frac{h}{\kappa t} \tag{1.2}$$

where $f''(u) \geq \kappa > 0$ is the uniform convexity of f [22]. In particular u is locally BV.

A generalization of the above condition is given in [15]: the positive part of the i-th component v_i of $\partial_x u$ satisfies

$$v^+(T,A) \leq \frac{1}{c_0} \frac{\mathcal{L}^1(A)}{T-t} + C_0\big(Q(t) - Q(T)\big),$$

where Q is the Glimm interaction functional.

We will study this regularity more deeply later on, since it is strictly related to the SBV regularity.

1.2. Differentiability along characteristics.

In the uniformly convex scalar case, since

$$x \mapsto -\lambda(u(t,x))$$

is a quasi-monotone vector field by (1.2), one can consider the unique Filippov solution to the differential inclusion

$$\dot{x} \in \big[-\lambda(u(t,x+), -\lambda(u(t,x-)\big].$$

The solutions to this inclusion outside the jump set of u are called characteristics curves.

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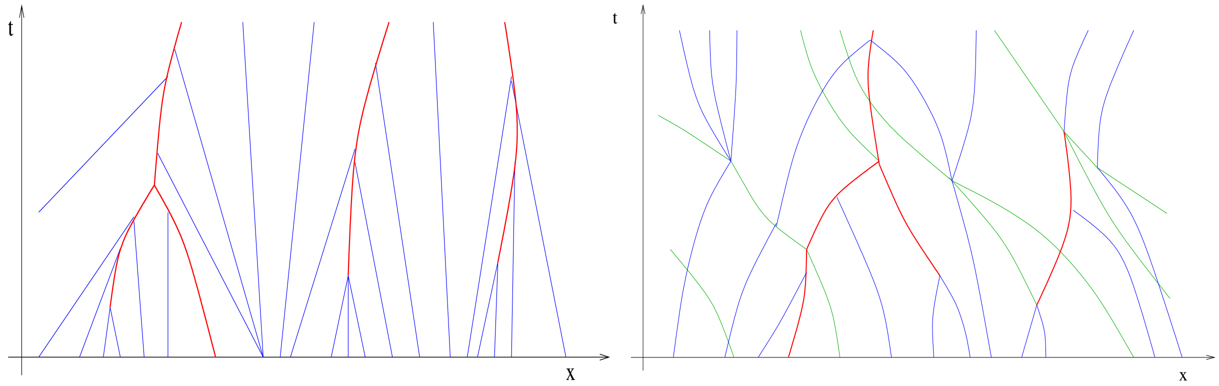

FIGURE 1. As the characteristics curves and the jump set fro the solution of a scalar uniformly convex conservation law are usually presented (left), and the characteristics and wave pattern for the system case (right).

As for C^1 solutions one can then prove that the solution is constant along the characteristics, i.e. if $\gamma(t)$ is a characteristic then $t \mapsto u(t,\gamma(t))$ is constant, and thus γ is a segment: these properties are easy to verify in the case $u \in C^1$.

It is thus possible to ask if the same conditions holds for solutions to scalar balance laws

$$u_t + f(u)_x = g(t,x,u),$$

where one expects that the following holds:

$$\frac{d}{dt} u(t,\gamma(t)) = g(t,\gamma(t),u(t,\gamma(t))).$$

In general this is not true, but it is known to holds for convex f [16]. The vector case of this result is still completely open.

1.3. **Differentiability properties of L^∞-solutions.** For L^∞-solutions to conservation laws where no BV estimates can be proved, the structure of the solution is in general not clear: for example, solutions in more than one dimension, or non convex scalar equations. It is possible however to prove that the nonlinearity of the flux f implies that some sort of BV structure survives: there is a rectifiable jump set, where left and right limits of the solution exists, and outside this set the solution has vanishing mean oscillation [20].

The proof of similar results for systems is an open problem.

1.4. **Fractional differentiability.** By means of the kinetic representation, it is possible to prove that the solution belongs to a compact space in L^1, in particular [21].

1.5. **SBV regularity.** For solutions of strictly hyperbolic systems of conservation laws in one space dimension one expects the following structure: countably many shock curves and regularity of the solution in the remaining set. In the system case, however, the structure is much more complicated, due to the presence of waves of the other families: indeed, the characteristic curves are not straight lines any more, and the interaction among waves complicates the wave pattern (see Fig. 1).

One way of interpreting this structure is to say that *the solution u has a rectifiable jump part, and in the remaining set the derivative of u is absolutely continuous.* This means that in the decomposition of $\partial_x u$ as a derivative of a BV function, the Cantor part of the derivative is 0. This fact has been verified in the scalar case in [1], while in the vector case it has been proved in [25].

All the fundamental ideas can be understood in the scalar case:

$$u_t + f(u)_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \; u \in \mathbb{R},$$

so we will restrict to this case in this paper.

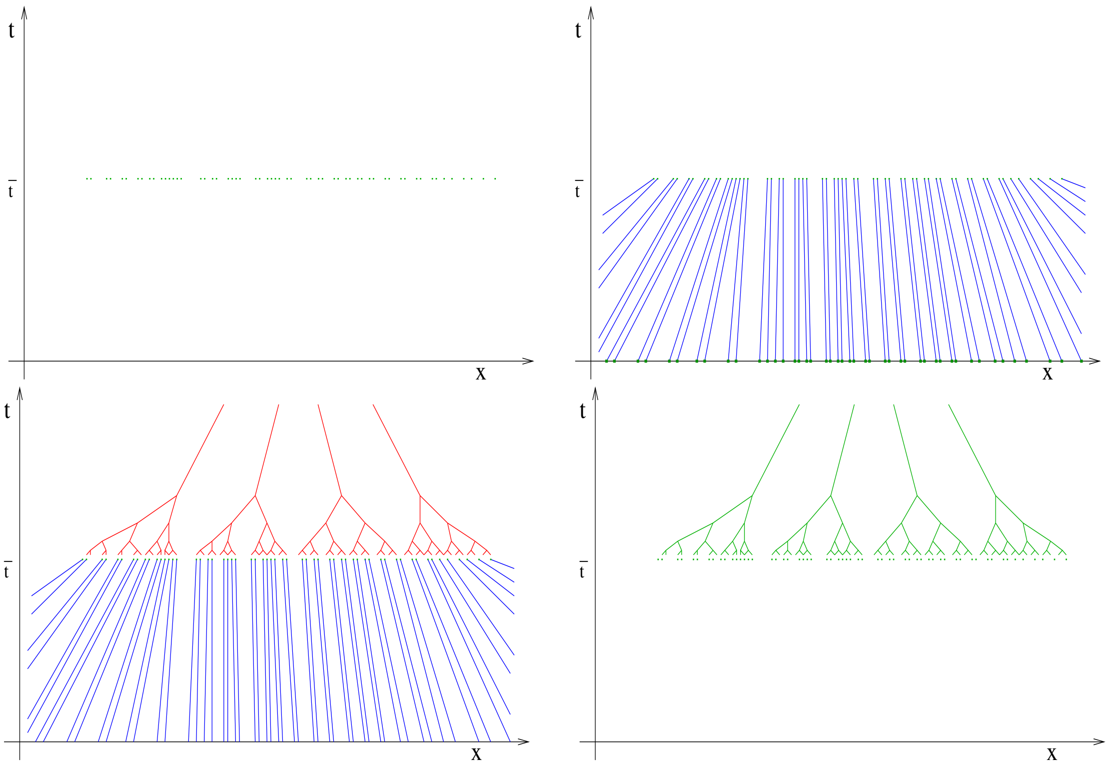

FIGURE 2. The analysis of SBV regularity in the scalar case, and where the measure μ defined in (2.1) is concentrated.

2. Proof of SBV regularity in the scalar case

The interpretation of Fig. 1 can be interpreted as

- shocks are concentrated on countably many Lipschitz curves (with first derivative in BV),
- decay of positive and negative waves as t^{-1},
- no other terms in the derivative, i.e. no Cantorian part.

The idea of the proof in the scalar case given in [1] is as follows, see Fig. 2.

Let $\bar{t}$ be a time where the spatial derivative of $u(t)$ has a Cantor part concentrated on the $\mathcal{L}^1$-negligible set C. Then since $u(\bar{t})\llcorner_C$ is continuous, for each $\bar{x} \in C$ there exists only one characteristics starting at $t = 0$ and arriving at $(\bar{t}, \bar{x})$. Then we can consider the set of initial points $C(0)$ of C.

Since the slopes of the characteristics are related to u by the function $\lambda(u) = f'(u)$, then we have that the opening is of $\geq \kappa |\partial_x u|(C)$, where $\kappa \leq f''(u)$ is the constant of uniform convexity. In particular, the $\mathcal{L}^1$-measure of $C(0)$ is $\geq \kappa \bar{t} |\partial_x u|(C)$.

Using the fact that characteristics do not intersect outside the end points, one can prove that if A is Borel and the characteristics starting from A arrives at time t, then for all $0 < s < t$ it holds

$$\mathcal{L}^1 \Big\{ \gamma(s), \gamma(0) \in A, \gamma \text{ characteristic} \Big\} \geq \left(1 - \frac{s}{t}\right) \mathcal{L}^1(A).$$

Hence if the characteristics arriving in C at $\bar{t}$ can be prolonged, then C has positive measure, since $\mathcal{L}^1(C(0)) > 0$.

It thus follows that if we define the functional

$$H(t,R) := \mathcal{L}^1 \Big\{ x \in B(0,R) : \text{the characteristic leaving } x \text{ can be prolonged up to } t \Big\},$$

then this functional is decreasing (since in the scalar case the characteristic equation has forward uniqueness), and has a downward jump at $\bar{t}$.

We conclude that the number of times where a Cantor part in the derivative $\partial_x u$ appears is countable. Then as a function of two variable, $\partial_x u$ is SBV, and using the equation $u_t = -f(u)_x$ also $\partial_t u$ is SBV.

2.1. **A reformulation of the above proof.** Since $x \mapsto -f'(u(t,x))$ is a quasi-monotone operator, it follows that the ODI

$$\dot{x} \in -f'(u(t,x))$$

generates a unique Lipschitz semigroup $X(t,x)$ [13, 8]. In particular we can consider the transport solution of

$$\rho_t + (f'(u(t))\rho)_x = 0, \quad \rho(0) = \mathcal{L}^1,$$

which can be represented as $X(t)_\sharp \mathcal{L}^1$, i.e. the Jacobian of $X^{-1}(t)$.

If we split $\rho(t) = \rho^c(t) + \rho^a(t)$, ρ^a atomic part, then

$$\rho^c + (f'(u)\rho^c)_x = -\mu, \quad \rho^a + (f'(u)\rho^a)_x = \mu, \tag{2.1}$$

where μ is a distribution. Using the fact that the atomic part of ρ can only increase (because of monotonicity), then μ is a positive Radon measure.

The previous proof shows that if a Cantor part appears in ρ^c, then

$$\mu(\{t\} \times A) \geq \rho^{\text{cantor}}(A),$$

and the local boundedness of μ allows to conclude as in the previous proof. In this model case the measure μ is concentrated on the Cantor set and in the jump set.

2.2. **The equation for $\partial_x u$.** The measure $v := \partial_x u(t)$ satisfies the same transport equation in conservation form

$$v_t + (f'(u(t))v)_x = 0, \quad v(0) = D_x u(0),$$

but since it has a sign the equations for its atomic and non atomic part are a little more complicated. In fact cancellation among negative and positive waves should be considered.

By using the wavefront tracking approximation, one can prove that if $v = v^c + v^a$, v^a atomic part of v, then

$$v^c_t + (f'(u(t))v^c)_x = -\mu^{CJ}, \quad v^a_t + (f'(u(t))v^a)_x = \mu^{CJ},$$

with μ^{CJ} signed locally bounded measure such that

$$\mu^{CJ} - \big\{\text{measure of cancellation of waves}\big\} \leq 0.$$

Summing up, we have 3 equations

$$\begin{aligned} v_t + (f'(u(t))v)_x &= 0 \\ |v|_t + (f'(u(t))|v|)_x &= -\mu^C \leq 0, \\ v^a_t + (f'(u(t))v^a)_x &= \frac{1}{2}\mu^C + \mu^J, \end{aligned}$$

with $\mu^J \leq 0$. The proof of SBV regularity can be thus restated as

$$\mu^J(\{t\} \times A) \leq v^{\text{cantor}}(t, A).$$

2.3. **Decay estimates.** We have seen that for convex conservation laws the decay of positive waves reads as

$$v(t,A) \leq \frac{1}{c_0}\frac{\mathcal{L}^1(A)}{t}, \quad f'' \geq c_0.$$

The measure μ^J allows to obtain the corresponding decay estimate for the negative part v^c:

$$v^c(T,A) \geq -\frac{1}{c_0}\frac{\mathcal{L}^1(A)}{t-T} + \mu^J\big(\text{domain of influence of A}\big).$$

In fact, the measure μ^J controls exactly the points where the characteristics collide and generate jumps. Observe that for the positive waves in convex scalar conservation laws no new centered rarefaction waves are created, and that for the system case the decay estimate has a form very similar to the one above.

Using now the fact that $u(t)$ is absolutely continuous outside the jump part, one can write the equation for v^c along each ray γ:

$$v^c_t + (f'(u(t))v^c)_x = 0, \quad \frac{d}{dt}v^c(t,\gamma(t)) = -f''(u)(v^c)^2.$$

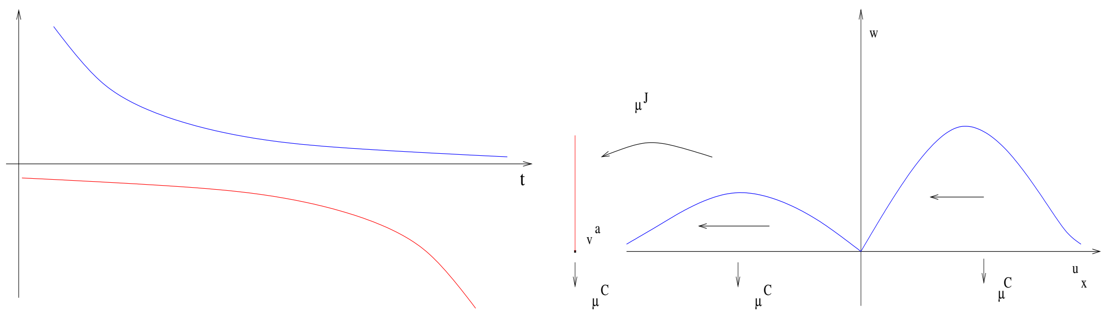

FIGURE 3. The decay estimate along a characteristic (left) and the dynamic interpretation of the scalar conservation law (right).

This yields that if the ray $\gamma(t)$ has a life span of $[0,T]$, then

$$-\frac{1}{c_0}\frac{1}{T-t} \leq v^c(t,\gamma(t)) \leq \frac{1}{c_0}\frac{1}{t}.$$

2.3.1. *Dynamical interpretation.* We can thus give the following dynamic representation of the evolution of the derivative $D_x u$.

If we consider the measures

$$\omega^c(t) := v^c_\sharp(v^c\mathcal{L}^1), \quad \omega^a(t) := v^a(t,\mathbb{R}^1)$$

then it follows that

$$\omega^c_t + y^2\omega^c = -\tilde{\mu}, \quad \omega^a_t = \tilde{\mu},$$

with (formally)

$$\tilde{\mu} = v(t)_\sharp\left(\frac{1}{2}\mu^C + \mu^J\right).$$

We can thus give the dynamic representation of the evolution of the derivative $D_x u$ of Fig. 3.

3. SBV estimates for systems

We now review the main idea in the system case.

3.1. **Decomposition into wave measures.** We consider the hyperbolic system

$$u_t + f(u)_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \ u \in \mathbb{R}^n,$$

and we assume that the $\bar{\imath}$-eigenvalue λ_i of $Df(u)$ is g.n.l.: by choosing the direction of the unit eigenvector $r_{\bar{\imath}}$,

$$D\lambda_{\bar{\imath}}(u)r_{\bar{\imath}}(u) \leq c_0 < 0.$$

We moreover decompose the derivative of the solution as [14]

$$u_x(t) = \sum v_i(t)\tilde{r}_i,$$

with $\tilde{r}_i = r_i$ where u is continuous, otherwise is the direction of the jump of the i-th family. Each $v_i(t)$ is a bounded measure.

Our aim is to prove that $v_{\bar{\imath}}(t)$ has a Cantor part only at countably many times. In general the situation is more complicated than in the scalar case, due to the presence and the interaction of the waves of different families.

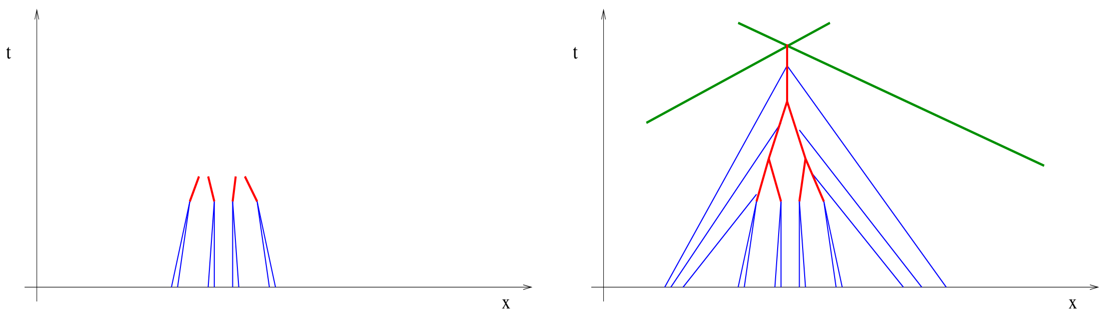

FIGURE 4. Possible evolution of jumps created by a Cantor part.

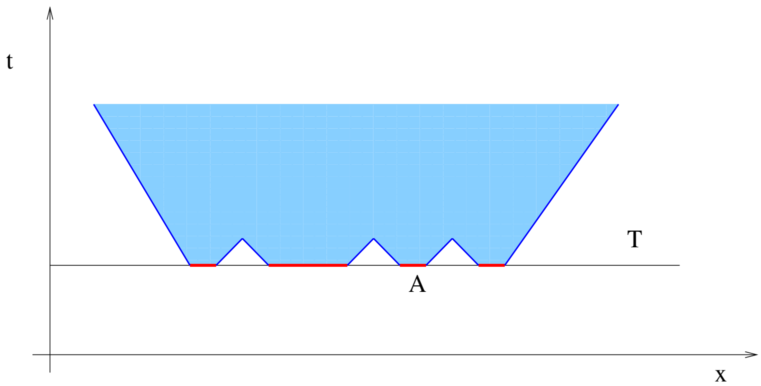

FIGURE 5. Domain of influence of A.

3.2. **Equation for wave measures.** Let $\tilde{\lambda}_i$ be the i-th eigenvector if u is continuous or the speed of the i-th shock. By the wavefront approximation, one obtain the following balance equation

- conservation of v_i:
$$(v_i)_t + (\tilde{\lambda}_i v_i)_x = \mu_i^I$$
where μ_i^I is a signed measure bounded by the decrease of the interaction potential $Q(u)$;
- conservation of $|v_i|$:
$$(|v_i|)_t + (\tilde{\lambda}_i |v_i|)_x = \mu_i^{IC}$$
where μ_i^{IC} is a signed measure bounded by the decrease of the potential $\text{Tot.Var.}(u) + CQ(u)$.

3.2.1. *Equation for the atomic part.* If $\bar{i}$ is genuinely nonlinear, the equation for the atomic part $v_{\bar{i}}^a$ is
$$(v_{\bar{i}}^a)_t + (\tilde{\lambda}_{\bar{i}} v_{\bar{i}}^a)_x = \mu_{\bar{i}}^{ICJ},$$
where $\mu_{\bar{i}}^{ICJ}$ is a distribution satisfying
$$\mu^J := \mu_{\bar{i}}^{ICJ} - |\mu_{\bar{i}}^I| - |\mu_{\bar{i}}^{IC}| \leq 0.$$
Hence $\mu_{\bar{i}}^J$ is a bounded measure (*jump measure*), which measures the amount of jumps created.

The fact that μ^J is a measure (signed distribution) follows from the fact that it is easy to create a jump because of nonlinearity, but to cancel it you have to use cancellation or interaction, see Fig. 4.

3.3. **Proof of SBV regularity.** The continuous part $v_{\bar{i}}^c$ of $v_{\bar{i}}$ thus satisfies
$$(v_{\bar{i}}^c)_t + (\lambda_{\bar{i}} v_{\bar{i}}^c)_x = \mu_{\bar{i}}^c, \quad \mu_{\bar{i}}^c := \mu_{\bar{i}}^I - \mu_{\bar{i}}^{ICJ}.$$
As argument similar to the estimate of the decay of positive waves yields now
$$v_{\bar{i}}^c(T, A) \geq -\frac{1}{c_0} \frac{\mathcal{L}^1(A)}{t - T} - |\mu_{\bar{i}}^c| \Big(\text{Domain of influence of } A, \text{ Fig. } 5 \Big).$$

In particular, if A is a set of measure 0 where the Cantor part is concentrated, then by taking a sequence $t_n \searrow T$ we obtain

$$|\mu_i^c|(A) > 0.$$

Since μ_i^c is a bounded measure, then the set of times where a Cantor part appears is countable. These times corresponds to:

(1) strong interactions among waves;
(2) generation of shock with the same strength of the Cantor part.

4. Final remarks on some related cases

The SBV regularity can be proved for other kind of systems or equations. Here we list some interesting cases.

- SBV regularity for fluxes with countably many inflection points [23], or SBV regularity for $v_i(D\lambda_i r_i)$ [12]
- SBV regularity for u solutions to HJ equation
$$u_t + H(t, x, \nabla u) = 0$$
with uniformly convex Hamiltonian [9, 11] or with simple degeneracies [10]
- SBV regularity for Temple class systems with source terms

A very interesting open problem is the presence of Cantor part in the measure div$d$, where d is the direction of the optimal ray for the solution

$$u_t + H(\nabla u) = 0,$$

with H only smooth, convex. Some advances have been obtained in [10].

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