GLOBAL STRUCTURE OF ADMISSIBLE BV SOLUTIONS TO PIECEWISE GENUINELY NONLINEAR, STRICTLY HYPERBOLIC CONSERVATION LAWS IN ONE SPACE DIMENSION

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Abstract. The paper describes the qualitative structure of an admissible BV solution to a strictly hyperbolic system of conservation laws whose characteristic families are piecewise genuinely nonlinear. More precisely, we prove that there are a countable set of points Θ and a countable family of Lipschitz curves $\mathscr{T}$ such that outside $\mathscr{T} \cup \Theta$ the solution is continuous, and for all points in $\mathscr{T} \setminus \Theta$ the solution has left and right limit. This extends the corresponding structural result in [7] for genuinely nonlinear systems.

An application of this result is the stability of the wave structure of solution w.r.t. L^1_{loc}-convergence.

The proof is based on the introduction of subdiscontinuities of a shock, whose behavior is qualitatively analogous to the discontinuities of the solution to genuinely nonlinear systems.

1. Introduction

This paper is concerned with the qualitative structure of admissible solutions to the strictly hyperbolic $N \times N$ system of conservation laws in one space dimension of the form

$$\begin{cases} u_t + f(u)_x = 0 & u : \mathbb{R}^+ \times \mathbb{R} \to \Omega \subset \mathbb{R}^N,\ f \in C^2(\Omega, \mathbb{R}), \\ u_{|t=0} = u_0 & u_0 \in \mathrm{BV}(\mathbb{R}; \Omega). \end{cases} \tag{1.1}$$

We assume that the system is strictly hyperbolic in $\Omega \subset \mathbb{R}^N$: this means that the eigenvalues $\{\lambda_i(u)\}_{i=1}^N$ of the Jacobian matrix $A(u) := Df(u)$ satisfy

$$\lambda_1(u) < \cdots < \lambda_N(u), \qquad u \in \Omega.$$

Furthermore, as we only consider the solutions with small total variation and thus they live in a neighborhood of a point, it is not restrictive to assume that Ω is bounded and there exist constants $\{\check{\lambda}_i\}_{i=0}^N$, such that

$$\check{\lambda}_{i-1} < \lambda_i(u) < \check{\lambda}_i, \qquad \forall u \in \Omega,\ i = 1, \dots, N. \tag{1.2}$$

Let $\{r_i(u)\}_{i=1}^N$ and $\{l_j(u)\}_{j=1}^N$ be a basis of right and left eigenvectors, depending smoothly on u, such that

$$l_j(u) \cdot r_i(u) = \delta_{ij} \text{ and } |r_i(u)| \equiv 1, \quad i, j = 1, \dots, N.$$

The integral curve of the vector field $r_i(u)$ with initial datum u_0

$$\frac{du}{d\omega} = r_i(u(\omega)), \quad u(0) = u_0.$$

will be denoted by $R_i[u_0](\omega)$, and it is called the *i-th rarefaction curve* through u_0.

The Rankine-Hugoniot condition

$$f(u_1) - f(u_0) = \sigma(u_1 - u_0) \quad \text{if} \quad u(t,x) = u_0 + (u_1 - u_0)\chi_{x \geq \sigma t} \quad \text{is a weak solution,}$$

generates N distinct smooth curves $S_i[u_0]$ starting from any $u_0 \in \Omega$ and N smooth functions $\sigma_i[u_0]$ such that

$$\sigma_i[u_0](s)[S_i[u_0](s) - u_0] = f(S_i[u_0](s)) - f(u_0),$$

and moreover

$$S_i[u_0](0) = u_0, \qquad \sigma_i[u_0](0) = \lambda_i(u_0), \qquad \frac{d}{ds} S_i[u_0](0) = r_i(u_0).$$

The curve $S_i[u_0]$ is called *the i-th Hugoniot curve* issuing from u_0; we will also say that $[u_0, u_1]$ is an *i-th discontinuity* with speed $\sigma_i(u_0, u_1)$.

We are now ready to introduce the definition of piecewise genuinely nonlinear systems.

Key words and phrases. Hyperbolic conservation laws, Wave-front tracking, Global structure of solution.

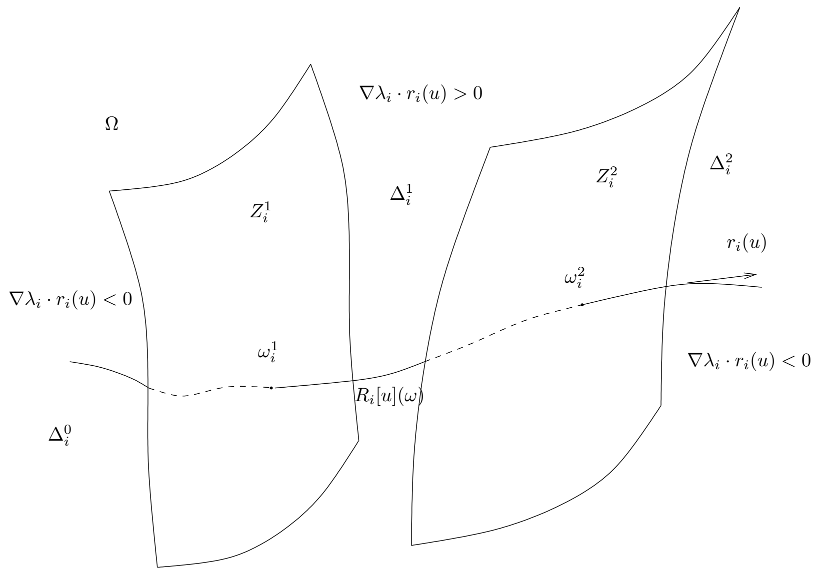

FIGURE 1.

Definition 1.1. We say that i-th characteristic field of the system (1.1) is *piecewise genuinely nonlinear* if the set $Z_i := \{u : \nabla\lambda_i \cdot r_i(u) = 0\}$ is the union of $(N-1)$-dimensional distinct manifolds Z_i^j, $j = 1, \dots, J_i$ transversal to the vector field $r_i(u)$ and such that each rarefaction curve $R_i[u_0]$ crosses all the Z_i^j.

This implies that along R_i, the function λ_i has J_i critical points (see Figure 1). Without loss of generality we can also assume that the points $\omega^j[u_0]$ given by

$$R_i[u_0](\omega^j[u_0]) \in Z_i^j,$$

are monotone increasing w.r.t. $j = 1, \dots, J_i$.

We will denote by $\Delta_i^j \subset \mathbb{R}^N$ the region between Z_i^j and Z_i^{j+1}:

$$\Delta_i^j := \big\{u \in \Omega : \ \omega^j[u] < 0 < \omega^{j+1}[u]\big\}, \quad \Delta_i^0 := \big\{u \in \Omega : \omega^1[u] > 0\big\}, \ \Delta_i^{J_i} := \big\{u \in \Omega : \omega^{J_i}[u] < 0\big\}. \tag{1.3}$$

Without any loss of generality (the analysis of the other case being completely similar), we set

$$\nabla\lambda_i \cdot r_i(u) < 0 \ \text{ if } j \text{ is even}, \ u \in \Delta_i^j,$$

$$\nabla\lambda_i \cdot r_i(u) > 0 \ \text{ if } j \text{ is odd}, \ u \in \Delta_i^j.$$

In what follow, we assume that *each characteristic field of* (1.1) *is piecewise genuinely nonlinear*. We will thus call the hyperbolic system *piecewise genuinely nonlinear*.

Remark 1.2. From the above definitions it follows that we do not allow characteristic families to be linearly degenerate. Thus our assumptions are slightly stricter than the natural extension of the setting of [5], where the families are either genuinely nonlinear or linearly degenerate.

It is however immediate to verify that the proof of regularity for linearly degenerate characteristic families does not depends on the properties of the remaining families, so that the results which we state in this paper are valid also if some family is linearly degenerate. In fact, the regularity results we state are valid for a piecewise genuinely nonlinear family i, even if the system is not piecewise genuinely nonlinear.

Let $[u^-, u^+]$, $u^+ = S_i[u^-](s)$, be an admissible i-discontinuity. For us, this means that it is the limit of the vanishing viscosity approximation, and it can be shown to be equivalent to the following stability condition (used in [8]):

$$\forall\, 0 \le |\tau| \le s \ \Big(\sigma_i[u^-](\tau) \ge \sigma_i(u^-, u^+)\Big).$$

The construction of these admissible discontinuities will be presented in Section 2, as a consequence of the construction of the solution to a Riemann problem.

Following the notation of [9], we call the jump $[u^-, u^+]$ *simple* if

$$\forall\, |\tau| \in]0, |s|[\ \Big(\sigma_i[u^-](\tau) > \sigma_i(u^-, u^+)\Big).$$

If $[u^-, u^+]$ is not simple, then we call it a *composition of the waves* $[u_0, u_1], [u_1, u_2], \cdots, [u_\ell, u_{\ell+1}]$ with $u_0 = u^1$ and $u_{\ell+1} = u^+$, if

$$u_k = S_i[u^-](s_k) \quad \text{and} \quad \sigma_i(u_{k-1}, u_k) = \sigma_i(u^-, u^+),\ k = 1, \dots, \ell+1, \tag{1.5}$$

where

$$0 = s_0 < s_1 < s_2 < \cdots < s_\ell < s \quad (\text{or } s < s_\ell < \cdots < s_1 < s_0 = 0),$$

and there are no other points τ such that (1.5) holds. (For general f, it may happens that the set where $\sigma(u, u^-) = \sigma(u^+, u^-)$ is not finite, but this does not happen for piecewise genuinely nonlinear systems, as it will be shown as a consequence of Lemma 4.3).

In [9], under assumption of piecewise genuinely nonlinearity, by using Glimm scheme it is proved that if the initial data has small total variation, there exists a weak admissible BV solution of (1.1). Therefore, this solutions enjoys the usual regularity properties of BV function: u either is approximately continuous or has an approximate jump at each point $(x, t) \in \mathbb{R}^+ \times \mathbb{R} \setminus \mathcal{N}$, where $\mathcal{N}$ is a subset whose one-dimensional Hausdorff measure $\mathcal{H}^1$ is zero. In the same paper, the author shows much stronger regularity that u holds. The set $\mathcal{N}$ contains at most countably many points, and u is continuous (not just approximate continuous) outside $\mathcal{N}$ and countably many Lipschitz continuous curves.

In [7], the authors adopt wave-front tracking approximation to prove the similar result for (1.1) with the assumption that each characteristic field is genuinely nonlinear. Moreover, the authors were able to prove that outside the countable set Θ there exist right and left limits u^-, u^+ on the jump curves, and these limits are stable w.r.t. wavefront approximate solutions: more precisely, for each jump point $(\bar{t}, \bar{x})$ not belonging to the countable set Θ (the points where a strong interaction occurs, see the definition at page 15), there exists a shock curve $x = y_\epsilon(t)$ for the approximate solution u_ϵ converging to it and such that its left and right limit converge to u^-, and u^+ uniformly. This means that

$$\lim_{r\to 0+}\Bigg(\limsup_{\epsilon\to\infty}\ \sup_{\substack{x<y_\epsilon(t)\\(x,t)\in B(P,r)}} \big|u_\epsilon(x,t) - u^-\big|\Bigg) = 0, \qquad \lim_{r\to 0+}\Bigg(\limsup_{\epsilon\to\infty}\ \sup_{\substack{x>y_\epsilon(t)\\(x,t)\in B(P,r)}} \big|u_\epsilon(x,t) - u^+\big|\Bigg) = 0.$$

In [5] (Theorem 10.4), the author generalizes this result to the case when some characteristic field may be linearly degenerate.

In our setting, in order to prove this additional regularity estimates on shocks, some additional difficulties arise: in fact the proof in [5] is based on the wave structure of the solution to genuinely nonlinear or linearly degenerate systems, where only one shock curve passes through the discontinuous point (which is not a point where a strong interaction occurs, i.e. not in Θ). In our case, instead, it may happen that the shock is composed by several waves as in (1.5), and these waves separate even if the point does not belong to the countable Θ.

For example, consider a scalar equation where f has two inflection points. It is thus clearly piecewise genuinely nonlinear (see Figure 2). Let u_0 be the initial data

$$u_0 = \begin{cases} u_1 & \text{if } x < x_1, \\ u_2 & \text{if } x_1 < x < x_2, \\ u_3 & \text{if } x_2 < x < x_3, \\ u_4 & \text{if } x > x_3. \end{cases}$$

By carefully choosing f and the points x_1, x_2, x_3 and the value $u_1, \dots, u_4$, one can obtain the wave pattern shown in Figure 3: the point where the two jumps meet is not a strong interaction point, however the waves join together.

In a similar way, one can construct examples where the shock splits, even without a strong interaction. Clearly such wave pattern can not be reproduced if f is convex or concave.

In this paper we prove the following theorem:

Theorem 1.3. *Let u be an admissible BV solution of the Cauchy problem (1.1) with f piecewise genuinely nonlinear. Then there exist a countable set Θ of interaction points and a countable family $\mathscr{T}$ of Lipschitz continuous curves such that u is continuous outside Θ and $Graph(\mathscr{T})$.*

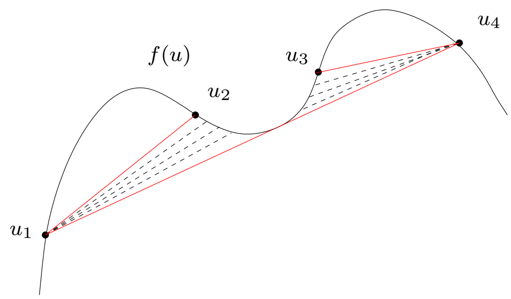

FIGURE 2.

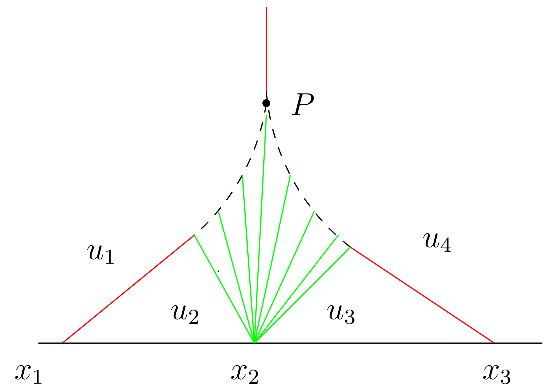

FIGURE 3.

Moreover, suppose $u(t_0, x)$ *is discontinuous at* $x = x_0$ *as a function of* x*, and* $(t_0, x_0) \notin \Theta$*. Write* $u^L = u(t_0, x_0-)$*,* $u^R = u(t_0, x_0+)$ *and suppose that* $u^{\mathrm{R}} = S_i[u^{\mathrm{L}}](s)$ *with* $s > 0$ $(s < 0)$*.*

- *If* $[u^{\mathrm{L}}, u^{\mathrm{R}}]$ *is simple, there exists a Lipschitz curve* $y(t) \in \mathscr{T}$*, s.t.* $y(t_0) = x_0$ *and*
$$u^{\mathrm{L}} = \lim_{\substack{x<y(t)\\(x,t)\to(t_0,x_0)}} u(x,t), \qquad u^{\mathrm{R}} = \lim_{\substack{x>y(t)\\(x,t)\to(t_0,x_0)}} u(x,t) \quad \text{and} \quad \dot y(t_0) = \sigma(u^{\mathrm{L}}, u^{\mathrm{R}}).$$
- *If* $[u^{\mathrm{L}}, u^{\mathrm{R}}]$ *is a composition of the waves* $[u^{\mathrm{L}}, u_1]$*,* $[u_1, u_2], \cdots, [u_\ell, u^{\mathrm{R}}]$*, then there exist* p *Lipschitz continuous curves* $y_1, \cdots, y_p \in \mathscr{T}$*,* $p \le \ell + 1$ *satisfying*
  - $y_1(t_0) = \cdots = y_p(t_0) = x_0$,
  - $y_1'(t_0) = \cdots = y_p'(t_0) = \sigma(u^{\mathrm{L}}, u^{\mathrm{R}})$,
  - $y_1(t) \le \cdots \le y_p(t)$ *for all* t *in a neighborhood of* t_0*.*

  Moreover,
$$u^{\mathrm{L}} = \lim_{\substack{x<y_1(t)\\(x,t)\to(t_0,x_0)}} u(x,t), \qquad u^{\mathrm{R}} = \lim_{\substack{x>y_p(t)\\(x,t)\to(t_0,x_0)}} u(x,t), \tag{1.6}$$
  and if in a small neighborhood of (t_0, x_0)*,* y_j *and* y_{j+1} *are not identical, one has*
$$u_j = \lim_{\substack{y_j(t)<x<y_{j+1}(t)\\(x,t)\to(t_0,x_0)}} u(x,t). \tag{1.7}$$

As in [5], the above result is based on the following convergence result for approximate wave-front solutions, which implies the stability of the wave pattern w.r.t. L^1_{loc}-convergence of solutions (see Remark 5.1).

Theorem 1.4. *Consider a sequence of wave-front tracking approximate solutions* u_ϵ *converging to* u *in* $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$*. Suppose* $P = (\tau, \xi)$ *is a discontinuity point of* u *and write* $u^L = u(\tau, \xi-)$*,* $u^R = u(\tau, \xi+)$*. Assume* $[u^{\mathrm{L}}, u^{\mathrm{R}}]$ *is the composition of* ℓ *waves, and let* $\mathscr{T} \ni y_j : [t_j^-, t_j^+] \to \mathbb{R}$*,* $j = 1, \cdots, \ell$*, be* ℓ *Lipschitz continuous curves (not necessarily distinct) passing through the point* P*, such that* u *is discontinuous across* y_j *and*
$$y_1(t) \le \cdots \le y_\ell(t) \qquad \text{in a small neighborhood of } \tau.$$
Then up to a subsequence, there exist $y_{j,\epsilon} : [t^-_{j,\epsilon}, t^+_{j,\epsilon}] \to \mathbb{R}$*,* $j = 1, \cdots, \ell$*, which are discontinuity curves of* u_ϵ *not necessarily distinct, such that* $t^-_{j,\epsilon} \to t^-_j$*,* $t^+_{j,\epsilon} \to t^+_j$ *and*
$$y_{j,\epsilon}(t) \to y_j(t) \qquad \text{for every } t \in [t_j^-, t_j^+].$$
Moreover, one has
$$\lim_{r\to 0+} \left(\limsup_{\epsilon\to\infty} \sup_{\substack{x<y_{1,\epsilon}(t)\\(x,t)\in B(P,r)}} \left| u_\epsilon(x,t) - u^L \right| \right) = 0,$$
$$\lim_{r\to 0+} \left(\limsup_{\epsilon\to\infty} \sup_{\substack{x>y_{\ell,\epsilon}(t)\\(x,t)\in B(P,r)}} \left| u_\epsilon(x,t) - u^R \right| \right) = 0.$$

Note that it is possible that the curve y_j coincide for all j, while the curves $y_{\epsilon,j}$ do not have any common point $\epsilon > 0$.

A brief outline of this paper follows.

In Section 2, we recall the construction of Riemann solvers introduced in [3]. The construction is based on the requirement that the self similar solution is the limit of the vanishing viscosity approximation, and this criterion implies the other well known stability criteria introduced in the literature. Moreover, only strict t is needed in this construction. The structure of the solution to the Riemann problem is needed in the next section to construct the wavefront approximations, and in Section 4 will be used to identify the subdiscontinuity curves.

In Section 3, we briefly describe the wave-front tracking approximate scheme for general strictly hyperbolic system, as presented in [1]. In particular, we introduce the definition of interaction and cancellation measures. Even if this part is well known in the literature, we reproduce the essential ideas for reader's convenience.

Section 4 contains the main idea of the paper: the definition of subdiscontinuity curves and (δ, k)-approximate subdiscontinuity curves in the approximate wavefront solution u_ϵ. In this section we show that their qualitative behavior resembles the shocks in genuinely nonlinear systems (in particular they cannot be split after an interaction), and thus these curves are suitable in order to extend the analysis of [5]. In particular, we prove that their number is uniformly bounded with respect to approximation parameter ϵ, a property particularly useful when passing to the limit of the approximation solution u_ϵ.

In Section 5, we give the proofs of Theorem 1.3 and Theorem 1.4, by proving that the approximate subdiscontinuity curves converge to the curves in the family $\mathcal{T}$ defined in the statements. For the interesting case of shocks, the proof works as follows: if the statements of the theorems were false, then waves not supported by the curves y_i would exist in the approximating solution in a neighborhood of the point $(t_0, x_0) \notin \Theta$. These waves cannot be shocks (otherwise they will converge to some of the limiting curves y_i) Thus by the structure of the system they must interact in the vicinity of the shock. In the limit $\epsilon \searrow 0$, this will imply that the point under consideration is in Θ.

In Section 6, we construct a strictly hyperbolic 2×2 system of conservation laws, which is not piecewise genuinely nonlinear and whose admissible solution to a particular initial datum does not have the structural properties described in Theorem 1.3. In fact, it is not possible to find finitely many curves supporting a shock of the second characteristic family in a small neighborhood of any point, even if the set of times t where the discontinuities of the second characteristic family are present has positive Lebesgue measure. In particular, it is not possible to even state (1.6). This shows that the assumption of piecewise genuinely nonlinearity cannot be removed.

Notation. Throughout the paper, we write $A \lesssim B$ ($A \gtrsim B$) if there exists a constant $C > 0$ which only depends on the system (1.1) such that $A \leq CB$ ($A \geq CB$).

2. Solution of Riemann problem

In this section we describe the construction of the solution to the Riemann problem for general hyperbolic systems of conservation laws, and this construction is taken from [2]. The procedure is divided in three steps:

(1) find Riemann problems which can be solved using only waves of the i-th family,
(2) give the explicit solution of these elementary Riemann problems,
(3) show how to piece together these functions in order to obtain the solutions to general Riemann problems.

The first two points actually goes together.

The starting point is that for a fixed point $u^- \in \Omega$ and $i \in \{1, \cdots, N\}$, there are smooth vector valued maps $\tilde{r}_i = \tilde{r}_i(u, v_i, \sigma_i)$ for $(u, v_i, \sigma_i) \in \Omega \times \mathbb{R} \times \mathbb{R}$, v_i and σ_i sufficiently small, with $\tilde{r}_i(u, 0, \sigma) = r_i(u)$ for all u, σ_i. These functions describe the center manifold of traveling profiles. Setting $l_i^0 := l_i(u^0)$, with $u^0 \in \Omega$ fixed, we can normalize $\tilde{r}_i$ such that

$$l_j^0 \cdot \tilde{r}_i(u, v_i, \sigma_i) = 1. \tag{2.1}$$

Define the speed function

$$\tilde{\lambda}_i(u, v_i, \sigma_i) := l_i^0 \cdot Df(u)\tilde{r}_i(u, v_i, \sigma_i).$$

Next, one constructs the Riemann problems

$$u_0(x) = \begin{cases} u^{\mathrm{L}} & x < 0, \\ u^{\mathrm{R}} & x > 0, \end{cases} \tag{2.2}$$

which can be solved by using only waves of the i-th family: in doing this, clearly one has also to find the correct right state u^{R} given u^{L}. For some constants $\delta_0, C_0 > 0$ fixed and $s > 0$, consider the subset of $\mathrm{Lip}([0,s], \mathbb{R}^{N+2})$ given by

$$\begin{aligned} \Gamma_i(s,u^-) := \Big\{ \gamma : \gamma(\xi) = & (u(\xi), v_i(\xi), \sigma_i(\xi)) \\ & u(0) = u^-, |u(\xi) - u^-| = \xi, v_i(0) = 0, \\ & |v_i(\xi)| \leq \delta_1, |\sigma_i(\xi) - \lambda_i(u^0)| \leq 2C_0\delta_1 \leq 1 \Big\}, \end{aligned} \tag{2.3}$$

and given a curve $\gamma \in \Gamma_i(s,u^-)$, define the scalar flux function

$$\tilde{f}_i(\tau;\gamma) = \int_0^\tau \tilde{\lambda}_i(u(\xi), v_i(\xi), \sigma_i(\xi)) d\xi. \tag{2.4}$$

Recall that the lower convex envelope of $\tilde{f}_i$ in the interval $[a,b] \subset [0,s]$ is given by

$$\operatorname*{conv}_{[a,b]} \tilde{f}_i(\tau;\gamma) := \inf \Big\{ \theta f_i(\tau';\gamma) + (1-\theta) f_i(\tau'';\gamma) : \theta \in [0,1], \tau', \tau'' \in [a,b], \tau = \theta\tau' + (1-\theta)\tau'' \Big\}.$$

Finally define the nonlinear operator $\mathcal{T}_{i,s} : \Gamma_i(s,u^-) \to \mathrm{Lip}([0,s], \mathbb{R}^{N+2})$, $\mathcal{T}_{i,s}(\gamma) = \check{\gamma} = (\check{u}, \check{v}_i, \check{\sigma}_i)$, by

$$\begin{cases} \check{u}(\tau) & = & u^- + \int_0^\tau \tilde{r}_i(u(\xi), v_i(\xi), \sigma(\xi)) d\xi, \\ \check{v}_i(\tau) & = & \tilde{f}_i(\tau;\gamma) - \mathrm{conv}_{[0,s]} \tilde{f}_i(\tau;\gamma), \\ \check{\sigma}_i(\tau) & = & \frac{d}{d\tau} \mathrm{conv}_{[0,s]} \tilde{f}_i(\tau;\gamma). \end{cases} \tag{2.5}$$

In [2] it is shown that, for $0 \leq s \leq \bar{s} \ll 1$, $\delta_1 \ll 1$ and $C_0 \gg 1$, $\mathcal{T}_{i,s}$ maps $\Gamma_i(s,u^-)$ into itself, and it is a contraction in $\Gamma_i(s,u^-)$ with respect to the distance

$$D(\gamma,\gamma') := \delta_1 ||u - u'||_{L^\infty} + ||v_i - v_i'||_{L^1} + ||v_i\sigma_i - v_i'\sigma_i'||_{L^1}, \tag{2.6}$$

where δ_1 is given in the definition of $\Gamma_i(s,u^-)$, formula (2.3), and

$$\gamma = (u, v_i, \sigma_i), \gamma' = (u', v_i', \sigma_i') \in \Gamma_i(s,u^-).$$

Hence, given u^- and $0 \leq s \leq \bar{s}$, let us denote the fixed point of $\mathcal{T}_{i,s}$ by

$$\bar{\gamma}(\tau; s, u^-) = \big(\bar{u}(\tau;s,u^-), \bar{v}_i(\tau;s,u^-), \bar{\sigma}_i(\tau;s,u^-)\big), \quad \tau \in [0,s].$$

We will give a short sketch of the proof later on.

For $s > 0$ the elementary curve $T_i[u^-] : [0,\bar{s}] \to \mathbb{R}^N$ for i-th family is defined by

$$u^{\mathrm{R}} = T_i[u^-](s) := \bar{u}(s; u^-, s). \tag{2.7}$$

This is the set of end points of solutions to (2.5).

For the case when $s < 0$, a right state $u^{\mathrm{R}} = T_i[u^{\mathrm{L}}](s)$ can be constructed in the same way as before, except that one replaces $\mathrm{conv}_{[0,s]} \tilde{f}_i(\tau;\gamma)$ in (2.5) with the upper concave envelope of $\tilde{f}_i(\tau;\gamma)$ on $[s,0]$, that is

$$\operatorname*{conc}_{[a,b]} \tilde{f}_i(\tau;\gamma) := \sup \Big\{ \theta f_i(\tau';\gamma) + (1-\theta) f_i(\tau'';\gamma) : \theta \in [0,1], \tau', \tau'' \in [a,b], \tau = \theta\tau' + (1-\theta)\tau'' \Big\},$$

and looks at the fixed point of of the integral system (2.5) on the interval $[s,0]$, and the elementary curve $T_i[u^-]$ for $s < 0$ is defined accordingly to (2.7).

Because of the assumption (2.1) and the definition (2.7), the elementary curve $T_i[u^{\mathrm{L}}]$ is parameterized by its i-th component relative to the basis $r_1(u^0), \cdots, r_N(u^0)$ i.e.

$$s = l_i^0 \cdot \big(T_i[u^{\mathrm{L}}](s) - u^{\mathrm{L}}\big). \tag{2.8}$$

We will also write the notation

$$\sigma_i[u^-](s,\tau) := \bar{\sigma}_i(\tau; u^-, s), \quad \tilde{f}_i[u^-](s,\tau) := \tilde{f}_i(\tau; \bar{\gamma}). \tag{2.9}$$

One has thus the following theorem [3].

Theorem 2.1. *For every $u \in \Omega$ and $|s| \leq \bar{s}$ sufficiently small, there are*

(1) *N Lipschitz continuous curves $s \mapsto T_i[u](s) \in \Omega$, $i = 1, \dots, N$, satisfying $\lim_{s\to 0} \frac{d}{ds} T_i[u](s) = r_i(u)$,*
(2) *N Lipschitz continuous functions $(s,\tau) \mapsto \sigma_i[u](s,\tau)$, with $0 \leq |\tau| \leq |s|$, $\operatorname{sgn}\tau = \operatorname{sng} s$, and $i = 1, \dots, N$, satisfying $\tau \mapsto \sigma_i[u](s,\tau)$ increasing,*

with the following properties.
When $u^{\mathrm{L}} \in \Omega$, $u^{\mathrm{R}} = T_i[u^{\mathrm{L}}](s)$, for some s sufficiently small, the admissible solution of the Riemann problem (1.1)-(2.2) is given by

$$u(x,t) := \begin{cases} u^{\mathrm{L}} & x/t \leq \sigma_i[u^{\mathrm{L}}](s,0), \\ T_i[u^{\mathrm{L}}](\tau) & x/t = \sigma_i[u^{\mathrm{L}}](s,\tau), |\tau| \in [0,|s|], \operatorname{sgn}\tau = \operatorname{sng} s, \\ u^{\mathrm{R}} & x/t > \sigma_i[u^{\mathrm{L}}](s,s). \end{cases} \tag{2.10}$$

We give a short sketch of the proof of the construction of the curves $T_i[u](s)$ for readers convenience.

Proof. It is clear that

$$u^- + \int_0^\tau \tilde{r}_i(u(\xi), v_i(\xi), \sigma(\xi)) d\xi$$

is $C^{1,1}$ if γ is Lipschitz, as well as $\tilde{f}_i(\tau;\gamma)$ given by (2.4). Moreover, for $|s| \leq \bar{s} \ll 1$, one obtains $|v_i| \leq \delta_1$ and $|\sigma_i - \lambda_i(u^-)| \leq 2\|f''\|_{L^\infty}\delta_1$ for some constant δ_1, by using the trivial estimates:

$$|\breve{v}_i(\xi)| \lesssim s \cdot \sup_{\varsigma \in [0,s]} \Big| \tilde{\lambda}_i(u(\varsigma), v_i(\varsigma), \sigma_i(\varsigma)) - \lambda_i(u^-) \Big| \lesssim s^2, \tag{2.11}$$

$$|\breve{\sigma}(\xi)| \lesssim \sup_{\varsigma \in [0,s]} \Big| \tilde{\lambda}_i(u(\varsigma), v_i(\varsigma), \sigma_i(\varsigma)) - \lambda_i(u^-) \Big| \lesssim s. \tag{2.12}$$

Therefore, the operator $\mathcal{T}_{i,s}$ maps $\Gamma_i(s,u^-)$ into itself.

If $\gamma_1, \gamma_2 \in \Gamma_i(s,u^-)$ are two curves, then

$$\big|\tilde{r}_i(\gamma_1) - \tilde{r}_i(\gamma_2)\big|, \big|\tilde{\lambda}_i(\gamma_1) - \tilde{\lambda}_i(\gamma_2)\big| \leq \mathcal{O}(1)\big(|u_1 - u_2| + |v_1 - v_2| + |v_1\sigma_1 - v_2\sigma_2|\big), \tag{2.13}$$

where we have used $\tilde{r}_i(u,0,\sigma) = r_i(u)$. The above estimates imply that

$$\begin{aligned} \delta_1 \bigg| \int_0^\tau \tilde{r}_i(\gamma_1) d\xi - \int_0^\tau \tilde{r}_i(\gamma_2) d\xi \bigg| &\leq \mathcal{O}(1)\delta_1 \Big(\bar{s}\|u_1 - u_2\|_{L^\infty} + \|v_1 - v_2\|_{L^1} + \|v_1\sigma_1 - v_2\sigma_2\|_{L^1} \Big) \\ &\leq \frac{1}{2} D(\gamma_1,\gamma_2), \end{aligned}$$

for $\bar{s} \leq \delta_1 \ll 1$.

By using the elementary estimates

$$\Big\| f - \operatorname{conv} f - (g - \operatorname{conv} g) \Big\|_{L^\infty} \leq \frac{1}{2} \bigg\| \frac{df}{dx} - \frac{dg}{dx} \bigg\|_{L^1},$$

$$\bigg\| \frac{d}{d\tau} \operatorname{conv} f - \frac{d}{d\tau} \operatorname{conv} g \bigg\|_{L^1} \leq \bigg\| \frac{df}{dx} - \frac{dg}{dx} \bigg\|_{L^1},$$

we obtain also from (2.13)

$$\begin{aligned} &\Big\| \tilde{f}(\gamma_1) - \operatorname{conv} \tilde{f}(\gamma_1) - \big(\tilde{f}(\gamma_2) - \operatorname{conv} \tilde{f}(\gamma_2)\big) \Big\|_{L^1} \leq \bar{s} \Big\| \tilde{f}(\gamma_1) - \operatorname{conv} \tilde{f}(\gamma_1) - \big(\tilde{f}(\gamma_2) - \operatorname{conv} \tilde{f}(\gamma_2)\big) \Big\|_{L^\infty} \\ &\leq \frac{\bar{s}}{2} \bigg\| \frac{d}{dx}\tilde{f}(\gamma_1) - \frac{d}{dx}\tilde{f}(\gamma_2) \bigg\|_{L^1} \leq \mathcal{O}(1)\frac{\bar{s}}{2}\big(\|u_1 - u_2\|_{L^\infty} + \|v_1 - v_2\|_{L^1} + \|v_1\sigma_1 - v_2\sigma_2\|_{L^1} \big) \leq \frac{1}{2} D(\gamma_1,\gamma_2), \end{aligned}$$

and similarly, since $|v_i| \leq \delta_1$,

$$\bigg\| v_i(\gamma_1) \frac{d}{d\tau} \operatorname{conv} \tilde{f}(\gamma_1) - v_i(\gamma_2) \frac{d}{d\tau} \operatorname{conv} \tilde{f}(\gamma_2) \bigg\|_{L^1} \leq \mathcal{O}(1)\delta_1 D(\gamma_1,\gamma_2) \leq \frac{1}{2} D(\gamma_1,\gamma_2).$$

These estimates show that the map $\mathscr{T}_{i,s}$ is a contraction in $\Gamma_i(s,u^-)$, and thus the fixed point $\bar{\gamma}$ is well defined.

In order to prove that the curve $T_i[u^-](s)$ is Lipschitz, if $s' > s$ and $\bar{\gamma}'$ is the fixed point to (2.5) on the interval $[0,s']$, one just estimate the distance of $\mathcal{T}_{i,s}(\bar{\gamma}')$ from $\bar{\gamma}'$. The only components which can vary are

v_i and σ_i, simply because we are restricting the convex envelope to the interval $[0,s] \subsetneq [0,s']$. Again by the estimates

$$\Big\| \operatorname*{conv}_{[0,s]} f - \operatorname*{conv}_{[0,s']} f \Big\|_{L^\infty(0,s)}, \bigg\| \frac{d}{d\tau} \operatorname*{conv}_{[0,s]} f - \frac{d}{d\tau} \operatorname*{conv}_{[0,s']} f \bigg\|_{L^\infty(0,s)} \leq \mathcal{O}(1) \|f''\|_{L^\infty} s'(s'-s),$$

one concludes that

$$D(\bar\gamma, \bar\gamma') \leq 2D\big(\mathcal{T}_{i,s}(\bar\gamma'), \bar\gamma'\big) \leq \mathcal{O}(1) s'(s'-s),$$

where we used the contraction factor $1/2$. Being $\bar s \leq \delta_1$, the above estimate yields the Lipschitz regularity as well as the existence of the derivative at $s = 0$, which can be also shown to be equal to $r_i(u)$ from the definition of $\mathcal{T}_{i,s}(u^-)$.

The final observation is that if γ' is a fixed point to $\mathcal{T}_{i,s'}(u^-)$ such that for some $s \in (0,s')$ it holds $v_i(\gamma', s) = 0$, then $\gamma' \llcorner [0,s]$ is a fixed point to $\mathcal{T}_{i,s}(u^-)$, as well as $\gamma' \llcorner [s,s']$ is a fixed point to $\mathcal{T}_{i,s'-s}[u(\gamma', s)]$: this is consequence of the fact that if $v_i(\gamma', s) = 0$, then

$$\operatorname*{conv}_{[0,s]} \tilde f_i(\gamma') = \operatorname*{conv}_{[0,s']} \tilde f_i(\gamma') \llcorner [0,s].$$

Thus shows that the fixed point given by (2.10) can be rewritten as

$$u(x,t) := \begin{cases} u(\gamma, 0) & x/t \leq \sigma_i(\gamma, 0), \\ u(\gamma, \tau) & x/t = \sigma_i(\gamma, \tau), |\tau| \in [0, |s|], \operatorname{sgn} \tau = \operatorname{sng} s, \\ u(\gamma, s) & x/t > \sigma_i(\gamma, s), \end{cases}$$

where $\gamma = (u(\gamma), v_i(\gamma), \sigma_i(\gamma))$: this latter formulation is the limit solution to the Riemann problem constructed by vanishing viscosity. $\square$

Remark 2.2. In [3] it is proved that if $u^{\mathrm L}, u^{\mathrm R} \in \Delta_i^j$ with some j odd (even) and $u^{\mathrm R} = T_i[u^{\mathrm L}](s),\ s > 0\ (s < 0)$, the solution u of the Riemann problem with the initial date (2.2) is a *centered rarefaction wave*, that is for $t > 0$,

$$u(x,t) = \begin{cases} u^{\mathrm L} & \text{if } x/t < \lambda_i(u^{\mathrm L}), \\ R_i[u^{\mathrm L}](\tau) & \text{if } x/t \in [\lambda_i(u^{\mathrm L}), \lambda_i(u^{\mathrm R})],\ x/t = \lambda_i(R_i[u^{\mathrm L}](\tau)), \\ u^{\mathrm R} & \text{if } x/t > \lambda_i(u^{\mathrm R}), \end{cases}$$

where $\tau \in [0,s]$ ($\tau \in [s,0]$) such that $s = l_i^0 \cdot (R_i[u^{\mathrm L}](s) - u^{\mathrm L})$. This is a consequence of the fact that $\nabla \lambda_i \cdot r_i(u) > 0$ in Δ_i^j. Notice that u is Lipschitz continuous for $t > 0$.

Remark 2.3. As shown in [3] (see also Remark 4 in [1] and Section 4 of [8]), under the assumption of piecewise genuine nonlinearity, the solution of the Riemann problem provided by (2.10) is a composed wave of the i-th family containing a finite number of rarefaction waves and admissible discontinuities. Recalling Theorem 2.1, one knows that the open intervals where the v_i-component of the solution to (2.5) vanishes correspond to rarefaction waves, while the closed intervals where the v_i-component of the solution to (2.5) is different from zero correspond to admissible discontinuities.

Finally, using the curves $T_i[u]$ and the solutions to the Riemann problem $[u^{\mathrm L}, T_i[u^{\mathrm L}](s)]$, one can construct the solution to a general Riemann problem. The idea is that, since the characteristic speeds are well separated, one can piece together the solution to elementary Riemann problems made only of i-th waves.

More precisely, the admissible solution [3] of a Riemann problem for (1.1)-(2.2), where now $u^{\mathrm R}$ satisfies only $|u^{\mathrm L} - u^{\mathrm R}| \ll 1$, is obtained by considering the Lipschitz continuous map

$$\mathbf{s} := (s_1, \dots, s_N) \mapsto T[u^L](\mathbf{s}) := T_N\big[T_{N-1}\big[\cdots\big[T_1[u^{\mathrm L}](s_1)\big]\cdots\big](s_{N-1})\big](s_N) = u^{\mathrm R}, \tag{2.14}$$

which, due to Point (1) of Theorem 2.1, is one to one from a neighborhood of the origin onto a neighborhood of $u^{\mathrm L}$. Then we can uniquely determine intermediate states $u^{\mathrm L} = \omega_0,\ \omega_1,\ \dots,\ \omega_N = u^{\mathrm R}$, and the *wave strength* $s_1,\ s_2,\ \dots,\ s_N$ such that

$$\omega_i = T_i[\omega_{i-1}](s_i), \quad i = 1, \dots, N,$$

provided that $|u^{\mathrm L} - u^{\mathrm R}|$ is sufficiently small.

By Theorem 2.1, each Riemann problem with initial data

$$u_0 = \begin{cases} \omega_{i-1} & x < 0, \\ \omega_i & x > 0, \end{cases} \tag{2.15}$$

admits a self-similar solution u_i, containing only i-waves. We call u_i the i-th *elementary composite wave* or simply *i-wave*. From the strict hyperbolicity assumption (1.2), the speed of each elementary i-th wave in the solution u_i is inside the interval $[\check{\lambda}_{i-1}, \check{\lambda}_i]$ if $s \ll 1$, so we can construct the function

$$u(x,t) = \begin{cases} u^{\mathrm{L}} & x/t \leq \check{\lambda}_0, \\ u_i(x,t) & \check{\lambda}_{i-1} < x/t \leq \check{\lambda}_i, i = 1, \dots, N, \\ u^{\mathrm{R}} & x/t > \check{\lambda}_N. \end{cases} \tag{2.16}$$

This function yields the admissible solution to the Riemann problem: it is clearly obtained by piecing together the self-similar solutions of the Riemann problems given by (1.1)-(2.15).

We end this section with a functional equivalent to the total variation of the solution: assuming for simplicity that $u : \mathbb{R} \to \mathbb{R}$ is piecewise constant with small L^∞-norm with jumps at x_α (as for wavefront approximate solutions), then

$$V(u) := \sum_\alpha \sum_i |s_{i,\alpha}|, \tag{2.17}$$

where $s_{i,\alpha}$ are the components of $\mathbf{s}_\alpha$ given by (2.14) for the Riemann problem in x_α. It is clear that $V(u)$ is equivalent to Tot.Var.(u), because $T[u^{\mathrm{L}}](\mathbf{s})$ is Lipschitz and invertible.

3. Description of wave-front tracking approximation

In this section we describe the construction of the wavefront tracking algorithm for general systems of conservation laws, following the approach of [1]. Since for piecewise genuinely nonlinear systems the solution to the Riemann problem is somehow simpler than in the general case, we slightly modify the algorithm to simplify our analysis.

In order to construct approximate wave-front tracking solutions, given a fixed $\epsilon > 0$, we first choose a piecewise constant function $u_{0,\epsilon}$ which is a good approximation to initial data u_0 such that

$$\text{Tot.Var.}(u_{0,\epsilon}) \leq \text{Tot.Var.}(u_0), \quad ||u_{0,\epsilon} - u_0||_{L^1} < \epsilon, \tag{3.1}$$

and $u_{0,\epsilon}$ only has finitely many jumps. Let $x_1 < \cdots < x_m$ be the jump points of $u_{0,\epsilon}$.

For each $\alpha = 1, \dots, m$, we approximately solve the Riemann problem with the initial data of the jump $[u_{0,\epsilon}(x_\alpha-), u_{0,\epsilon}(x_\alpha+)]$ by a function $w(x,t) = \phi(\frac{x-x_0}{t})$ where ϕ is a piecewise constant function which will be defined below.

The straight lines where the discontinuities are located are called *wave-fronts* (or just *fronts* for shortness). The wave-fronts travels with constant speed until they meet other wavefronts at a so-called interaction point, and then the corresponding new Riemann problem is approximately solved with a piecewise constant self similar solution. The procedure can be continued up to $t = +\infty$ if the choice of the approximate Riemann solutions produce only finitely many interactions in any compact set of times: for this aim, 3 types of approximate Riemann solutions are considered.

3.0.1. *The approximate i-th elementary wave.* The key step is to give a procedure to replace the solution to the elementary Riemann problem (2.15) with a piecewise constant self-similar function.

Suppose that $u_i(x/t)$ is an i-th elementary composite wave which is obtained by solving Riemann problem with initial data (2.15) where $\omega_i = T_i[\omega_{i-1}](s_i)$. For notational convenience, in this section we will write $\sigma_i(\tau) := \sigma_i[\omega_{i-1}](s_i, \tau)$, and for definiteness we consider $s_i > 0$, the other case being completely similar. Let

$$p := \left[\frac{\sigma_i(s_i) - \sigma_i(0)}{\epsilon} \right] + 1,$$

where $[\cdot]$ denotes the integer part, and let

$$\vartheta_{i,\ell} := \sigma_i(0) + \frac{\ell}{p} \left[\sigma_i(s_i) - \sigma_i(0) \right], \qquad \ell = 0, \cdots, p-1.$$

Since $\tau \mapsto \sigma_i(\tau)$ is increasing and continuous, we define the points

$$\tau_{i,\ell} := \min\big\{\tau \in [0, s_i], \sigma_i(s) = \vartheta_\ell\big\},$$

and set

$$\omega_{i-1,\ell} = T_i[\omega_{i-1}](\tau_{i,\ell}). \tag{3.2}$$

The i-th elementary composite wave $u_i(x/t)$ will be thus approximated by the function $\tilde{u}_i(x/t)$ given by

$$\tilde{u}_i(x,t) = \begin{cases} \omega_{i-1} & x/t < \vartheta_{i,0}, \\ \omega_{i-1,\ell} & \vartheta_{i-1,\ell-1} < x/t < \vartheta_{i,\ell}, \ \ell = 1, \cdots, p-1, \\ \omega_i & x/t > \vartheta_{i,p-1}. \end{cases} \tag{3.3}$$

Notice that $\tilde{u}_i$ consists of p fronts, hence it is piecewise constant. We moreover observe that since at each point $\tau_{i,\ell}$ it holds $\bar{v}_i(\tau_{i,\ell}) = 0$, then one has

$$T_i[\omega_{i-1}](\tau_{i,\ell}) = \bar{u}(\tau_{i,\ell}; \omega_{i-1}, s_i),$$

where $\bar{u}$ is the solution of (2.5). This shows that an equivalent interpretation of (3.2) is $\omega_{i-1,\ell} = \bar{u}(\tau_{i,\ell}; \omega_{i-1}, s_i)$.

Using the approximate i-th elementary wave we can construct the approximate Riemann solvers.

3.0.2. *Approximate Riemann solvers.* We present now three types of approximate Riemann solvers, and later we will specify the rule describing in which situation each one is used.

Accurate Riemann solver: in this case, one just replaces each i-th elementary composite wave of the exact Riemann solution with the approximate i-th elementary wave defined by (3.3) with discretization parameter ϵ: hence the fronts are separated if and only if their difference in speed is $\geq \epsilon$.

Simplified Riemann solver: assume that at the interaction point the wave $[u^{\mathrm{L}}, u^M]$ with strength s of the i-th family coming from the left interacts with the wave $[u^M, u^{\mathrm{R}}]$ with strength s' of the i'-th family coming from the right, with $i \leq i'$. The simplified Riemann solver is given by piecing together the elementary approximate solution (3.3) to the two Riemann problem

$$[u^L, T_i[u^L](s)] \quad \text{and} \quad [T_i[u^L](s), T_{i'}[T_i[u^L](s)](s')] \quad \text{if } i < i',$$
$$[u^L, T_i[u^L](s+s')] \quad \text{if } i = i',$$

where now the discretization parameter is 2ϵ: hence the fronts are separated if and only if their difference in speed is $\geq 2\epsilon$.

In order to match U^R, one also fix a parameter $\hat{\lambda} > \sup_\Omega \lambda_N(u)$ and consider a non-physical front traveling with speed $\hat{\lambda}$ and of size

$$[T_{i'}[T_i[u^L](s)](s'), u^R] \text{ if } i < i', \quad \text{or} \quad [T_i[u^L](s+s'), u^R] \text{ if } i = i'.$$

Crude Riemann solver: this describes the interactions with the nonphysical fronts introduced by the approximate Riemann solver and the i-waves. If $[u^L, u^M]$ is a nonphysical front coming from the left which interacts with an i-th front of strength s coming from the right, then the approximate solution consists of two wave fronts: a single jump $[u^L, T_i[u^L](s)]$ with speed computed by $\tilde{f}_i(s)/s$ and the remaining part of the discontinuity travels as a nonphysical front.

It is customary to think that the nonphysical front corresponds to the $(N+1)$-th characteristic field.

It is not restrictive to assume that at each time $t > 0$ at most one interaction occurs involving only two incoming fronts: in fact, it is enough to change the speed of the front by an arbitrarily small quantity. Since the algorithm provides solutions with uniformly bounded total variation, by letting this error go to 0 the solution still converges to the admissible solution. In this way we can use the Riemann solvers defined above to construct the solution.

Remark 3.1. We can divide the wavefronts in an approximate solution into 3 types:

Discontinuity front: these are fronts which are also admissible discontinuities;
Rarefaction front: these correspond to piecewise constant approximations of rarefactions;
Mixed front: these are discontinuities composed of admissible shocks and rarefaction fronts.

The last case, in which the shock is not admissible, can occur because of the definition of the approximate i-th elementary curve, as easily seen even in the scalar case.

3.0.3. *Interaction potential and BV estimates.* In this section we estimate the growth of total variation due to the nonlinear interaction of waves. We will introduce two quantities, namely the amount of interaction $\mathcal{I}$ and the Glimm interaction potential $\mathcal{Q}$.

Suppose that two wavefronts ς', ς'' interact at $(\bar{t}, \bar{x})$. For definiteness, let ς' be a wavefront of the i'-th family with strength s', and let ς'' be a wavefront of the i''-th family with strength s'', and assume that ς' is located at the left of ς'', so that $i'' \leq i'$. Without loss of generality, we can also assume that $s' > 0$. Denote with $\tilde{f}'_{i'}$, $\tilde{f}''_{i''}$ the corresponding scalar flux functions defined by (2.4).

The *amount of interaction* $\mathcal{I}(s', s'')$ between s' and s'' is defined as follows.

If ς', ς'' belong to different characteristic families $i' > i''$, then we define

$$\mathcal{I}(s', s'') := |s's''|. \tag{3.4}$$

In the case $i' = i''$, then we have 3 cases to consider, depending on the sign and size of s''. if $g' : [0, a] \to \mathbb{R}$, $g'' : [b, c] \to \mathbb{R}$ are two functions, then define

$$(g' \cup g'')(x) := \begin{cases} g'(x) & x \in [0, a], \\ g''(x - a + b) + g'(a) - g''(b) & x \in (a, a + c - b]. \end{cases}$$

(a) If $s'' > 0$, we set

$$\mathcal{I}(s', s'') := \int_0^{s'+s''} \Big| \big(\operatorname*{conv}_{[0,s']} \tilde{f}'_i \cup \operatorname*{conv}_{[0,s'']} \tilde{f}''_i \big)(\xi) - \operatorname*{conv}_{[0,s'+s'']} (\tilde{f}'_i \cup \tilde{f}''_i)(\xi) \Big| d\xi.$$

(b) If $-s' \leq s'' < 0$, we set

$$\mathcal{I}(s', s'') := \int_0^{s'} \Big| \operatorname*{conv}_{[0,s']} \tilde{f}'_i(\xi) - \big(\operatorname*{conv}_{[0,s'+s'']} \tilde{f}'_i \cup \operatorname*{conc}_{[s'',0]} \tilde{f}''_i \big)(\xi) \Big| d\xi.$$

(c) If $s'' < -s' < 0$, we set

$$\mathcal{I}(s', s'') := \int_{s''}^{0} \Big| \operatorname*{conc}_{[s'',0]} \tilde{f}''_i(\xi) - \big(\operatorname*{conc}_{[s'',-s']} \tilde{f}''_i \cup \operatorname*{conv}_{[0,s']} \tilde{f}' \big)(\xi) \Big| d\xi.$$

The form of the above amount of interaction $\mathcal{I}(s', s'')$ relies on the analysis of the scalar case, where in that case $\mathcal{I}(s', s'')$ is the area between the curves representing the solutions to the Riemann problems, see [**?**].

The key estimate proved in [2] (also see Lemma 1 in [1]) is that the quantity $\mathcal{I}(s', s'')$ controls how the wave pattern changes before and after the interaction: if $\mathbf{s}$ is given by solving the Riemann problem at the interaction as in (2.14), then

$$\sum_{i=1}^{N+1} |s_i - s'_i - s''_i| \lesssim \mathcal{I}(s', s''). \tag{3.5}$$

where $(s'_i, s''_i) = (\delta_{i,i'} s', \delta_{i,i''} s'')$. [1]

In particular the functional $V(t)$ given by (2.17) increases at most of $\mathcal{O}(1)\mathcal{I}(s', s'')$,

$$V(\bar{t}) - V(\bar{t}-) \lesssim \mathcal{I}(s', s''). \tag{3.6}$$

Observe that we also consider the nonphysical waves in the above estimate (3.5) as an additional $N+1$-th wave family.

Remark 3.2. Note that the form of of the amount of interaction given here is slightly different that the one given in [2], but it is fairly easy to prove that the two forms are equivalent.

In order to bound the increase of the functional $V(t)$, a second functional $\mathcal{Q}$, the *Glimm interaction potential*, is defined as follows: if in $u(t)$ the wavefronts are located at x_α with strength s_α, then

$$\mathcal{Q}(t) := \sum_{\substack{j>i \\ x_\alpha < x_\beta}} |s_{j,\alpha} s_{i,\beta}| + \frac{1}{4} \sum_{i_\alpha = i_\beta < N+1} \int_0^{|s_\alpha|} \int_0^{|s_\beta|} \big| \sigma_{i_\beta}[\omega_\beta](s_\beta, \tau'') - \sigma_{i_\alpha}[\omega_\alpha](s_\alpha, \tau') \big| d\tau' d\tau''.$$

The last term do not contains the $N+1$-th family because the speed is the constant $\hat{\lambda}$ fixed.

[1] $\delta_{i,j}$ is Kronecker delta.

If $\bar{t}$ is the time of interaction of s', s'', then one can prove that (Lemma 5 in [1])

$$\mathcal{Q}(\bar{t}) - \mathcal{Q}(\bar{t}-) \lesssim \mathcal{I}(s', s''), \tag{3.7}$$

The above estimate together with (3.6) allows to define the *Glimm functional*

$$\Upsilon(t) := V(t) + C_0 \mathcal{Q}(t) \tag{3.8}$$

with C_0 suitable constant, so that $\Upsilon(t)$ is monotone decreasing in t.

3.0.4. *Construction of wavefront approximate solutions.* The Glimm functional is used to show that one can choose the Riemann solvers defined in Section 3.0.2 in order to have

(1) a finite number of interactions points,
(2) a finite number of waves,
(3) a uniform bound of the total variation of the solution,
(4) the total variation of the nonphysical waves converging to 0,
(5) an error on the conservation equation converging to 0 weakly in measure.

Hence the limit function will be a solution to (1.1) with uniform bounded total variation, and a standard Riemann semigroup comparison technique yields the uniqueness of the limit. We will now sketch the procedure.

The construction starts at initial time $t = 0$ with a given $\epsilon > 0$, by taking $u_{0,\epsilon}$ as a suitable piecewise constant approximation of initial data u_0 satisfying (3.1).

At the jump points of $u_{0,\epsilon}$, we locally solve the Riemann problem by accurate Riemann solver. The approximate solution can be prolonged until a first time t_1 when two wavefronts s, s'' interact. Depending on the amount of interaction at this interaction point, one chooses the appropriate approximate Riemann solver and compute the solution until the next interaction points occurs.

The rule for choosing which Riemann solvers one uses is the following. Fix a parameter $\rho = \rho(\epsilon) > 0$. If s', s'' are physical waves, then one uses the accurate Riemann solver if $\mathcal{I}(s', s'') \geq \rho$, otherwise one applies the simplified Riemann solver. Finally, when one of the waves is nonphysical, then the crude Riemann solver is used.

In [1] it is proved that if $\rho = \rho(\epsilon)$ is chosen sufficiently small, then the construction yields an approximate wavefront solution u_ϵ satisfying the properties (1)-(5) listed above.

For definiteness, for any t we consider $x \mapsto u(t, x)$ right continuous.

3.0.5. *Further estimates.* To conclude this section, we consider some natural quantities related to the approximate solution u_ϵ.

We define the *measure of interaction* $\mu_\epsilon^{\mathrm{I}}$ and the *measure of interaction and cancellation* $\mu_\epsilon^{\mathrm{IC}}$ as purely atomic measures concentrated on the interaction points: if $P = (\bar{t}, \bar{x})$ is an interaction point, then the value of $\mu_\epsilon^{\mathrm{I}}(P)$, $\mu_\epsilon^{\mathrm{IC}}(P)$ are given by

$$\mu_\epsilon^{\mathrm{I}}(\{P\}) := \mathcal{I}(s', s''), \tag{3.9a}$$

$$\mu_\epsilon^{\mathrm{IC}}(\{P\}) := \mathcal{I}(s', s'') + \begin{cases} |s'| + |s''| - |s' + s''| & i' = i'', \\ 0 & i' \neq i''. \end{cases} \tag{3.9b}$$

Using these measures and the wave strength estimates (3.5), one can write balance of waves for approximate solutions, showing that the the wave measures s_i of u satisfy a balance equation with source $\mu_\epsilon^{\mathrm{I}}$, $\mu_\epsilon^{\mathrm{IC}}$. In fact, in each region Γ transversal to the wavefronts, setting

$$W_{\epsilon,\mathrm{in}}^i(\Gamma) := \sum_{\text{entering } \Gamma} s_i, \quad W_{\epsilon,\mathrm{out}}^i(\Gamma) := \sum_{\text{exiting } \Gamma} s_i,$$

$$W_{\epsilon,\mathrm{in}}^{i,\pm}(\Gamma) := \sum_{\text{entering } \Gamma} s_i^\pm, \quad W_{\epsilon,\mathrm{out}}^{i,\pm}(\Gamma) := \sum_{\text{exiting } \Gamma} s_i^\pm, \qquad s_i^\pm = \max\{\pm s_i, 0\},$$

then one has from the interaction estimates (3.5)-(3.7) that

$$|W_{\epsilon,\mathrm{out}}^i - W_{\epsilon,\mathrm{in}}^i|(\Gamma) \lesssim \mu_\epsilon^{\mathrm{I}}(\Gamma), \quad |W_{\epsilon,\mathrm{out}}^{i\pm} - W_{\epsilon,\mathrm{in}}^{i\pm}|(\Gamma) \lesssim \mu_\epsilon^{\mathrm{IC}}(\Gamma). \tag{3.10}$$

We observe that the uniform boundedness of Tot.Var.$(u(t))$ w.r.t. time t and parameter ϵ together with the Glimm interaction estimates imply that $\mu^{\mathrm{I}}_\epsilon$, $\mu^{\mathrm{IC}}_\epsilon$ are bounded measures for all ϵ. Hence, up to subsequences $\epsilon_\nu \searrow 0$, there exist bounded measures μ^{I} and μ^{IC} on $\mathbb{R}^+ \times \mathbb{R}$ such that the following weak convergence holds:

$$\mu^{\mathrm{I}}_{\epsilon_\nu} \rightharpoonup \mu^{\mathrm{I}}, \quad \mu^{\mathrm{IC}}_{\epsilon_\nu} \rightharpoonup \mu^{\mathrm{IC}}.$$

The key problem in passing to the limit of the balances (3.10) is that the map (2.14) in nonlinear, so that a stronger convergence of the derivatives of u should be proved. This will be a corollary of our regularity estimates.

In the following we will not relabel the subsequence ϵ_ν.

4. Construction of subdiscontinuity curves

In this section we define the family of approximate subdiscontinuity curves. The key point is that due to the piecewise genuine nonlinearity assumption, one can select finitely many subdiscontinuities of a given jumps where the flux f_i is convex (or concave, see below). These components behave very similarly to the genuinely nonlinear case: the main property is that they cannot be split by interactions, but only completely removed by cancellation. Thus for these components one can adapt the procedure used to define the discontinuity curves for genuinely nonlinear systems.

We now define the (i,j)-subdiscontinuities s_i^j of an i-th shock s_i. he index j refers to the regions Δ_i^j defined in (1.3). Let $[u^{\mathrm{L}}, u^{\mathrm{R}}]$, $u^{\mathrm{R}} = T_i[u^{\mathrm{L}}](s_i)$, be a wavefront of i-th family in the approximate solution u_ϵ. For definiteness, we assume $s_i > 0$. Since the derivative of the curve T_i is very close the i-eigenvector r_i, it follows that the curve $\bar u(\cdot; u^L, s_i)$ [2] intersects transversally the surfaces Z_i^j. Let thus $0 \le \tau^{j_1} \le \tau^{j_1+1} \le \cdots \le \tau^{j_2} \le s_i$ be the values such that

$$u^{j_1+k} = \bar u(\tau^{j_1+k}; s_i, u^L) \in Z_i^{j_1+k}, \quad k = 1, \dots, j_2 - j_1.$$

If $\tau^{j_1} > 0$, set $\tau^{j_1-1} = 0$ and if $\tau^{j_2} < s_i$, set $\tau^{j_2+1} = s_i$.

Definition 4.1. We say that the wavefront $[u^{\mathrm{L}}, u^{\mathrm{R}}]$ has a (i,j)-*subdiscontinuity* $[u^j, u^{j+1}]$ *of strength* $s_i^j = \tau^{j+1} - \tau^j$ when the latter is different from 0, with j odd for $s_i < 0$ and j is even for $s_i > 0$.

Notice that obviously only mixed fronts and discontinuity fronts can have (i,j)-subdiscontinuities s_i^j, because rarefaction fronts are contained in regions where the i-th eigenvalue is *increasing* across the discontinuity, while by the above definition the subdiscontinuities belong to the part of the wavefront in which the i-th eigenvalue is *decreasing*.

Observe moreover that the wave decomposition given by (1.5) are such that in each component there is at least a subdiscontinuity.

The above observation implies that the subdiscontinuities are quite stable, in the sense that they do not split when involved in an interaction: this is a direct consequence of the construction of the (approximate) Riemann solution, and it will be proved in Lemma 4.3.

The second step is to define for the subdiscontinuities s_i^k of a wavefront s_i the components which has a uniform strength in some time interval. The following definition is an adaptation of the definition of (δ, i)-approximate discontinuities [6].

Definition 4.2. For $\delta \neq 0$ fixed, a (δ, i, j)-*approximate subdiscontinuity curve* is a polygonal line in (x,t)-plane with nodes $(t_0, x_0), (t_1, x_1), \cdots, (t_n, x_n)$ such that

(1) (t_k, x_k) are interaction points with $0 \le t_0 < t_1 < \cdots < t_n$,
(2) j is odd for $s_i < 0$ or j is even for $s_i > 0$,
(3) for $1 \le k \le n$, the segment $[(t_{k-1}, x_{k-1}), (t_k, x_k))$ is the support of an (i,j)-subdiscontinuity front with strength $|s_i^j| \ge \delta/2$, and there is at least one time $t \in [t_0, t_n]$ such that $|s_i^j| \ge \delta$.

In order to count them, the following property of piecewise genuinely nonlinear system comes in handy.

Lemma 4.3. *The solution of a Riemann problem given by any approximate Riemann solver contains at most one subdiscontinuity* s_i^j *for all* $i = 1, \dots, N$, $j = 1, \dots, J_i$.

While the proof can be obtained directly from the analysis of [2], we repeat it.

[2] the fixed point of the operator $\mathcal{T}_{i,s}$, see (2.5).

Proof. For approximate solutions to a Riemann problem, the proof reduces in proving that the speed $\bar{\sigma}_i(\tau; u^-, s)$ obtained by solving the system (2.5) is constant in each subdiscontinuity component.

Assume that this is not the case, and for definiteness let $s_i > 0$, so that in the subdiscontinuities we have $D\lambda_i \cdot r_i < 0$. Then by inspection of system (2.5) one has that if $\bar{\tau} \in (\tau^j, \tau^{j+1})$, j even, is the point where $d\bar{\sigma}_i/d\tau > 0$, then $\bar{v}_i(\tau_\ell) = 0$ for a sequence $\tau_\ell \to \bar{\tau}$. Hence $\tilde{\lambda}_i(\bar{\gamma}(\tau_\ell)) = \lambda_i(\bar{u}(\tau_\ell))$, which implies

$$\frac{d\bar{\sigma}}{d\tau}(\bar{\tau}) = D\lambda_i \cdot \frac{d\bar{u}}{d\tau}(\bar{\tau}).$$

By using $d\bar{u}/d\tau = \tilde{r}_i(\bar{\gamma})$, since $v_i(\tau) = 0$ one obtains

$$\frac{d\bar{u}}{d\tau}(\bar{\tau}) = r_i(\bar{u}(\bar{\tau})),$$

so that

$$0 < \frac{d\bar{\sigma}}{d\tau}(\bar{\tau}) = D\lambda_i(\bar{u}(\bar{\tau})) \cdot r_i(\bar{u}(\bar{\tau})) \leq 0,$$

which is a contradiction. □

Remark 4.4. The same proof shows that a composite wave with strength s can have at most $[J_i/2] + 1$ components[3]. In fact, the extremal values of a component have $\bar{v}_i = 0$, and thus only one can be present in the regions Δ_i^j for j even if $s < 0$ or j odd for $s > 0$. Moreover it is clear that the points u_k of (1.5) are uniquely determined by the condition of being the unique point in some $\Delta_i^{j_k}$, j_k even for $s < 0$ or j_k odd for $s > 0$, such that $\lambda_i(u_k) = \sigma_i[u^{\mathrm{L}}](s, s)$.

Define the family of curves $\mathscr{T}_{\delta,i}^j(\epsilon)$ as follows: if $\{y_\ell : I_\ell \to \mathbb{R}\}_{\ell=1}^L$ have been chosen, for a jump point $(t, x) \notin \cup_\ell \mathrm{graph}\, y_\ell$ such that the subdiscontinuity s_i^j has strength $\geq \delta$, let y_{L+1} be the unique curve supporting an approximate (δ, i, j)-subdiscontinuity passing through (t, x) such that

(1) it is the leftmost among all approximate (δ, i, j)-subdiscontinuities passing trough (t, x),
(2) it is maximal w.r.t. set inclusion.
(3) it is disjoint from all the curves y_ℓ, $\ell = 1, \ldots, L$.

The uniqueness follows from the fact that the above lemma implies uniqueness of the curve y_{L+1} in the future. In particular, in the past the curve y_{L+1} never meets another wave y_ℓ, $\ell \leq L$.

The next proposition implies that the number $M_{\delta,i}^j(\epsilon)$ of curves in $\mathscr{T}_{\delta,i}^j(\epsilon)$ is finite, independently of ϵ.

Proposition 4.5. *For fixed j and δ, $M_{\delta,i}^j(\epsilon)$ is uniformly bounded w.r.t. ϵ.*

Proof. First of all, for all fixed times t the number of (δ, i, j)-subdiscontinuities is clearly bounded by $2\mathrm{Tot.Var.}(u(t))/\delta$. Suppose that there is a sequence of times $\{t_\ell\}_{\ell=1}^{L_\epsilon}$ such that at each t_ℓ there exists an approximate (δ, i, j)-subdiscontinuity curve γ_ℓ whose interval of definition does not contain $t_{\ell'}$, $\ell' < \ell$. Since we can take t_ℓ increasing and at a fixed time the number of subdiscontinuity curves is finite, we thus conclude that L_ϵ many of them are created and canceled.

Since the number of curves $M_{\delta,i}^j(\epsilon)$ is increasing with δ decreasing, we can assume that

$$\delta \leq \mathrm{dist}(Z_i^j, Z_i^{j+1}).$$

It follows that a subdiscontinuity s_i^j of $[u^{\mathrm{L}}, u^{\mathrm{R}}]$ can have size $< \delta/2$ only if $u^{\mathrm{L}} \in s_i^j$ or $u^{\mathrm{R}} \in s_i^j$.

As a consequence of Lemma 4.3, the only way to decrease the strength of an approximate (i, j)-subdiscontinuity γ_i^j from $s_i^j(t_1) \geq \delta$ to a $s_i^j(t_2) < \delta/2$ at a later time $t_2 > t_1$ is only by interaction and cancellation: this is a direct consequence of the fact that we cannot split the subdiscontinuities. Hence we can reduce a subcomponent s_i^j of $[u^{\mathrm{L}}, u^{\mathrm{R}}]$ only by varying the end points of the curve $T_i[u^{\mathrm{L}}](s)$.

Due to the Lipschitz dependence of the curve $T_i[u^{\mathrm{L}}](s)$ from u^{L} and s, and the transversality of the surfaces Z_i^j, it follows that to reduce the size of a subdiscontinuity from δ to $\delta/2$ we have to vary s or u^{L} of at least $\delta/2$. In both cases, from Glimm interaction estimate, it follows that the amount of interaction along the curve supporting s_i^j is of at least $\delta^2/4$: indeed, by direct inspection, interactions of the same family just increase s_i^j (remember that it belong to the end portions of the jump $[u^{\mathrm{L}}, u^{\mathrm{R}}]$).

[3] $[\cdot]$ is the integer part of a real number.

Hence, by Glimm interaction estimates, it follows that the amount of interaction/cancellation along $\gamma_i^j([t_1,t_2])$ is $\geq \mathcal{O}(1)\delta/2$.

From the uniform boundedness of Glimm functional and of the disjointness of the subdiscontinuity curves, we conclude that L_ϵ is uniformly bounded, which implies the uniformly boundedness of the number $M^j_{\delta,i}(\epsilon)$ w.r.t. ϵ. □

Remark 4.6. In Section 6, we show why the piecewise genuine nonlinearity is essential for the validity of the above proposition. In fact, an explicit example in a 2×2 hyperbolic system shows that the statement is false in the general case.

We will denote the curves of $\mathscr{T}^j_{\delta,i}(\epsilon)$ as $y^{j,\ell}_{\delta,i}(\epsilon)$, $\ell = 1,\dots,M^j_{\delta,i}$, with $M^j_{\delta,i}$ independent of ϵ. By standard compactness estimates, completely similar to the genuinely nonlinear case, one can fairly easily prove that up to subsequences we can assume that $y^{j,\ell}_{\delta,i}(\epsilon) \to y^{j,\ell}_{\delta,i}$ in the uniform topology, with $y^{j,\ell}_{\delta,\ell}$ non necessarily distinct curves in $\mathbb{R}^+\times\mathbb{R}$.

Let us denote by

$$\mathscr{T}^j_{\delta,i} := \left\{y^{j,\ell}_{\delta,i} : I^{j,\ell}_{\delta,i} \to \mathbb{R}, \ell = 1,\dots,M^j_{i,\delta}\right\}, \quad i \in 1,\dots,N, j = 1,\dots,J_i,$$

the collection of all these limiting curves for fixed δ, i, j, and set moreover

$$\mathscr{T}_{\delta,i} := \bigcup_j \mathscr{T}^j_{\delta,i}, \qquad \mathscr{T}_i := \bigcup_\delta \mathscr{T}_{\delta,i}.$$

With an abuse of notation, we will also write $\mathscr{T}_i$ for the graph in $\mathbb{R}^+\times\mathbb{R}$ of the curves in $\mathscr{T}_i$.

Definition 4.7. Let Θ consist of all jump points of initial data and the atoms of interaction and cancellation measure μ^{IC}.

Lemma 4.8. *Let $y^{j,\ell}_{\delta,i} : I_\ell \to \mathbb{R}$ be an (δ,i,j)-subdiscontinuity curve in $\mathscr{T}^j_{\delta,i}$. If $(t, y^{j,\ell}_{\delta,i}(t)) \notin \Theta$, then the derivative $\dot y^{j,\ell}_{\delta,i}(t)$ exists.*

Proof. By the definition of $\mathscr{T}^j_{\delta,i}$, there exists a curve $y^{j,\ell}_{\delta,i}(\epsilon) \in \mathscr{T}^j_{\delta,i}(\epsilon)$ converging uniformly to $y^{j,\ell}_{\delta,i}$. Since $(t, y^{j,\ell}_{\delta,i}(t))$ is not an atom for μ^{IC}, then for all $\eta > 0$ there exists $r > 0$ such that

$$\mu^{IC}_\epsilon\big(B((t, y^{j,\ell}_{\delta,i}(t)), r)\big) \leq \eta.$$

For a discontinuity of size $\delta > 0$, it follows from the Glimm estimate that its change in speed is proportional to μ^{IC}/δ, and thus the approximating curve $y^{j,\ell}_{\delta,i}(\epsilon)$ have a speed whose total variation is $\leq \eta/\delta$. The conclusion follows from the l.s.c. of the Lipschitz constant w.r.t. uniform convergence. □

To conclude this section, we give a definition of a partial order relation among subdiscontinuities s^j_i of the same family but with different j. For definiteness, we assume that $s^j_i > 0$ so that the index j is even.

Consider the calligraphic ordering $\prec$ in $\mathbb{R}^2$:

$$(x,y) \prec (x',y') \quad \Longleftrightarrow \quad \Big(x < x' \ \vee \ \big(x = x' \wedge y < y'\big)\Big).$$

Let $P_i(u) = u_i$ be the projection of the vector u on its i-th component and let $y^{j,\ell}_{\delta,i}$, $y^{j',\ell'}_{\delta',i}$ be two subdiscontinuity curves, corresponding to the subdiscontinuities

$$U := \big[u^{j,\ell}_{\delta,i}-, u^{j,\ell}_{\delta,i}+\big], \quad U' := \big[u^{j,\ell}_{\delta',i}-, u^{j',\ell'}_{\delta',i}+\big],$$

and with j even. Then we define

$$y^{j,\ell}_{\delta,i} \prec y^{j',\ell'}_{\delta',i} \quad \Longleftrightarrow \quad \exists t, u \in U, u' \in U' \ \big((y^{j,\ell}_{\delta,i}(t), P_i(u)) \prec (y^{j',\ell'}_{\delta',i}(t), P_i(u'))\big). \tag{4.1}$$

It is fairly easy to see that the above definition does not depend on the points u, u', but maybe it is not clear if it is independent of t. However a direct inspection on the Riemann solver formula implies that this monotonicity is preserved, so that $\prec$ is a partial ordering on $\cup_{j\text{ even}}\mathscr{T}^j_i$. The fact that it is not a linear order is due to the possibility that the interval of existence of the curves $y^{j,\ell}_{\delta,i}$ are disjoint.

A completely similar partial ordering can be introduced on $\cup_{j\text{ odd}}\mathscr{T}^j_i$, by taking

$$y^{j,\ell}_{\delta,i} \prec y^{j',\ell'}_{\delta',i} \quad \Longleftrightarrow \quad \exists t, u \in U, u' \in U' \ \big((y^{j,\ell}_{\delta,i}(t), -P_i(u)) \prec (y^{j',\ell'}_{\delta',i}, -P_i(u'))\big).$$

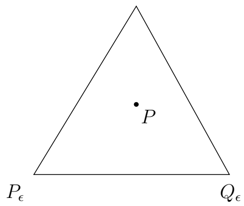

FIGURE 4.

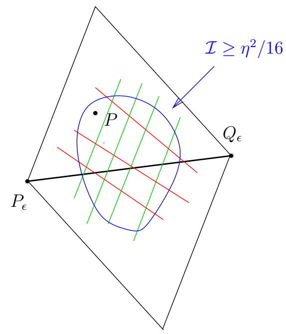

FIGURE 5.

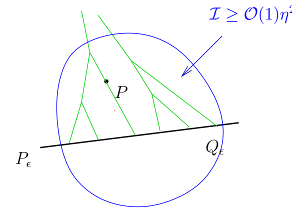

FIGURE 6.

5. Proof of the main theorems

In this section we give a proof of Theorems 1.3 and 1.4. The theorems contain 3 different statements:

(1) outside the interaction points Θ and the discontinuity curves $\cup_i \mathscr{T}_i$, the solution is continuous and the limit of the wavefront approximations converge pointwise,
(2) on the discontinuity points in $\cup_i \mathscr{T}_i$ which are not interaction point in Θ, the solution is right and left continuous, and there are curves converging to the discontinuity curve such that the wavefront approximations converges pointwise on both sides of these curves;
(3) if the discontinuity is a composed shock and the components split in a neighborhood of the point, a similar continuity and convergence result holds in the region between the two curves.

A consequence of the proof is that the stability of the wave structure is preserved under L^1-convergence of solutions: this result is contained in the remark ending this section.

First we prove that u is continuous outside the points of interactions Θ and the discontinuity curves $\cup_i \mathscr{T}_i$. Consider a point

$$P = (\tau, \xi) \notin \Theta \cup \bigcup_i \mathscr{T}_i,$$

and assume by contradiction that it is not a continuity point of u. Then, by the L^1-convergence of approximate solutions u_ϵ, there exists $\eta > 0$ and a sequence of points $P_\epsilon = (x_{P_\epsilon}, t_{P_\epsilon})$, $Q_\epsilon = (x_{Q_\epsilon}, t_{Q_\epsilon})$ converging to P such that

$$\big|u_\epsilon(Q_\epsilon) - u_\epsilon(P_\epsilon)\big| \geq \eta, \tag{5.1}$$

up to subsequences. Due to the finite finite speed of propagation, we can assume that the segment $[P_\epsilon, Q_\epsilon]$ is space-like, i.e. its slope $\check\lambda$ is higher that all the characteristic speeds (see Figure 4), otherwise by the estimate

$$\sup_{a+\check\lambda t < x < b - \check\lambda t} \big|u(t,x) - c\big| \leq \mathcal{O}(1) \sup_{a<x<b} \big|u(0,x) - c\big| \tag{5.2}$$

the inequality (5.1) is impossible in an arbitrarily small neighborhood of P.

Three cases have to be considered.

Case 1.1: If there exists $i < i'$ such that the total wave strength of the i-th and i'-th families crossing the segments $[P_\epsilon, Q_\epsilon]$ are uniformly larger than $\eta/4$, then it follows that in Γ_ϵ, a small neighborhood of P, these waves are either created, canceled or have interacted. In all these cases, the amount of interaction on the region Γ_ϵ is uniformly large, that is $\mu_\epsilon^{IC}(\Gamma_\epsilon) \geq \eta^2/16$, which implies the point $P \in \Theta$ against the assumption.

Case 1.2: If instead only one family i has total variation of order $\eta/2$ and there a large discontinuity, since $P \notin \Theta$ this discontinuity contains some subdiscontinuity which is not canceled in a neighborhood of P, contradicting $P \notin \cup_i \mathscr{T}_i$.

Case 1.3: If finally the discontinuities are arbitrarily small as $\epsilon \to 0$, then they must belong to one of the regions Δ_i^j. Since in these region the characteristic speed is genuinely nonlinear, then these waves must interact either in the future or in the past (depending from the sign of j, and they cannot be canceled or created because $P \notin \Theta$). In all cases, one concludes the $P \in \Theta$, yielding a contradiction.

Note that we have proved that at these points the convergence is pointwise, not in L^1.

Next, consider a point $P = (\tau, \xi) \in \cup_i \mathscr{T}_i \setminus \Theta$. It is clear that P belongs to $\mathscr{T}_i$ for only one family i, otherwise $P \in \Theta$. Since $x \mapsto u(\tau, x)$ has bounded variation in $\mathbb{R}$, the limits

$$u^L := \lim_{x\to\xi-} u(x,\tau), \qquad u^R := \lim_{x\to\xi+} u(x,\tau) \tag{5.3}$$

exist, and moreover $u^{\mathrm{R}} = T_i[u^{\mathrm{L}}](s)$. Without loss of generality we can assume $s > 0$, and let $\{y_i^j, j = j_1, \dots, j_P, j \text{ even}\}$ be the subdiscontinuity curves passing through P. Since $P \notin \Theta$, these curves are defined in a neighborhood of τ, and by the ordering we have that $j \mapsto y_i^j$ is increasing in the sense of (4.1). Let $y_i^j(\epsilon)$, $j = j_1, \dots, j_P$, j even, be the corresponding curves (for the approximate solutions u_ϵ) converging to y_i^j: their existence follows from the definition of $\mathscr{T}_i$ and the fact that $P \notin \Theta$.

The same analysis performed in the continuity points implies that on the left of $y_i^{j_1}$ the solution converges pointwise to a value $u^- \in \mathbb{R}^N$:

$$\lim_{r\to 0+} \left(\limsup_{\epsilon\to\infty} \sup_{\substack{x < y_i^{j_1}(\epsilon,t) \\ (x,t)\in B(P,r)}} \big| u_\epsilon(x,t) - u^- \big| \right) = 0, \tag{5.4a}$$

and the same on the right side of $y_1^{J_p}(\epsilon)$:

$$\lim_{r\to 0+} \left(\limsup_{\epsilon\to\infty} \sup_{\substack{x > y_i^{j_P}(\epsilon,t) \\ (x,t)\in B(P,r)}} \big| u_\epsilon(x,t) - u^+ \big| \right) = 0, \tag{5.4b}$$

for some $u^+ \in \mathbb{R}^N$.

In fact, if the equality (5.4a) is not true, there a sequence of points $P_\epsilon = (x_{P_\epsilon}, t_{P_\epsilon})$, $Q_\epsilon = (x_{Q_\epsilon}, t_{Q_\epsilon})$ converging to P such that

$$\big| u_\epsilon(Q_\epsilon) - u_\epsilon(P_\epsilon) \big| \geq \eta, \tag{5.5}$$

and each segment $[P_\epsilon, Q_\epsilon]$ is space-like. Let us consider three corresponding cases as for continuous points.

Case 2.1: If there exists $i \neq i'$ such that the total wave strength of the i-th and i'-th families crossing the segments $[P_\epsilon, Q_\epsilon]$ are uniformly larger than $\eta/4$, then it follows that either a uniformly large amount of cancellation occurs in a small neighborhood of P, or waves with a uniformly large total strength interact with the curve $y_\epsilon^{j_1}$. Both contradicts the assumption $P \in \Theta$.

Case 2.2: If instead only one family i has total variation of order $\eta/2$ and there a large discontinuity, since $P \notin \Theta$ this discontinuity contains some subdiscontinuity which is not canceled in a neighborhood of P. This implies that there exist a subdiscontinuity curve $y^{j_0} \in \mathscr{T}_i$ such that $y^{j_0} \prec y^{j_1}$, which contradicts the assumption that y^{j_1} is the most left (in the sense of order (4.1)) subdiscontinuity curve passing though the point P.

Case 2.3: As the same as the situation in Case 1.3.

The equality (5.4b) is analogous to prove.

Similarly, if two curves y_i^j, y_i^{j+1} split for $t < t_P$ (or $t > t_P$), then also the subdiscontinuity curves $y_i^j(\epsilon)$, $y_i^{j+1}(\epsilon)$ converging y_i^j, y_i^{j+1} to have different locations for $t < t_P - \delta$ (or $t > t_P + \delta$), and the same analysis done before implies that

$$\lim_{r\to 0+} \left(\limsup_{\epsilon\to\infty} \sup_{\substack{y_i^j(\epsilon,t) < x < y_i^{j+1}(\epsilon,t) \\ (x,t)\in B(P,r), t\leq t_P - \delta}} \big| u_\epsilon(x,t) - u^{j-} \big| \right) = 0,$$

or

$$\lim_{r\to 0+} \left(\limsup_{\epsilon\to\infty} \sup_{\substack{y_i^j(\epsilon,t) < x < y_i^{j+1}(\epsilon,t) \\ (x,t)\in B(P,r), t> t_P + \delta}} \big| u_\epsilon(x,t) - u^{j+} \big| \right) = 0,$$

for some $u^{j\pm}$.

The fact that $u^- = u^{\mathrm{L}}$, $u^+ = u^{\mathrm{R}}$ is a consequence of this convergence and equation (5.3) together with (5.2), while the fact that $u^{j-} = u^{j+}$ is given by the decomposition of shocks (1.5) follows from Lemma 4.8 and Remark 4.4.

Finally, the R-H conditions for the curves in $\cup_i \mathscr{T}_i$, that is

$$\dot{y}(t) = \sigma(u^L, u^R),$$

follows from the left and right limits (5.4) and the construction of wave-front tracking approximation.

Remark 5.1. If u_ν is a sequence of exact solutions to (1.1) with uniformly bounded total variation such that $u_\nu \to u$ in L^1_{loc}, then by a standard diagonal argument on the approximating wavefront solutions $u_{\nu,\epsilon}$ one obtains the following.

(1) If P is a continuity point of u but not an interaction point in Θ, then u_ν converges strongly to u, i.e. for all η there exists r such that

$$\big|u_\nu(B(P,r)) - u(P)\big| \leq \eta.$$

(2) If P is a discontinuity point but not an interaction point, then there exists discontinuities in u_ν converging to the discontinuity of u in P and such that the values of u_ν converges to the values of u on the left and on the right of the discontinuity in the sense of Theorem 1.3.

Remark 5.2. A fairly easy consequence of the convergence of the wave structure is that the wave strength s_i converges weakly. In fact, the convergence of u^{L}, u^{R} on each shock apart the point in Θ yields that the decomposition of the measures

$$u_x^\pm = \sum_i v_i^\pm \hat{r}_i(t,x), \quad \hat{r}_i(t,x) = \begin{cases} r_i(u(t,x)) & (t,x) \text{ continuity point of } u, \\ u^{\mathrm{R}} - u^{\mathrm{L}}/|u^{\mathrm{R}} - u^{\mathrm{L}}| & (t,x) \text{ discontinuity point in } \cup \mathscr{T}_i \setminus \Theta, \end{cases}$$

converges weakly: indeed, even if the decomposition is nonlinear, the convergence of u given by Theorem 1.4 yields that the vectors $\hat{r}_i$ converges in L^1 w.r.t. the measure $|u_x|(t)$ outside the countable number of times $P_t\Theta$.

Thus it is possible to pass to the limit to the wave balances (3.10) as in [4], obtaining as in [4] that

$$\big|\partial_t v^i + \partial_x(\hat{\lambda}_i v^i)\big| \leq \mu^I, \quad \big|\partial_t |v|^i + \partial_x(\hat{\lambda}_i |v|^i)\big| \leq \mu^{IC}.$$

6. A counterexample on general strict hyperbolic systems

In this last section we prove that the assumption of piecewise genuinely nonlinearity cannot be omitted: by an explicit example we show that the set of jump points of its admissible solution does not contain any segment, even if it is of positive $\mathcal{H}^1$-measure. Hence the pointwise convergence of the limits in the left and right of a discontinuity cannot the proved, since there is not a clear boundary.

Consider a 2×2 system of the following form

$$\begin{cases} u_t + f(u,v)_x = 0, \\ v_t - v_x = 0. \end{cases} \tag{6.1}$$

where f is a smooth function. The Jacobian matrix of flux function is

$$DF(u,v) = \begin{pmatrix} f_u & f_v \\ 0 & -1 \end{pmatrix},$$

and the eigenvalues, eigenvectors are

$$\lambda_1 = -1, \quad \lambda_2 = f_u, \qquad r_1(u,v) = (f_v, -f_u - 1)^T, \quad r_2 = (1,0)^T.$$

The system is strict hyperbolic if $f_u > -1$.

We will choose f in order to have

$$Z_2 = \{(u,v) : \nabla\lambda_2 \cdot r_2\} = \{(u,v) : f_{uu}(u,v) = 0\} = \{v = 0\}. \tag{6.2}$$

This yields that the vector field r_2 is tangent to the manifold Z_2, therefore the second characteristic family is not piecewise genuinely nonlinear or linearly degenerate.

Define $f(u,v) = e^{-1/v}u^2/2$ when $v > 0$ and $f(u,0) \equiv 0$. The value of f for $v < 0$ will be computed below, in order to have the wave pattern we desire.

Let the initial data to be

$$u_0(x) = \begin{cases} u_\ell & x < 0, \\ u_r & x > 0, \end{cases} \qquad v_0(x) = \begin{cases} -a & x < h, \\ a & x > h, \end{cases} \tag{6.3}$$

for some small constants $u_\ell > u_r$ and $a, h > 0$.

Since the second equation in (6.1) is a linear transport equation, one has

$$v(x,t) = \begin{cases} -a & x+t<h, \\ a & x+t>h. \end{cases} \tag{6.4}$$

Then one can solve the system (6.1) by regarding it as a scalar conservation laws of u

$$u_t + f(u,v)_x = 0$$

with discontinuous coefficient v. The definition of f for $v<0$ is chosen in order to have a solution whose wave pattern is given by Figure 8: a centered rarefaction waves at $t=0$ for u which after crossing the shock of v becomes a centered compressive waves, generating a shock.

If u^- is the value of u before crossing the jump of v of size $2a$, then by Rankine-Hugoniot conditions

$$-(u^+ - u^-) = f(u^+, a) - f(u^-, -a).$$

This yields

$$u^+ = e^{1/a}\Big(\sqrt{1+2e^{-1/a}(f(u^-,-a)+u^-)} - 1\Big). \tag{6.5}$$

The equation for the wave with value u^+ and converging to the point $(2h,0)$ is

$$x = e^{-1/a}u^+(t-2h),$$

while the equation for the wave u^- starting at 0 is

$$x = f_u(u^-,-a)t.$$

Since they have to meet at the same point along the line $x = h-t$, one obtains

$$e^{-1/a}u^+\big(-2f_u(u^-,-a)-1\big) = f_u(u^-,-a). \tag{6.6}$$

Hence, substituting (6.5) into the expression (6.6), we obtain the ODE defining $f(u,-a)$

$$f_u(u,-a) = -\frac{e^{-1/a}(e^{-1/a}u^2/2 - f(u,-a))}{1+2e^{-1/a}(e^{-1/a}u^2/2 - f(u,-a))} = \frac{1-g(u,-a)}{2g(u,-a)-1}, \tag{6.7}$$

where $g(u,-a) = \sqrt{1+2e^{-1/a}(f(u,-a)+u)}$. By setting $f(0,-a)=0$, we can solve this ODE obtaining a function $f(u,-a)$, in a neighborhood of $u=0$, smoothly depending on the parameter a: the explicit solution is given by

$$f(u,-a) = \frac{1}{4}e^{1/a}\Big(\sqrt{1+4e^{-1/4}u} - 1 - 2e^{-1/a}u\Big).$$

It is easy to see that $f(\cdot,a)$ is concave for $a<0$, because

$$f_{uu} = \frac{-g_u}{2g+1} < 0.$$

Finally, since $g(u,a)$ tends to 0 exponentially fast as $a \to 0$, one can also see that f is smooth across the line $v=0$.

Notice that there is shock of 2-th family starting from the point $(h,0)$. However, we can modify the initial data a little to get rid of this shock. In fact, recalling the formula (6.5) and letting

$$u_1 = \frac{-1+\sqrt{1+2e^{-1/a}(f(u_r,-a)+u_r)}}{e^{-1/a}}.$$

We can replace u_0 in the initial data by

$$\tilde{u}_0 = \begin{cases} u_l & x<0, \\ u_r & 0<x<h, \\ u_1 & x>h. \end{cases}$$

By the total variation estimates for the general system

$$\text{Tot.Var.}\{u(t,\cdot)\} \lesssim \text{Tot.Var.}\{u_0(\cdot)\},$$

it is not restrict to assume that the total variation of $\tilde{u}_0$ is sufficiently small.

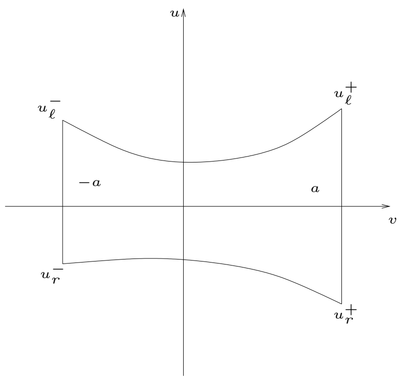

FIGURE 7.

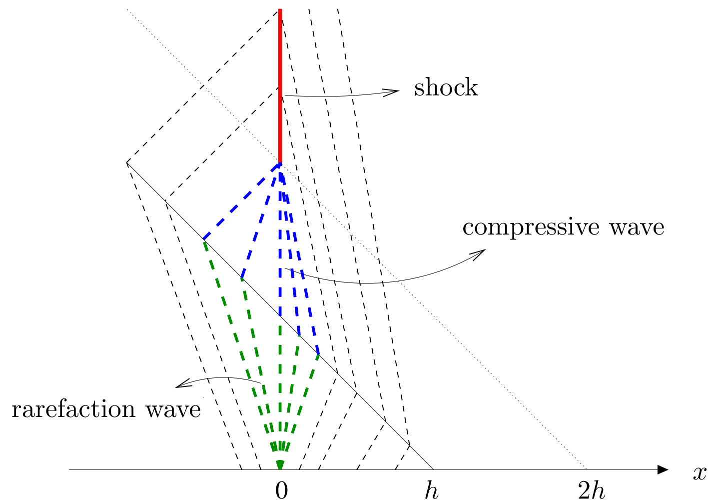

FIGURE 8.

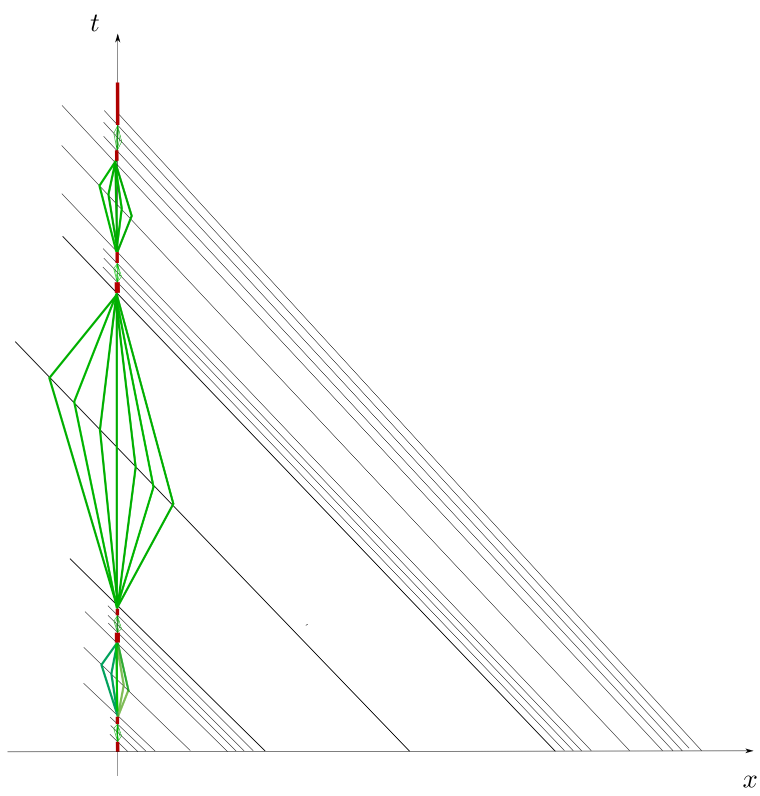

FIGURE 9.

Using this function f it is now easy to construct the example. In fact, if $\{(a_\ell, b_\ell)\}_\ell$ is a sequence of open sets in $[0, 1]$ whose complement is a Cantor set K of positive Lebesgue measure, take in fact initial data for u as

$$u(0, x) = u^- \chi_{x<0} + u^+ \chi_{x>0} + \sum_\ell u_{0,\ell} \Big(\chi_{(a_\ell, (a_\ell + b_\ell)/2)} - \chi_{(a_\ell + b_\ell)/2, b_\ell)} \Big).$$

where the sequence $\{u_{0,\ell}\}$ is chosen to get rid of extra shocks of 2-th family starting at points $(a_\ell, 0), (b_\ell, 0)$, and define

$$v(0,x) = \sum_\ell v_{0,\ell} \Big(\chi_{(a_\ell,(a_\ell+b_\ell)/2)} - \chi_{(a_\ell+b_\ell)/2,b_\ell)} \Big).$$

Then one can verify that the waves pattern is as in Figure 9.

Thus the times where $u(t)$ has a discontinuities are given exactly by K: the solution oscillates on the Riemann invariants of Figure 7.

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