SBV regularity for scalar conservation laws*

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Abstract

We outline a short proof of SBV regularity which can be extended to systems of conservation laws. The fundamental idea is the construction of an interaction measure which controls the creation of jumps.

1 Introduction

In recent years regularity estimates for nonlinear equation have received a lot of attention. A natural regularity question in hyperbolic conservation laws is the following: being the solution u in general BV, under which conditions it belongs to the smaller space Special BV (SBV)? We recall that the space SBV is the space of all BV function for which the measure derivative u_x does not contain a Cantor part.

Since solution u to Hamilton-Jacobi equations with uniformly convex Hamiltonian are known to be semiconcave, then a similar question can be stated in this cases: we would like to prove that the measure second derivative of u does not have a Cantor part.

In this introduction we first review the most important results concerning SBV regularity.

The first positive result has been given in [ADL], where it is shown that the solution $u(t)$ of a genuinely nonlinear scalar conservation law in one space dimension is SBV up to countably many times. In that paper, the authors considers the characteristic lines

$$\dot{x} = f'(u(t,x)), \quad u(0,x) = y,$$

and prove the following: every time a Cantor part in $u_x(t)$ appears, then there is a set of positive measure A such that all the characteristics starting from $y \in A$ are defined in the interval $[0,t]$ but cannot be prolonged more than t. By the σ-finiteness of $\mathcal{L}^1$, one can apply the same observation used to prove that the positive part of $u_x(t)$ is absolutely continuous up to countably many times, and deduce that up to countably many times the solution $u(t)$ is SBV.

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The use of the measure of the set $A(t)$ of initial points for characteristics which can be prolonged up to time t has been applied to obtain extension of the above result: in [Rog] the SBV estimate is used for scalar balance laws, later extended to Temple systems in [AnN] and in [BDR] to the case of Hamilton-Jacobi equation in several space dimension with uniformly convex Hamiltonian.

In [BC] a different approach is introduced. The authors shows that there is a bounded measure μ^{jump} which controls the creation of jumps in the entropic solution u to a genuinely nonlinear hyperbolic system of conservation laws. The idea is particularly simple to understand: the genuinely nonlinearity makes easy to create jumps, but only by interaction and cancellation one can remove them. This measure allows to make balances concerning only the continuous part of the measure u_x, and thus to recover SBV regularity as a consequence of decay properties of the solution: in fact, for Burgers equation, the fundamental estimate

$$-\frac{\mathcal{L}^1(B)}{t-T} + \mu^{\text{jump}}(T,t) \le u_x(T,B) \le \frac{\mathcal{L}^1(B)}{T} \tag{1}$$

resembles the decay estimate for positive waves. In (1), B is a Borel set and $0 \le T \le t$.

In this short paper we give the fundamental ideas in the particularly simple case of a scalar conservation law with convex flux, and we show how the quantity $f'(u)$ is in SBV, i.e. $\sigma := D_x f(u)$ does not have a Cantor part. The paper is organized as follows. After recalling some basic fact of wavefront tracking approximation, we introduce the wave balances of the quantities $\sigma = D_x f'(u)$ and it positive, negative, continuous and jump part. These balances allows to define the new interaction measure μ^{jump}, and to prove that it is a finite measure on $\mathbb{R}^+ \times \mathbb{R}$. Once this measure has been introduced, we use this to compute the wave balance in regions bounded by characteristics, i.e. solution to the differential inclusion $\dot{x} \in f'(u(t,x))$. The argument at this point is standard, the new ingredient is the use of the measure μ^{jump} to control the continuous part of σ.

2 Preliminaries

We consider the scalar conservation law

$$u_t + f(u)_x = 0, \qquad u(0,x) = u_0(x), \tag{2}$$

with $u_0(x) \in \mathrm{BV}(\mathbb{R})$: for simplicity we will assume $|u_0(x)| = 1$. The flux f is assumed smooth, at least $C^2(\mathbb{R})$, and we will denote with f', f“ its first and second derivatives.

A standard method for constructing a sequence of approximate solutions converging to the unique entropy solution of (2) is given in [Daf2]: let $\nu \in \mathbb{N}$ and consider the points $\{z2^{-\nu}\}_{z\in\mathbb{Z}}$. Let f^ν be the piecewise linear function whose nodes are at the points $\{(z2^{-\nu}, f(z2^{-\nu}))\}_{z\in\mathbb{Z}}$:

$$f^\nu(u) := (1-\alpha)f\big(z2^{-\nu}\big) + \alpha f\big((z+1)2^{-\nu}\big), \quad u = (1-\alpha)z2^{-\nu} + \alpha(z+1)2^{-\nu}.$$

The standard Riemann solver for f^ν takes values only in $\{z2^{-\nu}\}_{z\in\mathbb{Z}}$ if u^-, u^+ belongs to the same set $\{z2^{-\nu}\}_{z\in\mathbb{Z}}$, so that a solution can be constructed by starting with $u_0^\nu \in BV(\mathbb{R}, \{z2^{-\nu}\}_{z\in\mathbb{Z}})$. In fact, one solves the initial Riemann problem (finitely many!), and then each time two or more waves collide (an interaction point) the procedure starts again by solving this newly generated Riemann problem.

The key argument is that at each interaction either the Glimm interaction functional decreases (or equivalently the number of waves), or the total variation decreases, and in both cases of a fixed positive quantity depending only on ν. Hence the number of interaction is finite, and one has not to worry about the cases of infinite interaction points in finite time.

The convergence to the entropy solution follows because the approximated solutions generate a 1-Lipschitz semigroup in L^1, and the set $BV(\mathbb{R}, \{z2^{-\nu}\}_{z\in\mathbb{Z}})$ is dense in $L^\infty(\mathbb{R}, [0,1])$ as $\nu \to +\infty$ w.r.t. the L^1-norm.

Since the number of interactions are countable, we can perturb a little bit the speed of the waves in order to require that no multiple interaction occurs: only two waves at a time interact. This will simplify the computations in the next sections.

The fundamental assumption on the flux f is the following.

Assumption 1 *The flux f is a convex function.*

This implies that the entropy admissible jumps $[u^-, u^+]$ satisfy $u^+ < u^-$. As for notation, by C we will denote a sufficiently large constant.

3 Wave balances

In this section we develop some balances for particular waves measures. These are nonlinear functions of the derivative Du, which is a signed measure on $\mathbb{R}$.

Definition 3.1 *We define the* wave measure σ *by* $\sigma(t) = D_x f'(u(t))$, *i.e.*

$$\forall \phi \in C_c(\mathbb{R}) \ \left(\int \phi\sigma := -\int \frac{d}{dx}\phi(x) f'(u(t,x)) \mathcal{L}^1(dx) \right).$$

Since $t \mapsto u(t)$ is a Lipschitz function in $L^1_{\mathrm{loc}}(\mathbb{R})$, it follows easily that σ is continuous w.r.t. the weak* topology.

Similarly, we can consider the Jordan decomposition

$$\sigma(t) = \sigma^+(t) + \sigma^-(t), \qquad \sigma^+, -\sigma^- \geq 0, \ \sigma^+ \perp \sigma^-,$$

and the maps $t \mapsto \sigma^+(t), \sigma^-(t)$. It is a general fact [BC] that these maps are universally measurable if $t \mapsto \sigma(t)$ is continuous, but simple examples shows that the continuity is lost.

Define the speed of the each jump by the Rankine-Hugoniot conditions: outside of interaction times,

$$\tilde{\lambda}(t,x) := \begin{cases} \frac{f(u^+)-f(u^-)}{u^+-u^-} & u \text{ has the jump } [u^-, u^+] \text{ in } (t,x), \\ f'(u(t,x)) & u \text{ is continuous in } (t,x). \end{cases}$$

Since the set of times where $\tilde{\lambda}$ is not defined in finite, then this set is negligible for the measures

$$\int \sigma(t)\mathcal{L}^1(dt), \quad \int \sigma^+(t)\mathcal{L}^1(dt), \quad \int \sigma^-(t)\mathcal{L}^1(dt)$$

(it is negligible when integrating in times), so that the divergence forms

$$\partial_t \sigma(t) + \partial_x \big(\tilde{\lambda}\sigma(t)\big), \quad \partial_t \sigma^+(t) + \partial_x \big(\tilde{\lambda}\sigma^+(t)\big), \quad \partial_t \sigma^-(t) + \partial_x \big(\tilde{\lambda}\sigma^-(t)\big) \tag{3}$$

are distributions in $I\!\!R^2$.

Another way of interpreting (3) is that we ask if the maps $t \mapsto \sigma, \sigma^+, \sigma^-$ satisfy some PDE in the weak (distributional) sense.

In all three cases, it follows that the set where these divergence forms are not 0 are the interaction points $\{(t_i, x_i)\}_{i=1}^I$: in fact, the speed $\tilde{\lambda}$ has been chosen exactly to be the speed of the jump. Hence, a direct computation yields for example

$$\partial_t \sigma + \partial_x \big(\tilde{\lambda}\sigma\big) = \sum_i p_i \delta_{(t_i, x_i)},$$

where

$$p_i = \sum_{\text{exiting}} \sigma - \sum_{\text{entering}} \sigma,$$

i.e. the difference of the strength (with sign) of the waves after the interaction and before the interaction.

The same formula holds also for σ^+, σ^-, the difference being in the count of the entering and exiting waves in the interaction point: more precisely,

$$\partial_t \sigma^\pm + \partial_x \big(\tilde{\lambda}\sigma^\pm\big) = \sum_i p_i^\pm \delta_{(t_i, x_i)},$$

with

$$p_i^\pm = \sum_{\text{exiting}} \sigma^\pm - \sum_{\text{entering}} \sigma^\pm,$$

A final observation is that $p_i = p_i^+ + p_i^-$, simply because $\sigma = \sigma^+ + \sigma^-$.

3.1 Balance for σ

Using the fact that in the approximated solution u^ν only jumps are present, by the definition of σ we have that

$$\sigma(t) = \sum_{j \text{ jumps}} \Big(f'\big(u(t, x_j(t)+\big) - f'\big(u(t, x_j(t-))\big)\Big)\delta_{x_j(t)}$$

where $x_j(t)$ are piecewise linear functions defined in some time interval.

For the interaction point (t_i, x_i) we obtain

$$\sum_{\text{entering}} \sigma = f\big(u(t_i, x_i-)\big) - f\big(u(t_i, x_i+)\big) = \sum_{\text{exiting}} \sigma,$$

and hence the measure $\sigma(t)$ satisfies the conservation equation in divergence form

$$\partial_t \sigma + \partial_x \big(\tilde{\lambda} \sigma\big) = 0.$$

We note here that this equation has many more solutions, and we are selecting one particular solution, namely $\sigma(t) = D_x f'(u(t))$.

3.2 Balance for $\sigma^\pm$

Recall that by our assumption on the approximation, only two waves collide at a time. We thus consider two cases. For shortness we will remove the index i from the quantity p.

3.2.1 Waves with the same sign

This case can happen only for σ^-, and by the assumption on convexity from the interaction one single wave exist which is the sum of the two entering: hence $p^+ = p^- = 0$. Another way of seeing it is that since the waves are negative, we conclude that $p^+ = 0$ and thus

$$p^- = p - p^+ = 0.$$

3.2.2 Waves with different sign

In this case a positive waves is cancelled by a negative one, so that the quantity $2^{1-\nu}$ of total variation disappears. The computation for p^+ is thus

$$p^+ = -\sigma^+, \quad \sigma^+ \text{ entering}.$$

If we denote with u^-, u^m, u^+ the values of u before the interaction, then using the regularity of f we have that for

$$\sigma^+ = f'(u^+) - f'(u^m) \leq \int_{u^m}^{u^+} f''(s) ds \leq C\big(u^+ - u^-\big) = C 2^{-\nu}.$$

This shows that the measure $\sum_i p^+ \delta_{(t_i,x_i)}$ is bounded by

$$\bigg\| \sum_i p^+ \delta_{(t_i,x_i)} \bigg\| = \sum_i |p_i| \leq C \text{Tot.Var.}(u).$$

We call it the *cancellation measure* $\mu^{C,\nu}$.

Since $p_i^- = -p_i^+$, we recover the same bound for $\sum_i p^- \delta_{(t_i,x_i)}$.

4 The jump measure

Fix $0 < \epsilon_0 = k_0 2^{-\nu} \leq \epsilon_1/2 = k_1 2^{-\nu-1}$. We assume that ϵ_0 is a multiple of ν.

Definition 4.1 *A Lipschitz curve $\gamma : [t_1, t_2] \to I\!R^2$, $\gamma(t) = (t, x(t))$, is an (ϵ_0, ϵ_1)-shock if*

1. *$\sigma(\gamma(t)) \geq \epsilon_0$ for all $t \in [t_1, t_2]$;*
2. *$\sigma(\gamma(\bar{t})) \geq \epsilon_1$ for at least one $\bar{t} \in [t_1, t_2]$.*

Since the flux function f is smooth, one has

$$\sigma(t,x) = f'(u(t,x+)) - f'(u(t,x-)) = \int_{u^-}^{u+} f''(s)ds \leq C|u^+ - u^-|,$$

and from $u \in BV$ it follows that there are at most finitely many (ϵ_0, ϵ_1)-jumps.

Definition 4.2 *The set $J^{\epsilon_0,\epsilon_1}$ of the (ϵ_0, ϵ_1)-shocks is*

$$J^{\epsilon_0,\epsilon_1} := \bigcup \Big\{ \gamma_i([t_{1,i}, t_{2,i}]) : \gamma_i \;\; (\epsilon_0, \epsilon_1)\textit{-shock in } [t_{1,i}, t_{2,i}] \Big\}.$$

Denote with σ^ϵ the measure

$$\sigma^\epsilon := \sigma|_{J^{\epsilon_0,\epsilon_1}}.$$

We compute the distribution

$$\partial_t \sigma^\epsilon + \partial_x \big(\tilde{\lambda} \sigma^\epsilon \big) = \sum_i p_i^\epsilon \delta_{(t_i,x_i)}.$$

At each interaction, the following cases happens:

1. two negative waves collide, generating the beginning point of an (ϵ_0, ϵ_1)-shock: then $p_i^\epsilon \leq 0$;
2. two (ϵ_0, ϵ_1)-jumps collinde: hence $p_i^\epsilon = 0$;
3. an (ϵ_0, ϵ_1)-jump of size $> k_0 2^{-\nu}$ collides with a positive wave of size $2^{-\nu}$: then its size remains above ϵ_0 and $p_i^\epsilon = C2^{-\nu}$;
4. an (ϵ_0, ϵ_1)-jump of size $k_0 2^{-\nu}$ collides with a positive wave of size $2^{-\nu}$, and thus it disappears as an (ϵ_0, ϵ_1)-shock: then $p_i^\epsilon = k_0 2^{-\nu}$. Then along the shock at least a cancellation of order $\epsilon_1 - \epsilon_0$ occurred, and thus

$$\begin{aligned} \epsilon_0 \sharp \Big\{ \text{number of terminal points} \Big\} \;\; &\leq \;\; \frac{\epsilon_0}{\epsilon_1 - \epsilon_0} \Big\{ \text{cancellation measure} \Big\} \\ &\leq \;\; \frac{k_0}{k_1 - k_0} \text{Tot.Var.}(u(0)). \end{aligned}$$

We conclude that the measure

$$\mu^{\text{jump},\epsilon,\nu} := \partial_t \sigma^\epsilon + \partial_x \big(\tilde{\lambda} \sigma^\epsilon \big)$$

is a uniformly bounded measure, because it positive part is uniformly bounded and

$$\begin{aligned}
\mu^-(I\!R^+ \times I\!R) &= \mu^+(I\!R^+ \times I\!R) - \mu(I\!R^+ \times I\!R) \\
&\leq \left(1 + \frac{k_0}{k_1 - k_0}\right)\mu^{C,\nu}(I\!R^+ \times I\!R) \\
&\quad + \int_0^{+\infty} \int_I R\partial_t \sigma^\epsilon + \partial_x\big(\tilde{\lambda}\sigma^\epsilon\big) dx dt \\
&\leq \left(1 + \frac{k_0}{k_1 - k_0}\right)\mu^{C,\nu}(I\!R^+ \times I\!R) + \text{Tot.Var.}(u(0)).
\end{aligned}$$

5 Decay estimates

The key argument here is the balance of positive waves in backward cones and the balance of negative waves in forward cones. For simplicity we will consider these balances for the entropic solution and not for the approximated solution: in the latter case one has to add an $\mathcal{O}(\nu + \epsilon_1)$ to the estimates.

5.1 Decay of positive waves

Consider an interval $I = [a, b]$, and let $a(t)$, $b(t)$ be the minimal backward characteristics such that $a(T) = a$, $b(T) = b$. The definition of wavefront solution implies that no waves can enter in the interval $I(t) = [a(t), b(t)]$, so that the definition of σ implies

$$\frac{d}{dt}\big(b(t) - a(t)\big) = \sum \sigma(t, I(t)) = \sigma(T, I).$$

Since $b(0) - a(0) = \sigma(T, I)T \geq 0$, we deduce immediately that $\sigma(T, I) \leq \frac{b-a}{T}$.

5.2 Decay of negative waves

In this case the analysis is similar, but negative waves can enter in the future, and we need only to consider the continuous part: the balance is measured by μ^{jump}. For this, we have that the difference in speed satisfies

$$\frac{dz}{dt} \leq \sigma^{\text{cont}}(t, I(t)) = \sigma^{\text{cont}}(T, I) - \mu^{\text{jump}}(T, t).$$

Integrating in time and using that $z(t) > 0$ we obtain

$$\sigma^{\text{cont}}(T, I) \geq -\frac{z(T)}{t - T} + \mu^{\text{jump}}(T, t).$$

Bot estimates can be extended to finitely many intervals, and with standard methods to Borel subsets. Collecting the two estimates we arrive to the following proposition.

Proposition 5.1 *If u is the entropy BV solution to a scalar conservation law with convex flux, then the measure $\sigma = f'(u)_x$ satisfies the estimate*

$$-\frac{\mathcal{L}^1}{t-T} + \mu^{\text{jump}}(T,t) \leq \sigma^{\text{cont}}(t,B) \leq \frac{\mathcal{L}^1(B)}{t}$$

for all B Borel.

In particular, if B is of Lebesgue measure 0 but $\sigma^{\text{cont}}(B) > 0$, we obtain that

$$\mu^{\text{jump}}(t) \leq \sigma^{\text{cont}}(t,B) \leq 0,$$

from which we deduce that the measure $\mu^{\text{jump}}(t)$ is not 0 every time a Cantor part appears in σ^{cont}: this means that immediately this Cantor measure is transformed into jumps.

Corollary 5.2 *Up to countably many times, the function $f'(u(t))$ is SBV.*

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