A NOTE ON SINGULAR LIMITS TO HYPERBOLIC SYSTEMS

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ABSTRACT. In this note we consider two different singular limits to hyperbolic system of conservation laws, namely the standard backward schemes for non linear semigroups and the semidiscrete scheme.

Under the assumption that the rarefaction curve of the corresponding hyperbolic system are straight lines, we prove the stability of the solution and the convergence to the perturbed system to the unique solution of the limit system for initial data with small total variation. The method used here to estimate the source terms is based of the calculus of residues.

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S:intro

1. Introduction

Consider a hyperbolic system of conservation laws in one space variable

E:hcl1 (1.1)
$$\begin{cases} u_t + f(u)_x = 0 \\ u(0,x) = u_0(x) \end{cases}$$

where $u \in \mathbb{R}^n$ and f is a smooth function from an open set $\Omega \subseteq \mathbb{R}^n$ with values in $\mathbb{R}^n$. Let K_0 be a compact set contained in Ω, and let δ_1 sufficiently small such that the compact set

E:compact1 (1.2)
$$K_1 \doteq \Big\{ u \in \mathbb{R}^n : \mathrm{dist}\big(u, K_0\big) \leq \delta_1 \Big\}$$

is entirely contained in Ω.

We assume that the Jacobian matrix $A = Df$ is uniformly strictly hyperbolic in K_1, i.e.

E:strhyp1 (1.3)
$$\min_{i<j} \Big\{ \lambda_j(u) - \lambda_i(v) \Big\} \geq c > 0, \qquad \forall u, v \in K_1,$$

where we denote by $\lambda_i(u)$ the eigenvalues of $A(u)$, $\lambda_i < \lambda_j$. Let $r_i(u)$, $l^i(u)$ be the its right, left eigenvectors, normalized such that

$$\big|r_i(u)\big| = 1, \qquad \big\langle l^j(u), r_i(u) \big\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

In this setting it is proved that if $u_0(-\infty) \in K_0$ and Tot.Var.(u_0) is sufficiently small, there exists a unique "entropic" solution $u : [0, +\infty) \mapsto \mathrm{BV}(\mathbb{R}, \mathbb{R}^n)$ in the sense of [6]. Moreover these solutions can be constructed as limits of wave front tracking approximations and they depend Lipschitz continuously on the initial data in the L^1_{loc} topology.

For a special class of systems, called in [8] *Straight Line Systems*, i.e. systems such that the integral curves of the right eigenvectors $r_i(u)$ are straight lines, or equivalently

E:straight1 (1.4)
$$\big(Dr_i\big) r_i = 0,$$

very recently it has been proved that solutions to (1.1) can be constructed as L^1 limits of solutions to different singular approximations of the hyperbolic system:

- Vanishing viscosity approximation [4], [5]. This is the limit as $\epsilon \to 0$ of the solution $u^\epsilon(t)$ of the system
$$u_t + f(u)_x - \epsilon u_{xx} = 0.$$

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- Relaxation approximation [2, 9]. While in case 1) the perturbation is parabolic, in this case we consider a hyperbolic perturbation, namely
$$u_t + f(u)_x = \epsilon\big(\Lambda^2 u_{xx} - u_{tt}\big),$$
where, for linear stability, Λ is strictly bigger than all the eigenvalues of $Df(u)$.
- Godunov scheme [8]. This is a discrete scheme obtained from (1.1) by considering differential ratio instead of derivatives:
$$u(n+1,j+1) = u(n,j+1) + \frac{\Delta t}{\Delta x}\Big[f\big(u(n,j)\big) - f\big(u(n,j+1)\big)\Big],$$
where, for stability of the scheme, it is assumed that $0 < \lambda_1 < \cdots < \lambda_n < \Delta x/\Delta t$.

The main task in showing the convergence of these approximations is to obtain uniform BV estimates for $t \geq 0$, if the initial data u_0 are of sufficiently small total variation.

This task is achieved by decomposing the equations satisfied by u_x, or u_x and u_t in [2], or $u(n,j) - u(n,j-1)$ in [8], as n scalar perturbed conservation laws, coupled by terms of higher order. These terms are then considered as the source of total variation. For the special case of straight line systems and for the vanishing viscosity approximation, a decomposition of u_x which makes the source terms integrable is the projection along the eigenvectors r_i of the Jacobian $Df(u)$:

E:decomp01 (1.5)
$$u_x = \sum_i v^i r_i.$$

Once it is proved that the L^1 norm of the component v^i is bounded, by Helly's theorem there exists a subsequence u^{ϵ_k} converging to a weak solution $\bar{u}(t)$ of (1.1) as $k \to \infty$.

This result can be understood if one thinks to the source of total variation for solutions to (1.1). In fact, due to the assumption (1.4), the shock and the rarefactions curves coincide: this implies that the total variation increases only when waves of different families interact. In the approximations considered here, the same condition (1.4) implies that the travelling profiles lies on the rarefaction curves, so that a decomposition of the form (1.5) generates coupling terms of the form $v^i v^j$, $i \neq j$: in fact the source terms can be different from 0 only when two waves of different families are present at the same point. Finally, since the speeds of the components are different due to (1.3), one can show that these coupling terms are of squared order w.r.t. the L^1 norm of v^i.

To prove the uniqueness of the limit $\bar{u}(t)$, one consider the equation for a perturbation h of the singular approximations. We observe that $h = u_x$ is a particular solution of such system. A generalization of the arguments used to prove an a priori bound on the total variation of u shows the boundedness of the L^1 norm of the components h^i, where

E:linear0 (1.6)
$$h = \sum_i h^i r_i(u).$$

By a standard homotopy argument [4], this yields the stability of all solutions of the approximating system. Since the Lipschitz continuous dependence on the initial data is uniform w.r.t. both ϵ and t, in the limit we obtain a uniform Lipschitz semigroup $\mathcal{S}$.

Finally it is well known that a uniform Lipschitz semigroup of solutions to (1.1) is uniquely defined if we know the jumps conditions of the entropic shocks, see [7]. In our case, one can analyze the Green kernel of the linearized equation (1.6) when ϵ tends to 0 to prove that in the hyperbolic limit there exists a constant $\hat{\lambda} > 0$ such that
$$\int_a^b \big|\mathcal{S}_t u(x) - \mathcal{S}_t v(x)\big| dx \leq \int_{a-\hat{\lambda}}^{b+\hat{\lambda}} \big|u(x) - v(x)\big| dx.$$
The above equation implies a local dependence on the initial data for the limiting semigroup. This result and the fact that in the scalar case the solution u^ϵ converges to the entropic solution prove that the jump conditions coincide with the scalar jumps along the eigenvectors r_i, see [4].

Thus, under the assumption (1.4), the limit semigroup is independent on the approximation and coincides with the solution constructed by wave front tracking using the classical Lax Riemann solver.

In this note we want to extend the previous approach to the following cases:

(1) Semigroup approximation [10, 11]. This is obtained as limit of the system

$$\frac{u(t,x)-u(t-\epsilon,x)}{\epsilon}+A\big(u(t,x)\big)u(t,x)_x=0.$$

This is the standard backward scheme for non-linear semigroups.

(2) Semi-discrete schemes [1], i.e. infinite dimensional ODE defined by

$$\frac{\partial}{\partial t}u(t,x)+\frac{1}{\epsilon}\Big(f\big(u(t,x)\big)-f\big(u(t,x-\epsilon)\big)\Big)=0.$$

We will prove that as $\epsilon \to 0$ the limits of the respective solutions converge to a unique solution to (1.1), and that this limits defines a Lipschitz continuous semigroup $\mathcal{S}$ on the space of function with small TV. Moreover, using the same arguments of [4], this semigroup is perfectly defined by a Riemann solver which, as explained above, coincides with the Riemann solver obtained in [4].

The same can be proved for quasilinear systems as in [8], but for simplicity we consider only systems in conservation forms. Without any loss of generality we assume that

E:genass1 (1.7)
$$\min_i\big\{\lambda_i(u)\big\}=\kappa>0,\qquad \max_i\big\{\lambda_i(u)\big\}=K<1,$$

for all u in the compact set K_1.

We now give a sketch of the proof. Using the decomposition (1.5), one obtains the equations for the components of the form

$$L_iv^i=\sum_{jnot=k}Q_i(v^j,v^k),\qquad i=1,\dots,n,$$

where

$$L_iv^i=\frac{v^i(t,x)-v^i(t-\epsilon,x)}{/\epsilon}+\big(\lambda_i(u)v^i(t,x)\big)_x$$

for the semigroup approximations, or

$$L_iv^i=v^i(t,x)_t+\frac{1}{\epsilon}\Big(\lambda_i\big(u(t,x),u(t,x-\epsilon)\big)v^i(t,x)-\lambda_i\big(u(t,x-\epsilon),u(t,x-2\epsilon)\big)v^i(t,x-\epsilon)\Big),$$

for the semidiscrete scheme. We have used the notation $\lambda_i(u,z)$ as the eigenvalue of the average matrix

$$A(u,z)=\int_0^1 A\big(\theta u+(a-\theta)z\big)d\theta.$$

In both cases, L_i generates a semigroup $t\mapsto v_i(t)$ such that

$$\big\|v_i(t)\big\|_{L^1}\le\big\|v_i(0)\big\|_{L^1}.$$

The BV bound follows if we can estimate the source terms Q_i, $i=1,\dots,n$. The computation of the source Q_i reduces to a model problem: there are two linear equations, $L_1v^1=0$, $L_2v^2=0$, with $\lambda_1<\lambda_2$ by the strictly hyperbolicity assumption, and we must estimates the quantities

E:intggg1 (1.8)
$$\sum_{n=0}^{+\infty}\int_{\mathbb{R}}\big|v^1(n,x)v^2(n,x)\big|dx,\quad\text{or}\quad\sum_{n=-\infty}^{+\infty}\int_0^{+\infty}\big|v^1(t,n)v^2(t,n)\big|dt,$$

respectively. This computation is thus a linear problem, which can be solved by estimating the above integrals for the Green kernels of the equations. Following [2], we use a simpler approach, based on the Fourier components of the solutions v^1, v^2. In the Fourier coordinates, the integrals (1.8) reduces to an integral in the complex plane, hence to a calculus of residues.

Similar computations can be applied to different schemes, if the equations for the scalar components v^i are in conservation form and the system is a Straight Line system.

S:apperox1

2. Approximation by semigroup theory

We consider in this section the case 1) of Section 1, i.e. the following singular approximation to system of conservation laws:

E:singappr1

$$\frac{u(t,x)-u(t-\epsilon,x)}{\epsilon} + A\big(u(t,x)\big)u(t,x)_x = 0, \tag{2.1}$$

where we recall that $u \in \mathbb{R}^n$ and $A(u) = Df(u)$. By the rescaling $t \to t/\epsilon$, $x \to x/\epsilon$ and setting for simplicity $u_n(x) = u(n,x)$, we obtain the evolutionary equations

E:singappr2

$$u_n - u_{n-1} + A(u_n)u_{n,x} = 0. \tag{2.2}$$

It is easy to prove that if the BV norm of u_{n-1} is sufficiently small, then u_n exists: in fact the solution can be represented as

$$u_n(x) = \int_{-\infty}^{x} \exp\left\{ \int_x^y A^{-1}\big(u_n(z)\big)dz \right\} A^{-1}\big(u_n(y)\big)u_{n-1}(y)dy,$$

and since the eigenvalues of A are positive we have that

$$\big\|u_n\big\|_\infty \leq C\text{Tot.Var.}\big(u_{n-1}\big),$$

C being a uniform constant of $\|A^{-1}\|_\infty$ in the compact set K_0.

SS:proj1

2.1. Projection on rarefaction curves.

We now start the procedure explained in Section 1. By projecting the derivative along the eigenvectors $r_i(u_n)$ of $A(u_n)$

E:proj1

$$u_{n,x} = \sum_i v_n^i r_i(u_n) = \sum_i v_n^i r_{i,n}, \tag{2.3}$$

the equations for the components v^i are

$$\sum_i v_n^i r_{i,n} - \sum_i v_{n-1}^i r_{i,n-1} + \sum_i \big(\lambda_{i,n} v_n^i r_{i,n}\big)_x = 0.$$

This can be rewritten as

E:compeq2

$$\sum_i \Big(v_n^i - v_{n-1}^i + \big(\lambda_{i,n}v_n^i\big)_x\Big) r_{i,n} = \sum_i v_{n-1}^i \big(r_{i,n-1} - r_{i,n}\big) - \sum_{i,j} \lambda_{i,n} v_n^i v_n^j \big(Dr_{i,n}\big) r_{j,n} \tag{2.4}$$

The left-hand side is in conservation form, and we consider the right-hand side as the source of total variation. If we assume as in the introduction that $(Dr_i)r_i(u) = 0$, the function $r_i(u) - r_i(v)$ is zero when $u - v$ is parallel to $r_i(u) = r_i(v)$. Thus we have

E:expas1

$$r_i(u) - r_i(v) = \sum_{j \neq i} \alpha_j(u,v) \big\langle l^j(u), u - v\big\rangle, \tag{2.5}$$

where $\alpha_j(u,u) = r_j(u)$. Using (2.2), the expansion (2.4) thus becomes

E:compeq3

$$\begin{aligned} v_n^i - v_{n-1}^i + \big(\lambda_{i,n}v_n^i\big)_x &= \sum_{j\neq k} \Big(\lambda_{k,n} v_{n-1}^j v_n^k \big\langle l_n^i, \alpha_j(u_n, u_{n-1})\big\rangle - v_n^i v_n^j \big\langle l_n^i, \big(Dr_{i,n}\big) r_{j,n}\big\rangle \Big) \\ &= \sum_{j\neq k} H_{jk}^i(n) v_{n-1}^j v_n^k + \sum_{j \neq k} K_{jk}^i(n) v_n^j v_n^k. \end{aligned} \tag{2.6}$$

To estimate the source terms in (2.6), we first consider the case of two linear equations.

S:linear

2.2. Analysis of the linear case.

Consider a single linear equation

E:linear1

$$v_n - v_{n-1} + \lambda v_{n,x} = 0, \qquad \lambda > 0. \tag{2.7}$$

We can find the fundamental solution to the previous equation by means of Fourier transform: we have

$$v_n(x) = \int_{\mathbb{R}} c(n,\xi) e^{-i\xi x} d\xi,$$

and substituting

$$c(n,\xi) - c(n-1,\xi) - i\lambda\xi c(n,\xi) = 0 \quad \Longrightarrow \quad c(n,\xi) = \frac{c_0(\xi)}{\big(1 - i\lambda\xi\big)^n}.$$

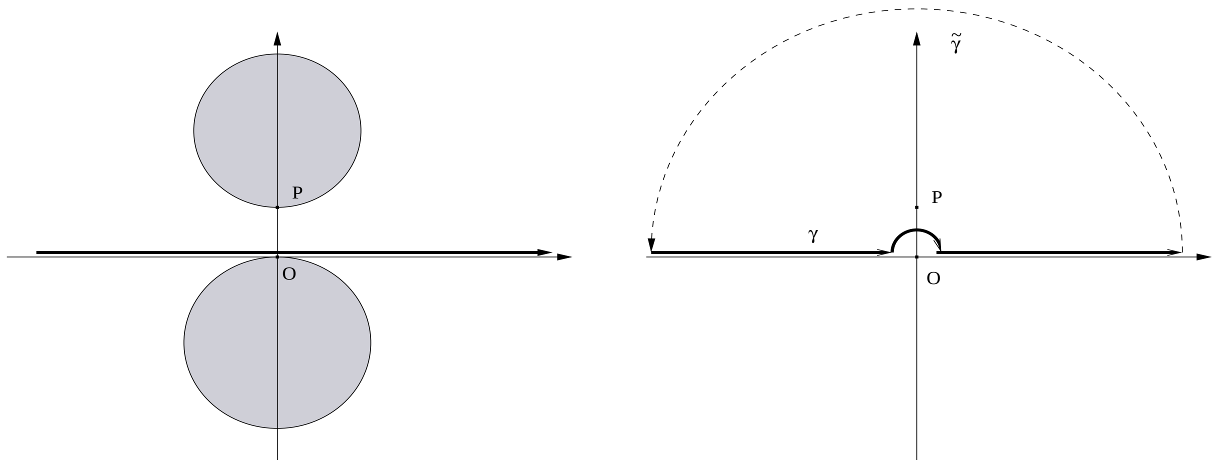

FIGURE 1. Integration path on the complex plane, where $P = i(\lambda - \mu)/\lambda\mu$. Fi:integ1

In particular the fundamental solution has $c_0(\xi) \equiv 1/2\pi$, so that

E:fundsol1
$$v_n(x) = \frac{1}{\lambda}\left(\frac{x}{\lambda}\right)^{n-1} \frac{e^{-x/\lambda}}{(n-1)!}\chi_{[0,+\infty)}(x). \tag{2.8}$$

Consider two equations of the form (2.7),

E:linear2
$$\begin{aligned} v_n - v_{n-1} + \lambda v_{n,x} &= 0 \\ z_n - z_{n-1} + \mu z_{n,x} &= 0 \end{aligned} \tag{2.9}$$

with initial data $v_0(x) = \delta(x)$ and $z_0(x) = \delta(x - x_0)$, and assume without any loss of generality that $\lambda > \mu > 0$. We can compute the intersection integrals: denoting with $d(n,\xi)$ the Fourier coefficients of $z_n(x)$ we have

E:transcomp1
$$\begin{aligned} \sum_{n=0}^{N} \int_{\mathbb{R}} v_n(x) z_n(x) dx &= \sum_{n=0}^{N} 2\pi \int_{\mathbb{R}} c(n,\xi) d(n,-\xi) e^{-i\xi x_0} d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \sum_{n=0}^{N} \frac{1}{\big(1 - i\lambda\xi\big)^n \big(1 + i\mu\xi\big)^n} e^{-i\xi x_0} d\xi. \end{aligned} \tag{2.10}$$

If ξ is considered as a complex variable, we can let $N \to +\infty$ only in the region where

$$Z \doteq \Big\{\xi \in \mathbb{Z} : \big|(1 - i\lambda\xi)(1 + i\mu\xi)\big| < 1\Big\}$$

i.e. outside the regions depicted in Figure 1. Deforming the path to avoid the region Z, we can pass to the limit:

$$\begin{aligned} \sum_{n=0}^{+\infty} \int_{\mathbb{R}} v_n(x) z_n(x) dx &= \frac{1}{2\pi}\int_{\gamma} \frac{e^{-i\xi x_0}}{1 - \frac{1}{(1-i\lambda\xi)(1+i\mu\xi)}} d\xi \\ &= \frac{1}{2\pi}\int_{\gamma} \frac{(1 - i\lambda\xi)(1 + i\mu\xi)}{(1 - i\lambda\xi)(1 + i\mu\xi) - 1} e^{-i\xi x_0} d\xi. \end{aligned}$$

By means of complex analysis we have finally that

E:transcomp2
$$P(x_0) \doteq \sum_{n=0}^{+\infty} \int_{\mathbb{R}} v_n(x) z_n(x) dx = \begin{cases} 1/(\lambda - \mu) \cdot \exp\Big((\lambda - \mu)/(\lambda\mu) x_0\Big) & x_0 < 0 \\ 1/(\lambda - \mu) & x_0 \geq 0 \end{cases} \tag{2.11}$$

In fact, depending on the sign of x_0, the integration along the line γ is equivalent to the integration around the pole 0 or the pole $P = i(\lambda - \mu)/\lambda\mu$.

S:bvest

2.3. **BV estimates.** Now to prove that (2.2) has a solution with uniformly bounded total variation. Define the functional

E:interpot1

$$Q(n) = Q(u_n, u_{n-1}) \doteq \sum_{i<j} \int_{\mathbb{R}} P_0(x-y) \Big\{ \big|v_n^i(x) v_n^j(y)\big| + \big|v_{n-1}^i(x) v_n^j(y)\big| + \big|v_n^i(x) v_{n-1}^j(y)\big| \Big\} dxdy, \tag{2.12}$$

where P is computed substituting to $\lambda - \mu$ the constant of separation of speeds c, and taking the minimal value of the exponent $(\lambda - \mu)/\lambda\mu$:

$$P_0(x) \doteq \begin{cases} 1/c \cdot \exp\Big(c/\big(K(K-c)\big) x_0\Big) & x_0 < 0 \\ 1/c & x_0 \geq 0 \end{cases}$$

We recall that c and K are defined in the introduction.

Using the same analysis of [2], we see immediately that

E:derivpot1

$$\begin{aligned} Q(n) - Q(n-1) = & \sum_{i<j} \int_{\mathbb{R}} P_0(x-y) \Big\{ \big|v_n^i(x) v_n^j(y)\big| - \big|v_{n-1}^i v_{n-1}^j\big| \Big\} \\ & + \sum_{i<j} \int_{\mathbb{R}} P_0(x-y) \Big\{ \big|v_{n-1}^i(x) v_n^j(y)\big| - \big|v_{n-2}^i v_{n-1}^j\big| \Big\} \\ & + \sum_{i<j} \int_{\mathbb{R}} P_0(x-y) \Big\{ \big|v_n^i(x) v_{n-1}^j(y)\big| - \big|v_{n-1}^i v_{n-2}^j\big| \Big\} dxdy \\ \leq & - \Big(1 - C \max_{m=1,\dots,n} \text{Tot.Var.}\big(u_m\big)\Big) \left[\sum_{j \neq k} \big|v_{n-1}^j v_n^k\big| + \sum_{j \neq k} \big|v_n^j v_n^k\big| \right], \end{aligned} \tag{2.13}$$

where C is c constant depending only on H_{jk}^i, K_{jk}^i, κ, K and c. Thus if δ_0 is sufficiently small, using (2.13) we have

$$\text{Tot.Var.}(u_1) + C_0 Q(u_1, u_0) \leq \delta_1 \quad \text{and} \quad \frac{d}{dt} \Big\{ \text{Tot.Var.}(u) + C_0 Q(u) \Big\} \leq 0,$$

where the constant C_0 is big enough, independent on δ_0. This proves that the solution u_n has uniformly bounded total variation for all $n \in \mathbb{N}$.

S:stabil

2.4. **Stability estimates.** We now consider the stability estimates of (2.2). The equations for a perturbation $u + \delta h$ as $\delta \to 0$ are

E:pertueq1

$$h_n - h_{n-1} + \big(A(u_n) h_n\big)_x = \big(DA(u_n) u_{n,x}\big) h - \big(DA(u_n) h_n\big) u_{n,x}. \tag{2.14}$$

Using the same projection of (2.3), i.e.

$$h_n = \sum_i h_n^i r_{i,n},$$

we have that the equations for the components h_n^i are

E:perteq2

$$\begin{aligned} h_n^i - h_{n-1}^i + \big(\lambda_{i,n} h_n^i\big)_x = & \sum_{j \neq k} \Big(\lambda_{k,n} h_{n-1}^j v_n^k \langle l_n^i, \alpha_j(u_n, u_{n-1}) \rangle - h_n^i v_n^j \langle l_n^i, (D r_{i,n}) r_{j,n} \rangle \Big) \\ & + \sum_{j \neq k} h_n^i v_n^j \langle l_n^i, \big(A(u_n) r_{j,n}\big) r_{i,n} - \big(A(u_n) r_{i,n}\big) r_{j,n} \rangle \\ = & \sum_{j \neq k} H(n) h_{n-1}^j v_n^k + \sum_{j \neq k} K'(n) h_n^j v_n^k. \end{aligned} \tag{2.15}$$

Using the same analysis of the above section and following the same approach of [2], one can prove that the functional

$$Q(n) = Q(u_n, u_{n-1}) \doteq \sum_{i<j} \int_{\mathbb{R}} P_0(x-y) \Big\{ \big|h_n^i(x) v_n^j(y)\big| + \big|h_{n-1}^i(x) v_n^j(y)\big| + \big|h_n^i(x) v_{n-1}^j(y)\big| \Big\} dxdy,$$

gives the estimate

$$\int_{\mathbb{R}} \big|h_n^i(x)\big| dx \leq C \int_{\mathbb{R}} \big|h_0^i(x)\big| dx,$$

hence by a standard homotopy argument the stability of the solution u_n.

S:apperox2

3. Approximation by semi-discrete scheme

We now consider the case 2) of Section 1, i.e. the following singular approximation to system of conservation laws:

E:semidiscr1

$$\frac{\partial}{\partial t}u(t,x) + \frac{1}{\epsilon}\Big(f\big(u(t,x)\big) - f\big(u(t,x-\epsilon)\big)\Big) = 0, \tag{3.1}$$

where $u \in \mathbb{R}^n$. By the rescaling $t \to t/\epsilon$, $x \to x/\epsilon$, we obtain the evolutionary equations

E:semidiscr2

$$\dot u_n(t) + f\big(u_n(t)\big) - f\big(u_{n-1}(t)\big) = 0. \tag{3.2}$$

The equation for the "derivative" $v_n \doteq u_n - u_{n-1}$ are

E:deriveq1

$$\dot v_n(t) + f\big(u_n(t)\big) - 2f\big(u_{n-1}(t)\big) + f\big(u_{n-2}(t)\big) = 0. \tag{3.3}$$

SS:proj2

3.1. **Projection on rarefaction curves.** The vector v_n is now decomposed along the eigenvectors $r_{i,n}$ of the Riemann problem u_{n-1}, u_n: we have

$$\dot u_n(t) + \sum_i \lambda_{i,n} v_n^i r_{i,n} = 0,$$

E:deriveq2

$$\begin{aligned}
\sum_i \Big(\dot v_n^i + \lambda_{i,n} v_n^i - \lambda_{i,n-1} v_{n-1}^i\Big) r_{i,n} = &- \sum_{i,j} v_n^i v_n^j \big(Dr_{i,n}\big) r_{j,n} - \sum_{i,j} v_n^i v_{n-1}^j \big(Dr_{i,n}\big) r_{j,n-1} \\
&+ \sum_i \lambda_{i,n-1} v_{n-1}^i \big(r_{i,n-1} - r_{i,n}\big) \\
= &- \sum_{i,j} v_n^i v_n^j \big(Dr_{i,n}\big) r_{j,n} - \sum_{i,j} v_n^i v_{n-1}^j \big(Dr_{i,n}\big) r_{j,n} \\
&+ \sum_{i,j} v_n^i v_{n-1}^j \big(Dr_{i,n}\big)\big(r_{j,n} - r_{j,n-1}\big) \\
&+ \sum_i \lambda_{i,n-1} v_{n-1}^i \big(r_{i,n-1} - r_{i,n}\big),
\end{aligned} \tag{3.4}$$

where $\lambda_{i,n}$ and $r_{i,n}$ are the eigenvalues and right eigenvectors of the average matrix

$$A\big(u_n, u_{n-1}\big) \doteq \int_0^1 Df\Big(u_{n-1} + \big(u_n - u_{n-1}\big)s\Big) ds.$$

If we assume the condition (1.4), the functions $(Dr_{i,n})r_{j,n}$ and $r_{i,n} - r_{i,n-1}$ are zero when $u_n - u_{n-1}$ and $u_{n-1} - u_{n-2}$ are parallel to $r_{i,n} = r_{i,n-1}$. Thus we have

E:expans1

$$\begin{aligned}
r_{j,n} \bullet r_{i,n} &= \sum_{j \neq i} \alpha_{j,n} v_n^j, \\
r_{i,n} - r_{i,n-1} &= \sum_{j \neq i} \beta_{j,n} v_n^j + \sum_{j \neq i} \gamma_{j,n-1} v_{n-1}^j,
\end{aligned} \tag{3.5}$$

as in Section 2.1. Using (3.5), the expansion (3.4) thus becomes

E:deriveq3

$$\dot v_n^i + \lambda_{i,n} v_n^i - \lambda_{i,n-1} v_{n-1}^i = \sum_{j \neq k} H_n(t) v_n^j v_n^k + \sum_{j \neq k} G_n(t) v_n^j v_{n-1}^k. \tag{3.6}$$

To estimate the source terms in (3.6), we consider the case of two linear equations.

S:linear2

3.2. **Analysis of the linear case.** Consider a single linear equation

E:linear3
$$\dot v^i_n + \lambda v^i_n - \lambda v^i_{n-1} = 0, \qquad \lambda > 0. \tag{3.7}$$

We can find the fundamental solution to the previous equation by means of Fourier transform: defining the periodic function

$$c(t,x) \doteq \sum_n v_n(t) e^{inx},$$

we have that the equation satisfied by c is

$$\begin{aligned} c_t &= \sum_n \dot v_n e^{inx} = \lambda \sum_n \big(v_{n-1} - v^n \big) e^{inx} \\ &= \lambda \big(e^{ix} - 1 \big) c, \end{aligned}$$

whose general solution is

$$c(t,x) = c(0,x) \exp \Big(\lambda \big(e^{ix} - 1 \big) t \Big).$$

In particular the fundamental solution starting at n_0 has $c(0,x) = \exp(in_0 x)$, so that if $n_0 = 0$

E:fundsol2
$$v_n(t) = \begin{cases} 0 & n < 0 \\ \big(\lambda t\big)^n / n! \cdot \exp\big(-\lambda t\big) & n \geq 0 \end{cases} \tag{3.8}$$

If now we consider two equations of the form (3.7),

E:linear4
$$\begin{aligned} \dot v_n + \lambda \big(v_n - v_{n-1} \big) &= 0, \\ \dot z_n + \mu \big(z_n - z_{n-1} \big) &= 0. \end{aligned} \tag{3.9}$$

we can compute the intersection integrals: denoting with $d(t,x)$ the Fourier transform of $z_n(t)$ and assuming that $\lambda > \mu > 0$, we have

E:transcomp3
$$\begin{aligned} \int_0^{+\infty} \sum_{n=-\infty}^{+\infty} v_n(t) z_n(t) dt &= \lim_{T \to +\infty} \int_0^T \frac{1}{2\pi} \int_0^{2\pi} c(t,x) d(t,-x) e^{-in_0 x} dx dt \\ &= \lim_{T \to +\infty} \frac{1}{2\pi} \int_0^{2\pi} \int_0^T \exp \Big(\lambda \big(e^{ix} - 1 \big) t + \mu \big(e^{-ix} - 1 \big) t \Big) e^{-in_0 x} dx dt \\ &= \lim_{T \to +\infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-in_0 x} \Big(\exp \Big(\lambda \big(e^{ix} - 1 \big) t + \mu \big(e^{-ix} - 1 \big) t \Big) - 1 \Big)}{\lambda \big(e^{ix} - 1 \big) + \mu \big(e^{-ix} - 1 \big)} dx \\ &= \frac{1}{2\pi i} \oint_\gamma \frac{z^{-n_0}}{\big(z - 1 \big) \big(\lambda z - \mu \big)} dz, \end{aligned} \tag{3.10}$$

where γ is the path represented in Figure 2.

By means of complex analysis we have that

E:transcomp4
$$P(n_0) \doteq \int_0^{+\infty} \sum_{n=-\infty}^{+\infty} v_n(t) z_n(t) dx = \begin{cases} 1/(\lambda - \mu) \cdot \big(\lambda / \mu \big)^{n_0} & n_0 < 0 \\ 1/(\lambda - \mu) & n_0 \geq 0 \end{cases} \tag{3.11}$$

S:bvest2

3.3. **BV estimates.** Now to prove that (3.1) has a solution with uniformly bounded total variation. By defining the functional

E:interpot2
$$Q\big(u(t)\big) \doteq \sum_{i<j} \sum_{n,m=-\infty}^{+\infty} P(n-m) \Big\{ v^i_n(t) v^j_m(t) + v^i_{n-1}(t) v^j_m(t) + v^i_n(t) v^j_{m-1}(t) \Big\} dx dy, \tag{3.12}$$

where P is computed using the constant of separation of speeds c instead of $\lambda - \mu$ and $\min \lambda_i / \max \lambda_j$ instead of λ/μ, since the left hand side of (3.6) is in conservation form, we conclude immediately that

$$\text{Tot.Var.}\big(u(0)\big) + C_0 Q\big(u(o)\big) \leq \delta_1, \qquad \frac{d}{dt} \Big\{ \text{Tot.Var.}(u) + C_0 Q(u) \Big\} \leq 0.$$

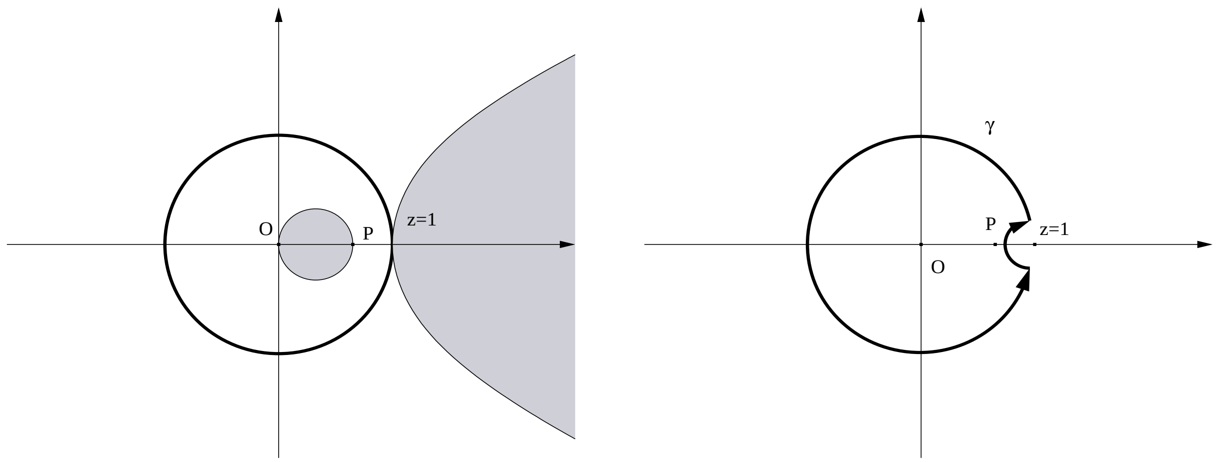

FIGURE 2. Integration path on the complex plane, where $P = \mu/\lambda$. Fi:integ2

In fact we have

$$\begin{aligned}
\frac{dQ}{dt} &= \sum_{m,n} P(m-n)\Big(\big|v^{m,i}\big|_t\big|v^{n,j}\big| + \big|v^{m,i}\big|\big|v^{n,j}\big|_t\Big) \\
&\leq \sum_{m,n} P(m-n)\Big(\Big(-\lambda_i^m\big|v^{m,i}\big| + \lambda_i^{m-1}\big|v^{m-1,i}\big|\Big)\big|v^{n,j}\big| + \big|v^{m,i}\big|\Big(-\lambda_j^n\big|v^{n,j}\big| + \lambda_j^{n-1}\big|v^{n-1,j}\big|\Big)\Big) \\
&= \frac{1}{c}\sum_{m,n}\Big(\lambda_i^m P(m-n+1) - \big(\lambda_i^n + \lambda_j^m\big)P(m-n) + \lambda_j^n P(m-n-1)\Big)\big|v^{m,i}\big|\big|v^{n,j}\big| \\
&= \frac{1}{c}\sum_{m-n\leq -1} k^{m-n-1}\big(k-1\big)\big(\lambda_i^m k - \lambda_j^n\big)\big|v^{m,i}\big|\big|v^{n,j}\big| - \frac{1}{c}\sum_n \lambda_j^n\big(1-1/k\big)\big|v^{n,i}\big|\big|v^{n,j}\big| \\
&\leq -\sum_i \big|v^{n,i}\big|\big|v^{n,j}\big|
\end{aligned}$$

This concludes the proof of bounded total variation.

S:stabil2

3.4. **Stability estimates.** Finally we consider the stability estimates of (3.1). The equations for a perturbation $u + \delta h$ as $\delta \to 0$ are

E:pertueq3

$$\dot{h}_n(t) + Df\big(u_n\big)h_n - Df\big(u_{n-1}\big)h_{n-1} = 0. \tag{3.13}$$

Considering the projection

$$h_n(t) = \sum_i h_n^i(t) r_i(u_n),$$

we have that the equations for the components h_n^i are

$$\dot{h}_n^i + \lambda_i(u_n)h_n^i - \lambda_i(u_{n-1})h_{n-1}^i = \sum_{j\neq k} H'(n)h_{n-1}^j v_n^k + \sum_{j\neq k} G'(n)h_n^j v_n^k.$$

It is clear that a functional of the form

$$Q(t) \doteq \sum_{i<j}\sum_{n,m=-\infty}^{+\infty} P(n-m)\Big\{\big|h_n^i(t)v_m^j(t)\big| + \big|h_{n-1}^i(t)v_m^j(t)\big| + \big|h_n^i(t)v_{m-1}^j(t)\big|\Big\}dxdy,$$

gives the estimate

$$\sum_n \big|h_n^i(t)\big| \leq \sum_n \big|h_n^i(0)\big|,$$

hence by a standard homotopy argument the stability in ℓ^1 of the solution $u_n(t)$.

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