STABILITY OF $L^\infty$ SOLUTIONS FOR HYPERBOLIC SYSTEMS WITH COINCIDING SHOCKS AND RAREFACTIONS

STEFANO BIANCHINI

Abstract. We consider a hyperbolic system of conservation laws

$$\begin{cases} u_t + f(u)_x = 0 \\ u(0, \cdot) = u_0 \end{cases}$$

where each characteristic field is either linearly degenerate or genuinely nonlinear. Under the assumption of coinciding shock and rarefaction curves and the existence of a set of Riemann coordinates $w$, we prove that there exists a semigroup of solutions $u(t) = S_t u_0$, defined on initial data $u_0 \in L^\infty$. The semigroup $S$ is continuous w.r.t. time and the initial data $u_0$ in the $L^1_{loc}$ topology. Moreover $S$ is unique and its trajectories are obtained as limits of wave front tracking approximations.

S.I.S.S.A. Ref. 65/2000/M

1. Introduction

Consider the Cauchy problem for a strictly hyperbolic system of conservation laws

$$\begin{cases} u_t + f(u)_x = 0 \\ u(0, \cdot) = u_0 \end{cases}$$

where $u \in \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ is sufficiently smooth, $\Omega$ open. If the initial data $u_0$ is of small total variation, the global existence was proved first in [18]. Moreover a series of papers [6, 7, 9, 10, 15] establishes the uniqueness and well posedness of the Cauchy problem (1.1). However, when $u_0$ has large total variation or even more generically $u_0$ belongs to $L^1$, the solution $u$ may not exist globally in $L^\infty$ [20]: only for special system it is possible to consider initial data with large total variation. We recall some of the results available in this direction.

1) For scalar conservation laws, the entropy solution to (1.1) generates a contracting semigroup w.r.t the $L^1$ distance, on a domain of $L^\infty$ data [21].

2) For general Temple class system, in [3, 5, 24] it is proved the existence and stability of the entropy solution for initial data with arbitrarily large but bounded total variation.

3) If all characteristic families are genuinely nonlinear and the system is Temple class, the existence and stability for initial data in $L^\infty$ is proved in [12].

4) For special $2 \times 2$ systems, in which one of the equation is autonomous, various results have been proved in [4, 16], with initial data with unbounded total variation.

An open question is if the semigroup of solutions to the systems of case 2), defined on all the initial data $u_0$ with total variation arbitrary large but bounded, can be extended to data in $L^\infty$. In many systems, in fact, some of the characteristic fields are linearly degenerate, so that the results of [12] do not apply.

An example is the $2 \times 2$ traffic model considered in [2],

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ (\rho(v + p(v)))_t + (\rho v(v + p))_x = 0 \end{cases}$$

where $\rho(t, x)$ is the density of cars in the point $(t, x)$ and $v(t, x)$ is their velocity. In this model, the first eigenvalue is genuinely nonlinear and the integral curves of the corresponding right eigenvector are straight lines. The second eigenvector is linearly degenerate, so that the assumption of coinciding shock

Date: June 13, 2000.
1991 Mathematics Subject Classification. 35L65.
Key words and phrases. Hyperbolic systems, conservation laws, well posedness.
I wish to thank Prof. Alberto Bressan for having suggested the problem and his careful reading of the manuscript.
and rarefaction curves is verified for this system. The existence of a set of Riemann coordinates follows by the fact that the system is $2 \times 2$.

Another example is a simple $2 \times 2$ model for chromatography,

$$
\begin{align*}
    u_1^t + \left( \frac{u_1}{1 + u_1 + u_2} \right)_x &= 0 \\
    u_2^t + \left( \frac{u_2}{1 + u_1 + u_2} \right)_x &= 0
\end{align*}
$$

where all characteristic fields are linearly degenerate and the integral curves of the eigenvalues are straight lines. The major difficulty here in applying the results of [12] is the fact that the total variation of the solution does not decay in time.

The aim of this paper is to prove that, at least in the case where the eigenvalues are genuinely nonlinear or linearly degenerate and shocks and rarefactions coincide, the solution to (1.1) can be defined for $u_0 \in L^\infty$.

This result is particular interesting from the point of view of control theory. Consider for example the traffic model (1.2) in the quarter plane $t \geq 0, x \geq 0$: this system describes the flow of cars in a highway, given a boundary condition $\hat{u}(t)$ on the line $x = 0$. The function $\hat{u}$ can be thought as a control on the system: we are allowed to choose $\hat{u}$ in order to minimize some prescribed cost functional, for example the average time spent by a car to arrive from $x = 0$ to $x = \bar{x}$. As shown in [1], in general the compactness of the attainable set can be obtained only with $L^\infty$ boundary data.

To illustrate the heart of the matter, we assume that the system (1.1) admits a system of Riemann coordinates $w \in \mathbb{R}^n$, and that shock and rarefaction curves coincide in $\Omega$. Moreover we assume that each characteristic field is linearly degenerate or genuinely nonlinear. Differently form [11], we do not assume that rarefaction curves are straight lines. We consider a set $E$ of the form

$$
E = \left\{ u \in \Omega : w(u) \in [a_i, b_i], \ i = 1, \ldots, n \right\}.
$$

With $L^\infty(\mathbb{R}; E)$ we denote the space of $L^\infty$ functions with values in $E$. The main result of this paper is the following:

**Theorem 1.1.** There exists a unique semigroup $S : [0, +\infty) \times L^\infty(\mathbb{R}; E) \mapsto L^\infty(\mathbb{R}; E)$ such that the following properties are satisfied:

i) for all $u_n, u \in L^\infty(\mathbb{R}; E)$, $t_n, t \in [0, +\infty)$, with $u_n \to u$ in $L^1_{loc}$, $|t - t_n| \to 0$ as $n \to +\infty$,

$$
\lim_{n \to +\infty} S_{t_n} u_n = S_t u \quad \text{in } L^1_{loc};
$$

ii) the trajectory $S_t u_0$ is a weak entropy solution to the Cauchy problem (1.1) for every $u_0 \in L^\infty(\mathbb{R}; E)$;

iii) if $u_0$ is piecewise constant, then, for $t$ sufficiently small, $S_t u_0$ coincides with the function obtained by piecing together the solutions of the corresponding Riemann problems.

From the results of [11], [14], any solution to (1.1) satisfying Lax entropy conditions and a weak regularity assumption is unique. Theorem 1.1 proves that it is possible to define a weak solution $u(t)$ when the initial data are in $L^\infty$ so that $u(t)$ depends continuously w.r.t. the initial data $u_0$. The uniqueness follows because $S$ satisfies iii) and it is limit of wave front approximations.

As it is shown in the last example of [12], the semigroup $S$ cannot be uniformly continuous: thus we cannot apply any compactness argument to construct the solution $u(t) = S_t u_0$. The fundamental problem is that, differently from [12], the total variation of the Riemann invariants corresponding to linearly degenerate families does not decrease in time.

The main idea of this paper is to study how the solution to the characteristic equation

$$
\dot{x}(t) = \lambda_i(u(t, x(t))), \quad x(0) = y,
$$

depends on the solution $u$ of (1.1). Denote with $x(t, y)$ the solution of (1.3).

It will be shown that, for a fixed time $\tau$, the map $y \mapsto x(\tau, y)$ depends Lipschitz continuously on the initial data $u_0$, and moreover the Lipschitz constant is independent of the total variation of $u_0$. Since the Riemann invariant $w_i$ is the broad solution to

$$
(w_i)_t + \lambda_i(u(t, x))(w_i)_x = 0,
$$

the trajectory $S_t u_0$ is a weak entropy solution to the Cauchy problem (1.1) for every $u_0 \in L^\infty(\mathbb{R}; E)$;
a simple argument gives the convergence of the wave front tracking approximations. We recall that a
broad solution of (1.4) with initial data \( \tilde{w}_0(\cdot) \) is given by \( w_i(x(t, y)) = \tilde{w}_0(y) \), where \( x(t, x) \) is the solution to (1.3). In other words, the value of \( w_i \) is constant along the integral lines of (1.3).

We note that the stability of the map \( y \mapsto x(t, y) \) implies also the well posedness of the ODE (1.3)
when \( u(t, x) \) is an \( L^\infty \) solution of the system (1.1). This result is quite surprising because, as noted in
[16], for general hyperbolic systems the solution to (1.3) does not exist or it is not unique. In our case,
the assumption on the existence of Riemann invariants and the conservation form of the equations (1.1)
implies the continuous dependence of \( x(t, y) \) on the initial data \( u_0 \), and then we can extend the notion of
solutions to (1.3) when \( u_0 \) is in \( L^\infty \).

The paper is organized as follows. Section 2 contains the basic assumptions on the system (1.1). Moreover we construct the wave front approximation of the solution \( u(t) \). In Section 3 we analyze
carefully the shift differential map, i.e. the evolution of a perturbation in \( u_0 \) in which only the position of
the initial jumps has changed. The method we use is essentially the one in [12], with slight modifications
due to the fact that in our system the rarefaction curves do not need to be straight lines. The main result
is here the explicit computation of the shift differential map.

Section 4 is concerned with the equation for characteristics (1.3). We prove the Lipschitz dependence
of \( x(t) \) w.r.t. both the initial data \( u_0 \) and \( y \). Moreover we will show that the Lipschitz
constant is independent from the total variation of \( u_0 \). Finally, in Section 5, we prove Theorem 1.1.

2. Basic assumptions and wave front approximations

We consider a strictly hyperbolic system of conservation laws

\[
(2.1) \quad u_t + f(u)_x = 0,
\]

where \( f : \Omega \to \mathbb{R}^n \) is a smooth vector field defined on some open set \( \Omega \subseteq \mathbb{R}^n \). Let \( A(u) = Df(u) \)
be the Jacobian matrix of \( f \) and denote with \( \lambda_i(u) \) its eigenvalues and with \( r_i(u) \), \( l_i(u) \) its right and
left eigenvectors, respectively. We assume that the eigenvalues \( \lambda_i \) can be either genuinely nonlinear or
linearly degenerate. In the following the \( i \)-th rarefaction curve through \( u \in \Omega \) will be written as \( R_i(s)u \),
with \( R_i(0)u = u \), while the \( i \)-th shock curve will be denoted by \( S_i(s)u \), and its speed by \( \sigma_i(s, u) \). The
directional derivative of a function \( \phi(u) \) in the direction of \( r_i(u) \) will be denoted as

\[
\phi_i = \lim_{h \to 0} \frac{\phi(u + hr_i(u)) - \phi(u)}{h},
\]

while the left and right limit of a BV function \( f \) in a point \( x \) will be written as

\[
\lim_{y \to x^+} f(y) = \lim_{y \to x^-} f(y).
\]

We assume that the rarefaction curves \( R_i \) generate a system of Riemann coordinates \( w(u) \). We recall
that a necessary and sufficient condition for the local existence of Riemann coordinates is the Frobenius
involutive condition: if \([X, Y] \) denotes the Lie bracket of the vector fields \( X, Y \), the condition is

\[
[r_i, r_j] \in \text{span}\{r_i, r_j\} \quad \text{for all } i, j = 1, \ldots, n.
\]

In the following we will use indifferently the conserved coordinates \( u \) or the Riemann coordinates \( w \).

Fix a domain

\[
E = \left\{ u \in \Omega : w(u) \in [a_i, b_i], \; i = 1, \ldots, n \right\}.
\]

Since \( E \) is compact, there is a constant \( c > 0 \) such that

\[
r_i \cdot \lambda_i(u) > c \quad \forall u \in E, \quad \text{if } \lambda_i \text{ is genuinely nonlinear}.
\]

We suppose that the system (2.1) is uniformly strictly hyperbolic in \( \Omega \): this means that there exists a
constant \( d \) such that

\[
\lambda_{i+1}(u) - \lambda_i(u) \geq d, \quad \forall u, v \in E, \; i = 1, \ldots, n - 1.
\]

We also assume that in the system (2.1) shock and rarefaction curves coincide: this implies [27] that
either the rarefaction curve \( R_i(s)u \) is a straight line or the eigenvalue is linearly degenerate. In fact, one
can prove that

\[ \frac{d^2}{ds^2} \sigma_i(s, u) \bigg|_{s=0} = \frac{1}{6} (r_i \cdot \lambda_i(u)) \langle \nabla^2 (u), r_i \cdot r_i(u) \rangle + \frac{1}{3} r_i \cdot (r_i \cdot \lambda_i(u)), \]

and for the shock curve \( S_i(s)u \) we have

\[ \langle \nabla^2 (u), S''(0)u - R''(0)u \rangle = \frac{1}{2(\lambda_j(u) - \lambda_i(u))} (r_i \cdot \lambda_i(u)) \langle \nabla^2 (u), r_i \cdot r_i(u) \rangle. \]

If \( \lambda_i \) is genuinely nonlinear, the left hand side of (2.6) is zero if and only if the rarefaction curve is a straight line, because \( r_i \cdot r_i(u) \) is orthogonal to \( r_i(u) \).

The flux function \( f \) thus satisfies the following assumptions:

H1) the eigenvalues \( \lambda_i \) of \( Df \) are linearly degenerate or genuinely nonlinear;

H2) the rarefaction curves form a system of coordinates;

H3) shock and rarefaction curves coincide.

The system (2.1) has thus \( n_{ld} \) linearly degenerate fields \( \lambda_i \), corresponding to the Riemann invariants \( w_i \), and \( n_{gnl} = n - n_{ld} \) genuinely nonlinear fields \( \lambda_k \), corresponding to the Riemann invariants \( w_k \). In the latter case we have \( r_k \cdot r_k(u) = 0 \) for all \( u \in E \).

Remark 2.1. If \( \Omega \subseteq \mathbb{R}^2 \), then the rarefaction curves \( R_i(s)u \) always generate a system of Riemann coordinates. Thus our assumptions are satisfied by the following classes of systems:

i) both eigenvalues are linearly degenerate;

ii) one eigenvalue is linearly degenerate, the other genuinely nonlinear and the rarefaction curves of the latter are straight lines;

iii) both eigenvalues are genuinely nonlinear and the system is of Temple class.

The various situations are shown in fig. 1. Case ii) corresponds to the traffic model considered in [2], while case i) corresponds to \( 2 \times 2 \) chromatography.

Given the two points \( u^-, u^+ \in E \), with coordinates \( u^- = u(w_1^-, \ldots, w_n^-) \) and \( u^+ = u(w_1^+, \ldots, w_n^+) \), with \( w_i^+ \neq w_i^- \), consider the intermediate states \( u(\omega_i) \), where

\[ \omega_0 = w(u^-), \quad \omega_i = (w_1^+, \ldots, w_i^+, w_{i+1}^-, \ldots, w_n^-), \quad i = 1, \ldots, n. \]

For all \( i = 1, \ldots, n \), we denote with \( v_i(u^-, u^+) \) the vectors defined as \n
\[ v_i(u^-, u^+) = u(\omega_i) - u(\omega_{i-1}), \]
and we define $r_i(u^-, u^+)$ as

$$
(2.9)\quad r_i(u^-, u^+) = \begin{cases} 
  v_i(u^-, u^+) = \frac{u(\omega_i) - u(\omega_{i-1})}{|u(\omega_i) - u(\omega_{i-1})|} & \text{if } u_i^- \neq u_i^+ \\
  r_i(\omega_i) = r_i(\omega_i) & \text{if } u_i^- = u_i^+
\end{cases}
$$

where $r_i(u)$ is the $i$-th eigenvector of $DF(u)$. We assume that the vectors $r_i(u^-, u^+)$ are linearly independent for all $u^-, u^+ \in E$. This condition is satisfied for data in a sufficiently small neighborhood of a given point $\bar{u} \in \Omega$. We denote also with $\{\bar{v}^i(u^-, u^+), i = 1, \ldots, n\}$ the dual base.

We now define an approximated semigroup of solutions $S^\nu$ on a set $E^\nu \subseteq E$. The construction is similar to the one in [3]. For any integer $\nu \in \mathbb{N}$, set

$$
(2.10)\quad E^\nu \doteq \{ u \in E : w_i(u) \in 2^{-\nu}\mathbb{Z}, i = 1, \ldots, n \},
$$

and let $D^{\nu, M}$ be the domain defined as

$$
(2.11)\quad D^{\nu, M} \doteq \{ u : \mathbb{R} \rightarrow E^\nu : u \text{ piecewise constant and } \text{Tot.Var.}(u) \leq M \}.
$$

Given $\bar{u} \in E^\nu$, we construct a solution $u(t)$ by wave front tracking. We first define how to solve the Riemann problem $[u^-, u^+]$, with $u^-, u^+ \in E^\nu$.

The solution to the Riemann problem $u^-, u^+$ is constructed by piecing together the solutions to the simple Riemann problems $[\omega_{i-1}, \omega_i]$, where $\omega_i$ is defined in (2.7). If the $i$-th field is linearly degenerate, then $[\omega_{i-1}, \omega_i]$ is solved by a contact discontinuity travelling with speed $\lambda_i(\omega_i)$. If the $i$-th field is genuinely nonlinear and $w_i^+ < w_i^-$, then $[\omega_{i-1}, \omega_i]$ is solved by a shock travelling with the Rankine-Hugoniot speed $\sigma_i(\omega_{i-1}, \omega_i)$. Finally, if the $i$-th field is genuinely nonlinear and $w_i^+ > w_i^-$, then $[\omega_{i-1}, \omega_i]$ is solved by a rarefaction fan: if $w_i^+ = w_i^- + p_i 2^{-\nu}$, $p_i \in \mathbb{N}$, consider the states

$$
\omega_{0, i} = \omega_{i-1}, \quad \omega_{i, \ell} = (w_i^+, \ldots, w_{i-1}^-, w_i^- + \ell 2^{-\nu}, w_{i+1}^-, \ldots, w_n^-), \quad \ell = 1, \ldots, p_i.
$$

The solution will consist of $p_i$ shock waves $[\omega_{i, \ell-1}, \omega_{i, \ell}]$, travelling with the corresponding shock speed $\sigma_i(\omega_{i, \ell-1}, \omega_{i, \ell})$.

At time $t = 0$ we solve the initial Riemann problems of $\bar{u}$. Note that the number of wave fronts is bounded by $2^\nu \cdot \text{Tot.Var.}(\bar{u})$. When two or more fronts interact, we solve again the Riemann problem they generate, and so on. It is easy to show that at each interaction at least one of the following alternatives holds:

i) the number of waves decreases at least by 1;
ii) the total variation of the solution $u(t)$ decreases by $2^{1-\nu}$;
iii) the interaction potential $Q(t)$, defined as

$$
Q(t) \doteq \sum_{\alpha, \beta \text{ approaching}} |\sigma_\alpha| |\sigma_\beta| \leq M^2,
$$

decreases by $2^{-\nu}$. We recall that two waves $\sigma_\alpha, \sigma_\beta$ of the families $k_\alpha, k_\beta$, located at points $x_\alpha, X_\beta$, are considered as approaching if $x_\alpha < x_\beta$ and $k_\alpha > k_\beta$.

This implies that there are at most a finite number of interactions, so that we can construct our approximate solution for all $t \geq 0$. Note that $S^\nu u = u(t)$ is a semigroup of solutions, but not entropic due to the presence of rarefaction fronts.

If the $i$-th family is linearly degenerate, the $i$-th Riemann coordinate $w_i(t, \cdot)$ of the solution can be constructed by solving the semilinear system

$$
(2.13)\quad \begin{cases} 
  (w_i)_t + \lambda_i(u(t, x))(w_i)_x = 0 \\
  w_i(0, x) = w_{i,0}(x)
\end{cases}
$$

Since $u$ is a piecewise constant solution, with a finite number of jumps, the broad solution to (2.13) is well defined [8]: if we denote with $x(t, y)$ the solution to the ODE

$$
(2.14)\quad \dot{x} = \lambda_i(u(t, x)), \quad x(0) = y,
$$

then the solution to (2.13) is given by

$$
(2.15)\quad w_i(t, x(t, y)) = w_{i,0}(y).
$$
In the following sections we will consider the dependence on the initial data \( u_0 \) of the genuinely nonlinear Riemann coordinates \( w_k(t, \cdot) \) and the map \( h^i_t(y) \) defined as
\[
h^i_t(y) = x_i(t, y),
\]
where \( x_i(t, y) \) is the solution to (2.14).

3. Estimates on the shift differential map

In this section we prove some properties of the shift differential map. These properties are closely related to the structure of (2.1), i.e. the conservation form, the coinciding shock and rarefaction assumption, which prevents the creation of shock when two jumps of the same family collide, and the existence of Riemann invariants, which prevents the creation of shock when two jumps of different families interact.

Consider a wave front solution \( u(t, \cdot) \) of (2.1), and assume that the initial datum \( u(0, \cdot) \) has a finite number \( N \) of jumps, located in \( y_a \):
\[
u(0, x) = \sum_{\alpha=1}^{N} \sigma_{\alpha} \chi_{\left[y_a, +\infty\right)}(x).
\]
If \( \xi_\alpha \) is the shift rate of the jump \( \sigma_\alpha \), define \( u^\theta(t, \cdot) \) as the front tracking solution with initial datum
\[
u^\theta(0, x) = \sum_{\alpha=1}^{N} \sigma_{\alpha} \chi_{\left[y_a + \theta \xi_\alpha, +\infty\right)}(x).
\]
In the following, we will use the integral shift function, defined by
\[
v(t, x) = \lim_{\theta \to 0} \left\{ -\theta \int_{-\infty}^{x} u^\theta(t, y) - u(t, y) dy \right\}.
\]
If \( u(t, \cdot) \) has a shock \( \sigma_\beta \), located in \( y_\beta \), and if \( \xi_\beta \) is its shift rate, it is clear that the following relation holds:
\[
\sigma_\beta \xi_\beta = v(t, y_\beta) - v(t, y_\beta^-).
\]

We first recall the following result in [12], obtained using the conservation form of the equations:

**Lemma 3.1.** Consider a bounded, open region \( \Gamma \) in the \( t-x \) plane. Call \( \sigma_\alpha, \alpha = 1, \ldots, N \), the fronts entering \( \Gamma \) and let \( \xi_\alpha \) be their shifts. Assume that the fronts leaving \( \Gamma \), say \( \sigma'_\beta, \beta = 1, \ldots, N' \), are linearly independent. Then their shifts \( \xi'_\beta \) are uniquely determined by the linear relation
\[
\sum_{\beta=1}^{N'} \xi'_\beta \sigma'_\beta = \sum_{\alpha=1}^{N} \xi_\alpha \sigma_\alpha.
\]

**Remark 3.2.** As observed in [12], formula (3.4) implies that the shift rates of the outgoing fronts depend only on the shift rates of the incoming ones, and not on the order in which these wave-fronts interact inside \( \Gamma \). In particular we can perform the following operations, without changing the shift rates of the outgoing fronts:

- **O1) switch the order of which three or more fronts interact;**
- **O2) invert the order of two fronts at time 0, if they have zero shift rate.**

The second lemma is concerned with a configuration where a sequence of contact discontinuities interacts with a wave of another family.

**Lemma 3.3.** Consider a family of parallel contact discontinuities \( \sigma_\alpha, \alpha = 1, \ldots, N \) of the \( i \)-th linearly degenerate family and a single wave-front \( \sigma \) of the \( k \)-th family, \( k \neq i \). Let \( \xi_\alpha \) and \( \xi \) be their initial shifts, respectively, and let \( \xi'_\alpha \), \( \xi' \) be their shifts after interaction. Assume that \( \xi'_\alpha = \xi \) for all \( \alpha \). Then after the interactions all the shift rates \( \xi'_\alpha \) of the \( i \)-th family have the same value \( \xi' \) and
\[
ex'_\alpha = \frac{\xi'(N' - \Lambda) - \xi(N' - \Lambda)}{\Lambda - \Lambda}, \quad \xi'(N' - \Lambda) = \frac{\xi(N' - \Lambda) - \xi(N' - \Lambda)}{\Lambda - \Lambda},
\]
where \( \bar{\Lambda}, \Lambda \) and \( \bar{N}', N' \) are the speeds of the shocks \( \sigma_\alpha \) and \( \sigma \), before and after interaction, respectively.
Proof. Define the vector $v$ in the $t$-$x$ plane as the shift of the first collision point. By a direct computation one finds
\begin{equation}
(3.6) \quad v = \left( \frac{\xi - \xi}{\Lambda - \Lambda} - \frac{\Lambda \xi - \Lambda \xi}{\Lambda - \Lambda} \right).
\end{equation}

Since all the incoming shock of the linearly degenerate family have the same speed $\Lambda$, by simple geometrical considerations it follows that the vector $v$ is constant during all interactions (fig. 2). Formula (3.5) follows easily. \hfill \square

Remark 3.4. Note that this lemma allows us to perform the following new operation, without changing the shift rates:

- **O3** replace a family of contact discontinuities $\sigma_\alpha$ of a linearly degenerate, all with the same shift rate $\xi$, by a single wave $\sigma = \sum \sigma_\alpha$ with shift rate $\xi$.

In the next lemma we will show that the existence of Riemann coordinates $w$ implies a strong relation among shocks of different families.

**Lemma 3.5.** Consider two adjacent jumps belonging to different families, $\sigma_i$ and $\sigma_j$, $i < j$, located at $x_i > x_j$. Let $\sigma'_i$, $\sigma'_j$ be their strength after interaction. Then the following holds:
\begin{equation}
(3.7) \quad \text{span}\{\sigma_i, \sigma_j\} = \text{span}\{\sigma'_i, \sigma'_j\}.
\end{equation}

Proof. If $\xi_i$, $\xi_j$ are the shift rates before interaction, and $\xi'_i$, $\xi'_j$ after interaction, then (3.7) follows easily from the conservation relation
\begin{equation}
(3.8) \quad \sigma_i \xi_i + \sigma_j \xi_j = \sigma'_i \xi'_i + \sigma'_j \xi'_j \quad \forall \xi_i, \xi_j \in \mathbb{R},
\end{equation}

because by assumptions no waves of other families are generated. \hfill \square

Remark 3.6. Note that the previous lemma implies that the conservation relation (3.8) is bidimensional, i.e. the shocks $\sigma_i$, $\sigma_j$ and $\sigma'_i$, $\sigma'_j$ lie on a two dimensional plane (fig. 3). We can obtain then an identity which relates the the strengths $\sigma$ with the speeds $\Lambda$: substituting (3.5) in (3.8), since $\xi$, $\xi$ are arbitrary, we get
\begin{equation}
(3.9) \quad \sigma_i (\Lambda_j - \Lambda_i) = \sigma'_i (\Lambda'_j - \Lambda_i) + \sigma'_j (\Lambda'_j - \Lambda_j),
\end{equation}
\begin{equation}
\sigma_j (\Lambda_i - \Lambda_j) = \sigma'_i (\Lambda'_i - \Lambda_i) + \sigma'_j (\Lambda'_j - \Lambda_j).
\end{equation}

One can show that if a Riemann solver verifies (3.9) for all couple of waves $i,j$, then there exists a flux function $f$ such that the wave front approximation is a weak solution to (2.1).

An important property of the shift differential map for Temple class systems is the fact that a perturbation to the initial data, initially localized in $[a,b]$, remains in the neighborhood of the set $\bigcup_i [x_i(t,a), x_i(t,b)]$, where $x_i(t,y)$ is the solution of the $i$-th characteristic equation starting at $y$. We now extend this property to hyperbolic systems satisfying the hypotheses H1), H2), H3) of section 2.
Consider $N$ jumps $\sigma_\alpha$, $\alpha = 1, \ldots, N$, of some linearly degenerate family $i$, located at $x_\alpha$ and corresponding to the jumps $c(\alpha)e_i$ in the Riemann coordinates $w$:

$$\sigma_\alpha = u(w(x_\alpha -) + c(\alpha)e_i) - u(w(x_\alpha -)),$$

for some constants $c(\alpha)$, $\alpha = 1, \ldots, N$.

**Definition 3.7.** We say that the jumps $\sigma_\alpha$ defined in (3.10) are in involution if

$$\sum_{\alpha=1}^{N} c(\alpha) = 0,$$

i.e. the initial and final Riemann coordinate $w_i$ is the same: $w_i(x_1-) = w_i(x_N+)$.

Note that, by the existence of Riemann coordinates, this relation does not depend on the positions and strength of the shocks of the other families. We can now extend Lemma 2 in [12] to our systems:

**Lemma 3.8.** Consider a wave front tracking solution $u$. Assume that there are $N$ shocks $\sigma_\alpha$

i) either of the $i$-th linearly degenerate family in involution,

ii) or of the $k$-th genuinely nonlinear family,

and let $x_\alpha(t)$, $0 \leq t \leq T$, be the position of the shock $\sigma_\alpha$, $\alpha = 1, \ldots, N$. Then it is possible to assign at time $t = 0$ shift rates to all shocks such that $\xi_1 = 1$ and the shift of all fronts outside the strip $\Gamma \equiv \{(t, x); t \in [0, T], x_1(t) \leq x \leq x_N(t)\}$ is zero.

**Proof.** We consider only the case of linearly degenerate family $i$, since in the other case the proof is exactly the one given in [12].

Let $x_\alpha(t)$, $\alpha = 1, \ldots, N$, be the position of the shock $\sigma_\alpha$ of the $i$-th family in involution, and let $\tilde{w}_i$ be the value of the Riemann coordinate at $x_1(t) = x_1(0)-$. For $w \in E$, define $\tilde{w}$ as the projection of $w$ on the hyperplane $\{w_i = \tilde{w}_i\}$, and $\tilde{u} = u(\tilde{w})$.

We choose the shift rates such that

$$-\frac{d}{dt} \int_{-\infty}^{x} u^0dy = \sum_{x_i(t) \leq x} \xi_i(t)\sigma_i(t) = c(t, x)(u(t, x) - \tilde{u}(t, x)),$$

where $c(t, x)$ is a scalar function different from 0 only in $[x_1(t), x_N(t)]$, and we recall that $\tilde{u}(t, x) = u(\tilde{w}(t, x))$. By imposing the value $\xi_1 = 1$, i.e. $c(0, x_1(0)-) = 0$, $c(0, x_1(0)+) = 1$, we need to prove that (3.12) can be satisfied at time $t = 0$. We have two cases.

1) If the jump $\sigma_1$ belongs to the $i$-th family and is inside $[x_0(0), x_N(0)]$, then set $\xi = c(t, x_1-)$. 

**Figure 3.** Vector relations among shocks.
2) If the jump $\sigma_i$ belong to the $k$-th family with $k \neq i$, then by assumption (2.8) and by (3.7) there exists a unique shift $\xi_i$ and a unique constant $c(0,x+)$ such that
\[
\xi_i \sigma_i + c(0,x-)(u(0,x-) - \dot{u}(0,x-)) = c(0,x+)(u(0,x+) - \dot{u}(0,x)).
\]
Since we assume that the shocks are in involution, setting $N = \xi_i$ we have that (3.12) holds at time $t=0$: in fact the last jump has size $\dot{u}(0,x_N(0)) - u(0,x_N(0))$.

We now show that this property is conserved for all $t \geq 0$. This follows easily from conservation and Lemma 3.5. The proof is exactly the same as in [12]: we repeat it for completeness. Consider the interaction between two shocks $\sigma_i$ and $\sigma_j$ in the point $(\tau,y)$, see Fig. 4. By inductive assumption, we have for the states $u_l, u_m$ and $u_l$ that
\[
\sum_{\gamma,(\tau) < y} \sigma_\gamma(\tau) \xi_\gamma(\tau) = c_l(u_l - \ddot{u}_l),
\]
\[
c_l(u_l - \ddot{u}_l) + \sigma_i \xi_i = c_m(u_m - \ddot{u}_m),
\]
\[
c_m(u_m - \ddot{u}_m) + \sigma_j \xi_j = c_r(u_r - \ddot{u}_r).
\]
Using conservation we have
\[
\xi_i \sigma_i + \xi_j \sigma_j = \xi_i' \sigma_j' + \xi_j' \sigma_i',
\]
so that for the new middle state $u'_m$ we have
\[
c_l(u_l - \ddot{u}_l) + \sigma_j' \xi_j' = c'_m(u'_m - \ddot{u}'_m) = c_r(u_r - \ddot{u}_r) - \sigma_i' \xi_i',
\]
and using Lemma 3.5 we conclude
\[
\text{span}\{u_l - \ddot{u}_l, \sigma_i\} \cap \text{span}\{u_r - \ddot{u}_r, \sigma_i'\} = \text{span}\{u_m - \ddot{u}_m\}.
\]
The same relation proves that they vanish outside $\Gamma$: in fact, assume for example that $c_l = 0$ and $j < i$. Then from (3.15) we get
\[
\sigma_j' \xi_j' = c'_m(u'_m - \ddot{u}'_m),
\]
which implies that $c'_m = 0$. This concludes the proof.

Remark 3.9. Note that for discontinuities of a linearly degenerate family all shift rates has the same sign. Note moreover that if no waves of other families are present, then we shift all jumps $\sigma_i$ by unit rate 1. This corresponds to the case considered in Lemma 3.3, i.e. to the substitution of a family of contact discontinuities with a single jump, whose strength in this case is 0 by the involution assumption.
Using conservation and the previous lemmas, we obtain explicitly the shift differential map at a given time $\tau$. We recall that, given the states $u^-, u^+ \in E$, we denote with $r_i(u^-, u^+)$ the vectors defined in (2.9), and with $l^i(u^-, u^+)$ its dual base. Let $P_j(u^-, u^+)$ be the projection operator on $\text{span}\{r_i(u^-, u^+), i = 1, \ldots, j\}$:

$$P_j(u^-, u^+)v = \sum_{i=1}^{j} \langle l^i(u^-, u^+), v \rangle r_i(u^-, u^+),$$

where $\langle \cdot, \cdot \rangle$ denotes the the scalar product in $\mathbb{R}^n$.

Given a point $(t, x)$, with $u(t, x)$ continuous in $x$, define $x_i$ the intersection of the backward $i$-th characteristics starting at $(t, x)$ with the real axis $(0, x)$, and for all $(0, y)$ let $j(y)$ the index such that $x_{j(y)} \leq y < x_{j(y)-1}$, $j(y) = 1, \ldots, n + 1$, with $x_0 = +\infty$ and $x_{n+1} = -\infty$. Without any loss of generality, we can assume that in $(0, y)$ there is a jump $\sigma$ of the $k$-th family.

Define the points $w_l, w_r \in E$ by

$$w_l(x, y) = \begin{cases} w(t, x) & j(y) = 1 \\ (w_l(0, y-), \ldots, w_{j(y)-1}(0, y-), w_{j(y)}(t, x), \ldots, w_n(t, x)) & 2 \leq j(y) \leq n \\ w(0, y-) & j(y) = n + 1 \end{cases}$$

$$w_r(x, y) = \begin{cases} w(0, y+) & j(y) = 1 \\ (w_l(t, x), \ldots, w_{j(y)-1}(t, x), w_{j(y)}(0, y+), \ldots, w_n(0, y+)) & 2 \leq j(y) \leq n \\ w(t, x) & j(y) = n + 1 \end{cases}$$

Moreover define the point $w_m \in E$ by

$$w_m(x, y) = (w_l(t, x), \ldots, w_k(0, y+), \ldots, w_{j(y)-1}(t, x), w_{j(y)}(0, y+), \ldots, w_n(0, y+)) \quad 2 \leq j(y) \leq n,$$

if $k < j(y)$, and in a similar way, if $k \geq j(y)$,

$$w_m(x, y) = (w_l(t, x), \ldots, w_{j(y)-1}(t, x), w_{j(y)}(0, y+), \ldots, w_k(0, y-), \ldots, w_n(0, y+)) \quad 2 \leq j(y) \leq n.$$

Define $P(x, y)$ as the vector

$$P(x, y) = \begin{cases} 0 & j(y) = 1 \\ P_{j(y)-1}(w_l, w_m)\sigma + P_{j(y)-1}(w_m, w_r)(\sigma - P_{j(y)-1}(w_l, w_m)\sigma) & 2 \leq j(y) \leq n + 1, k < j(y) \\ P_{j(y)-1}(w_l, w_m)(P_{j(y)-1}(w_m, w_r)\sigma) & 2 \leq j(y) \leq n, k \geq j(y) \end{cases}$$

where $w_l = w_l(x, y)$, $w_m = w_m(x, y)$, $w_r = w_r(x, y)$ and $\sigma$ is the initial jump in $(0, y)$. Consider now a front tracking solution $u^\theta$, obtained by shifting the initial jumps $\sigma_\alpha$ in $y_\alpha$ with rates $\xi_\alpha$.

**Theorem 3.10.** If $v(t, x)$ is the integral shift function of $u^\theta(t, \cdot)$, defined in (3.2), then

$$v(t, x) = \lim_{\theta \to 0} \left\{ -\frac{1}{\theta} \int_{-\infty}^{t} u^\theta(t, y) - u(t, y)dy \right\} = \sum_{\alpha} P(x, y_\alpha)\xi_\alpha.$$

**Proof.** The theorem will be proved outside the times of interaction, because the Lipschitz dependence in $L^1$ of the approximate semigroup implies the validity of (3.21) for all $t \geq 0$.

If is sufficient to show that $\sum_{y_\alpha} P(x, y_\alpha)\xi_\alpha$ is piecewise constant, with jumps only at the points $x_\beta$ where $u(t, \cdot)$ has a shock $\sigma_\beta$, and the following relation holds:

$$\sum_{y_\alpha} (P(x_\beta+, y_\alpha) - P(x_\beta-, y_\alpha))\xi_\alpha = \sigma_\beta\xi_\beta, \quad \lim_{x \to -\infty} \sum_{y_\alpha} P(x, y_\alpha)\xi_\alpha = 0,$$

where $\xi_\beta$ is the shift rate of $\sigma_\beta$, located in $x$. Note that by (3.20) the second equality of (3.22) is trivially satisfied.

By linearity in the shift rates $\xi_\alpha$, we can consider the case in which a single shock is shifted, let us say $\sigma$ at $y$: (3.21) becomes

$$v(t, x) = P(x, y)\xi.$$
Formula (3.16) follows from the following considerations: consider a wave front pattern, fig. 5, where for simplicity we assume that \( k < j(y) \). The states \( w_l, w_m \) are computed considering the Riemann problem generated by adding to the \( k \)-jump \( \sigma \) in \((0, y)\) all the \( i \)-waves starting from the left of \((0, y)\) and ending in the right of \((t, x)\) and all the \( i \)-waves, with \( i \neq k \), starting from the right of \((0, y)\) and ending in the left of \((t, x)\). The jump \( w_m, w_r \) is a single wave of the \( k \)-th family formed by adding all the \( k \)-waves between \((0, y)\) and \((t, x)\). Using the definition of \( v(t, x) \) given (3.2), one obtains easily the second case of (3.20): in fact the shift rates of the shocks in the left of \((t, x)\) is given by the shift rates of the jumps of the Riemann problem \( w_l, w_m \) ending in the left of \((t, x)\), \( P_{j(y)-1}(w_l, w_m)\sigma \), plus the shift rate of the shock \( w_m, w_r \), \( P_k(w_m, w_r)(\sigma - P_{j(y)-1}(w_l, w_m)\sigma) \). Since only the \( i \)-waves with \( i \geq j(y) > k \) are present in \( \sigma - P_{j(y)-1}(w_l, w_m)\sigma \), then \( P_k(w_m, w_r)(\sigma - P_{j(y)-1}(w_l, w_m)\sigma) = P_{j(y)-1}(w_m, w_r)(\sigma - P_{j(y)-1}(w_l, w_m)\sigma) \). The other cases can be computed in a similar way: in this case one solves the Riemann problem \( w_m, w_r \) in \((0, y)\), and consider the \( k \)-wave \( w_l, w_m \) starting in the left of \((0, y)\) and ending in the right of \((t, x)\).

From the above considerations it is clear that \( P(x, y) \) is piecewise constant, with jumps only when in \((t, x)\) there is a \( i \)-shock \( \sigma^i \); in fact otherwise the wave front pattern used to compute \( P(x, y) \) remains the same. Let \( \{z_p: p = 1, \ldots, M\} \) be the set of the starting points of all shocks arriving in \((t, x)\), and define

\[
(3.24) \quad z^- = \min_p z_p, \quad z^+ = \max_p z_p.
\]

We consider two cases:

1) the shocks arriving in \((t, x)\) start on both sides of \((0, y)\): \( z^- \leq y \leq z^+ \). In this case, \((P(x+, y) - P(x-, y))\xi \) is the shift rate of the \( i \)-shock starting in the Riemann problem \( w_l(x-, y), w_m(x+, y) \) if \( i > k \) ( \( w_m(x-, y), w_r(x+, y) \) if \( i < k \)) which collides with a \( k \)-shock \( w_m(x+, y), w_r(x+, y) \) \( (w_l, w_m \) if \( i < k \)): in fact the only difference is that in \( w_m(x-, y), w_m(x+, y) \) there is a shock of the \( i \)-th family starting in \((0, y)\), and \( i \) is genuinely nonlinear. Finally, using \( r_i \bullet r_i(u) = 0 \) and Lemma 3.5, one can change position to the \( i \)-wave and the remaining \( k \)-wave \( w_m, w_r \), whose strength does not change.

If \( i = k \), there are no \( k \)-shocks starting on the right (left) of \((0, y)\) and ending on the right (left) of \((t, x)\), so that \((P(x+, y) - P(x-, y))\xi \) is the shift rate of the \( i \)-shock of the Riemann problem \( w_l(x-, y), w_r(x+, y) \).

2) the shocks of the \( i \)-th family arriving in \((t, x)\) start either in \((\infty, y)\) or \((y, \infty)\): assume for definiteness that \( y < z^- \). In this case the difference \((P(x+, y) - P(x-, y))\xi \) is the shift rate of the shock \( \sigma^i \) colliding with the shifted shocks of the Riemann problem \( w_l(x-, y), w_m(x-, y) \) in \((0, y)\), crossing the jump \( w_m(x-, y), w_r(x-, y) \), and finally overtaking \( \sigma' \). In fact one can use Lemma 3.5 (and \( r_i \bullet r_i(u) = 0 \) if \( i \) is genuinely nonlinear) to obtain the wave pattern of fig. 7.

The various cases will be proved in the following lemmas.

**Lemma 3.11.** Assume that \( z^- \leq y \leq z^+ \), i.e. case 1). If the shock \( \sigma^i \) is of the \( i \)-th family, then its shift \( \xi' \) is

\[
(3.25) \quad \xi' \sigma^i = (P(x+, y) - P(x-, y))\xi.
\]
Proof. We follow closely the method of [12]. Assume for definiteness $k < j(y)$, the other cases being similar. The basic idea is to reduce the computation to the single Riemann problem $w_l(x-, y), w_m(x+, y)$, with eventually a single k-wave $w_m, w_r$.

Consider fig. 6. By Lemma 3.1, we can simplify the wave configuration considering only the fronts crossing starting in the right of $(0, y)$ and ending in the left of $(t, x)$: in fact we can move the other fronts to $\pm \infty$ without changing the shift rate of $\sigma'$. We can now shift the initial position of the waves of the $i$-th family merging in $x$ such that their initial position coincide with $y$, without changing the shift rate $0$. This operation can be repeated for all shocks of genuinely nonlinear families.

Finally, we can move the shocks of the linearly degenerate families such that they have the same sequence of interaction with the other shocks. This means that, if $x^i_l$ is the position of the $j$-th shock of the $i$-th linearly degenerate family, the only interactions among shocks occurring in the sector $[x^i_l(t), x^i_r(t)]$ are the one involving one $i$-th wave and one $k$-th wave, with $k \neq i$. Using Lemma 3.3, we can at this point substitute them with a single shock, whose strength is the sum of the strengths of the $i$-waves. Finally we move their position at $t = 0$ such that it coincides with $y$: we obtain the wave patterns of fig. 6. To conclude, we just need to prove that the Riemann problem obtained in this way is exactly $w_l(x-, y), w_m(x+, y)$ and that the remaining $k$-wave is $w_m(x+, y), w_r(x+, y)$.

By the previous argument, the strength of the shock of the $j$-th family $j < i$, $j \neq k$, is given by the $j$-waves starting in the right of $y$ and ending in the left of $x$: since they are the only $j$-wave crossing the segment $[(0, y-), (t, x+)]$, it follows

$$w_{l,j}(x, y) = w_j(0, y-), \quad w_{r,j}(x, y) = w_j(t, x+).$$

The other relations for $j = k$ and $j > i$ follows in the same way. Finally, for $j = i$ the jump is $w_i(t, x+) - w_i(t, x-)$. Note that the wave pattern is the same obtained in 1).

We consider only the case $y < z^-$, since the other is entirely similar.

**Lemma 3.12.** Assume that $y < z^-$. Then the shift $\xi'$ of $\sigma'$ is given by

$$\xi' \sigma' = (P(x+, y) - P(x-, y)) \xi.$$  

*Proof.* The hypothesis implies that the $i$-th shocks ending at $x$ starts in the right of $y$. With the same simplification considered in Lemma 3.11, we reduce to the Riemann problem $w_l(x, y), w_r(x, y)$ in $\tilde{y}$, such that the waves of the $j$-th families, $j > i$, generated at $\tilde{y}$ collide with the $i$-wave in $x_n$ (see fig. 7), after overtaking the $k$-wave $w_m, w_r$. The conclusion follows easily, since the wave pattern is the same considered in 2).

This concludes the proof of Theorem 3.10.

Finally we extend to our case the following result proved in [12]:

**Figure 6.** Computation of the shift rate in the case of Lemma 3.11.
Figure 7. Computation of the shift rate in the case of Lemma 3.12.

Figure 8. Cancellation among contact discontinuities.

**Proposition 3.13.** Let $u$ be a wave-front tracking solution, and consider two wave-fronts, $x(t)$ and $y(t)$, $t \in [0, T]$. Then there exists a second front tracking solution $\tilde{u}$ such that the initial and final positions of the two shocks is the same, and Tot.Var.(\tilde{u}) is uniformly bounded.

**Proof.** For genuinely nonlinear fields, the proof is the same as in [12]. We then restrict the proof to the case of a linearly degenerate fields $i$.

Assume that there exists two jumps $\sigma_1, \sigma_2$ of the $i$-th family, with positions $z_1(t) < z_2(t)$, such that

$$x(0) \notin [z_1(0), z_2(0)] \quad \text{and} \quad y(0) \notin [z_1(0), z_2(0)], \quad x(T) \notin [z_1(T), z_2(T)] \quad \text{and} \quad y(T) \notin [z_1(T), z_2(T)].$$

For definiteness, assume $w_i(0, z_1-) < w_i(0, z_1+)$, and the following conditions is satisfied:

$$w_i(0, z_1-) \in [w_i(0, z_2-), w(0, z_2+)].$$

Let $\sigma_\alpha$, $\alpha = 1, \ldots, N$ be the jumps of linearly degenerate family $i$ in the strip $[z_1(0), z_2(0)]$: if we define

$$\sigma_{N+1} = u(w_i(0, z_1-)) - u(w_i(0, z_1-)),$$

it is easy to verify that the shocks $\sigma_\alpha$, $\alpha = 1, \ldots, N + 1$, are in involution. By Lemma 3.8, we can then moving the jumps to the left until either $z_1(t)$ meets the wave fronts $x(t)$, or $z_1(t)$ coincides with another shock of the $i$-th family (fig. 8). It is clear that we can repeat the same procedure also in the following cases:

i) $w_i(0, z_1-) > w_i(0, z_1+)$ and $w_i(0, z_1-) \in [w_i(0, z_2+), w(0, z_2-)]$;

ii) $w_i(0, z_2-) < w_i(0, z_2+)$ and $w_i(0, z_2-) \in [w_i(0, z_1+), w(0, z_1-)]$;

iii) $w_i(0, z_2-) > w_i(0, z_2+)$ and $w_i(0, z_2-) \in [w_i(0, z_1-), w(0, z_1+)]$.

It is now easy to prove that the total variation of the jumps of the $i$-th family satisfying (3.27) can at most be $3\|w\|_\infty$. Since $x(t)$, $y(t)$ divide the lines $t = 0$ and $t = \tau$ in three regions, the total variation of $w_i$ is bounded by $27\|w\|_\infty$. \qed
4. Estimates on characteristics

In this section we prove some estimates on the solution \( x_i(t, y) \) of the characteristic equation:

\[
\begin{cases}
\dot{x}_i = \lambda_i(u(t, x_i)) \\
x_i(0) = y
\end{cases}
\]

(4.1)

We assume for simplicity that the \( i \)-th family is linearly degenerate, however the same results are valid for characteristics of a genuinely nonlinear family if the following condition holds: for all \( \tau \) there exists an \( \epsilon \) such that in the strip \( \{(t, x); \tau \leq t \leq T, x_i(t, y) - \epsilon \leq x \leq x_i(t, y) + \epsilon\} \) there are no shock waves of the \( i \)-th family. Given front tracking approximation \( u, x_i(t, y) \) is unique, since it crosses only a finite number of transversal jumps, and it depends Lipschitz continuously on the initial data \( y \) (see [8]).

We want to give uniform estimates on this dependence. The idea is to suppose that in \( y \) there is a shock \( \sigma^\epsilon \) of the \( i \)-family of size \( \epsilon \): \( w_i(0, y+) - w_i(0, y-) = \epsilon \). Since by assumption no shocks of the \( i \)-family collide with \( \sigma^\epsilon \), it is easy to construct a wave front solution: for \( x < x(t, y) \), the solution \( u^\epsilon(t, \cdot) \) takes values in

\[
E^{\epsilon^-} \doteq \left\{ u : w_j(u) \in [a_j, b_j] \cap 2^{-\nu} \mathbb{Z}, j = 1, \ldots, n \right\},
\]

while for \( x > x(t, y) \), enlarging \( E \) and assuming \( \epsilon \) sufficiently small,

\[
E^{\epsilon^+} \doteq \left\{ u : w_j(u) \in [a_j, b_j] \cap 2^{-\nu} \mathbb{Z}, j \neq i, w_i(u) \in [a_i, b_i] \cap 2^{-\nu} \mathbb{Z} + \epsilon \right\}.
\]

The following lemma proves the continuous dependence of the solution \( u^\epsilon(t) \) and the position \( x_i^\epsilon(t, y) \) of the shock \( \sigma^\epsilon \) w.r.t. \( \epsilon \).

Lemma 4.1. Consider a front tracking solution \( u \), with initial data \( u_0 \) and the characteristic lines \( x_i(t, y_1) < x_i(t, y_2) \), defined in (4.1) for a linearly degenerate family \( i \). Let \( u^\epsilon \) the wave front solution with initial data \( u(w_0^\epsilon) \), where \( w_0^\epsilon \) is defined as

\[
w_0^\epsilon(x) \doteq \begin{cases} 
 w(u_0(x)) & x \leq y_1 \\
 w(u_0(x)) + \epsilon e_i & y_1 < x \leq y_2 \\
 w(u_0(x)) & x > y_2
\end{cases}
\]

(4.2)

Then there exists constants \( L, L' \), depending only on the total variation of the initial data \( u_0 \), such that for all \( t \geq 0 \)

\[
\int_{\mathbb{R}} |u(t, x) - u^\epsilon(t, x)| dx \leq L \epsilon |y_1 - y_2| \quad \text{and} \quad |x_i^\epsilon(t, y_j) - x_i(t, y_j)| \leq L' \epsilon t |y_1 - y_2|, \quad j = 1, 2,
\]

(4.3)

where \( x_i^\epsilon(t, y_j) \) is the position of the shock \( \sigma^\epsilon \) starting in \((0, y_j)\).

Proof. The first inequality is an easy consequence of the \( L^1 \) continuous dependence for front tracking solutions, see [3]. For the second one, note that all the shocks different from \( \sigma^\epsilon \) have size uniformly bigger than 0, so that their position is shifted of the order \( \epsilon \). Thus the second inequality follows by standard ODE perturbation estimates, see [8].

An easy application of the previous lemma together with Proposition 3.13 implies that to compute \( x_1(t, y_1) \) and \( x_2(t, y_2) \), we can actually consider in (4.1) a solution \( \bar{u} \) with uniformly bounded total variation, so that the constant \( L' \) in (4.3) is independent on the total variation of \( u_0 \).

We now estimate the dependence of \( x_i(t, y) \) w.r.t. \( u \).

Proposition 4.2. Let \( \xi_\alpha \) be the shift rate of the jump \( \sigma_\alpha \) in \( u(0, \cdot) \), and denote with \( x_i^\theta \) the solution to

\[
\begin{cases}
\dot{x}_i^\theta = \lambda_i(u^\theta(t, x_i^\theta)) \\
x_i^\theta(0) = y
\end{cases}
\]

where \( u^\theta(t) \) is the shifted front tracking solution. Then there exists a constant \( D \) independent of the total variation of \( u \) such that

\[
\lim_{\theta \to 0} \frac{x_i^\theta(t, y) - x_i(t, y)}{\theta} \leq D \sum_\alpha |\sigma_\alpha \xi_\alpha|.
\]

(4.4)
As in the previous proposition, let
\begin{equation}
\xi^\epsilon \sigma^\epsilon = \sum_{\alpha} (P(x^+, y_\alpha) - P(x^-, y_\alpha)) \xi_\alpha.
\end{equation}
Proof. If \( \epsilon \) is the size of the shock \( \sigma^\epsilon \) located in \((0, y)\), then we can apply Theorem 3.10 to compute its shift \( \xi^\epsilon \): by formula (3.21) we obtain
\begin{equation}
(4.5)
\xi^\epsilon = \frac{x_{\theta}^\epsilon(t, y) - x_i^\epsilon(t, y)}{\theta},
\end{equation}
where \( x_{\theta}^\epsilon(t, y) \) is the position of the shifted shock and \( x_i^\epsilon(t, y) \) is its original position. Note that \((P(x^+, y_\alpha) - P(x^-, y_\alpha)) \xi_\alpha \) is the shift rate of the shock \( \sigma^\epsilon \), after colliding with the shocks of the Riemann problems \( w_t, w_m, \) and \( w_m, w_r \). Their total shift is proportional to \( |\sigma_\alpha \xi_\alpha| \), and after the interaction with \( \sigma^\epsilon \), the shift of the latter is proportional to \( |\sigma^\epsilon| |\sigma_\alpha \xi_\alpha| \). Thus taking the limit as \( \epsilon \) tends to 0 of (4.5), we obtain for \( \epsilon \) sufficiently small
\begin{equation}
||x_{\theta}^\epsilon(t, y) - x_i^\epsilon(t, y)|| \leq D \sum_{\alpha} |\sigma_\alpha \xi_\alpha|,
\end{equation}
which implies (4.4).

We prove now the uniform Lipschitz continuity of the map \( y \mapsto x_i(t, y) \) for all \( t \geq 0 \).

**Proposition 4.3.** Consider two characteristic lines \( x_i(t, y_1), x_i(t, y_2) \), solution to (4.1). There exists \( C > 0 \), depending only on the system and the set \( E \), such that
\begin{equation}
1/C \leq \frac{x_i^2(t, y_2) - x_i^2(t, y_1)}{y_2 - y_1} \leq C.
\end{equation}
Proof. As in the previous proposition, let \( \epsilon \) be the size of the shock \( \sigma^\epsilon(t) \) located in \((0, y)\) in Riemann coordinates. If \( \xi(t) \) is its shift rate, then for \( \theta \) sufficiently small by Theorem 3.10 we obtain
\begin{equation}
(4.7)
\frac{x_i(t, y + \theta \xi) - x_i(t, y)}{\theta} \sigma^\epsilon(t) = \xi^\epsilon(t) \sigma^\epsilon(t) = r_i(w_t, w_r) \langle l_i(w_t, w_r), \sigma(0) \rangle \xi(0).
\end{equation}
In fact, by assumption, in the simplified wave patterns to compute the shift rate of \( \sigma^\epsilon \), there are no waves of the \( i \)-th family different from \( \sigma^\epsilon \). Dividing by \( \epsilon \) and taking the limit as \( \epsilon \) tends to 0, we obtain
\begin{equation}
\frac{x_i(t, y + \theta \xi) - x_i(t, y)}{\theta} \frac{\partial}{\partial w_i} u(t, x) = \frac{\partial}{\partial w_i} u(0, y) \langle l_i(w_t, w_r), r_i(0, y) \rangle \xi(0),
\end{equation}
which implies
\begin{equation}
\frac{d}{dy} x_i(t, y) = \frac{\partial u(0, y)/\partial w_i}{\partial u(t, x)/\partial w_i} \langle l_i(w_t, w_r), r_i(0, y) \rangle.
\end{equation}
We use the fact that \( \sigma^\epsilon(t)/\epsilon \) tends to \( \partial u(0, y)/\partial w_i \cdot r_i(w_t, w_r) \) as \( \epsilon \to 0 \). Since \( E \) is compact, the conclusion (4.6) follows easily.

**Remark 4.4.** The above proposition implies that the map \( h^\epsilon_l \) defined in (2.16) is uniformly Lipschitz, independent on the total variation of \( u_0 \), together with its inverse map \( (h^\epsilon_l)^{-1} \).

To end this section, we give a different proof of the following result given in [12]:

**Proposition 4.5.** If \( x(t), y(t) \) are the positions of two adjacent \( k \)-rarefaction waves, then for some constant \( \kappa > 0 \) one has
\begin{equation}
y(\tau) - x(\tau) \geq \kappa \tau 2^{-\nu},
\end{equation}
where \( \kappa > 0 \) is the constant defined in (2.3). Thus for all \( \tau > 0 \) the total variation of the Riemann invariant \( w_k \) of the \( k \)-th genuinely nonlinear family with \( N \) shocks at \( t = 0 \) is bounded by
\begin{equation}
\text{Tot.Var.} \{w_k(\tau, \cdot); [a, b]\} \leq \frac{2(b - a)}{\kappa \tau} + \|w_k\|_{L^\infty} + (N + 1)2^{1-\nu}.
\end{equation}
Proof. Consider two adjacent $k$-rarefaction fronts $x(t)$ and $y(t)$, and let $t_\alpha$, $\alpha = 1, \ldots, N$, be the interaction times of $x(t)$, $y(t)$ with other waves in the interval $[0, \tau]$. Fixed $t_i \in (t_\alpha, t_{\alpha+1})$ for some $\alpha$, let $z(t, x(t_i))$ be the characteristic line of the $k$-th genuinely nonlinear family starting in $(t_i, x(t_i))$ (see fig. 9). Assume $t_{i+1} > t_i$ sufficiently close to $t_i$ such that $t_{i+1} \in (t_\alpha, t_{\alpha+1})$ and $z(t, x(t_i))$ does not collide with shocks of other families for $t \in [t_i, t_{i+1}]$. Let $z(t, x(t_{i+1}))$ be the characteristic curve starting in $(t_{i+1}, x(t_{i+1}))$. By the assumption of genuinely nonlinearity, at time $t_{i+1}$ we have
\[
z(t_{i+1}, x(t_i)) - z(t_{i+1}, x(t_{i+1})) \geq c(t_{i+1} - t_i)2^{\nu-1},
\]
for some constant $c$, depending only on $E$. Using Proposition 4.3, at time $\tau$ we have
\[
z(\tau, x(t_i)) - z(\tau, x(t_{i+1})) \geq \frac{c}{C}(t_{i+1} - t_i)2^{\nu-1}.
\]
Repeating the process, it is possible to find a countable number of times $t_i$ such that
\[
\lim_{i \to -\infty} t_i = t_\alpha, \quad \lim_{i \to +\infty} t_i = t_{\alpha+1},
\]
and using (4.11) we get
\[
z(\tau, x(t_\alpha)) - z(\tau, x(t_{\alpha+1})) \geq \frac{c}{C}(t_{\alpha+1} - t_\alpha)2^{\nu-1}.
\]
Repeating the process for $y(t)$ and for all intervals $(t_{\alpha+1}, t_\alpha)$, we obtain (4.9) where $\kappa = c/C$. 

The second equation follows noticing that the total amount of positive jumps in the interval $[a, b]$ is bounded by $(1 + N)2^{-\nu} + (b - a)/\kappa \tau$. □

5. Proof of the main theorem

In this section we construct the semigroup $S$ on $L^\infty(\mathbb{R}; E)$. In [3] it is shown that for all $M$, there exists a semigroup $S^M$ defined on the domain
\[
D^M = \left\{ u : \mathbb{R} \to E : \text{Tot.Var.}(u) \leq M \right\},
\]
which is the only limit of the wave front tracking approximations constructed in section 2. We study now the dependence of the solution on the initial data $u \in D^M$. We consider separately the case for genuinely nonlinear and linearly degenerate families.

Proposition 5.1. Consider a front tracking solution $u$, such that $u(0, \cdot)$ has $N$ jumps $\sigma_\alpha$, $\alpha = 1, \ldots, N$, and let $\xi_\alpha$ be their shift rates. Given $\tau \geq 0$, denote with $w_\beta(\tau, \cdot)$ the jumps in the Riemann invariant $w_\beta(\tau, \cdot)$ of the $k$-th genuinely nonlinear family. Then there exists a constant $K$, depending only on the system and the domain $E$ such that
\[
\sum_{\beta} |\xi_\beta \sigma_\beta| \leq K(1 + N2^{-\nu}) \sum_{\alpha=1}^N |\xi_\alpha \sigma_\alpha|.
\]
This concludes the proof.

implies that have by Proposition 5.1

This shows that (5.5)

\[ N \]
on the number of initial jumps

If now (5.8) We can assume that u

Theorem 5.3. Consider two initial data \( u_1 \) and \( u_2 \), and denote with \( w_{j,k}(t,\cdot) \), the \( k \)-th Riemann coordinate of \( S^{M} u_j \), \( j = 1, 2 \), corresponding to the \( k \)-th genuinely non linear family. Moreover, let \( h_{j,i}^{\nu}, \ j = 1, 2 \), the map defined in (4.1) for the \( i \)-th linearly degenerate family. Then there exists a constant \( K' \), independent of \( M \), such that the following estimates hold:

\[ \int_{\mathbb{R}} |w_{1,k}(t,x) - w_{2,k}(t,x)| \, dx \leq K' \int_{\mathbb{R}} |u_1(x) - u_2(x)| \, dx, \]

\[ \sup_{t \geq 0, x \in \mathbb{R}} |h_{1,i}^{\nu}(x) - h_{2,i}^{\nu}(x)| \leq K' \int_{\mathbb{R}} |u_1(x) - u_2(x)| \, dx. \]

We can assume that \( u_\nu \) has a finite number \( N \) of jumps. If we denote with \( \gamma_\nu \) the path \( \theta \mapsto S^{\nu} u_\theta \), we have by Proposition 5.1

\[ \|\gamma_\nu\|_{L^1} \leq E\|u_\nu - u_\theta\|_{L^1} \]

If now \( \nu \to +\infty \), since \( w_{j,k}(\tau) \) converges to \( x_{j,k}(\tau) \), we obtain (5.6). Since this estimate does not depend on the number of initial jumps \( N \), we can extend it uniformly on \( M \).

Using the same pseudo polygonal path, in a similar way we can prove that

\[ \|x_{2,i}(\tau,y) - x_{1,i}(\tau,y)\|_{L^1} \]

This shows that \( x_{\nu}(\tau,\cdot) \) converges uniformly to the solution \( x_i(\tau,\cdot) \) as \( \nu \to +\infty \) and \( u^\nu \to u \). It also implies that

\[ \|x_{2,i}(\tau,y) - x_{1,i}(\tau,y)\|_{L^1} \]

This concludes the proof.

Proof. This is a corollary of Proposition 4.2.

Using the above propositions, we can prove the following theorem:

**Theorem 5.3.** Consider two initial data \( u_1 \) and \( u_2 \), and denote with \( w_{j,k}(t,\cdot) \), the \( k \)-th Riemann coordinate of \( S^{M} u_j \), \( j = 1, 2 \), corresponding to the \( k \)-th genuinely non linear family. Moreover, let \( h_{j,i}^{\nu}, \ j = 1, 2 \), the map defined in (4.1) for the \( i \)-th linearly degenerate family. Then there exists a constant \( K' \), independent of \( M \), such that the following estimates hold:

\[ \int_{\mathbb{R}} |w_{1,k}(t,x) - w_{2,k}(t,x)| \, dx \leq K' \int_{\mathbb{R}} |u_1(x) - u_2(x)| \, dx, \]

\[ \sup_{t \geq 0, x \in \mathbb{R}} |h_{1,i}^{\nu}(x) - h_{2,i}^{\nu}(x)| \leq K' \int_{\mathbb{R}} |u_1(x) - u_2(x)| \, dx. \]

We can assume that \( u_\nu \) has a finite number \( N \) of jumps. If we denote with \( \gamma_\nu \) the path \( \theta \mapsto S^{\nu} u_\theta \), we have by Proposition 5.1

\[ \|\gamma_\nu\|_{L^1} \leq E\|u_\nu - u_\theta\|_{L^1} \]

If now \( \nu \to +\infty \), since \( w_{j,k}(\tau) \) converges to \( x_{j,k}(\tau) \), we obtain (5.6). Since this estimate does not depend on the number of initial jumps \( N \), we can extend it uniformly on \( M \).

Using the same pseudo polygonal path, in a similar way we can prove that

\[ \|x_{2,i}(\tau,y) - x_{1,i}(\tau,y)\|_{L^1} \]

This shows that \( x_{\nu}(\tau,\cdot) \) converges uniformly to the solution \( x_i(\tau,\cdot) \) as \( \nu \to +\infty \) and \( u^\nu \to u \). It also implies that

\[ \|x_{2,i}(\tau,y) - x_{1,i}(\tau,y)\|_{L^1} \]

This concludes the proof.

Proof. The proof follows by Theorem 3.10 and Proposition 4.5. In fact, fixed a shock \( \sigma_\alpha \), using Theorem 3.10, we have that at time \( \tau \) for a shock \( \sigma_\beta \) of the \( i \)-th family there exist \( D' \)

\[ |\xi_\beta \sigma_\beta| \leq D'|\xi_\alpha \sigma_\alpha|, \]

if the shock \( \sigma_\beta \) starts on both sides of \( \sigma_\alpha \), or, using the same estimate of Proposition 4.2,

\[ |\xi_\beta| \leq D|\sigma_\alpha \xi_\alpha|, \]

if \( \sigma_\beta \) start on one side of \( \sigma_\alpha \). Since there is at most 1 shocks such that (5.3) holds, and the interval of influence is \([x_\alpha - \lambda \tau, x_\alpha + \lambda \tau]\), using Proposition 4.5 together with (5.3) and (5.4) we obtain

\[ \sum_\beta |\xi_\beta \sigma_\beta| \leq D'|\xi_\alpha \sigma_\alpha| + D|\sigma_\alpha \xi_\alpha| \cdot \text{Tot.Var.} \left\{ w_k, [x_\alpha - \lambda \tau, x_\alpha + \lambda \tau] \right\} \leq F(1 + 2^{-\nu})|\xi_\alpha \sigma_\alpha|. \]

The conclusion follows the linearity of the shift differential map.

Proof. This is a corollary of Proposition 4.2.
**Definition 5.4.** For all \( u \in L^\infty(\mathbb{R}, E) \), let \( u^M \in D^M \) be such that
\[
\lim_{M \to +\infty} u^M = u \quad \text{in } L^1_{\text{loc}}.
\]
Define \( S_t u \) as
\[
S_t u = \lim_{M \to +\infty} S^M_t u,
\]
where the limit is in \( L^1_{\text{loc}} \).

It is easy to prove that the right hand side of (5.10) is a Cauchy sequence in every compact set \([a, b]\); in fact, using the finite speed of propagation, we can consider \( u \) with compact support \([a - \lambda t, b + \lambda t]\). For the components \( w_k \) of the \( k \)-th genuinely nonlinear family, it follows directly from (5.6), while for a linearly degenerate component \( w_i \), let \( \tilde{w} \) be a Lipschitz continuous function such that
\[
\int_{\mathbb{R}} |w_i(0, x) - \tilde{w}(x)| dx \leq \epsilon.
\]
By Theorem 5.3 we have for \( u_1, u_2 \in D^M \) such that \( \|u - u_i\|_{L^1} < \delta, i = 1, 2, \)
\[
\sup_{t \geq 0, x \in \mathbb{R}} |h_{1,i}(x) - h_{2,i}(x)| < K' \delta,
\]
and it follows by easy computations that
\[
\|w_{1,i}(t) - w_{2,i}(t)\|_{L^1} \leq \|w_{1,i}(t) - \tilde{w} \circ (h_{1,i})^{-1}\|_{L^1} + \|w_{2,i}(t) - \tilde{w} \circ (h_{2,i})^{-1}\|_{L^1} + \|\tilde{w} \circ (h_{1,i})^{-1} - \tilde{w} \circ (h_{2,i})^{-1}\|_{L^1}
\]
\[
\leq C \|w_{1,i}(0) - \tilde{w}\|_{L^1} + C \|w_{2,i}(0) - \tilde{w}\|_{L^1} + L(b - a)G\|u_2 - u_1\|_{L^1}
\]
\[
\leq 2C(\epsilon + \delta) + L(b - a)G \delta;
\]
where \( L \) is the Lipschitz constant of \( \tilde{w} \). This shows that \( w^M_{1,i}(t) \) is a Cauchy sequence for all \( t \geq 0 \), because the right hand side of (5.11) can be made arbitrarily small. We can now prove the main theorem:

**Theorem 5.5.** The semigroup \( S : [0, +\infty) \otimes L^\infty(\mathbb{R}; E) \mapsto L^\infty(\mathbb{R}; E) \) defined in (5.10) is the only continuous semigroup on \( L^\infty(\mathbb{R}; E) \) such that the following properties are satisfied:

i) for all \( \bar{u}^n, \bar{u} \in L^\infty(\mathbb{R}; E), t_n, t \in [0, +\infty), \) with \( \bar{u}^n \to \bar{u} \) in \( L^1_{\text{loc}}, |t - t_n| \to 0 \) as \( n \to +\infty, \)
\[
\lim_{n \to +\infty} S_{t_n} \bar{u}^n = S_t \bar{u} \quad \text{in } L^1_{\text{loc}};
\]
ii) each trajectory \( t \mapsto S_t u_0 \) is a weak entropic solution to the Cauchy problem
\[
\begin{cases}
  u_t + f(u)_x = 0 \\
  u(0, x) = u_0(x)
\end{cases}
\]
with \( u_0 \in L^\infty(\mathbb{R}; E); \)
iii) if \( u_0 \) is piecewise constant, then, for \( t \) sufficiently small, \( S_t u_0 \) coincides with the function obtained by piecing together the solutions of the corresponding Riemann problems.

**Proof.** The statement follows easily, since we proved that \( S_t u \) is the unique limit of wave front approximations, and for data with bounded total variation we can apply the results in [3]. \( \square \)

**Remark 5.6.** Note that what we also proved that the characteristic equation (4.1) is well posed for \( L^\infty \) data: the solution \( x_i(t, y) \) is Lipschitz continuous w.r.t. both variables. In fact, if \( u^n_0 \) converges to \( u_0 \) in \( L^1_{\text{loc}} \), Proposition 4.2 implies that \( x^n_i(t, y) \), solution to the \( i \)-characteristic equation, tends to \( x(t, y) \) uniformly for all \( t, y \): it is then easy to prove that \( x(t, y) \) satisfies the equation
\[
x_i(t, y) = y + \int_0^t \lambda_i(s, x_i(s, y)) ds.
\]
The above equation implies uniqueness of \( x_i(t, y) \) in the sense of Caratheodory, and Proposition 4.3 prove continuous dependence on \( y \).

This is not trivial, since even for \( 2 \times 2 \) systems not in conservation form the dependence is Hölder continuous, while for general \( n \times n \) the solution does not exist [16].
Note moreover that semigroup $S$ is continuous, but not uniformly continuous. However if the initial data takes values is a compact set of $L^1 \cap L^\infty$, then the semigroup becomes uniformly continuous. This extend the Lipschitz continuity when the initial data have bounded total variation.

Acknowledgment. This research was partially supported by the European TMR Network on Hyperbolic Conservation Laws ERBFMRXCT960033.

REFERENCES


S.I.S.S.A. - I.S.A.S., VIA BEIRUT 2-4, 34014 TRIESTE (ITALY)
E-mail address: bianchin@sissa.it
URL: http://www.sissa.it/~bianchin/