BV Solutions for a Class of Viscous Hyperbolic Systems

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Abstract. The paper is concerned with the Cauchy problem for a nonlinear, strictly hyperbolic system with small viscosity:

$$u_t + A(u)u_x = \varepsilon\, u_{xx}, \qquad\qquad u(0,x) = \bar u(x). \tag{$*$}$$

We assume that the integral curves of the eigenvectors r_i of the matrix A are straight lines. On the other hand, we do not require the system $(*)$ to be in conservation form, nor do we make any assumption on genuine linearity or linear degeneracy of the characteristic fields.

In this setting we prove that, for some small constant $\eta_0 > 0$ the following holds. For every initial data $\bar u \in \mathbf{L}^1$ with Tot.Var.$\{\bar u\} < \eta_0$, the solution u^ε of $(*)$ is well defined for all $t > 0$. The total variation of $u^\varepsilon(t,\cdot)$ satisfies a uniform bound, independent of t, ε. Moreover, as $\varepsilon \to 0+$, the solutions $u^\varepsilon(t,\cdot)$ converge to a unique limit $u(t,\cdot)$. The map $(t,\bar u) \mapsto S_t\bar u \doteq u(t,\cdot)$ is a Lipschitz continuous semigroup on a closed domain $\mathcal{D} \subset \mathbf{L}^1$ of functions with small total variation. This semigroup is generated by a particular Riemann Solver, which we explicitly determine.

The above results can also be applied to strictly hyperbolic systems on a Riemann manifold. Although these equations cannot be written in conservation form, we show that the Riemann structure uniquely determines a Lipschitz semigroup of "entropic" solutions, within a class of (possibly discontinuous) functions with small total variation. The semigroup trajectories can be obtained as the unique limits of solutions to a particular parabolic system, as the viscosity coefficient approaches zero.

The proofs rely on some new a priori estimates on the total variation of solutions for a parabolic system whose components drift with strictly different speeds.

1 - Introduction

Consider a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension:

$$u_t + f(u)_x = 0. \tag{1.1}$$

For initial data with small total variation, the global existence of weak solutions was proved in [8]. Moreover, the uniqueness and stability of entropy admissible BV solutions was recently established in a series of papers [3,4,5,6]. A long standing open question is whether these discontinuous solutions can be obtained as vanishing viscosity limits. More precisely, given a smooth initial data $\bar{u} : I\!R \mapsto I\!R^n$ with small total variation, consider the parabolic Cauchy problem

$$u(0, x) = \bar{u}(x). \tag{1.2}$$

$$u_t + A(u)u_x = \varepsilon\, u_{xx}. \tag{1.3}$$

Here $A(u) \doteq Df(u)$ is the Jacobian matrix of f and $\varepsilon > 0$. It is then natural to expect that, as $\varepsilon \to 0$, the solution u^ε of (1.2)-(1.3) converges to the unique entropy weak solution u of (1.1)-(1.2). Unfortunately, no general theorem in this direction is yet known. Some of the main results available in the literature are listed below.

1) In the case of a scalar conservation law, the entropic solutions of (1.1) determine a semigroup which is contractive w.r.t. the $\mathbf{L}^1$-distance. In this case, a general convergence theorem for vanishing viscosity approximations was proved in the classical work of Kruzhkov [12].

2) For various 2×2 systems, if a uniform $\mathbf{L}^\infty$–bound on all functions u^ε is available, one can consider a weak limit $u^\varepsilon \rightharpoonup u$. By a compensated compactness argument introduced by DiPerna [7], it then follows that u is actually a weak solution of the nonlinear system (1.1). For a comprehensive discussion of the compensated compactness method and its applications to conservation laws, see [18].

3) For $n \times n$ Temple class systems, a proof of the convergence of the viscous solutions u^ε to a solution of (1.1) can be found in [17,18].

4) Assume that all characteristic fields of the system (1.1) are linearly degenerate. Then every solution with small total variation which is initially smooth remains smooth for all positive times [2]. Clearly such solution can be obtained as limit of vanishing viscosity approximations. By a density argument it follows that every weak solution of (1.1) with sufficiently small total variation is a limit of viscous approximations.

5) For a general $n \times n$ strictly hyperbolic system, let u be a piecewise smooth entropic solution of (1.1) with jumps along a finite number of smooth curves in the t-x plane. Thanks to this

additional regularity assumptions on u, it was proved in [10] that there exists a family of viscous solutions u^ε converging to u in $\mathbf{L}^1_{\rm loc}$ as $\varepsilon \to 0$.

From our point of view, the major difficulty toward a general proof of the convergence $u^\varepsilon \to u$ lies in deriving an a priori estimate on the total variation of the solution of (1.2)-(1.3), uniformly valid as $\varepsilon \to 0$. To fix the ideas, assume $\bar u \in \mathcal{C}^\infty_c$ with Tot.Var.$(\bar u)$ sufficiently small. Performing the rescalings $t \mapsto t/\varepsilon,\ x \mapsto x/\varepsilon$, the Cauchy problem becomes

$$u_t + A(u)u_x = u_{xx}, \tag{1.4}$$

$$u(0,x) = \bar u(\varepsilon x). \tag{1.5}$$

Observe that, as $\varepsilon \to 0$, the initial data $u(0,\cdot)$ has constant total variation, all of its derivatives approach zero, but its $\mathbf{L}^1$-norm approaches infinity. We thus need estimates on the total variation of a solution $u(t,\cdot)$ of (1.4) which are independent of the $\mathbf{L}^1$-norm of the initial data.

To illustrate the heart of the matter, let us denote by $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of the $n \times n$ Jacobian matrix $A(u) \doteq Df(u)$, and call $l^1,\ldots,l^n,\ r_1,\ldots,r_n$, its left and right eigenvectors, normalized so that

$$\big|r_i(u)\big| \equiv 1, \qquad\qquad l^i(u)\cdot r_j(u) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{1.6}$$

The directional derivative of a function $\phi = \phi(u)$ in the direction of the eigenvector r_i is written

$$r_i \bullet \phi(u) \doteq \lim_{h\to 0} \frac{\phi\big(u + h r_i(u)\big) - \phi(u)}{h}.$$

Moreover, by $u_x^i \doteq l^i(u)\cdot u_x$ we denote the i-th component of the gradient u_x w.r.t. the basis of right eigenvectors $\{r_1,\ldots,r_n\}$. Recalling (1.6), this implies

$$u_x = \sum_i u_x^i r_i. \tag{1.7}$$

With the above notations, (1.4) becomes

$$u_t + \sum_i \lambda_i u_x^i r_i = \sum_i (u_x^i r_i)_x = \sum_i (u_x^i)_x r_i + \sum_{i,j} u_x^i u_x^j (r_j \bullet r_i). \tag{1.8}$$

Differentiating (1.7) w.r.t. t and (1.8) w.r.t. x and equating the results one obtains

$$\begin{aligned} u_{xt} = &\sum_i (u_x^i)_t r_i - \sum_{i,j} u_x^i u_x^j \lambda_j (r_j \bullet r_i) \\ &+ \sum_{i,j} u_x^i (u_x^j)_x (r_j \bullet r_i) + \sum_{i,j,k} u_x^i u_x^j u_x^k \big((r_j \bullet r_k)\bullet r_i\big), \end{aligned}$$

$$
\begin{aligned}
u_{tx}+&\sum_i(\lambda_i u_x^i)_x r_i+\sum_{i,j}\lambda_i u_x^i u_x^j(r_j\bullet r_i)\\
&=\sum_i (u_x^i)_{xx}r_i+\sum_{i,j}(u_x^i)_x u_x^j(r_j\bullet r_i)+\sum_{i,j}(u_x^i)_x u_x^j(r_j\bullet r_i)\\
&\qquad+\sum_{i,j}u_x^i(u_x^j)_x(r_j\bullet r_i)+\sum_{i,j,k}u_x^i u_x^j u_x^k\big(r_k\bullet(r_j\bullet r_i)\big),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_i (u_x^i)_t r_i+\sum_i(\lambda_i u_x^i)_x r_i+\sum_{j\neq k}\lambda_j u_x^j u_x^k[r_k,r_j]\\
&\qquad=\sum_i (u_x^i)_{xx}r_i+2\sum_{i,j}(u_x^i)_x u_x^j(r_j\bullet r_i)+\sum_{i,j,k}u_x^i u_x^j u_x^k\Big(r_k\bullet(r_j\bullet r_i)-(r_j\bullet r_k)\bullet r_i\Big).
\end{aligned}
\tag{1.9}
$$

Here $[r_k,r_j]\doteq r_k\bullet r_j-r_j\bullet r_k$ is the usual Lie bracket. Taking the inner product of (1.9) with $l^i(u)$ one obtains

$$
\begin{aligned}
&(u_x^i)_t+(\lambda_i u_x^i)_x-(u_x^i)_{xx}\\
&=l^i\cdot\left\{\sum_{j\neq k}\lambda_k[r_k,r_j]u_x^j u_x^k+2\sum_{j,k}(r_k\bullet r_j)(u_x^j)_x u_x^k+\sum_{j,k,\ell}\Big(r_\ell\bullet(r_k\bullet r_j)-(r_\ell\bullet r_k)\bullet r_j\Big)u_x^j u_x^k u_x^\ell\right\}\\
&=\sum_{j\neq k}G^i_{j,k}(u)u_x^j u_x^k+\sum_{j,k}H^i_{j,k}(u)(u_x^j)_x u_x^k+\sum_{j,k,\ell}K^i_{j,k,\ell}(u)u_x^j u_x^k u_x^\ell.
\end{aligned}
\tag{1.10}
$$

Setting $v^i\doteq u_x^i$, we thus seek an estimate on the $\mathbf{L}^1$-norm of solutions to

$$
\begin{aligned}
v_t^i+\big(\lambda_i(u)v^i\big)_x-v_{xx}^i&=\sum_{j\neq k}G^i_{j,k}(u)v^j v^k+\sum_{j,k}H^i_{j,k}(u)v_x^j v^k+\sum_{j,k,\ell}K^i_{j,k,\ell}(u)v^j v^k v^\ell\\
&\doteq\phi^i(u,v^1,\ldots,v^n).
\end{aligned}
\tag{1.11}
$$

We regard (1.11) as a parabolic system of n scalar equations, coupled through the terms G,H,K. These coupling terms can be split in two groups:

- *Transversal terms* involving at least two distinct components, such as v^jv^k, $v_x^jv^k$, $v^jv^kv^\ell$ with $j\neq k$,
- *Non-transversal terms* involving one single component, such as $v^jv_x^j$, $v^jv^jv^j$.

In the present paper we perform a careful study of transversal terms, and show that their total contribution is of quadratic order. Hence, for small initial data, they cannot produce a substantial amplification of the solution of (1.11). As a consequence, if the geometry of the system is such that the diffusion operator yields only transversal terms, then the total variation of solutions to (1.3) remains uniformly bounded as $\varepsilon\to 0$. This is the case of systems, not necessarily of Temple class or not even in conservation form, where the integral curves of the eigenvectors r_i are straight lines, namely:

$$
r_i\bullet r_i(u)=0\qquad\qquad \text{for all } i,u. \tag{1.12}
$$

We conjecture that uniform bounds on the total variation remain valid also in the presence of non-transversal terms. The analysis of these terms however seems to require substantially different techniques, and will be taken up elsewhere.

Our main results can be stated as follows. We first consider the Cauchy problem for the parabolic system (1.4). Here $u \mapsto A(u)$ is a smooth map defined on an open set $\Omega \subseteq I\!R^n$, with values in the set of $n \times n$ matrices. We assume that each $A(u)$ is *uniformly strictly hyperbolic* in Ω, i.e. it has n real distinct eigenvalues $\lambda_1(u_1) < \cdots < \lambda_n(u_n)$ for all $u_1, \ldots, u_n \in \Omega$. One can thus choose bases of right and left eigenvectors of $A(u)$, normalized as in (1.6). For a BV function $\bar{u} : I\!R \mapsto I\!R^n$ we write $\bar{u}(-\infty) \doteq \lim_{x \to -\infty} \bar{u}(x)$.

Theorem 1. *Assume that all matrices $A(u)$ in (1.4) are uniformly strictly hyperbolic and that their normalized eigenvectors r_i satisfy (1.12). Then, for every compact set $K_0 \subset \Omega$ there exist constants $0 < \delta_0 < \delta_1$ so that the following holds. For every initial data $\bar{u}$ such that*

$$Tot.Var.\{\bar{u}\} \leq \delta_0, \qquad \bar{u}(-\infty) \in K_0, \tag{1.13}$$

the Cauchy problem (1.2), (1.4) has a unique, global solution u, which satisfies

$$Tot.Var.\big\{u(t,\cdot)\big\} \leq \delta_1 \qquad \text{for all } t \geq 0. \tag{1.14}$$

Moreover there exist constants L, L' such that, for every $\bar{u}, \bar{w}$ satisfying (1.13), the corresponding solutions satisfy

$$\big\|u(t) - w(s)\big\|_{\mathbf{L}^1} \leq L'\big(|t-s|^{1/2} + |t-s|\big) + L\,\big\|\bar{u} - \bar{w}\big\|_{\mathbf{L}^1}, \qquad \text{for all } t, s \geq 0. \tag{1.15}$$

We remark that the bound (1.14) depends only on the total variation of the initial data $\bar{u}$, not on its $\mathbf{L}^1$-norm. The above result thus yields an a priori bound on the total variation of solutions of (1.2)-(1.3), independent of the parameter ε. Our second main result shows that, as $\varepsilon \to 0+$, these solutions u^ε converge to a unique limit u, depending continuously on the initial data $\bar{u}$. Namely, the map $(t, \bar{u}) \mapsto S_t\bar{u} \doteq u(t,\cdot)$ is a Lipschitz continuous semigroup on a closed domain $\mathcal{D}$ of functions with small total variation.

By the analysis in [3], a Lipschitz semigroup S on a domain of BV functions is uniquely determined by its local behavior on piecewise constant initial data. In other words, if we assign a procedure for solving each Riemann problem, then the entire semigroup is completely determined. In the present case, our semigroup of "vanishing viscosity solutions" is generated by the following

Riemann Solver. Consider the Riemann problem

$$u_t + A(u)u_x = 0, \tag{1.16}$$

$$u(0,x) = \begin{cases} u^+ & \text{if} \quad x > 0, \\ u^- & \text{if} \quad x < 0, \end{cases} \tag{1.17}$$

with $|u^+ - u^-|$ suitably small. For $u \in \Omega$, $i = 1, \ldots, n$, define the i-rarefaction curve $\sigma \mapsto R_i(\sigma)(u)$ as the integral curve of r_i through u, parametrized so that

$$R_i(0)(u) = u, \qquad \frac{d}{d\sigma} R_i(\sigma)(u) = r_i\big(R_i(\sigma)(u)\big). \tag{1.18}$$

By the implicit function theorem there exist unique states $\omega_0 = u^-, \omega_1, \ldots, \omega_n = u^+$ and wave sizes σ_i such that

$$\omega_i = R_i(\sigma_i)(\omega_{i-1}) \qquad i = 1, \ldots, n. \tag{1.19}$$

Moreover, by strict hyperbolicity, there exist constants $\bar\lambda_1 < \ldots < \bar\lambda_{n-1}$ such that

$$\lambda_i\big(R_i(\theta\sigma_i)(\omega_{i-1})\big) \in \,]\bar\lambda_{i-1},\ \bar\lambda_i[\qquad \theta \in [0,1], \quad i = 1, \ldots, n.$$

with the convention $\bar\lambda_0 \doteq -\infty$, $\bar\lambda_n \doteq \infty$. For each i, consider the scalar function

$$F_i(\sigma) \doteq \int_0^\sigma \lambda_i\big(R_i(s)(\omega_{i-1})\big)\, ds, \tag{1.20}$$

and let $z_i(t,x)$ be the unique entropic solution to the Riemann problem for the scalar conservation law

$$z_t + F_i(z)_x = 0, \qquad z(0,x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ \sigma_i & \text{if} \quad x > 0. \end{cases} \tag{1.21}$$

A "solution" of the Riemann problem (1.16)-(1.17) is now defined by the assignment

$$u(t,x) \doteq R_i\big(z_i(t,x)\big)(\omega_{i-1}) \qquad \text{if} \ \ \frac{x}{t} \in [\bar\lambda_{i-1},\ \bar\lambda_i]. \tag{1.22}$$

We remark that, in general, the function in (1.22) is not a classical solution of (1.16). If the system is not in conservation form, this function cannot even be regarded as a solution in distributional sense. Yet, motivated by our next result it is appropriate to regard (1.22) as the unique solution of (1.16)-(1.17) in the vanishing viscosity sense.

Theorem 2. *Assume that all matrices $A(u)$ in (1.4) are unifomly strictly hyperbolic and that their eigenvectors satisfy (1.12). Then, for every compact set $K_0 \subset \Omega$ there exist constants $L, L', \delta_0 > 0$, a closed domain $\mathcal{D} \subset \mathbf{L}^1_{\rm loc}$ and a continuous semigroup $S : \mathcal{D} \times [0,\infty[\mapsto \mathcal{D}$ with the following properties.*

(i) Every function $\bar u$ satisfying (1.13) lies in the domain $\mathcal{D}$ of the semigroup.

(ii) For every $\bar u, \bar w \in \mathcal{D}$ with $\bar u - \bar w \in \mathbf{L}^1$ and every $t, s \geq 0$ one has

$$\|S_t\bar u - S_s\bar w\|_{\mathbf{L}^1} \leq L'|t-s| + L\|\bar u - \bar w\|_{\mathbf{L}^1}\,. \tag{1.23}$$

(iii) *For every piecewise constant initial data $\bar u \in \mathcal{D}$, there exists $\tau > 0$ such that the following holds. For $t \in [0, \tau]$, the semigroup trajectory $S_t \bar u$ coincides with the function $u(t, \cdot)$ obtained by piecing together the solutions (1.22) of the Riemann problems determined by the jumps in $\bar u$.*

(iv) *For every $\bar u \in \mathcal{D}$, the trajectory $u(t, \cdot) = S_t \bar u$ is the unique limit in $\mathbf{L}^1_{\rm loc}$ of the corresponding solutions $u^\varepsilon(t, \cdot)$ of the viscous Cauchy problem (1.2)-(1.3), as $\varepsilon \to 0+$.*

The proofs of the two above theorems are outlined in Section 2. Details are then worked out in Sections 3 to 6. The last section is concerned with a strictly hyperbolic system on a Riemann manifold. By a minor modification of the previous arguments, we show that the Riemann structure uniquely determines a viscosity operator and a continuous semigroup, whose trajectories are obtained as limits of solutions to the corresponding parabolic system, as the viscosity coefficient tends to zero.

2 - Outline of the proofs

In this section we describe the main steps in the proofs of Theorems 1 and 2. The total variation of a solution u will be estimated by deriving an a priori bound on the $\mathbf{L}^1$-norms of the gradient components $v^i \doteq l^i(u) \cdot u_x$, $i = 1, \ldots, n$. Denoting by ϕ^i the right hand side of (1.11), the Cauchy problem (1.4), (1.2) can be rewritten as

$$\begin{cases} v^i_t + [\lambda_i(u) v^i]_x - v^i_{xx} = \phi^i \\ v^i(0,x) = \bar v^i(x) \doteq l^i\big(\bar u(x)\big) \cdot \bar u_x(x), \end{cases} \tag{2.1}$$

where, for each t, the function $u = u(t,x)$ can be recovered from $v = (v^1, \ldots, v^n)$ by solving the system of O.D.E's

$$\begin{cases} u_x = \sum_{i=1}^n v^i(t,x) r_i(u), \\ \lim_{x \to -\infty} u(t,x) = \bar u(-\infty). \end{cases} \tag{2.2}$$

It is well known that if the initial data of (2.1), (2.2) are sufficiently smooth, then there exists a solution at least for a small time interval $[0, \bar t]$. We define the quantity

$$\eta_0 \doteq \sup_{i=1,\ldots,n} \Big\{ \|v^i(0,x)\|_{\mathbf{L}^1} \Big\}, \tag{2.3}$$

Since the left hand side of the evolution equation in (2.1) is in conservation form, one clearly has

$$\big\|v^i(t,\cdot)\big\|_{\mathbf{L}^1} \leq \big\|\bar v^i\big\|_{\mathbf{L}^1} + \int_0^t \int_{I\!\!R} \big|\phi^i(t,x)\big| \, dxdt, \tag{2.4}$$

therefore

$$\left\|v^i(t,\cdot)\right\|_{\mathbf{L}^1} \leq \eta_0 + \sup_{i=1,\ldots,n} \int_0^t \int_{I\!R} \left|\phi^i(s,x)\right| \, dxds \doteq \eta(t). \tag{2.5}$$

Therefore, if the total strength of the source term ϕ^i is bounded and of quadratic order w.r.t. $\eta(t)$, the solution of (2.1) is well defined for all $t \geq 0$ if η_0 is sufficiently small. The main goal of this section is then to prove an a priori bound on the terms ϕ^i of the form

$$\sup_{i=1,\ldots,n} \int_0^t \int_{I\!R} \left|\phi^i(s,x)\right| \, dxds \leq \kappa\eta(t)^2, \tag{2.6}$$

for some constant $\kappa > 0$.

We start by choosing constants $c, \delta_1 > 0$ small enough so that the compact set

$$K_1 \doteq \left\{u \in I\!R^n; \;\; \mathrm{dist}(u,\, K_0) \leq \delta_1\right\} \tag{2.7}$$

is entirely contained inside Ω, and moreover

$$\lambda_j(u) - \lambda_i(v) \geq c \qquad \text{whenever } i < j, \quad u, v \in K_1, \; |u - v| \leq \delta_1. \tag{2.8}$$

We also choose constants C_0, $\widehat{C}$ such that

$$\left|r_j \bullet \lambda_i(u)\right| \leq C_0, \qquad \left|G^i_{j,k}(u)\right|, \; \left|H^i_{j,k}(u)\right|, \; \left|K^i_{j,k,\ell}(u)\right| \leq C_0 \qquad \text{for all } u \in K_1 \tag{2.9}$$

$$\widehat{C} \geq \max\left\{2(n+1)^2 C_0, \;\; \frac{8}{\sqrt{\pi}}\right\}. \tag{2.10}$$

In the first part of our analysis we shall assume that, for all $i = 1, \ldots, n$, the initial data $\bar{v}^i$ in (2.1) satisfy

$$\int_{I\!R} \left|\bar{v}^i(x)\right| dx \leq \eta_0, \tag{2.11}$$

$$\int_{I\!R} \left|\bar{v}^i_x(x)\right| dx \leq \frac{4}{\sqrt{\pi}} \widehat{C}^2 \eta_0^2, \tag{2.12}$$

for some $\eta_0 > 0$. We choose the constant η_0 small enough so that

$$\eta_0 \leq \min\left\{\frac{c}{4\widehat{C}}, \;\frac{\delta_1}{2n}\right\}. \tag{2.13}$$

At a later stage, using standard smoothing properties of parabolic equations, we will remove the assumption (2.12).

If $\eta(t) > 2\eta_0$ for some t, by continuity there exists a time $\bar{t} > 0$ such that $\eta(t) < 2\eta_0$ for all $t \in [0, \bar{t}[$ and moreover $\eta(\bar{t}) = 2\eta_0$. We will show that for all $0 \leq t \leq \bar{t}$ we have the estimates

$$\int_{I\!R} \left|\phi^i(t,x)\right| dx < \frac{8}{\sqrt{\pi}} \widehat{C}^3 (2\eta_0)^3 \qquad\qquad 0 \leq t \leq \bar{t}, \tag{2.14}$$

$$\int_0^{\bar t}\int_{I\!R}\big|\phi^i(t,x)\big|\,dxdt<\frac{\widehat C}{c}\big(2\eta_0\big)^2,\tag{2.15}$$

so that it follows

$$\eta(\bar t)<\eta_0+\frac{\widehat C}{c}\big(2\eta_0\big)^2\leq 2\eta_0.\tag{2.16}$$

The above formula implies that $\eta(t)<2\eta_0$ for all $t\geq 0$, i.e. the solution of (2.1) exists and has bounded $\mathbf{L}^1$ norm for all $t\geq 0$. In fact, recalling that the eigenvectors r_i were chosen with unit length, by (2.2) and the choice of η_0 this yields

$$\begin{aligned}\big|u(t,x)-\bar u(-\infty)\big|&\leq \text{Tot.Var.}\big\{u(t,\cdot)\big\}\\&\leq\sum_{i=1}^n\big\|v(t,x)\big\|_{\mathbf{L}^1}<n\cdot 2\eta_0\leq\delta_1.\end{aligned}\tag{2.17}$$

Hence the function u takes values inside the compact set K_1, and the bounds (2.8)-(2.9) hold.

A detailed proof of (2.13)-(2.14) will be worked out in Section 3, providing a priori bounds on the $\mathbf{L}^1$ norms of the terms v^jv^k, $v^j_xv^k$, $v^jv^kv^\ell$ in (1.11), for $j\neq k$. By (2.5), using (2.14) and (2.15) we conclude

$$\int_{I\!R}\big|v^i(t,x)\big|\,dx\leq\eta_0+\frac{\widehat C}{c}\big(\eta(t)\big)^2\leq 2\eta_0,\tag{2.18}$$

$$\int_{I\!R}\big|v^i_x(t,x)\big|\,dx\leq\frac{8}{\sqrt{\pi}}\widehat C^2\big(\eta(t)\big)^2\leq\frac{8}{\sqrt{\pi}}\widehat C^2\big(2\eta_0\big)^2.\tag{2.19}$$

The second inequality provides a regularity estimate on the solution to (2.1).

The bounds (2.18)–(2.19) thus yield an a uniform bound on the total variation of $u(t,\cdot)$. Indeed, for every $t\geq 0$ one has

$$\text{Tot.Var.}\big\{u(t)\big\}\;\leq\;\delta_1.\tag{2.20}$$

This establishes global BV bounds on every solution u of (1.2), (1.4) whose initial data lies in the domain

$$\begin{aligned}\mathcal{D}^*\doteq\Big\{\bar u:I\!R\mapsto I\!R^n\;;\quad u\text{ is absolutely continuous},\quad \bar u(-\infty)\in K_0,\\ \big\|l_i(\bar u)\cdot\bar u_x\big\|_{\mathbf{L}^1}\leq\eta_0,\quad\big\|(l_i(\bar u)\cdot\bar u_x)_x\big\|_{\mathbf{L}^1}\leq\frac{4}{\sqrt{\pi}}\widehat C^2\eta_0^2,\quad i=1,\dots,n\Big\}.\end{aligned}\tag{2.21}$$

In Section 4 we then observe that, by the smoothing properties of the parabolic system (1.4), one can choose $t_0,\delta_0>0$ so that the following holds. For every initial data $\bar u$ satisfying (1.13), the corresponding solution of (1.2), (1.4) satisfies

$$u(t_0)\in\mathcal{D}^*,\qquad\text{Tot.Var.}\big\{u(t)\big\}\leq\delta_1\qquad\text{for all } t\in[0,t_0].\tag{2.22}$$

The previous analysis can thus be applied to the corresponding Cauchy problem on $[t_0, \infty[$. This yields the bounds (1.14), proving the first part of Theorem 1.

Concerning the estimates (1.15), the continuity of a BV solution as a function of time is a well known result for parabolic equations. To prove the Lipschitz continuous dependence w.r.t. the initial data, in Section 5 we first study the linearized equation describing the evolution of an infinitesimal perturbation. Replacing u by $u + \epsilon h$ in (1.4), letting $\epsilon \to 0$ and retaining terms of order ϵ we obtain

$$h_t + \big[DA(u) \cdot h\big] u_x + A(u) h_x = h_{xx}, \tag{2.23}$$

If the total variation of the reference solution u remains small, we show that every solution $h = h(t, x)$ of the linearized system (2.23) satisfies

$$\int_{\mathbb{R}} \big|h(t,x)\big|\, dx \le L \cdot \int_{\mathbb{R}} \big|h(0,x)\big|\, dx \qquad \text{for all } t \ge 0, \tag{2.24}$$

for some uniform constant L. If now two initial data $\bar u, \bar w$ are given, following [2] we construct the smooth path

$$\theta \mapsto \bar u^\theta \doteq \theta \bar u + (1-\theta)\bar w, \qquad\qquad \theta \in [0,1]. \tag{2.25}$$

Calling $t \mapsto u^\theta(t, \cdot)$ the solution of (1.4) with initial data $\bar u^\theta$, we can write

$$\begin{aligned} \big\|u(t) - w(t)\big\|_{\mathbf{L}^1} &\le \int_0^1 \left\| \frac{du^\theta(t)}{d\theta} \right\|_{\mathbf{L}^1} d\theta \\ &\le L \cdot \int_0^1 \left\| \frac{du^\theta(0)}{d\theta} \right\|_{\mathbf{L}^1} d\theta \\ &= L \cdot \|\bar u - \bar w\|_{\mathbf{L}^1}. \end{aligned} \tag{2.26}$$

Indeed, the tangent vector

$$h^\theta(t,x) \doteq \frac{du^\theta}{d\theta}(t,x)$$

is a solution of the linearized Cauchy problem

$$h^\theta_t + \big[DA(u^\theta) \cdot h^\theta\big] u^\theta_x + A(u^\theta) h^\theta_x = h^\theta_{xx}, \tag{2.27}$$

$$h^\theta(0,x) = \bar h^\theta(x) = \bar u(x) - \bar w(x), \tag{2.28}$$

hence it satisfies (2.24) for every θ. This completes the proof of Theorem 1.

We now give a proof of Theorem 2. For a fixed $\varepsilon > 0$, thanks to the coordinate rescaling $t \mapsto t/\varepsilon$, $x \mapsto x/\varepsilon$, the solution u^ε of (1.2)-(1.3) can be written in the form

$$u^\varepsilon(t,x) = U^\varepsilon(t/\varepsilon,\ x/\varepsilon),$$

where U^ε is the solution of (1.4)-(1.5). Clearly, for all $t \geq 0$ one has

$$\text{Tot.Var.}\big\{u^\varepsilon(t,\cdot)\big\} = \text{Tot.Var.}\big\{U^\varepsilon(t,\cdot)\big\}.$$

Therefore, by Theorem 1, for each initial condition $\bar{u}$ in the set

$$\mathcal{D}_0 \doteq \Big\{\bar{u} \in \mathbf{L}^1_{\text{loc}}\,; \quad \text{Tot.Var.}\{\bar{u}\} \leq \delta_0, \quad \bar{u}(-\infty) \in K_0\Big\} \tag{2.29}$$

the corresponding solution of (1.3) satisfies (1.14). Moreover, given two initial data $\bar{u}, \bar{w}$, from (1.15) we deduce

$$\big\|u(t) - w(s)\big\|_{\mathbf{L}^1} \leq L'\big(\sqrt{\varepsilon|t-s|} + |t-s|\big) + L\,\big\|\bar{u} - \bar{w}\big\|_{\mathbf{L}^1} \qquad \text{for all } t, s \geq 0. \tag{2.30}$$

For each $\bar{u} \in \mathcal{D}_0$ we can thus use Helly's compactness theorem and deduce the existence of a subsequence $\varepsilon_\nu \to 0$ such that the corresponding solutions u^{ε_ν} converge to some function u, namely

$$\lim_{\nu\to\infty} u^{\varepsilon_\nu}(t) = u(t) \qquad \text{in } \mathbf{L}^1_{\text{loc}}\,, \quad \text{for all } t \geq 0. \tag{2.31}$$

By a diagonalization argument, we can assume that, with the same sequence ε_ν, the convergence (2.31) holds for all solutions starting from a countable dense set of initial data in $\mathcal{D}_0$. In order to characterize this limit and show that the whole sequence u^ε converges as $\varepsilon \to 0$, we first derive a bound on the propagation speed of perturbations. By studying again the linear variational equation (2.23), in Section 6 we show that, for any interval $[a,b]$ and any two initial data $\bar{u}, \bar{w}$, the corresponding vanishing viscosity limit solutions satisfy

$$\int_{a+\tilde{\lambda}t}^{b-\tilde{\lambda}t} \big|u(t,x) - w(t,x)\big|\,dx \leq L \cdot \int_a^b \big|\bar{u}(x) - \bar{w}(x)\big|\,dx\,. \tag{2.32}$$

Here $\tilde{\lambda}$ is a suitably large constant providing an upper bound for all wave speeds, so that $\big|\lambda_i(\omega)\big| \leq \tilde{\lambda}$ for all $\omega \in K_1$, $i = 1, \ldots, n$. We can now define

$$\lim_{\nu\to\infty} u^{\varepsilon_\nu}(t,\cdot) = u(t,\cdot) \doteq S_t\bar{u} \qquad\qquad \bar{u} \in \mathcal{D}_0. \tag{2.33}$$

Indeed, since the limit (2.31) exists for a dense set of initial data $\bar{u}$, the uniform continuity property (2.32) implies the existence of the limit in (2.33) for all initial data in $\mathcal{D}_0$. By restricting this definition to a smaller, positively invariant domain $\mathcal{D} \subset \mathcal{D}_0$ we obtain a continuous semigroup $S : \mathcal{D} \times [0,\infty[\, \mapsto \mathcal{D}$. In the case where $\|\bar{u} - \bar{w}\|_{\mathbf{L}^1} < \infty$, from (2.30) and (2.32) it follows (1.23).

It remains to show that the flow of S is compatible with the Riemann Solver defined in Section 1. By the property (2.32), it suffices to consider the case where the initial data has a single jump as in (1.17), and show that the limit of viscous approximations converge to the solution defined at (1.22).

Consider first the case where the initial datum $\bar{u}$ lies on a single i-rarefaction curve, say

$$\bar{u}(x) = \bar{u}(-\infty) + \bar{z}(x) r_i\big(\bar{u}(-\infty)\big).$$

Define the scalar function F_i as in (1.20). By the standard theory of scalar conservation laws [12], it is well known that, as $\varepsilon \to 0$, the solution z^ε of the viscous Cauchy problem

$$z_t^\varepsilon + F_i(z^\varepsilon)_x = \varepsilon\, z_{xx}^\varepsilon, \qquad\qquad z^\varepsilon(0,x) = \bar{z}(x), \tag{2.34}$$

converges to the unique entropic solution z of the scalar conservation law

$$z_t + F_i(z)_x = 0, \qquad\qquad z^\varepsilon(0,x) = \bar{z}(x). \tag{2.34$'$}$$

Since, by assumptions, all rarefaction curves are straight lines, the solution of (1.3) with initial data (1.17) is given by

$$u^\varepsilon(t,x) = \bar{u}(-\infty) + z^\varepsilon(t,x) r_i\big(\bar{u}(-\infty)\big),$$

and thus converges as $\varepsilon \to 0$ to the function u given by

$$u(t,x) = \bar{u}(-\infty) + z(t,x) r_i\big(\bar{u}(-\infty)\big).$$

In particular, if u^-, u^+ in (1.17) lie on the same rarefaction curve, then u^ϵ converges to the solution u defined in (1.22).

Since the semigroup $\mathcal{S}$ can be constructed using wave front tracking, it can be shown that this case is sufficient to determine uniquely the semigroup. However, since we have in (2.32) a $\mathbf{L}^1_{\rm loc}$ dependence, it is easy to handle the general case.

Consider in fact a perturbed initial data of the form

$$\bar{u}^\delta(x) \doteq \begin{cases} u^- & \text{if} \quad x < \delta, \\ \omega_i & \text{if} \quad i\delta < x < (i+1)\delta,\ i = 1,\dots,n-1, \\ u^+ & \text{if} \quad x > n\delta. \end{cases} \tag{2.35}$$

Keeping $\delta > 0$ fixed and letting $\varepsilon \to 0$, by the property (2.32) and the analysis of the previous case, the corresponding viscous solutions $u^{\delta,\varepsilon}$ converge to the function

$$u^\delta(t,x) \doteq R_i\big(z_i(t,\ x - i\delta)\big)(\omega_{i-1}) \qquad \text{if}\ \ i\delta + t\bar{\lambda}\ \le\ x\ \le\ (i+1)\delta - t\bar{\lambda},$$

with $z_i, \bar{\lambda}_i$ as in (1.22). Note that, by the assumption (2.8) on the hyperbolicity of $A(u)$, at time $t_0 = \delta/2\bar{\lambda}$ the waves of the solution $u(t_0)$ have disjoint support at least of ct_0, where c is the constant in (2.8). Thus by the property (2.32) and the previous analysis we can extend u^δ to all $t \ge 0$, and letting $\delta \to 0$ we obtain the desired result.

Since a Lipschitz continuous semigroup is entirely determined by its local behavior on piecewise constant initial data, the previous analysis uniquely characterizes the limit of viscous approximations. In particular, this limit does not depend on the choice of the particular sequence ε_ν in (2.33). This completes the proof of Theorem 2.

3 - Estimate of the interaction terms

In this section we prove the key estimates (2.14)-(2.15) on the interaction terms ϕ^i. We shall denote by $\Gamma^i(t,x;s,y)$ the Green kernel for the linear equation

$$z^i_t + [\lambda_i(t,x)z^i]_x - z^i_{xx} = 0, \tag{3.1}$$

where $\lambda_i(t,x)$ is a bounded, sufficiently smooth function. In other words, $(t,x) \mapsto \Gamma^i(t,x;s,y)$ is the distributional solution of (3.1) such that

$$\lim_{t\to s+} \int_{I\!R} \Gamma^i(t,x;s,y)f(y)dy = f(x),$$

for every continuous functions $f \in \mathbf{L}^1$, see [1,9]. Since (3.1) is in conservation form, it is well known that $\Gamma^i(t,x;s,y)$ is positive and

$$\int_{I\!R} \Gamma^i(t,x;s,y)\,dx = 1 \qquad\qquad \text{for all } t > s, \quad y \in I\!R.$$

Taking $\lambda_i(t,x) = \lambda_i\big(u(t,x)\big)$ and calling Γ^i the corresponding Green kernel, the solution of the linear non-homogeneous Cauchy problem (2.5) can be represented as

$$v^i(t,x) = \int_{I\!R} \Gamma^i(t,x;0,y)\bar v^i(y)\,dy + \int_0^t \int_{I\!R} \Gamma^i(t,x;s,y)\phi^i(s,y)\,dyds. \tag{3.2}$$

The proof of our estimates will be given in three steps.

STEP 1: Estimate of the integral

$$\int_0^t \int_{I\!R} \big|v^i(t,x)\big|\,\big|v^j(t,x)\big|\,dxdt. \tag{3.3}$$

As a preliminary, we prove

Lemma 1. *Consider two scalar parabolic equations in conservation form:*

$$\begin{cases} z^i_t + \big[\lambda^i(t,x)z^i\big]_x - z^i_{xx} = 0,\\ z^j_t + \big[\lambda^j(t,x)z^j\big]_x - z^j_{xx} = 0. \end{cases} \tag{3.4}$$

Assume that there exists $c > 0$ such that

$$\inf_{t,x} \lambda_j(t,x) - \sup_{t,x} \lambda_i(t,x) \geq c.$$

Then for every initial data $\bar z^i$, $\bar z^j \in \mathbf{L}^1$, the corresponding solutions z^i, z^j satisfy

$$\int_0^\infty \int_{I\!R} \big|z^i(t,x)\big| \cdot \big|z^j(t,x)\big|\,dxdt \leq \frac{1}{c}\int_{I\!R} \big|\bar z^i(y_1)\big|\,dy_1 \int_{I\!R} \big|\bar z^j(y_2)\big|\,dy_2. \tag{3.5}$$

Proof. We first establish the bound

$$\sup_{\substack{(y_1,y_2)\in \mathbb{R}^2 \\ y_1\neq y_2}} \left\{ \int_0^\infty \int_{\mathbb{R}} \Gamma^i(t,x;0,y_1)\cdot \Gamma^j(t,x;0,y_2)\,dxdt \right\} \leq \frac{1}{c}, \tag{3.6}$$

where Γ^i, Γ^j denote the Green kernels corresponding to the two equations in (3.4). By possibly performing a change of variable of the form $x \mapsto x - \bar{\lambda} t$, which does not affect the value of the integral (3.6), we can assume

$$\lambda_i(t,x) \leq -\frac{c}{2} < 0 < \frac{c}{2} \leq \lambda_j(t,x).$$

This implies

$$\tilde{\lambda}_i(t,x) \doteq \lambda_i(t,x) + \frac{c}{2} \leq 0, \qquad \tilde{\lambda}_j(t,x) \doteq \lambda_j(t,x) - \frac{c}{2} \geq 0. \tag{3.7}$$

We now observe that the product function $K(t,x_1,x_2;\,s,y_1,y_2) \doteq \Gamma^i(t,x_1;\,s,y_1)\cdot\Gamma^j(t,x_2;\,s,y_2)$ provides the Green kernel for the linear equation in two space variables:

$$Z_t + \big[\lambda_i(t,x_1)Z\big]_{x_1} + \big[\lambda_j(t,x_2)Z\big]_{x_2} - \Delta Z = 0. \tag{3.8}$$

In the following we denote by $G^\lambda(t,x)$ the standard Gaussian kernel with constant drift λ and by $G^\lambda_x(t,x)$ its derivative w.r.t. x, i.e.

$$G^\lambda(t,x) \doteq \frac{1}{2\sqrt{\pi t}}\ \exp\left\{-\frac{(x-\lambda t)^2}{4t}\right\}, \qquad G^\lambda_x(t,x) \doteq \frac{-(x-\lambda t)}{4t\sqrt{\pi t}}\ \exp\left\{-\frac{(x-\lambda t)^2}{4t}\right\}.$$

Writing (3.8) in the form

$$Z_t - \frac{c}{2}Z_{x_1} + \frac{c}{2}Z_{x_2} - \Delta Z = -\big[\tilde{\lambda}_i(t,x_1)Z\big]_{x_1} - \big[\tilde{\lambda}_j(t,x_2)Z\big]_{x_2},$$

the corresponding Green kernel can be represented as

$$\begin{aligned}
K(t,x_1,x_2;0,y_1,y_2) &= G^{-c/2}(t,x_1-y_1)G^{c/2}(t,x_2-y_2) \\
&\quad - \int_0^t \iint_{\mathbb{R}^2} \Big\{ G_x^{-c/2}(t-s,x_1-z_1)G^{c/2}(t-s,x_2-z_2) \\
&\qquad\qquad \cdot K(s,z_1,z_2;0,y_1,y_2)\tilde{\lambda}_i(s,z_1) \Big\}\, dz_1dz_2\ ds \\
&\quad - \int_0^t \iint_{\mathbb{R}^2} \Big\{ G^{-c/2}(t-s,x_1-z_1)G_x^{c/2}(t-s,x_2-z_2) \\
&\qquad\qquad \cdot K(s,z_1,z_2;0,y_1,y_2)\tilde{\lambda}_j(s,z_2) \Big\}\, dz_1dz_2\ ds.
\end{aligned}$$

We now compute

$$
\begin{aligned}
-\int_{I\!R^+}\int_{I\!R} G^{-c/2}(t,x-z_1)G_x^{c/2}(t,x-z_2)\,dxdt &= \int_{I\!R^+}\int_{I\!R} G_x^{-c/2}(t,x-z_1)G^{c/2}(t,x-z_2)\,dxdt \\
&= \begin{cases} -\frac{1}{2}\,e^{(z_1-z_2)c/2} & \text{if} \quad z_1<z_2, \\[2ex] 0 & \text{if} \quad z_1>z_2. \end{cases}
\end{aligned}
$$

Using the above formula and assuming that $y_1 \neq y_2$ (in this case we can change the order of integration), recalling (3.7) and the fact that the kernel K is positive, we conclude

$$
\begin{aligned}
\int_{I\!R^+}\int_{I\!R} \Gamma^i(t,x;0,y_1)\Gamma^j(t,x;0,y_2)\,dxdt &= \int_{I\!R^+}\int_{I\!R} K(t,x,x;0,y_1,y_2)\,dxdt \\
&\leq \frac{1}{c} + \int_{I\!R^+}\iint_{z_1<z_2} e^{(z_1-z_2)c/2}K(s,z_1,z_2;0,y_1,y_2)\Big(\tilde\lambda_i(s,z_1)-\tilde\lambda_j(s,z_2)\Big)\,dz_1dz_2\,ds \\
&\leq \frac{1}{c}\,.
\end{aligned}
$$

This establishes (3.6).

Given any two initial conditions $\bar z^i, \bar z^j$, the corresponding solutions of (3.4) satisfy

$$
\begin{aligned}
&\int_{I\!R^+}\int_{I\!R} \big|z^i(t,x)\big|\,\big|z^j(t,x)\big|\,dxdt \\
&\quad\leq \int_{I\!R^+}\int_{I\!R}\iint_{I\!R^2} \big|\bar z^i(y_1)\big|\Gamma^i(t,x;0,y_1)\cdot\big|\bar z^j(y_2)\big|\Gamma^j(t,x;0,y_2)\;dy_1dy_2\,dxdt \\
&\quad\leq \int_{I\!R}\big|\bar z^i(y_1)\big|\,dy_1\cdot\int_{I\!R}\big|\bar z^j(y_2)\big|\,dy_2\;\cdot\sup_{\substack{(y_1,y_2)\in I\!R^2\\ y_1\neq y_2}}\left\{\int_{I\!R^+}\int_{I\!R}\Gamma^i(t,x;0,y_1)\cdot\Gamma^j(t,x;0,y_2)\,dxdt\right\}.
\end{aligned}
$$

By (3.6) this yields (3.5), proving the lemma.

Consider now two equations of the form (2.1). The solutions $v^i(t,x)$, $v^j(t,x)$ can be written in the form

$$
\begin{aligned}
v^i(t,x) &= \int_{I\!R}\Gamma^i(t,x;0,y)\bar v^i(y)\,dy + \int_0^t\int_{I\!R}\Gamma^i(t,x;s,y)\;\phi^i(s,y)\,dyds, \\
v^j(t,x) &= \int_{I\!R}\Gamma^j(t,x;0,y)\bar v^j(y)\,dy + \int_0^t\int_{I\!R}\Gamma^j(t,x;s,y)\;\phi^j(s,y)\,dyds.
\end{aligned}
\tag{3.9}
$$

Using (2.10) and the inductive assumptions (2.17), by (3.5) and (3.9) with easy calculations we obtain

$$
\begin{aligned}
\int_0^t\int_{I\!R}\big|v^i(t,x)\big|\,\big|v^j(t,x)\big|\,dxdt &\leq \frac{1}{c}\Big(\eta_0+\max_{i=1,\ldots,n}\int_0^t\int_{I\!R}\big|\phi^i(s,x)\big|dxds\Big)^2 \\
&= \frac{1}{c}\big(\eta(t)\big)^2 \leq \frac{1}{c}\big(2\eta_0\big)^2.
\end{aligned}
\tag{3.10}
$$

STEP 2: Estimate of the integral

$$\big\|v_x^i(t)\big\|_{\mathbf{L}^1} = \int_{I\!R} \big|v_x^i(t,x)\big|\,dx. \tag{3.11}$$

Define the quantity

$$\xi(t) \doteq \sup_{\substack{i=1,\dots,n\\ 0\le s\le t}} \big\|v_x^i(s)\big\|_{\mathbf{L}^1}. \tag{3.12}$$

Using (2.9), (2.10), we have

$$\begin{aligned}\int_{I\!R} \big|\phi^i(t,x)\big|\,dx &\le n^2 C_0 \eta(t)\xi(t) + n^2 C_0\big(\xi(t)\big)^2 + n^3 C_0 \eta(t)\big(\xi(t)\big)^2\\ &\le \frac{\widehat{C}}{2}\Big(\eta(t)\xi(t) + \xi^2(t) + n\eta(t)\xi^2(t)\Big).\end{aligned} \tag{3.13}$$

For convenience, for $i=1,\dots,n$, introduce the quantities

$$\lambda_i^* \doteq \lambda_i\big(\bar u(-\infty)\big), \qquad \|\lambda_i\|_\infty \doteq \max_{u\in K_1} \big|\lambda_i(u) - \lambda_i^*\big|, \qquad \|\lambda_i'\|_\infty \doteq \max_{\substack{1\le j\le n,\\ u\in K_1}} \big|r_j \bullet \lambda_i(u)\big|, \tag{3.14}$$

$$G^i(t,x) \doteq G^{\lambda_i^*}(t,x) = \frac{1}{2\sqrt{\pi t}}\ \exp\left\{-\frac{(x-\lambda_i^* t)^2}{4t}\right\}. \tag{3.15}$$

By (2.9), $\|\lambda_i'\|_\infty \le C_0$. We also observe that the heat kernel $G^i(t,x)$ satisfies

$$\int_{I\!R} \big|G_x^i(t,x)\big|\,dx = \frac{1}{\sqrt{\pi t}}, \qquad \int_{I\!R} \big|G_{xx}^i(t,x)\big|\,dx = \sqrt{\frac{2}{\pi e}}\cdot\frac{1}{t}. \tag{3.16}$$

Define the time $\hat t$ by

$$\sqrt{\hat t} \doteq \min\left\{\frac{\sqrt{\pi}}{32n\|\lambda_i'\|_\infty \eta_0},\ \frac{\sqrt{\pi}}{16\widehat{C}\eta_0}\right\} = \frac{\sqrt{\pi}}{16\widehat{C}\eta_0}. \tag{3.17}$$

Assume for simplicity that $\hat t \le \bar t$, where $\bar t$ is defined in section 2, the other case being completely similar. For $0 \le t \le \hat t$ we can write the solution as

$$\begin{aligned} v_x^i(t,x) = &\int_{I\!R} G^i(t,x-y)\bar v_y^i(y)\,dy\\ &- \int_0^t \int_{I\!R} G_x^i(t-s,x-y)\frac{\partial}{\partial y}\Big(\big(\lambda_i(u) - \lambda_i^*\big)v^i(s,y)\Big)\,dyds\\ &+ \int_0^t \int_{I\!R} G_x^i(t-s,x-y)\phi^i(s,y)\,dyds.\end{aligned}$$

Using (3.14), (3.16) and the assumption $\eta(t) \le 2\eta_0$, the $\mathbf{L}^1$-norm of the second integral on the right hand side of the above formula can be estimated as

$$\begin{aligned} &\int_{I\!R} \left|\int_0^t \int_{I\!R} G_x^i(t-s,x-y)\frac{\partial}{\partial y}\Big(\big(\lambda_i(u)-\lambda_i^*\big)v^i(s,y)\Big)\,dy\,ds\right|\,dx\\ &\le \int_0^t \big\|G_x^i(t-s,\cdot)\big\|_{\mathbf{L}^1}\cdot\left\|\frac{\partial}{\partial y}\Big(\big(\lambda_i(u)-\lambda_i^*\big)v^i(s,\cdot)\Big)\right\|_{\mathbf{L}^1}\,ds\\ &\le \int_0^t \frac{1}{\sqrt{\pi(t-s)}}\cdot\left[\|\lambda_i\|_\infty \xi(s) + \|\lambda_i'\|_\infty\left(\sum_k \big\|v^k(s)\big\|_{\mathbf{L}^1}\right)\big\|v^i(s)\big\|_{\mathbf{L}^\infty}\right]\,ds\\ &\le \frac{4n}{\sqrt{\pi}}\sqrt{\hat t}\|\lambda_i'\|_\infty \eta(t)\xi(t) \le \frac{2}{\sqrt{\pi}}\sqrt{\hat t}\widehat{C}(2\eta_0)\xi(t).\end{aligned}$$

Using the above estimate together with $\eta_0 \leq \eta(t) \leq 2\eta_0$, for $t \in [0, \hat{t}]$ we obtain

$$\sup_{t\in[0,\,\hat{t}\,]} \left\|v_x^i(t)\right\|_{\mathbf{L}^1} \leq \|\bar{v}_x^i\|_{\mathbf{L}^1} + \frac{1}{4}\xi(t) + \frac{2}{\sqrt{\pi}}\sqrt{\hat{t}}\,\frac{\widehat{C}}{2}\Big(\eta(t)\xi(t) + \xi^2(t) + n\eta(t)\xi^2(t)\Big),$$

so that

$$\xi(t) \leq \frac{4}{\sqrt{\pi}}\widehat{C}^2\eta_0^2 + \frac{1}{4}\xi(t) + \frac{1}{16\eta_0}\Big(\eta(t)\xi(t) + \xi^2(t) + n\eta(t)\xi^2(t)\Big).$$

By possibly reducing the value of η_0, we can assume that

$$\left(1 + 2\frac{20}{\sqrt{\pi}}\widehat{C}\,2\eta_0 + \frac{8n}{\sqrt{\pi}}\widehat{C}^2\,(2\eta_0)^2\right) < 2. \tag{3.18}$$

Using (3.18) and a comparison argument, it is easy to conclude that

$$\xi(t) < \frac{8}{\sqrt{\pi}}\widehat{C}^2\eta^2(t), \tag{3.19}$$

if $0 \leq t \leq \hat{t}$.

Consider now the case $\hat{t} \leq t \leq \bar{t}$, and assume that (3.19) holds in the interval $[t-\hat{t}, t)$. Assume moreover that t is the first time such that equality holds is (3.19). The x-derivative of the solution v^i of (2.1) can be written in the form

$$\begin{aligned} v_x^i(t,x) = &\int_{I\!R} G_x^i(\hat{t}, x-y)v^i(t-\hat{t}, y)\,dy \\ &- \int_{t-\hat{t}}^t \int_{I\!R} G_x^i(t-s, x-y)\frac{\partial}{\partial y}\Big(\big(\lambda_i(u) - \lambda_i^*\big)v^i(s,y)\Big)\,dy\,ds \\ &+ \int_{t-\hat{t}}^t \int_{I\!R} G_x^i(t-s, x-y)\phi^i(s,y)\,dy\,ds. \end{aligned}$$

With a computation similar to the one above and using (2.9), (2.10), (3.18) and the assumption $\eta_0 \leq \eta(t) \leq 2\eta_0$, for $t \in [0, \hat{t}]$ we obtain

$$\begin{aligned} \sup_{s\in[t-\hat{t},t]} \left\|v_x^i(s)\right\|_{\mathbf{L}^1} &\leq \frac{2}{\sqrt{\pi\hat{t}}}\eta(t) + \frac{1}{4}\sup_{s\in[t-\hat{t},t]} \left\|v_x^i(s)\right\|_{\mathbf{L}^1} + \frac{2}{\sqrt{\pi}}\sqrt{\hat{t}}\,\frac{\widehat{C}}{2}\Big(\eta(t)\xi(t) + \xi^2(t) + n\eta(t)\xi^2(t)\Big) \\ &\leq \frac{4}{\sqrt{\pi}}\widehat{C}^2\eta^2(t) + \frac{1}{4}\xi(t) + \frac{1}{16\eta_0}\Big(\eta(t)\xi(t) + \xi^2(t) + n\eta(t)\xi^2(t)\Big), \end{aligned}$$

hence a comparison argument shows that

$$\xi(t) < \frac{8}{\sqrt{\pi}}\widehat{C}^2\eta^2(t). \tag{3.20}$$

Using (3.19) and (3.20) we conclude that, for all $0 \leq t \leq \bar{t}$,

$$\|v_x^i(t)\|_{\mathbf{L}^1} < \frac{8}{\sqrt{\pi}}\widehat{C}^2\eta^2(t) \leq \frac{8}{\sqrt{\pi}}\widehat{C}^2\big(2\eta_0\big)^2.$$

In particular, substituting (3.19), (3.20) in (3.13) we recover (2.14) for $0 \leq t \leq \bar{t}$.

STEP 3: Evaluation of the double integral

$$\int_0^t \int_{I\!R} \left|v_x^i(t,x)\right| \left|v^j(t,x)\right| dxdt. \tag{3.21}$$

The idea is to use the representations of the first derivative v_x^i in terms of the heat kernel given in the previous step to show that the quantity in (3.21) can be bounded in terms of the product of $\mathbf{L}^1$-norms of v^i and v^j. Since these representations contain a convolution of v^i with the heat kernel G, we consider the family of functions $v^j(t+\tau, x+z)$, with $(\tau, z) \in [0, \bar{t}] \times I\!R$ and we estimate the quantity

$$\mathcal{I}(\sigma) \doteq \sup_{(\tau,z)\in[0,\sigma]\times I\!R} \int_0^{\sigma-\tau} \int_{I\!R} \left|v_x^i(t,x)\right| \left|v^j(t+\tau, x+z)\right| dxdt, \tag{3.22}$$

where $\sigma \geq 0$. As in the previous step, we compute the $\mathbf{L}^1$ norm of the source term: for all $0 \leq t \leq \bar{t}$ we have

$$\begin{aligned} Q(t) \doteq \int_0^t \int_{I\!R} \left|\phi^i(t,x)\right| dxdt &\leq \frac{n^2 C_0}{c}\eta^2(t) + n^2 C_0 \mathcal{I}(t) + \frac{n^3 C_0}{c}\frac{8}{\sqrt{\pi}} C^2 \big(2\eta_0\big)^2 \eta^2(t) \\ &\leq \frac{\widehat{C}}{2}\eta^2(t)\left(1 + \frac{8n}{\sqrt{\pi}} C^2 \big(2\eta_0\big)^2\right) + \frac{\widehat{C}}{c}\mathcal{I}(t). \end{aligned} \tag{3.23}$$

We first study the case $\sigma \leq \hat{t}$, with $\hat{t}$ defined at (3.17). We write the solution of (2.1) as

$$\begin{aligned} v_x^i(t,x) = &\int_{I\!R} G^i(t, x-y)\bar{v}_y^i(y)\, dy - \int_0^t \int_{I\!R} G_x^i(s,y)\big(\lambda_i(u) - \lambda_i^*\big) v_x^i(t-s, x-y)\, dyds \\ &- \int_0^t \int_{I\!R} G_x^i(s,y)\left(\sum_{k=1}^n r_k \bullet \lambda_i(u) v^k\right) v^i(t-s, x-y)\, dyds \\ &+ \int_0^t \int_{I\!R} G_x^i(s,y)\phi^i(t-s, x-y)\, dyds. \end{aligned}$$

We first estimate the following integrals, whose computation is carried out in appendix A:

$$\int_0^{\sigma-\tau} \iint_{I\!R^2} \left|G^i(t, x-y)\bar{v}_x^i(y) v^j(t+\tau, x+z)\right| dxdydt \leq \frac{4}{c\sqrt{\pi}} \widehat{C}^2 \eta^3(\sigma), \tag{3.24}$$

$$\int_0^{\sigma-\tau} \int_0^t \iint_{I\!R^2} \left|G_x^i(s,y)\big(\lambda_i(u) - \lambda_i^*\big) v_x^i(t-s, x-y) v^j(t+\tau, x+z)\right| dxdy\, dsdt \leq \frac{1}{8}\mathcal{I}(\sigma), \tag{3.25}$$

$$\begin{aligned} \int_0^{\sigma-\tau} \int_0^t \iint_{I\!R^2} &\left|G_x^i(s,y)(r_k \bullet \lambda_i) v^k v^i(t-s, x-y) v^j(t+\tau, x+z)\right| dxdy\, dsdt \\ &\leq \frac{16}{c\pi}\sqrt{\hat{t}} \|\lambda_i'\|_\infty \widehat{C}^2 \eta^4(\sigma), \end{aligned} \tag{3.26}$$

$$\int_0^{\sigma-\tau}\int_0^t\iint_{I\!R^2}\Big|G_x^i(s,y)\phi^i(t-s,x-y)v^j(t+\tau,x+z)\Big|\,dxdy\;dsdt \tag{3.27}$$
$$\leq \frac{2}{\sqrt{\pi}}\widehat{C}\eta(\sigma)\int_0^\sigma\int_{I\!R}\big|\phi^i(t,x)\big|dxdt.$$

Using the above estimates we obtain

$$\mathcal{I}(\sigma)\leq\frac{6}{c\sqrt{\pi}}\widehat{C}^2\eta^3(\sigma)+\frac{1}{8}\mathcal{I}(\sigma)+\frac{2}{\sqrt{\pi}}\widehat{C}\eta(\sigma)Q(\sigma). \tag{3.28}$$

Using (3.23), (3.18) and a comparison argument, this shows that $\mathcal{I}(\sigma)$ is uniformly bounded, in the case $\sigma\leq\hat{t}$ by

$$\mathcal{I}(\sigma)\leq\frac{8}{c\sqrt{\pi}}\,\widehat{C}^2\eta^3(\sigma).$$

If $\hat{t}\leq\sigma\leq\bar{t}$, we split the integrals in (3.22) in two parts:

$$\int_0^\sigma\int_{I\!R}\big|v_x^i(t,x)\big|\;\big|v^j(t+\tau,x+z)\big|\,dxdt$$
$$=\left\{\int_0^{\hat{t}}+\int_{\hat{t}}^\sigma\right\}\int_{I\!R}\big|v_x^i(t,x)\big|\;\big|v^j(t+\tau,x+z)\big|\,dxdt,$$

and write $v_x^i(t,x)$, $t\geq\hat{t}$, using the integral representation

$$\begin{aligned}v_x^i(t,x)=&\int_{I\!R}G_x^i(\hat{t},y)v^i(t-\hat{t},x-y)\,dy\\&-\int_0^{\hat{t}}\int_{I\!R}G_x^i(s,y)\big(\lambda_i(u)-\lambda_i^*\big)v_x^i(t-s,x-y)\,dyds\\&-\int_0^{\hat{t}}\int_{I\!R}G_x^i(s,y)\left(\sum_{k=1}^n r_k\bullet\lambda_i(u)v^k\right)v^i(t-s,x-y)\,dyds\\&+\int_0^{\hat{t}}\int_{I\!R}G_x^i(s,y)\phi^i(t-s,x-y)\,dyds.\end{aligned}$$

We now use the further estimates proved in Appendix A if $\sigma-\tau>\hat{t}$, the other case being entirely similar:

$$\int_{\hat{t}}^{\sigma-\tau}\int\int_{I\!R^2}\Big|G_x^i(\hat{t},y)v^i(t-\hat{t},x-y)v^j(t+\tau,x+z)\Big|\,dxdy\;dt\leq\frac{8}{c\pi}\widehat{C}\eta^3(\sigma), \tag{3.29}$$

$$\int_{\hat{t}}^{\sigma-\tau}\int_0^{\bar{t}}\iint_{I\!R^2}\Big|G_x^i(s,y)\big[\lambda_i(u)-\lambda_i^*\big]v_x^i(t-s,x-y)v^j(t+\tau,x+z)\Big|\;dxdy\,dsdt\leq\frac{1}{8}\mathcal{I}(\sigma), \tag{3.30}$$

$$\int_{\hat{t}}^{\sigma-\tau}\int_0^{\hat{t}}\iint_{I\!R^2}\Big|G_x^i(s,y)\big((r_k\bullet\lambda_i)v^k\big)v^i(t-s,x-y)v^j(t+\tau,x+z)\Big|\,dxdy\,dsdt \tag{3.31}$$
$$\leq\frac{16}{c\pi}\sqrt{\hat{t}}\|\lambda_i'\|_\infty\widehat{C}^2\eta^4(\sigma),$$

$$\int_{\hat{t}}^{\sigma-\tau}\int_0^{\hat{t}}\iint_{\mathbb{R}^2}\Big|G_x^i(s,y)\phi^i(t-s,x-y)v^j(t+\tau,x+z)\Big|\,dxdy\,dsdt \le \frac{2}{\sqrt{\pi}}\widehat{C}\eta(\sigma)Q(\sigma). \tag{3.32}$$

Using the above estimates, for all $(\tau,z)\in[0,\bar{t}]\times\mathbb{R}$ we obtain

$$\int_{\hat{t}}^{\sigma-\tau}\int_{\mathbb{R}}\big|v_x^i(t,x)\big|\,\big|v^j(t+\tau,x+z)\big|\,dxdt \le \frac{8}{\pi}\widehat{C}\frac{\eta^3(\sigma)}{c}+\frac{2}{c\sqrt{\pi}}\widehat{C}^2\eta^3(\sigma)+\frac{1}{8}\mathcal{I}_\sigma+\frac{2}{\sqrt{\pi}}\widehat{C}\eta(\sigma)Q(\sigma). \tag{3.33}$$

Adding the two expressions (3.28), (3.33), and recalling (2.10), we obtain

$$\begin{aligned}\mathcal{I}(\sigma) &\le \frac{1}{c}\left(\frac{8}{\pi}\widehat{C}+\frac{8}{\sqrt{\pi}}\widehat{C}^2\right)\eta^3(\sigma)+\frac{\mathcal{I}(\sigma)}{4}+\frac{2}{\sqrt{\pi}}\widehat{C}\eta(\sigma)Q(\sigma)\\ &\le \frac{10}{\sqrt{\pi}}\frac{\widehat{C}^2}{c}\eta^3(\sigma)+\frac{\mathcal{I}(\sigma)}{4}+\frac{2}{\sqrt{\pi}}\widehat{C}\eta^3(\sigma)Q(\sigma),\end{aligned}$$

from which using a comparison argument we deduce

$$\mathcal{I}(\sigma) \le \frac{20}{\sqrt{\pi}}\frac{\widehat{C}^2}{c}\eta^3(\sigma). \tag{3.34}$$

Note that (3.34) holds for every $\sigma \le \bar{t}$. This yields the desired bound for (3.22), namely

$$\int_0^t\int_{\mathbb{R}}\big|v_x^i(t,x)\big|\,\big|v^j(t,x)\big|\,dxdt \;\le\; \limsup_{\sigma\to t}\;\mathcal{I}(\sigma)\;\le\;\frac{20}{\sqrt{\pi}}\frac{\widehat{C}^2}{c}\eta^3(t). \tag{3.35}$$

Using (3.10), (3.18), (3.19)–(3.20) and (3.35), we now prove the estimates (2.6)

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\big|\phi^i(t,x)\big|\,dxdt &\le \frac{n^2C_0}{c}\eta^2(t)\left(1+2\frac{20}{\sqrt{\pi}}\widehat{C}\big(2\eta_0\big)+\frac{8n}{\sqrt{\pi}}\widehat{C}^2\big(2\eta_0\big)^2\right)\\ &< \frac{n^2C_0}{c}\,\eta^2(t)\cdot 2 \;\le\; \frac{\widehat{C}}{c}\,\eta^2(t),\end{aligned} \tag{3.37}$$

4 - Nonsmooth initial data

The estimate (3.37) proves that if $\bar{t}$ is the first time such that $\eta(\bar{t})=2\eta_0$, then (2.5) implies that

$$2\eta_0=\eta(\bar{t})<\eta_0+\frac{\widehat{C}}{c}\big(2\eta_0\big)^2\le 2\eta_0, \tag{4.1}$$

generating a contradiction. Thus, if $u(0,x)$ lies in the set

$$\begin{aligned}\mathcal{D}^* \doteq \Big\{u:\mathbb{R}\mapsto\mathbb{R}^n\;;\quad & u(-\infty)\in K_0,\quad \big\|l_i(u)\cdot u_x\big\|_{\mathbf{L}^1}\le\eta_0,\\ &\big\|(l_i(u)\cdot u_x)_x\big\|_{\mathbf{L}^1}\le\frac{4}{\sqrt{\pi}}\widehat{C}^2\eta_0^2,\quad i=1,\dots,n\Big\},\end{aligned} \tag{4.2}$$

the solution can be extended for all $t \geq 0$ and it has bounded variation: namely

$$\|v^i(t)\|_{\mathbf{L}^1} \leq 2\eta_0. \tag{4.3}$$

In this section we prove that the domain can be extended to all functions with suitably small total variation. The main argument in this last step of the proof is quite simple: If the total variation of the initial data $\bar{u}$ is very small, after a short interval of time the corresponding solution of (1.4) will be inside the domain $\mathcal{D}^*$, thanks to the smoothing properties of the parabolic system. Hence our previous analysis can be applied.

With a linear change of coordinates, we can assume that $A^* \doteq A\big(\bar{u}(-\infty)\big) = \text{diag}(\lambda_1^*, \ldots, \lambda_n^*)$. Denoting with $\mathbf{e}_i$ the i-th unit vector and writing $\hat{u}^i \doteq \mathbf{e}_i \cdot u$, we can write the solutions of (1.4) in the form

$$\hat{u}^i(t,x) = \int_{I\!R} G^i(t, x-y)\bar{u}^i(y)\, dy - \int_0^t \int_{I\!R} G^i(t-s, x-y)\ \mathbf{e}_i \cdot \big(A(u) - A^*\big) u_x(s,y)\, dyds, \tag{4.4}$$

$$\hat{u}^i_x(t,x) = \int_{I\!R} G^i(t, x-y) d\bar{u}^i(y) - \int_0^t \int_{I\!R} G^i_x(t-s, x-y)\ \mathbf{e}_i \cdot \big(A(u) - A^*\big) u_x(s,y)\, dyds, \tag{4.5}$$

$$\hat{u}^i_{xx}(t,x) = \int_{I\!R} G^i_x(t, x-y) d\bar{u}^i(y) - \int_0^t \int_{I\!R} G^i_x(t-s, x-y)\ \mathbf{e}_i \cdot \Big[\big(A(u) - A^*\big) u_x(s,y)\Big]_x dyds, \tag{4.6}$$

with obvious meaning of notations. Recall that $G^i \doteq G^{\lambda_i^*}$ is the Green kernel defined at (3.15). We look for an estimate of the form

$$\|u_{xx}(t)\|_{\mathbf{L}^1} \leq C\, \frac{\text{Tot.Var.}\{\bar{u}\}}{\sqrt{t}}. \tag{4.7}$$

Defining the time t_0 and the constant C as

$$\sqrt{t_0} \doteq \frac{\sqrt{\pi}}{32n\|DA\|_\infty \cdot \text{Tot.Var.}\{\bar{u}\}}, \qquad C \doteq \max\left\{\frac{9}{\sqrt{\pi}}, \|DA\|_\infty\right\},$$

we claim that (4.7) holds for $t \leq t_0$ if the initial datum is enough smooth. Clearly (4.7) is a strict inequality for t sufficiently small. Using the same techniques as in Section 3, step 2, if $0 < t < t_0$ we can estimate the $\mathbf{L}^1$ norms of (4.5) and (4.6) as

$$\begin{aligned}
\big\|u_x(t)\big\|_{\mathbf{L}^1} &\leq \text{Tot.Var.}\{\bar{u}\} + \frac{2n}{\sqrt{\pi}}\sqrt{t}\big\|A - A^*\big\|_\infty \sup_{0<s\leq t}\big\|u_x(s)\big\|_{\mathbf{L}^1} \leq 2\,\text{Tot.Var.}\{\bar{u}\} \\
\big\|u_{xx}(t)\big\|_{L^1} &< \frac{\text{Tot.Var.}\{\bar{u}\}}{\sqrt{\pi t}} + \frac{8n}{\sqrt{\pi}} C\|DA\|_\infty \cdot \text{Tot.Var.}\{\bar{u}\} \cdot \sup_{0<s\leq t}\big\|u_x(s)\big\|_{\mathbf{L}^1} \\
&\leq C\, \frac{\text{Tot.Var.}\{\bar{u}\}}{\sqrt{t}},
\end{aligned} \tag{4.8}$$

where

$$\|A - A^*\|_\infty \doteq \max_{u \in K_1}\big|A(u) - A^*\big|, \qquad \|DA\|_\infty \doteq \max_{u\in K_1}\left|\frac{d}{du}A(u)\right|. \tag{4.9}$$

Here the matrices $A(u)$ are regarded as linear operators in $L(\mathbb{R}^n, \mathbb{R}^n)$, and $\mathbb{R}^n$ has the norm $\|v\| \doteq \sum_{i=1}^n |v_i|$. This concludes the proof if the initial datum is sufficiently smooth. Since the estimates (4.7)-(4.8) depend only on the BV norm of u, by a density argument it follows that (4.7) holds for any enough small BV function. In particular, at time t_0 form (4.7) we have

$$\big\|u_{xx}(t_0)\big\|_{L^1} \leq \frac{32n}{\sqrt{\pi}} C^2 \big(\text{Tot.Var.}\{\bar{u}\}\big)^2. \tag{4.10}$$

With easy computations, from (1.7) it follows that the components v_x^i are given by

$$v^i = \sum_{j=1}^n (l^i \cdot \mathbf{e}_j)\, \hat{u}_x^j, \qquad v_x^i = \sum_{j=1}^n (l^i \cdot \mathbf{e}_j)\, \hat{u}_{xx}^j + \sum_{j,k=1}^n (\mathbf{e}_k \bullet l^i) \cdot \mathbf{e}_j \;\hat{u}_x^j \hat{u}_x^k. \tag{4.11}$$

Therefore, for a suitable constant C' depending on $\|DA\|_\infty$, by (4.10)-(4.11) we have

$$\begin{aligned} \big\|v^i(t_0)\big\|_{\mathbf{L}^1} &\leq nC' \big\|u_x(t_0)\big\|_{\mathbf{L}^1} \leq 2nC' \cdot \text{Tot.Var.}\{\bar{u}\} \leq \eta_0, \\ \big\|v_x^i(t_0)\big\|_{\mathbf{L}^1} &\leq nC' \Big(1 + n\big\|u_x(t_0)\big\|_{\mathbf{L}^1}\Big) \big\|u_{xx}(t_0)\big\|_{\mathbf{L}^1} \\ &\leq nC' \Big(1 + 2n \cdot \text{Tot.Var.}\{\bar{u}\}\Big) \frac{32n}{\sqrt{\pi}} C^2 \big(\text{Tot.Var.}\{\bar{u}\}\big)^2 \\ &\leq \frac{4}{\sqrt{\pi}} \widehat{C}^2 \eta_0^2, \end{aligned} \tag{4.12}$$

provided that Tot.Var.$\{\bar{u}\}$ is sufficiently small and $\widehat{C}$ big enough. All previous arguments can now be applied to the solution u on the interval $[t_0, \ \infty[$. This completes the proof of the first part of Theorem 1.

5 - $\mathbf{L}^1$ stability of viscous solutions

In this section we prove the stability estimate (1.15) for solutions u of (1.4). Toward this goal, we first study the linear variational equation satisfied by infinitesimal perturbations. The evolution equation (2.23) can be more conveniently written as

$$h_t + [A(u)h]_x + (h \bullet A(u))u_x - (u_x \bullet A(u))h = h_{xx}\,. \tag{5.1}$$

We first consider the case in which the initial data $\bar{u}$ is in D_W. Defining the components of h and u as

$$h = \sum h^j r_j(u), \qquad u_x = \sum v^j r_j(u), \tag{5.2}$$

with computations entirely similar to (1.7)–(1.10) we obtain

$$\begin{aligned} h_t &+ \sum_{j=1}^n (\lambda_j h^j)_x r_j + \sum_{j,k=1}^n \lambda_j h^j v^k r_k \bullet r_j \\ &\quad + \sum_{j,k=1}^n (v^j h^k - h^j v^k) \bigg(\lambda_j r_k \bullet r_j + (r_k \bullet \lambda_j) r_j - \sum_{\ell=1}^n \lambda_\ell \Big(l^\ell \cdot (r_k \bullet r_j)\Big) r_\ell\bigg) \\ &= \sum_{j=1}^n h_{xx}^j r_j + 2\sum_{j,k=1}^n h_x^j v^k r_k \bullet r_j + \sum_{j,k=1}^n h^j v_x^k r_k \bullet r_j + \sum_{j,k,\ell=1}^n h^j v^k v^\ell r_\ell \bullet (r_k \bullet r_j), \end{aligned}$$

$$\begin{aligned} h_t =& \sum_{j=1}^n h_t^j r_j - \sum_{j,k=1}^n \lambda_k h^j v^k (r_k \bullet r_j) + \sum_{j,k=1}^n h^j v_x^k (r_k \bullet r_j) \\ &+ \sum_{j,k,\ell=1}^n h^j v^k v^\ell (r_\ell \bullet r_k) \bullet r_j, \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^n h_t^j r_j \ &+ \sum_{j=1}^n (\lambda_j h^j)_x r_j + \sum_{j,k=1}^n (\lambda_j - \lambda_k) h^j v^k (r_k \bullet r_j) \\ &+ \sum_{j,k=1}^n (v^j h^k - h^j v^k) \Big(\lambda_j r_k \bullet r_j + (r_k \bullet \lambda_j) r_j - \sum_{\ell=1}^n \lambda_\ell \Big(l^\ell \cdot (r_k \bullet r_j) \Big) r_\ell \Big) \\ = & \sum_{j=1}^n h_{xx}^j r_j + 2 \sum_{j,k=1}^n h_x^j v^k r_k \bullet r_j + \sum_{j,k,\ell=1}^n h^j v^k v^\ell \Big(r_\ell \bullet (r_k \bullet r_j) - (r_\ell \bullet r_k) \bullet r_j \Big), \end{aligned} \tag{5.3}$$

The inner product of (5.3) with $l_i(u)$ yields

$$\begin{aligned} h_t^i + [\lambda_i h^i]_x - h_{xx}^i =& \sum_{j,k=1}^n (\lambda_k - \lambda_i) l^i \cdot [r_k, r_j] h^j v^k + 2 \sum_{j,k=1}^n l^i \cdot (r_k \bullet r_j) h_x^j v^k \\ &+ \sum_{j,k,\ell=1}^n l^i \cdot \Big(r_l \bullet (r_k \bullet r_j) - (r_\ell \bullet r_k) \bullet r_j \Big) h^j v^k v^\ell + \sum_{k=1}^n (h^i v^k - h^k v^i)(r_k \bullet \lambda_i) \\ \doteq & \sum_{\substack{j,k=1 \\ j \neq k}}^n G_{j,k}^i(u) h^j v^k + \sum_{\substack{j,k=1 \\ j \neq k}}^n H_{j,k}^i(u) h_x^j v^k + \sum_{\substack{j,k,\ell=1 \\ k \neq \ell}}^n K_{j,k,\ell}^i(u) h^j v^k v^\ell \\ &+ \sum_{\substack{j,k=1 \\ j \neq k}}^n J_{j,k}^i(u) h^j (v^k)^2 + \sum_{\substack{k=1 \\ k \neq i}}^n L_k^i(u)(h^i v^k - h^k v^i) \\ \doteq & \ \psi^i(u, v, h). \end{aligned} \tag{5.4}$$

The following estimates are quite similar to those in Sections 2 - 3.

Let the constant C_0 provide an upper bound for the absolute values of all functions $G_{j,k}^i$, $H_{j,k}^i$, $K_{j,k,l}^i$, $J_{j,k}^i$, L_k^i, as u ranges over the compact set K_1. For a given initial data $\bar{h}$, the functions h^i satisfy the equations

$$\begin{cases} h_t^i + \big[\lambda_i(u) h^i\big]_x - h_{xx}^i = \psi^i(t, x) \\ h^i(0, x) = \bar{h}^i(x) \doteq l^i\big(u(x)\big) \cdot \bar{h}(x). \end{cases} \tag{5.5}$$

Here $\psi^i \doteq \psi^i(u, v, h)$ is the right hand side of (5.4). We assume that there exists a constant ξ_0 such that

$$\|\bar{h}^i\|_{\mathbf{L}^1} \leq \xi_0, \qquad \int_{I\!R} \big|\bar{h}_x(x)\big| \, dx \leq \frac{4}{\sqrt{\pi}} \widehat{C}^2 (2\eta_0) \xi_0, \qquad i = 1, \ldots, n. \tag{5.6}$$

Since the right hand side of (5.5) is in conservation form, we have

$$\|h^i(t)\|_{\mathbf{L}^1} \leq \xi_0 + \max_{i=1,\ldots,n} \int_0^t \int_{I\!R} \big|\phi^i(s, x)\big| dx ds \doteq \xi(t), \tag{5.7}$$

so that if we can prove that for some constant $\kappa' < 1$

$$\max_{i=1,\dots,n} \int_0^t \int_{I\!R} \big|\phi^i(s,x)\big| dxds \le \kappa' \xi(t), \tag{5.8}$$

we are done.

The proof involves three steps, entirely similar to the ones in Section 3.

1. Writing the solutions $h^i(t,x)$, $v^j(t,x)$ in the integral form

$$\begin{aligned} h^i(t,x) &= \int_{I\!R} \Gamma^i(t,x;0,y)\bar h^i(y) + \int_0^t \int_{I\!R} \Gamma^i(t,x;s,y)\psi^i(s,y)\,dyds, \\ v^j(t,x) &= \int_{I\!R} \Gamma^j(t,x;0,y)\bar v^j(y)\,dy + \int_0^t \int_{I\!R} \Gamma^j(t,x;s,y)\phi^j(s,y)\,dyds, \end{aligned}$$

after some calculations we obtain

$$\begin{aligned} \int_0^\infty \int_{I\!R} \big|h^i(t,x)\big|\,\big|v^j(t,x)\big|\,dxdt &\le \frac{1}{c}\left(\eta_0 + \frac{\widehat C}{c}(2\eta_0)^2\right)\left(\xi_0 + \int_0^t \int_{I\!R} \big|\phi(s,x)\big| dxds\right) \\ &\le \frac{1}{c}\,2\eta_0\xi(t), \end{aligned} \tag{5.9}$$

because $\eta(t) \le 2\eta_0$.

2. Concerning $\big\|h^i_x(t)\big\|_{\mathbf{L}^1}$, for every $t \ge 0$ the same calculations as in Section 3, step 2, yield

$$\int_{I\!R} \big|h^i_x(t,x)\big|\,dx \le \frac{8}{\sqrt{\pi}}\widehat C^2 2\eta_0\xi(t), \tag{5.10}$$

$$\int_{I\!R} \big|\psi(t,x)\big| dx \le \frac{8}{\sqrt{\pi}}\widehat C^3 \big(2\eta_0\big)^2\xi(t). \tag{5.11}$$

3. Furthermore, the same computation as in Section 3, step 3 yields

$$\int_0^\infty \int_{I\!R} \big|h^i_x(t,x)\big|\,\big|v^j(t,x)\big|\,dxdt \le \frac{20}{\sqrt{\pi}}\frac{\widehat C^2}{c}\,(2\eta_0)^2\xi(t). \tag{5.12}$$

Using the above estimates (5.8)–(5.12) and recalling that η_0 also satisfies (3.18), we obtain

$$\begin{aligned} \int_0^t \int_{I\!R} \big|\psi^i(s,x)\big|\,dxds &\le \frac{C_0}{c}\,2\eta_0\xi(t)\left[n^2\left(1 + 2\frac{20}{\sqrt{\pi}}\widehat C\,(2\eta_0) + \frac{8n}{\sqrt{\pi}}\widehat C^2\,(2\eta_0)^2\right) + 2n\right] \\ &\le \frac{\widehat C}{c}\,2\eta_0\xi(t), \end{aligned} \tag{5.13}$$

By (2.13), the quantity $\xi(t)$ is bounded by

$$\lim_{t\to+\infty} \xi(t) = \left(1 - \frac{\widehat C}{c}\,2\eta_0\right)^{-1} \xi_0 \le 2\xi_0, \tag{5.14}$$

so that the functions h^i have uniformly bounded $\mathbf{L}^1$ norm:

$$\|h(t)\|_{\mathbf{L}^1} \leq 2\xi_0. \tag{5.15}$$

We consider now the case $\bar{u} \in \mathcal{D}$ and $\bar{h} \in BV$. Denoting the components $\hat{h}^i \doteq \mathbf{e}_i \cdot h$, we can write the solution of (5.1) in the form

$$\begin{aligned} \hat{h}^i(t,x) = & \int_{I\!R} G^i(t,x-y)d\bar{h}^i(y) - \int_0^t \int_{I\!R} G^i_x(t-s,x-y)\mathbf{e}_i \cdot \Big[\big(A(u)-A^*\big)h(s,y)\Big]\,dyds \\ & + \int_0^t \int_{I\!R} G^i(t-s,x-y)\mathbf{e}_i \cdot \Big[\big(u_x \bullet A(u)\big)h(s,y) - \big(h \bullet A(u)\big)u_x(s,y)\Big]\,dyds, \end{aligned} \tag{5.16}$$

$$\begin{aligned} \hat{h}^i_x(t,x) = & \int_{I\!R} G^i_x(t,x-y)\,d\bar{h}^i - \int_0^t \int_{I\!R} G^i_x(t-s,x-y)\mathbf{e}_i \cdot \Big[\big(A(u)-A^*\big)h(s,y)\Big]_x\,dyds \\ & + \int_0^t \int_{I\!R} G^i_x(t-s,x-y)\mathbf{e}_i \cdot \Big[\big(u_x \bullet A(u)\big)h(s,y) - \big(h \bullet A(u)\big)u_x(s,y)\Big]\,dyds. \end{aligned} \tag{5.17}$$

Using the same techniques of Section 4, in this case relying on the assumptions

$$\begin{aligned} \big\|h_x(t)\big\|_{\mathbf{L}^1} \leq C\frac{\|\bar{h}\|_{\mathbf{L}^1}}{\sqrt{t}}, \qquad & C = \max\left\{\frac{9}{\sqrt{\pi}},\ \|DA\|_\infty\right\}, \\ \sqrt{t} < \sqrt{t_0} \doteq & \frac{1}{9nC\|DA\|_\infty \cdot \text{Tot.Var.}\{\bar{u}\}}, \end{aligned} \tag{5.18}$$

for h, and recalling (4.17), we can estimate the $\mathbf{L}^1$-norms of (5.16) and (5.17) as

$$\begin{aligned} \big\|h(t)\big\|_{\mathbf{L}^1} &\leq \|\bar{h}\|_{\mathbf{L}^1} + \left(4C + \frac{4}{\sqrt{\pi}}\right)n\sqrt{t}\|DA\|_\infty \cdot \text{Tot.Var.}\{\bar{u}\} \cdot \sup_{0<s\leq t}\big\|h(s)\big\|_{\mathbf{L}^1} \leq 2\|\bar{h}\|_{\mathbf{L}^1}\,, \\ \big\|h_x(t)\big\|_{\mathbf{L}^1} &< \frac{\|\bar{h}\|_{\mathbf{L}^1}}{\sqrt{\pi t}} + \frac{16n}{\sqrt{\pi}}C\|DA\|_\infty \cdot \text{Tot.Var.}\{\bar{u}\} \cdot \sup_{0<s\leq t}\big\|h(s)\big\|_{\mathbf{L}^1} \leq C\frac{\|\bar{h}\|_{\mathbf{L}^1}}{\sqrt{t}}. \end{aligned}$$

In particular at time t_0, repeating the arguments at (4.22), for some constant C' we have

$$\begin{aligned} \big\|h^i(t_0)\big\|_{\mathbf{L}^1} &\leq nC'\,\big\|h(t_0)\big\|_{\mathbf{L}^1} \leq 2nC' \cdot \|\bar{h}\|_{\mathbf{L}^1} \leq \xi_0, \\ \big\|h^i_x(t_0)\big\|_{\mathbf{L}^1} &\leq nC'\,\big(1 + 2n\text{Tot.Var.}\{\bar{u}\}\big)\|h_x(t_0)\|_{\mathbf{L}^1} \\ &\leq nC'\,\big(1 + 2n\text{Tot.Var.}\{\bar{u}\}\big)9nC^3\text{Tot.Var.}\{\bar{u}\}\|\bar{h}\|_{\mathbf{L}^1} \leq \frac{4}{\sqrt{\pi}}\widehat{C}^2\,2\eta_0\xi_0, \end{aligned} \tag{5.19}$$

provided that $\widehat{C}$ is big enough and the total variation of $\bar{u}$ sufficiently small. The previous estimates on the norms $\big\|h^i(t)\big\|$ can thus be applied for $t \in [t_0,\ \infty[$, proving the estimates (2.24) on tangent vectors. By (2.26), this completes the proof of Theorem 1.

6 - The vanishing viscosity limit

To complete the proof of Theorem 2 given in Section 2, it only remains to prove the bound (2.32), for some constants L, $\tilde{\lambda}$. Toward this goal, we first establish an estimates on the size of a solution h to the linear variational equation (5.1). Using the integral representation

$$\begin{aligned} h(t,x) = & \int_{I\!R} G(t,x-y)\bar{h}(y)\,dy - \int_0^t \int_{I\!R} G_x(t-s,x-y)A(u)h(s,y)\,dyds \\ & + \int_0^t \int_{I\!R} G(t-s,x-y)\Big[\big(u_x \bullet A(u)\big)h(s,y) - \big(h \bullet A(u)\big)u_x(s,y)\Big]\,dyds, \end{aligned} \tag{6.1}$$

we obtain the bound

$$\begin{aligned} \big|h(t,x)\big| \leq & \int_{I\!R} G(t,x-y)\big|\bar{h}(y)\big|\,dy + \|A\|_\infty \int_0^t \int_{I\!R} \big|G_x(t-s,x-y)\big|\,\big|h(s,y)\big|\,dyds \\ & + 2\|DA\|_\infty \int_0^t \int_{I\!R} \big\|u_x(s)\big\|_{\mathbf{L}^\infty} \cdot G(t-s,x-y)\big|h(s,y)\big|\,dyds\,. \end{aligned} \tag{6.2}$$

Assuming that the initial data satisfies

$$\big|h(0,x)\big| \leq e^{-x} \qquad \text{for all } x \in I\!R, \tag{6.3}$$

we claim that

$$\big|h(t,x)\big| \leq E^+(t,x) \doteq f(t)\exp\left\{4\|DA\|_\infty \cdot \int_0^t \big\|u_x(s)\big\|_{\mathbf{L}^\infty}\,ds + t - x\right\} \tag{6.4}$$

for all $t \geq 0$, $x \in I\!R$, where f is an increasing function such that

$$f(t) \geq 1 + \int_0^t 2\|A\|_\infty f(s)\left(\frac{1}{\sqrt{t-s}} + \sqrt{\pi}\right)ds, \qquad f(0) = 1.$$

It is easy to show that we can take a function $f(t)$ satisfying $f(t) \leq 2\exp\{Ct\}$, if the constant C is big enough. The bound (6.4) follows from the estimates (see appendix B)

$$\int_{I\!R} G(t,x-y)E^+(0,y)\,dy \ \leq \ e^{t-x}, \tag{6.5}$$

$$\|A\|_\infty \cdot \int_0^t \int_{I\!R} \big|G_x(t-s,x-y)\big|E^+(s,y)\,dyds \ \leq \ \frac{1}{2}E^+(t,x) - \frac{1}{2}e^{t-x}, \tag{6.6}$$

$$2\|DA\|_\infty \cdot \int_0^t \int_{I\!R} \big\|u_x(s)\big\|_{\mathbf{L}^\infty} \cdot G(t-s,x-y)E^+(s,y)\,dyds \ \leq \ \frac{1}{2}E^+(t,x) - \frac{1}{2}e^{t-x}, \tag{6.7}$$

by a standard comparison argument. An entirely similar computation shows that, if $\big|h(0,x)\big| \leq e^x$, then for all t, x one has

$$\big|h(t,x)\big| \leq E^-(t,x) \doteq f(t)\exp\left\{4\|DA\|_\infty \cdot \int_0^t \big\|u_x(s)\big\|_{L^\infty}\,ds + t + x\right\}. \tag{6.8}$$

Observe that, by (4.19), we can choose a constant $\tilde{\lambda}$ large enough so that

$$E^+(t,x) \le 2e^{\tilde{\lambda}(\sqrt{t}+t)-x}, \qquad\qquad E^-(t,x) \le 2e^{\tilde{\lambda}(\sqrt{t}+t)+x}. \tag{6.9}$$

The estimates (6.9) are very rough bounds on the solutions of (6.1): in fact we just need a bound on the speed of propagation of perturbation. Now consider any two initial data $\bar{u}, \bar{w}$ satisfying the assumption (1.13) of Theorem 1. For every interval $[a,b]$ and any $\lambda > \tilde{\lambda}$, defining the path $\theta \mapsto \bar{u}^\theta$ as in (2.25), we compute

$$\begin{aligned} \int_{a+\lambda t}^{b-\lambda t} \big|u(t,x) - w(t,x)\big|\,dx &\le \int_0^1 \int_{a+\lambda t}^{b-\lambda t} \left|\frac{du^\theta}{d\theta}(t,x)\right| dx\,d\theta \\ &= \int_0^1 \int_{a+\lambda t}^{b-\lambda t} \big|h^\theta(t,x)\big|\,dx\,d\theta\,, \end{aligned} \tag{6.10}$$

where h^θ is the solution of the Cauchy problem (2.27)-(2.28). By linearity, we can write this solution as a sum:

$$h^\theta = h_1^\theta + h_2^\theta + h_3^\theta,$$

where h_j^θ, $j = 1,2,3$ denote respectively the solutions of (2.27) with initial data

$$h_1^\theta(0,x) = \begin{cases} \bar{u}(x) - \bar{w}(x) & \text{if } \ x \in [a,b], \\ 0 & \text{otherwise}, \end{cases}$$

$$h_2^\theta(0,x) = \begin{cases} \bar{u}(x) - \bar{w}(x) & \text{if } \ x < a, \\ 0 & \text{otherwise}, \end{cases} \qquad h_3^\theta(0,x) = \begin{cases} \bar{u}(x) - \bar{w}(x) & \text{if } \ x > b, \\ 0 & \text{otherwise}. \end{cases}$$

Applying the estimates (2.24) to h_1^θ, (6.4) to h_2^θ and (6.8) to h_3^θ we obtain

$$\begin{aligned} \int_{a+\lambda t}^{b-\lambda t} \big|u(t,x) - w(t,x)\big|\,dx &\le \int_0^1 \int_{a+\lambda t}^{b-\lambda t} \Big(\big|h_1^\theta(t,x)\big| + \big|h_2^\theta(t,x)\big| + \big|h_3^\theta(t,x)\big|\Big)\,dxd\theta \\ &\le L \cdot \|h_1^\theta(0)\|_{\mathbf{L}^1} + 2(b-a-2\lambda)\big\|h^\theta(0,\cdot)\big\|_{\mathbf{L}^\infty} \cdot \Big(\sup_{x<b-\lambda t} e^{\tilde{\lambda}(\sqrt{t}+t)-b+x} + \sup_{x>a+\lambda t} e^{a+\tilde{\lambda}(\sqrt{t}+t)-x}\Big) \\ &\le L \cdot \int_a^b \big|\bar{u}(x) - \bar{w}(x)\big|\,dx + 2(b-a)\|\bar{u} - \bar{w}\|_{\mathbf{L}^\infty} \cdot e^{(\tilde{\lambda}-\lambda)t + \tilde{\lambda}\sqrt{t}}. \end{aligned} \tag{6.11}$$

Given $\varepsilon > 0$, consider now two solutions $u^\varepsilon, w^\varepsilon$ of (1.3), with initial data $\bar{u}, \bar{w}$. Applying the previous estimates to the corresponding solutions obtained via the rescalings $t \mapsto t/\varepsilon$, $x \mapsto x/\varepsilon$, and observing that the distance $\big|\bar{u}(x) - \bar{w}(x)\big|$ is always bounded by the diameter of the compact set K_1, from (6.11) we deduce

$$\int_{a+\lambda t}^{b-\lambda t} \big|u^\varepsilon(t,x) - w^\varepsilon(t,x)\big|\,dx \le L \cdot \int_a^b \big|\bar{u}(x) - \bar{w}(x)\big|\,dx + 2(b-a)\cdot \mathrm{diam}(K_1) \cdot e^{(\tilde{\lambda}-\lambda)t/\varepsilon + \tilde{\lambda}\sqrt{t/\varepsilon}}.$$

Letting $\varepsilon \to 0$, the vanishing viscosity limits $u^\varepsilon \to u$, $w^\varepsilon \to w$ thus satisfy

$$\int_{a+\lambda t}^{b-\lambda t} \big|u(t,x) - w(t,x)\big|\,dx \le L \cdot \int_a^b \big|\bar{u}(x) - \bar{w}(x)\big|\,dx. \tag{6.12}$$

Since (6.12) is valid for every $\lambda > \tilde{\lambda}$, it implies (2.32).

7 - Hyperbolic systems on manifolds

Let $\mathcal{M}$ be a smooth n-dimensional Riemann and call T_u the tangent space at $u \in \mathcal{M}$. At each point u, let $A(u) : T_u \mapsto T_u$ be a linear mapping, smoothly depending on $u \in \mathcal{M}$. Assume that each $A(u)$ is *strictly hyperbolic*, i.e. it has n real distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$. If $u : [0, \infty[\, \times I\!\!R \mapsto \mathcal{M}$ is a smooth map, at any given (t, x) the partial derivatives u_t, u_x are vectors in $T_{u(t,x)}$. It is thus meaningful to consider the system

$$u_t + A(u)u_x = 0 \tag{7.1}$$

In a given set of coordinates, this yields a standard quasilinear hyperbolic system. Smooth solutions are thus well defined. On the other hand. since the equations are not in conservation form, there is no canonical way for defining discontinuous solutions.

Toward this goal, a possible approach is to consider a Riemannian structure on $\mathcal{M}$. In this case, one can choose bases of right and left eigenvectors $r_1(u), \ldots, r_n(u) \in T_u$ and $l^1(u), \ldots, l^n(u) \in T_u^*$ normalized as in (1.6):

$$\big\langle r_i,\ r_i\big\rangle = 1, \qquad\qquad l^i \cdot r_j = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{7.2}$$

For any $\varepsilon > 0$ we can now consider the parabolic system

$$u_t + A(u)u_x = \varepsilon \sum_i \big(l^i(u) \cdot u_x\big)_x r_i(u). \tag{7.3}$$

Definition. If there exists a sequence of solutions u^ε of (7.3) which converges to a function $u = u(t, x)$ in $\mathbf{L}^1_{\mathrm{loc}}$, we say that u is a *vanishing viscosity solution* of (7.1).

The same computations as in (1.7)–(1.9) now yield

$$\sum_i (u_x^i)_t r_i + \sum_i (\lambda_i u_x^i)_x r_i + \sum_{j\neq k} \lambda_j u_x^j u_x^k [r_k, r_j] = \varepsilon \left\{ \sum_i (u_x^i)_{xx} r_i + 2 \sum_{j \neq k} (u_x^j)_x u_x^k [r_k, r_j] \right\}. \tag{7.4}$$

Taking the products of (7.4) with $l^i(u)$ and calling $v^i \doteq l^i(u) \cdot u_x^i$, $i = 1, \ldots, n$, we obtain a system similar to (1.11), where the quadratic terms involve only products of waves of distinct families:

$$v_t^i + (\lambda_i v^i)_x - v_{xx}^i = \sum_{j\neq k} G_{ijk} v^j v^k + \sum_{j \neq k} H_{ijk} v_x^j v^k. \tag{7.5}$$

All of our previous analysis can thus be applied. In the following, for simplicity we assume that a compact $K_0 \subset \mathcal{M}$ is given, which lies in the domain of a single chart. In this case, the total variation of a function $\bar u : I\!\!R \mapsto \mathcal{M}$ and its $\mathbf{L}^1$-norm can be referred to one particular system of coordinates. The extension to the general case is straightforward.

Theorem 3. *Let (7.1) be a strictly hyperbolic system on a Riemannian manifold $\mathcal{M}$. For every compact set $K_0 \subset \mathcal{M}$, there exist constants $L, L', \eta_0 > 0$, a closed domain $\mathcal{D} \subset \mathbf{L}^1_{\rm loc}$ and a continuous semigroup $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ with the following properties.*

(i) Every function $\bar{u}$ satisfying (1.13) lies in the domain $\mathcal{D}$ of the semigroup.

(ii) For every $\bar{u}, \bar{w} \in \mathcal{D}$ with $\bar{u} - \bar{w} \in \mathbf{L}^1$ and every $t, s \geq 0$ one has

$$\|S_t\bar{u} - S_s\bar{w}\|_{\mathbf{L}^1} \leq L'|t - s| + L\|\bar{u} - \bar{w}\|_{\mathbf{L}^1} \, . \tag{1.23}$$

(iii) For every piecewise constant initial data $\bar{u} \in \mathcal{D}$, there exists $\tau > 0$ such that the following holds. For $t \in [0, \tau]$, the semigroup trajectory $S_t\bar{u}$ coincides with the function $u(t, \cdot)$ obtained by piecing together the solutions of the Riemann problems determined by the jumps in $\bar{u}$, constructed by the Riemann Solver at (1.16)–(1.22).

(iv) Every trajectory of the semigroup is a vanishing viscosity solution of (7.1).

Observe that all steps in the construction of the Riemann Solver at (1.18)–(1.22) remain meaningful also in the case of a Riemann manifold.

To prove the theorem, we observe that all the estimates derived in the previous sections remain valid in this case. In particular, for any smooth initial data $\bar{u} : I\!R \mapsto \mathcal{M}$ with small total variation, the corresponding solution of (7.3) is well defined for all $t \geq 0$, and its total variation remains uniformly small. Moreover, by the analysis in Sections 5-6, these solutions depend Lipschitz continuously on the initial data, with a Lipschitz constant independent of $\varepsilon > 0$, and converge to a limit as $\varepsilon \to 0$.

On the other hand, by the results in [14], the Riemann Solver described at (1.18)–(1.22) generates a unique Lipschitz semigroup S. To show that the trajectories of this semigroup coincide with our vanishing viscosity limits, by [3] it suffices to check the case of a Riemann initial data. Assume first that both states u^-, u^+ in (1.17) lie on a single i-rarefaction curve, say

$$u^+ = R_i(\sigma_i)(u^-).$$

Then the particular form of the diffusion operator implies that the solution of (7.3) remains on the same i-rarefaction curve for all positive times. We can thus write

$$u(t, x) = R_i\big(z(t, x)\big)(u^-),$$

where z is the solution of the scalar Cauchy problem (1.21).

The general case is proved by considering small perturbations $\bar{u}^\delta$ of the initial data, as in (2.35), and using the Lipschitz continuous dependence of vanishing viscosity solutions.

Appendix A

By the same computations as in Section 3, step 1, it follows that (3.10) holds also if we translate v^j:

$$\int_0^{\sigma-u}\int_{I\!R}\left|v^i(t,x)\right|\left|v^j(t+u,x+y)\right|dxdt \le \frac{1}{c}\big(\eta(t)\big)^2 \le \frac{1}{c}\big(2\eta_0\big)^2. \qquad (A.1)$$

A.1. Estimation of (3.24): from (2.10) and (A.1) it follows

$$\begin{aligned}
&\int_0^{\sigma-\tau}\iint_{I\!R^2}\left|G^i(t,x-y)\,\bar v^i_x(y)\,v^j(t+\tau,x+z)\right|dxdy\,dt\\
&\quad\le \int_{I\!R}\left|\bar v^i_x(y)\right|\int_0^{\sigma-\tau}\int_{I\!R}G^i(t,x-y)\left|v^j(t+\tau,x+z)\right|dx\,dt\,dy\\
&\quad\le \frac{4}{\sqrt\pi}\widehat C^2\eta_0^2\frac{\eta(\sigma)}{c}\le\frac{4}{\sqrt\pi}\widehat C^2\frac{\eta^3(\sigma)}{c}.
\end{aligned}$$

A.2. Estimation of (3.25): the definitions of $\mathcal{I}_\sigma$ in (3.22) and of $\hat t$ in (3.17) yield

$$\begin{aligned}
&\int_0^{\sigma-\tau}\int_0^t\iint_{I\!R^2}\left|G^i_x(s,y)\big[\lambda_i(u)-\lambda_i^*\big]v^i_x(t-s,x-y)v^j(t+\tau,x+z)\right|dxdy\,dsdt\\
&\le\|\lambda_i\|_\infty\int_0^{\sigma-\tau}\int_{I\!R}\left|G^i_x(s,y)\right|\int_s^{\sigma-\tau}\int_{I\!R}\left|v^i_x(t-s,x-y)v^j(t+\tau,x+z)\right|dxdt\,dyds\\
&\le\|\lambda_i\|_\infty\int_0^{\sigma-\tau}\int_{I\!R}\left|G^i_x(s,y)\right|\int_0^{\sigma-\tau-s}\int_{I\!R}\left|v^i_x(t,x)v^j(t+\tau+s,x+z+y)\right|dxdt\,dyds\\
&\le\frac{2}{\sqrt\pi}\sqrt{\sigma-\tau}\;\|\lambda_i\|_\infty\mathcal{I}(\sigma)\;\le\;\frac{2}{\sqrt\pi}\sqrt{\hat t}\|\lambda_i\|_\infty\mathcal{I}(\sigma)\;\le\;\frac{2n}{\sqrt\pi}\sqrt{\hat t}\|\lambda_i'\|_\infty\eta(\sigma)\mathcal{I}(\sigma)\;\le\;\frac{1}{8}\mathcal{I}(\sigma).
\end{aligned}$$

A.3. Evaluation of (3.26): by (3.19)–(3.20) and (A.1) we have

$$\begin{aligned}
&\int_0^{\sigma-\tau}\int_0^t\iint_{I\!R^2}\left|G^i_x(s,y)\Big((r_k\bullet\lambda_i)v^k\Big)v^i(t-s,x-y)v^j(t+\tau,x+z)\right|dydx\,dsdt\\
&\le\|\lambda_i'\|_\infty\sup_{0\le t\le\bar t}\left\|v^k(t)\right\|_{\mathbf{L}^\infty}\\
&\qquad\cdot\int_0^{\sigma-\tau}\int_{I\!R}\left|G^i_x(s,y)\right|\int_s^{\sigma-\tau}\int_{I\!R}\left|v^i(t-s,x-y)\cdot v^j(t+\tau,x+z)\right|dxdt\,dyds\\
&\le\|\lambda_i'\|_\infty\sup_{0\le t\le\bar t}\left\|v^k_x(t)\right\|_{\mathbf{L}^1}\\
&\qquad\cdot\int_0^{\sigma-\tau}\int_{I\!R}\left|G^i_x(s,y)\right|\int_0^{\sigma-\tau-s}\int_{I\!R}\left|v^i(t,x)\cdot v^j(t+s+\tau,x+z)\right|dxdt\,dyds\\
&\le\|\lambda_i'\|_\infty\frac{8}{\sqrt\pi}\widehat C^2\eta^2(\sigma)\frac{2}{\sqrt\pi}\sqrt\sigma\left(\frac{1}{c}\eta^2(\sigma)\right)\\
&\le\frac{16}{c\pi}\sqrt{\hat t}\|\lambda_i'\|_\infty\widehat C^2\eta^4(\sigma)\,.
\end{aligned}$$

A.4. Estimation of (3.27): using (3.19)–(3.20) we obtain

$$\begin{aligned}
&\int_0^{\sigma-\tau}\int_0^t\iint_{I\!R^2}\Big|G_x^i(s,y)\phi^i(t-s,x-y)v^j(t+\tau,x+z)\Big|\,dydx\,dsdt\\
&\le \sup_{0\le t\le\bar t}\big\|v^i(t)\big\|_{\mathbf{L}^\infty}\int_0^{\sigma-\tau}\int_{I\!R}\big|G_x^i(s,y)\big|\int_s^{\sigma-\tau}\int_{I\!R}\big|\phi^i(t-s,x-y)\big|\,dxdt\,dyds\\
&\le \frac{8}{\sqrt\pi}\widehat C^2\eta^2(\sigma)\frac{2}{\sqrt\pi}\sqrt{\sigma-\tau}\int_0^{\sigma-s-\tau}\int_{I\!R}\big|\phi^i(t,x)\big|\,dxdt\ \le\ \frac{2}{\sqrt\pi}\widehat C\eta(\sigma)\int_0^\sigma\int_{I\!R}\big|\phi^i(t,x)\big|\,dxdt.
\end{aligned}$$

A.5. Estimation of (3.29): from (2.10) and (A.1) it follows

$$\begin{aligned}
&\int_{\hat t}^{\sigma-\tau}\iint_{I\!R^2}\Big|G_x^i(\hat t,y)v^i(t-\hat t,x-y)v^j(t+\tau,x+z)\Big|\,dydx\,dt\\
&=\int_{I\!R}\big|G_x^i(\hat t,y)\big|\int_0^{\sigma-\tau-\hat t}\int_{I\!R}\big|v^i(t,x-y)\big|\,\big|v^j(t+\tau+\hat t,x+z)\big|\,dxdt\,dy\\
&\le \frac{1}{\sqrt{\pi\hat t}}\frac{\eta^2(\sigma)}{c}\ =\ \frac{8}{c\pi}\widehat C\eta^3(\sigma)\,.
\end{aligned}$$

A.6. Estimation of (3.30): the definitions of $\mathcal I(\sigma)$ in (3.22) and of $\hat t$ in (3.17) yield

$$\begin{aligned}
&\int_{\hat t}^{\sigma-\tau}\int_0^{\hat t}\iint_{I\!R^2}dxdy\Big|G_x^i(s,y)\big[\lambda_i(u)-\lambda_i^*\big]v_x^i(t-s,x-y)v^j(t+\tau,x+z)\Big|\,dxdy\,dsdt\\
&\le\|\lambda_i\|_\infty\int_0^{\hat t}\int_{I\!R}\big|G_x^i(s,y)\big|\int_{\hat t-s}^{\sigma-\tau-s}\int_{I\!R}\big|v_x^i(t,x)\big|\,\big|v^j(t+\tau+s,x+z+y)\big|\,dxdt\,dyds\\
&\le\frac{2n}{\sqrt\pi}\sqrt{\hat t}\|\lambda_i'\|_\infty(2\eta_0)\mathcal I(\sigma)\ \le\ \frac18\mathcal I(\sigma).
\end{aligned}$$

A.7. Estimation of (3.31): by (3.19)–(3.20) and (A.1) we have

$$\begin{aligned}
&\int_{\hat t}^{\sigma-\tau}\int_0^{\hat t}\iint_{I\!R^2}\Big|G_x^i(s,y)\big((r_k\bullet\lambda_i)v^k\big)v^i(t-s,x-y)v^j(t+\tau,x+z)\Big|\,dxdy\,dsdt\\
&\le\|\lambda_i'\|_\infty\sup_{0\le t\le\bar t}\big\|v^k(t)\big\|_{\mathbf{L}^\infty}\\
&\quad\cdot\int_0^{\hat t}\int_{I\!R}\big|G_x^i(s,y)\big|\int_{\hat t}^{\sigma-\tau}\int_{I\!R}\big|v^i(t-s,x-y)\big|\,\big|v^j(t+\tau,x+z)\big|\,dxdt\,dyds\\
&\le\frac{2}{\sqrt\pi}\sqrt{\hat t}\|\lambda_i'\|_\infty\frac{8}{\sqrt\pi}\widehat C^2\eta^2(\sigma)\left(\frac1c\eta^2(t)\right)\ =\ \frac{16}{c\pi}\sqrt{\hat t}\|\lambda_i'\|_\infty\widehat C^2\eta^4(\sigma).
\end{aligned}$$

A.8. Estimation of (3.32): using (3.19)–(3.20) we obtain

$$\begin{aligned}
&\int_{\hat t}^{\sigma-\tau}\int_0^{\hat t}\iint_{I\!R^2}\Big|G_x^i(s,y)\phi^i(t-s,x-y)v^j(t+\tau,x+z)\Big|\,dxdy\,dsdt\\
&\le\sup_{0\le t\le\bar t}\big\|v^i(t)\big\|_{\mathbf{L}^\infty}\int_0^{\hat t}\int_{I\!R}\big|G_x^i(s,y)\big|\int_{\bar t}^{\sigma-\tau}\int_{I\!R}\big|\phi^i(t-s,x-y)\big|\,dxdt\,dyds\\
&\le\frac{8}{\sqrt\pi}\widehat C^2\eta^2(\sigma)\frac{2}{\sqrt\pi}\sqrt{\hat t}\,Q(\sigma)\ \le\ \frac{2}{\sqrt\pi}\widehat C\eta(\sigma)\int_0^\sigma\int_{I\!R}\big|\phi^i(t,x)\big|\,dxdt.
\end{aligned}$$

Appendix B

B.1. Estimation of (6.5):

$$\int_{\mathbb{R}} G(t,x-y)E^+(0,y)\,dy \le \frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}} \exp\left\{\frac{(y+2t-x)^2}{4t}+t-x\right\}dy = e^{t-x}\,.$$

B.2. Estimation of (6.6):

$$\begin{aligned}
&\|A\|_\infty \cdot \int_0^t \int_{\mathbb{R}} \big|G_x(t-s,x-y)\big| E^+(s,y)\,dy\,ds \\
&= \|A\|_\infty \int_0^t \int_{\mathbb{R}} \frac{|x-y|}{4(t-s)\sqrt{\pi(t-s)}} f(s) \\
&\qquad\qquad \cdot \exp\left\{-\frac{(x-y)^2}{4(t-s)} + 4\|DA\|_\infty \int_0^s \big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty} d\varsigma + s - y\right\} dyds \\
&\le \|A\|_\infty \exp\left\{4\|DA\|_\infty \int_0^t \big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty} d\varsigma + t - x\right\} \\
&\qquad\qquad \cdot \int_0^t \frac{f(s)}{4(t-s)\sqrt{\pi(t-s)}} \cdot \int_{\mathbb{R}} |x-y| \exp\left\{-\frac{(y+2(t-s)-x)^2}{4(t-s)}\right\} dyds \\
&= \exp\left\{4\|DA\|_\infty \int_0^t \big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty} d\varsigma + t - x\right\} \cdot \int_0^t \frac{\|A\|_\infty f(s)}{\sqrt{\pi(t-s)}} \int_{\mathbb{R}} \big|\zeta - \sqrt{t-s}\big| e^{-\zeta^2}\,d\zeta ds \\
&\le \exp\left\{4\|DA\|_\infty \int_0^t \big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty} d\varsigma + t - x\right\} \cdot \int_0^t \|A\|_\infty f(s)\left(\frac{1}{\sqrt{t-s}} + \sqrt{\pi}\right) ds \\
&\le \exp\left\{4\|DA\|_\infty \int_0^t \big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty} d\varsigma + t - x\right\} \cdot \left(\frac{f(t)}{2} - \frac{1}{2}\right) \\
&= \frac{1}{2}E^+(t,x) - \frac{1}{2}\exp\left\{4\|DA\|_\infty \int_0^t \big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty} d\varsigma + t - x\right\} \\
&\le \frac{1}{2}E^+(t,x) - \frac{1}{2}e^{t-x}\,.
\end{aligned}$$

B.3. Estimation of (6.7):

$$\begin{aligned}
&2\|DA\|_\infty \cdot \int_0^t \big\|u_x(s)\big\|_{\mathbf{L}^\infty} \int_{\mathbb{R}} G(t-s,x-y)E^+(s,y)\,dyds \\
&= 2\|DA\|_\infty \cdot \int_0^t f(s) \cdot \exp\left\{4\|DA\|_\infty \int_0^s \big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty}\,d\varsigma + s\right\} \\
&\qquad\qquad \cdot \frac{\big\|u_x(s)\big\|_{\mathbf{L}^\infty}}{2\sqrt{\pi(t-s)}} \int_{\mathbb{R}} \exp\left\{-\frac{(x-y)^2}{4(t-s)} - y\right\} dy\,ds \\
&\le f(t) \cdot e^{t-x} \cdot \int_0^t 2\|DA\|_\infty \big\|u_x(s)\big\|_{\mathbf{L}^\infty} \cdot \exp\left\{4\|DA\|_\infty \cdot \int_0^s \big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty} d\varsigma\right\} ds \\
&\le \frac{1}{2}E^+(t,x) - \frac{1}{2}f(t)\cdot e^{t-x} \\
&\le \frac{1}{2}E^+(t,x) - \frac{1}{2}e^{t-x}.
\end{aligned}$$

Acknowledgment. This research was partially supported by the European TMR Network on Hyperbolic Conservation Laws ERBFMRXCT960033.

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