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SBV REGULARITY OF SYSTEMS OF CONSERVATION LAWS AND HAMILTON-JACOBI EQUATION

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ABSTRACT. We review the SBV regularity for solutions to hyperbolic systems of conservation laws and Hamilton-Jacobi equations. We give an overview of the techniques involved in the proof, and a collection of related problems concludes the paper.

CONTENTS

1. Introduction

Consider a strictly hyperbolic system of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R},\ u \in \mathbb{R}^n, \tag{1.1}$$

It is now a classical result that if the initial data

$$u(0,x) = u_0(x)$$

has a small BV norm, then the solution remains in BV for all $t > 0$. For a proof, one can use different methods: Glimm scheme [18, 3], wavefront tracking [2], vanishing viscosity [7] or other singular limits methods ([6, 5] for example).

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For special systems, L^∞-solutions can be constructed, by means of uniform stability estimates [4], compensated compactness [17] or uniform decay estimates [19, 24].

All these results can be see as regularity properties of solutions, yielding some compactness in $L^\infty(\mathbb{R})$. It is important to notice that continuous solutions in general do not exists, as it is taught at every basic PDE course.

Other kinds of regularity can be considered. We here give a short list.

1.1. Decay of positive waves. In the case $n = 1$, i.e. of a scalar conservation laws, Oleinik proved that the solution satisfies the one-sided Lipschitz bound

$$u(t, x + h) - u(t, x) \leq \frac{h}{\kappa t} \tag{1.2}$$

where $f''(u) \geq \kappa > 0$ is the uniform convexity of f [22]. In particular u is locally BV.

A generalization of the above condition is given in [15]: the positive part of the i-th component v_i of $\partial_x u$ satisfies

$$v^+(T, A) \leq \frac{1}{c_0} \frac{\mathcal{L}^1(A)}{T - t} + C_0 \big(Q(t) - Q(T) \big),$$

where Q is the Glimm interaction functional.

We will study this regularity more deeply later on, since it is strictly related to the SBV regularity.

1.2. Differentiability along characteristics. In the uniformly convex scalar case, since

$$x \mapsto -\lambda(u(t, x))$$

is a quasi-monotone vector field by (1.2), one can consider the unique Filippov solution to the differential inclusion

$$\dot{x} \in \big[-\lambda(u(t, x+), -\lambda(u(t, x-) \big].$$

The solutions to this inclusion outside the jump set of u are called characteristics curves.

As for C^1 solutions one can then prove that the solution is constant along the characteristics, i.e. if $\gamma(t)$ is a characteristic then $t \mapsto u(t, \gamma(t))$ is constant, and thus γ is a segment: these properties are easy to verify in the case $u \in C^1$.

It is thus possible to ask if the same conditions holds for solutions to scalar balance laws

$$u_t + f(u)_x = g(t, x, u),$$

where one expects that the following holds:

$$\frac{d}{dt} u(t, \gamma(t)) = g(t, \gamma(t), u(t, \gamma(t))).$$

In general this is not true, but it is known to holds for convex f [16]. The vector case of this result is still completely open.

1.3. Differentiability properties of L^∞-solutions. For L^∞-solutions to conservation laws where no BV estimates can be proved, the structure of the solution is in general not clear: for example, solutions in more than one dimension, or non convex scalar equations. It is possible however to prove that the nonlinearity of the flux f implies that some sort of BV structure survives: there is a rectifiable jump set, where left and right limits of the solution exists, and outside this set the solution has vanishing mean oscillation [20].

The proof of similar results for systems is an open problem.

1.4. Fractional differentiability. By means of the kinetic representation, it is possible to prove that the solution belongs to a compact space in L^1, in particular [21].

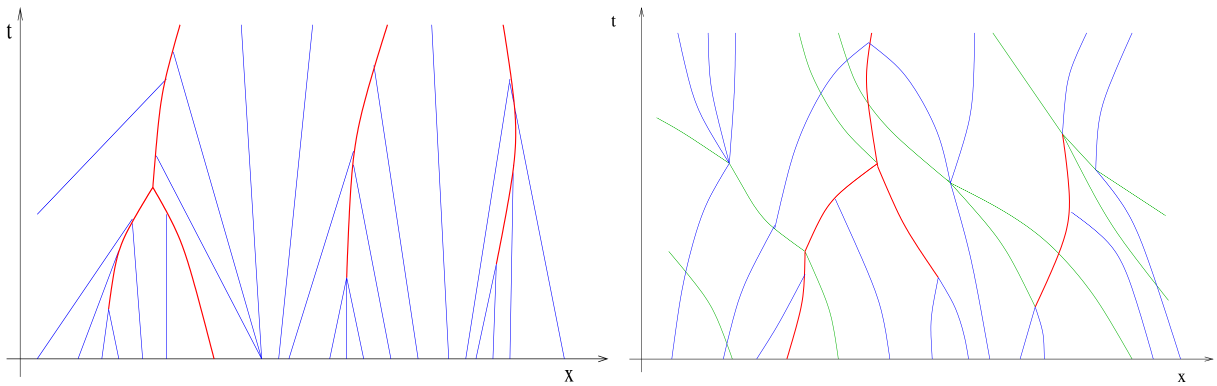

Fig. 1: As the characteristics curves and the jump set fro the solution of a scalar uniformly convex conservation law are usually presented (left), and the characteristics and wave pattern for the system case (right).

1.5. SBV regularity. For solutions of strictly hyperbolic systems of conservation laws in one space dimension one expects the following structure: countably many shock curves and regularity of the solution in the remaining set. In the system case, however, the structure is much more complicated, due to the presence of waves of the other families: indeed, the characteristic curves are not straight lines any more, and the interaction among waves complicates the wave pattern (see Fig. 1).

One way of interpreting this structure is to say that *the solution u has a rectifiable jump part, and in the remaining set the derivative of u is absolutely continuous.* This means that in the decomposition of $\partial_x u$ as a derivative of a BV function, the Cantor part of the derivative is 0. This fact has been verified in the scalar case in [1], while in the vector case it has been proved in [25].

All the fundamental ideas can be understood in the scalar case:

$$u_t + f(u)_x = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R},\ u \in \mathbb{R},$$

so we will restrict to this case in this paper. At the end we will consider the case of the Hamilton-Jacobi equation

2. Proof of SBV regularity in the scalar case

The interpretation of Fig. 1 can be interpreted as

- shocks are concentrated on countably many Lipschitz curves (with first derivative in BV),
- decay of positive and negative waves as t^{-1},
- no other terms in the derivative, i.e. no Cantorian part.

The idea of the proof in the scalar case given in [1] is as follows, see Fig. 2.

Let $\bar{t}$ be a time where the spatial derivative of $u(t)$ has a Cantor part concentrated on the $\mathcal{L}^1$-negligible set C. Then since $u(\bar{t})\llcorner_C$ is continuous, for each $\bar{x} \in C$ there exists only one characteristics starting at $t = 0$ and arriving at $(\bar{t}, \bar{x})$. Then we can consider the set of initial points $C(0)$ of C.

Since the slopes of the characteristics are related to u by the function $\lambda(u) = f'(u)$, then we have that the opening is of $\geq \kappa|\partial_x u|(C)$, where $\kappa \leq f''(u)$ is the constant of uniform convexity. In particular, the $\mathcal{L}^1$-measure of $C(0)$ is $\geq \kappa\bar{t}|\partial_x u|(C)$.

Using the fact that characteristics do not intersect outside the end points, one can prove that if A is Borel and the characteristics starting from A arrives at time t, then for all $0 < s < t$ it

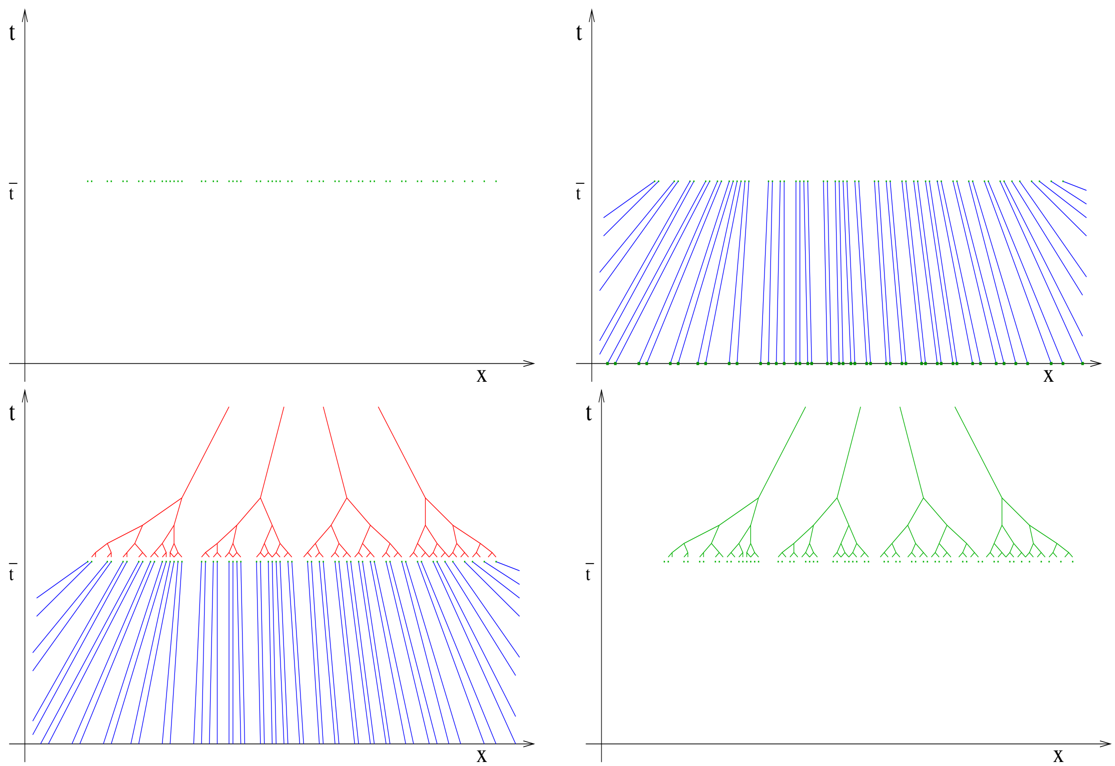

Fig. 2: The analysis of SBV regularity in the scalar case, and where the measure μ defined in (2.1) is concentrated.

holds

$$\mathcal{L}^1\Big\{\gamma(s), \gamma(0) \in A, \gamma \text{ characteristic}\Big\} \geq \bigg(1 - \frac{s}{t}\bigg)\mathcal{L}^1(A).$$

Hence if the characteristics arriving in C at $\bar{t}$ can be prolonged, then C has positive measure, since $\mathcal{L}^1(C(0)) > 0$.

It thus follows that if we define the functional

$$H(t, R) := \mathcal{L}^1\Big\{x \in B(0, R) : \text{the characteristic leaving } x \text{ can be prolonged up to } t\Big\},$$

then this functional is decreasing (since in the scalar case the characteristic equation has forward uniqueness), and has a downward jump at $\bar{t}$.

We conclude that the number of times where a Cantor part in the derivative $\partial_x u$ appears is countable. Then as a function of two variable, $\partial_x u$ is SBV, and using the equation $u_t = -f(u)_x$ also $\partial_t u$ is SBV.

2.1. A reformulation of the above proof. Since $x \mapsto -f'(u(t,x))$ is a quasi-monotone operator, it follows that the ODI

$$\dot{x} \in -f'(u(t,x))$$

generates a unique Lipschitz semigroup $X(t,x)$ [13, 8]. In particular we can consider the transport solution of

$$\rho_t + (f'(u(t))\rho)_x = 0, \quad \rho(0) = \mathcal{L}^1,$$

which can be represented as $X(t)_\sharp \mathcal{L}^1$, i.e. the Jacobian of $X^{-1}(t)$.

If we split $\rho(t) = \rho^c(t) + \rho^a(t)$, ρ^a atomic part, then

$$\rho^c + (f'(u)\rho^c)_x = -\mu, \quad \rho^a + (f'(u)\rho^a)_x = \mu, \tag{2.1}$$

where μ is a distribution. Using the fact that the atomic part of ρ can only increase (because of monotonicity), then μ is a positive Radon measure.

The previous proof shows that if a Cantor part appears in ρ^c, then

$$\mu(\{t\} \times A) \geq \rho^{\text{cantor}}(A),$$

and the local boundedness of μ allows to conclude as in the previous proof. In this model case the measure μ is concentrated on the Cantor set and in the jump set.

2.2. The equation for $\partial_x u$. The measure $v := \partial_x u(t)$ satisfies the same transport equation in conservation form

$$v_t + (f'(u(t))v)_x = 0, \quad v(0) = D_x u(0),$$

but since it has a sign the equations for its atomic and non atomic part are a little more complicated. In fact cancellation among negative and positive waves should be considered.

By using the wavefront tracking approximation, one can prove that if $v = v^c + v^a$, v^a atomic part of v, then

$$v_t^c + (f'(u(t))v^c)_x = -\mu^{CJ}, \quad v_t^a + (f'(u(t))v^a)_x = \mu^{CJ},$$

with μ^{CJ} signed locally bounded measure such that

$$\mu^{CJ} - \big\{\text{measure of cancellation of waves}\big\} \leq 0.$$

Summing up, we have 3 equations

$$\begin{aligned} v_t + (f'(u(t))v)_x &= 0 \\ |v|_t + (f'(u(t))|v|)_x &= -\mu^C \leq 0, \\ v_t^a + (f'(u(t))v^a)_x &= \frac{1}{2}\mu^C + \mu^J, \end{aligned}$$

with $\mu^J \leq 0$. The proof of SBV regularity can be thus restated as

$$\mu^J(\{t\} \times A) \leq v^{\text{cantor}}(t, A).$$

2.3. Decay estimates. We have seen that for convex conservation laws the decay of positive waves reads as

$$v(t, A) \leq \frac{1}{c_0}\frac{\mathcal{L}^1(A)}{t}, \quad f'' \geq c_0.$$

The measure μ^J allows to obtain the corresponding decay estimate for the negative part v^c:

$$v^c(T, A) \geq -\frac{1}{c_0}\frac{\mathcal{L}^1(A)}{t - T} + \mu^J\big(\text{domain of influence of A}\big).$$

In fact, the measure μ^J controls exactly the points where the characteristics collide and generate jumps. Observe that for the positive waves in convex scalar conservation laws no new centered rarefaction waves are created, and that for the system case the decay estimate has a form very similar to the one above.

Using now the fact that $u(t)$ is absolutely continuous outside the jump part, one can write the equation for v^c along each ray γ:

$$v_t^c + (f'(u(t))v^c)_x = 0, \quad \frac{d}{dt}v^c(t, \gamma(t)) = -f''(u)(v^c)^2.$$

This yields that if the ray $\gamma(t)$ has a life span of $[0, T]$, then

$$-\frac{1}{c_0}\frac{1}{T - t} \leq v^c(t, \gamma(t)) \leq \frac{1}{c_0}\frac{1}{t}.$$

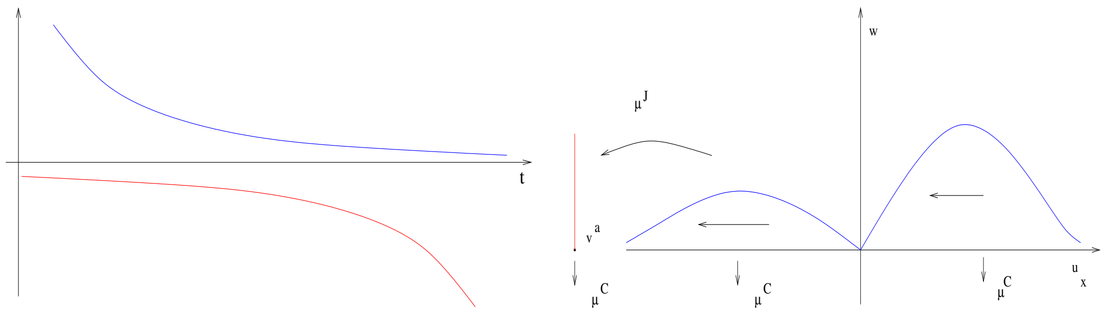

Fig. 3: The decay estimate along a characteristic (left) and the dynamic interpretation of the scalar conservation law (right).

2.3.1. Dynamical interpretation. We can thus give the following dynamic representation of the evolution of the derivative $D_x u$.

If we consider the measures

$$\omega^c(t) := v^c_\sharp\big(v^c \mathcal{L}^1\big), \quad \omega^a(t) := v^a(t, \mathbb{R}^1)$$

then it follows that

$$\omega^c_t + y^2 \omega^c = -\tilde{\mu}, \quad \omega^a_t = \tilde{\mu},$$

with (formally)

$$\tilde{\mu} = v(t)_\sharp \bigg(\frac{1}{2} \mu^C + \mu^J \bigg).$$

We can thus give the dynamic representation of the evolution of the derivative $D_x u$ of Fig. 3.

3. SBV estimates for systems

We now review the main idea in the system case.

3.1. Decomposition into wave measures. We consider the hyperbolic system

$$u_t + f(u)_x = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \; u \in \mathbb{R}^n,$$

and we assume that the $\bar{i}$-eigenvalue λ_i of $Df(u)$ is g.n.l.: by choosing the direction of the unit eigenvector $r_{\bar{i}}$,

$$D\lambda_{\bar{i}}(u) r_{\bar{i}}(u) \leq c_0 < 0.$$

We moreover decompose the derivative of the solution as [14]

$$u_x(t) = \sum v_i(t) \tilde{r}_i,$$

with $\tilde{r}_i = r_i$ where u is continuous, otherwise is the direction of the jump of the i-th family. Each $v_i(t)$ is a bounded measure.

Our aim is to prove that $v_{\bar{i}}(t)$ has a Cantor part only at countably many times. In general the situation is more complicated than in the scalar case, due to the presence and the interaction of the waves of different families.

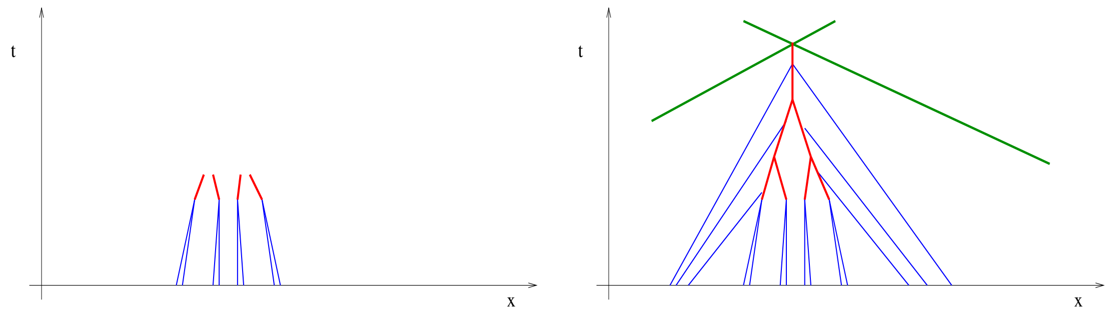

Fig. 4: Possible evolution of jumps created by a Cantor part.

3.2. Equation for wave measures. Let $\tilde{\lambda}_i$ be the i-th eigenvector if u is continuous or the speed of the i-th shock. By the wavefront approximation, one obtain the following balance equation

- conservation of v_i:
$$(v_i)_t + (\tilde{\lambda}_i v_i)_x = \mu_i^I$$
where μ_i^I is a signed measure bounded by the decrease of the interaction potential $Q(u)$;
- conservation of $|v_i|$:
$$(|v_i|)_t + (\tilde{\lambda}_i |v_i|)_x = \mu_i^{IC}$$
where μ_i^{IC} is a signed measure bounded by the decrease of the potential Tot.Var.(u) + $CQ(u)$.

3.2.1. Equation for the atomic part. If $\bar{i}$ is genuinely nonlinear, the equation for the atomic part $v_{\bar{i}}^a$ is
$$(v_{\bar{i}}^a)_t + (\tilde{\lambda}_{\bar{i}} v_{\bar{i}}^a)_x = \mu_{\bar{i}}^{ICJ},$$
where $\mu_{\bar{i}}^{ICJ}$ is a distribution satisfying
$$\mu^J := \mu_{\bar{i}}^{ICJ} - |\mu_{\bar{i}}^I| - |\mu_{\bar{i}}^{IC}| \leq 0.$$
Hence $\mu_{\bar{i}}^J$ is a bounded measure (*jump measure*), which measures the amount of jumps created.

The fact that μ^J is a measure (signed distribution) follows from the fact that it is easy to create a jump because of nonlinearity, but to cancel it you have to use cancellation or interaction, see Fig. 4.

3.3. Proof of SBV regularity. The continuous part $v_{\bar{i}}^c$ of $v_{\bar{i}}$ thus satisfies
$$(v_{\bar{i}}^c)_t + (\lambda_{\bar{i}} v_{\bar{i}}^c)_x = \mu_{\bar{i}}^c, \quad \mu_{\bar{i}}^c := \mu_{\bar{i}}^I - \mu_{\bar{i}}^{ICJ}.$$
As argument similar to the estimate of the decay of positive waves yields now
$$v_{\bar{i}}^c(T, A) \geq -\frac{1}{c_0}\frac{\mathcal{L}^1(A)}{t-T} - |\mu_{\bar{i}}^c|\Big(\text{Domain of influence of } A\text{, Fig. 5}\Big).$$

In particular, if A is a set of measure 0 where the Cantor part is concentrated, then by taking a sequence $t_n \searrow T$ we obtain
$$|\mu_{\bar{i}}^c|(A) > 0.$$
Since $\mu_{\bar{i}}^c$ is a bounded measure, then the set of times where a Cantor part appears is countable. These times corresponds to:

(1) strong interactions among waves;
(2) generation of shock with the same strength of the Cantor part.

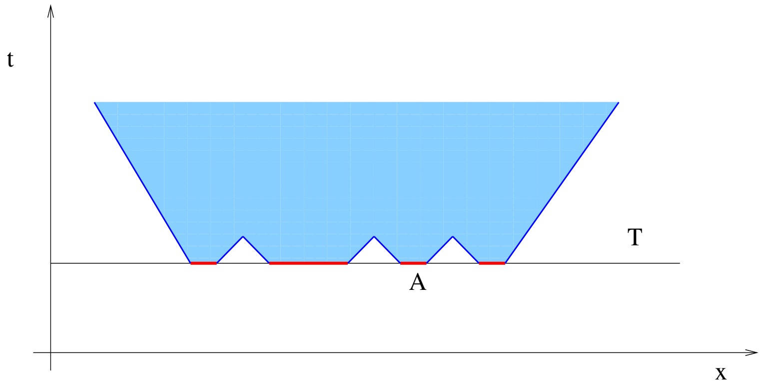

Fig. 5: Domain of influence of A.

4. SBV regularity for Hamilton-Jacobi

This part is taken from [11].

We consider a viscosity solution u to the Hamilton-Jacobi equation

$$\partial_t u + H(t, x, D_x u) = 0 \qquad \text{in } \Omega \subset [0, T] \times \mathbb{R}^n. \tag{4.1}$$

We prove the SBV regularity of $D_x u$ and $\partial_t u$ under hypotheses of differentiability and uniform convexity of H in the last variable, i.e.

(H1) $H \in C^3([0, T] \times \mathbb{R}^n \times \mathbb{R}^n)$ with bounded second derivatives and there exist positive constants a, b, c such that
  i) $H(t, x, p) \geq -c$,
  ii) $H(t, x, 0) \leq c$,
  iii) $|H_{px}(t, x, p)| \leq a + b|p|$,

(H2) there exists $c_H > 0$ such that

$$c_H^{-1} Id_n(p) \leq H_{pp}(t, x, p) \leq c_H Id_n(p)$$

for any t, x.

The proofs of the following statements can be found in Cannarsa and Sinestrari [26], Chapter 6.

The convexity of the Hamiltonian in the p-variable relates Hamilton-Jacobi equations to a variational problem.

Let L be the Lagrangian of our system, i.e. the Legendre transform of the Hamiltonian H with respect to the last variable, for any t, x fixed

$$L(t, x, v) = \sup_p \{\langle v, p \rangle - H(t, x, p)\}.$$

The Legendre transform inherits the properties of H, in particular L is $C^3([0, T] \times \mathbb{R}^n \times \mathbb{R}^n)$ and uniformly convex in the last variable.

In addition to the uniform convexity and C^3 regularity of L, the hypotheses on H, (H1) and (H2), ensure the existence of positive constants a, b, c such that

i) $L(t, x, v) \geq -c$,
ii) $L_x(t, x, 0) \leq c$,
iii) $|L_{vx}(t, x, v)| \leq a + b|v|$.

Define the value function $u(t,x)$ associated the the bounded Lipschitz function $u_0(x)$ for $(t,x) \in \Omega$

$$u(t,x) := \min \left\{ u_0(\xi(0)) + \int_0^t L(s,\xi(s),\dot{\xi}(s))ds \;\middle|\; \xi(t) = x,\ \xi \in [C^2([0,t])]^n \right\}. \tag{4.2}$$

Less regularity can be asked to ξ, but it is unnecessary since any minimizing curve exists and is smooth, due to the regularity of L, see [26].

Theorem 4.1. *Taken a minimizing curve ξ in (4.2), for the point (t,x), such that $\xi(s) \in \Omega_s$ for all $s \in [0,t]$, the following holds. (Recall $\Omega_s = \{x \in \mathbb{R}^n |\ (s,x) \in \Omega\}$.)*

i) *The map $s \mapsto L_v(s,\xi(s),\dot{\xi}(s))$ is absolutely continuous.*

ii) *ξ is a classical solution to the Euler-Lagrange equation*

$$\frac{d}{ds} L_v(s,\xi(s),\dot{\xi}(s)) = L_x(s,\xi(s),\dot{\xi}(s)),$$

and to the Du Bois-Reymond equation

$$\frac{d}{ds}[L(s,\xi(s),\dot{\xi}(s)) - \langle \dot{\xi}(s), L_v(s,\xi(s),\dot{\xi}(s))\rangle] = L_t(s,\xi(s),\dot{\xi}(s)),$$

for all $s \in [0,t]$, where $L_t(s,\xi(s),\dot{\xi}(s))$ is the derivative of L with respect to the first variable.

iii) *For any $r > 0$ there exists $K(r) > 0$ such that, if $(t,x) \in [0,r] \times B_r(0)$, then*

$$\sup_{s\in[0,t]} |\dot{\xi}(s)| \le K(r).$$

iv) *There exists a dual arc or co-state*

$$p(s) := L_v(s,\xi(s),\dot{\xi}(s)) \qquad s \in [0,t], \tag{4.3}$$

such that ξ, p solve the following system

$$\begin{cases} \dot{\xi}(s) = H_p(s,\xi(s),p(s)) \\ \dot{p}(s) = -H_x(s,\xi(s),p(s)). \end{cases}$$

v) *$(s,\xi(s))$ is regular, i.e. for any $0 < s < t$ ξ is the unique minimizer for $u(s,\xi(s))$, and $u(s,\cdot)$ is differentiable at $\xi(s)$.*

vi) *Let p be the dual arc associated to ξ as in (4.3) then we have*

$$p(t) \in D_x^+ u(t,x),$$

$$p(s) = D_x u(s,\xi(s)), \qquad s \in (0,t).$$

Theorem 4.2. *The value function u defined in (4.2) is a viscosity solution with bounded Lipschitz initial datum*

$$u(0,x) = u_0(x).$$

We present below some properties of the unique viscosity solution to the Hamilton-Jacobi equation (4.1), which follow from the representation formula we have just seen. These properties are taken from [26].

Theorem 4.3 (Dynamic Programming Principle)**.** *Fix (t,x), then for all $t' \in [0,t]$*

$$u(t,x) := \min \left\{ u(t',\xi(t')) + \int_{t'}^t L(s,\xi(s),\dot{\xi}(s))ds \;\middle|\; \xi(t) = x,\ \xi \in [C^2([t',t])]^n \right\}. \tag{4.4}$$

Moreover if ξ is a minimizer in (4.2) it is a minimizer also for (4.4) for any $t' \in [0,t]$.

Theorem 4.4 (Semiconcavity Theorem). *Suppose* (H1), (H2) *hold and* u_0 *belongs to* $C_b(\mathbb{R}^n)$*. Then for any* t *in* $(0,T]$*,* $u(t,\cdot)$ *is locally semiconcave with semiconcavity constant* $C(t) = \frac{C}{t}$*. Thus for any fixed* $\tau > 0$ *there exists a constant* $C = C(\tau)$ *such that* $u(t,\cdot)$ *is semiconcave with constant less than* C *for any* $t \geq \tau$*.*

Moreover u *is also locally semiconcave in both the variables* (t,x) *in* $(0,T] \times \mathbb{R}^n$*.*

4.1. Study of characteristics. We introduce the definition of generalized backward characteristics.

Definition 4.1. Given $x \in \Omega_t$ for t fixed in $[0,T]$, we call *generalized backward characteristic*, associated to u starting from x, the curve $s \mapsto (s, \xi(s))$, where $\xi(\cdot)$ and its dual arc $p(\cdot)$ solve the system

$$\begin{cases} \dot{\xi}(s) = H_p(s, \xi(s), p(s)) \\ \dot{p}(s) = -H_x(s, \xi(s), p(s)) \end{cases} \tag{4.5}$$

with final conditions

$$\begin{cases} \xi(t) = x \\ p(t) = p, \end{cases} \tag{4.6}$$

where $p \in D_x^+ u(t,x)$.

If $D_x^+ u(t,x)$ is single-valued then we call ξ a *classical backward characteristic.*

It is possible to show that the solutions of the above characteristic equation with final conditions

$$\begin{cases} \xi(t) = x \\ p(t) = p \in K \end{cases} \tag{4.7}$$

are very close to the autonomous case.

Proposition 4.1. *Consider a solution* ξ *to the system (4.5) with final conditions (4.7), let* $y := \xi(\tau)$ *and consider the straight line joining* x *to* y

$$\eta(s) = \frac{s-\tau}{t-\tau} x + \frac{t-s}{t-\tau} y. \tag{4.8}$$

Then we have the following estimates

$$\|\eta - \xi\|_{[C^0([\tau,t])]^n}, \|\eta_p - \xi_p\|_{[C^0([\tau,t])]^{n^2}}, \|\eta_{pp} - \xi_{pp}\|_{[C^0([\tau,t])]^{n^3}} \leq O((t-\tau)^2),$$

$$\|\dot{\eta} - \dot{\xi}\|_{[C^0([\tau,t])]^n}, \|\dot{\eta}_p - \dot{\xi}_p\|_{[C^0([\tau,t])]^{n^2}}, \|\dot{\eta}_{pp} - \dot{\xi}_{pp}\|_{[C^0([\tau,t])]^{n^3}} \leq O(t-\tau).$$

This allows the study of the function

$$\phi(\tau, y, t, x) := \min \left\{ \int_\tau^t L(s, \xi(s), \dot{\xi}(s)) ds \;\middle|\; \xi \in [C^2([\tau,t])]^n, \ \xi(\tau) = y, \ \xi(t) = x, \right\}.$$

Proposition 4.2. *It holds*

$$\left\| \phi(\tau, y(p), t, x) - (t-\tau) L\left(t, x, \frac{x - y(p)}{t-\tau}\right) \right\|_{C^2(K)} \leq O((t-\tau)^2).$$

In particular for $t-\tau$ *small enough* $y \mapsto \phi(\tau,y,t,x)$ *and* $x \mapsto \phi(\tau,y,t,x)$ *are convex with constant* $\frac{\tilde{C}}{t-\tau}$*.*

We will then restrict to a time interval for which the above propositions hold.

4.2. Proof of SBV regularity. We consider a ball $B_R(0) \subset \mathbb{R}^n$ and a bounded convex set $\Omega \subset [\tau, \tau+\varepsilon] \times \mathbb{R}^n$ with the properties that

- $\{s\} \times B_R(0) \subset \Omega$ for every $s \in [\tau, \tau+\varepsilon]$;
- for any $(t,x) \in \Omega$ and for any C^2 curve ξ which minimizes $u(t,x)$ in (4.2), the entire curve $\xi(s)$ for $s \in [\tau, t]$ is contained in Ω.

Indeed, from the fact that $\|Du\|_\infty < \infty$, it is enough to choose

$$\Omega := \big\{(t,x) \in [\tau, \tau+\varepsilon] \times \mathbb{R}^n \big| \ |x| \le R + C'(\tau+\varepsilon-t)\big\}$$

with C' sufficiently large and depending only on $\|Du\|_\infty$ and H.

The general idea of the proof is now standard, see [1, 9]. We construct a monotone bounded functional $F(t)$ defined on the interval $[\tau, \tau+\varepsilon]$. Then, we relate the presence of a Cantor part in the matrix $D^2_x u(t,\cdot)$ for a certain t in $[\tau, \tau+\varepsilon]$ with a jump of the functional F in t. Since this functional can have only a countable number of jumps, the Cantor part of $D^2_x u(t,\cdot)$ can be different from zero only for a countable number of t's.

4.2.1. Decreasing functional. Consider t belonging to $(\tau, \tau+\varepsilon]$ for a fixed $\tau > 0$ and $\varepsilon > 0$ small enough. For any $\tau \le s < t$ we define the set-valued map

$$X_{t,s}(x) := \{\xi(s) |\ \xi(\cdot) \text{ is a solution of (4.5), with } \xi(t) = x,\ p(t) = p \in D^+_x u(t,x)\}.$$

Moreover we will denote by $\chi_{t,s}$ the restriction of $X_{t,s}$ to the points where it is single-valued. The domain of $\chi_{t,s}$, $dom(\chi_{t,s}) =: U_t$, consists of those points where $D^+_x u(t,x)$ is single-valued, i.e. there exists a unique minimizer for $u(t,x)$ in the representation formula (4.2). For that reason $\chi_{t,s}$ is clearly defined a.e. in Ω_t. We will sometimes write $\chi_{t,s}(\Omega_t)$ meaning $\chi_{t,s}(U_t)$.

Define thus the functional

$$F(t) := \mathcal{H}^n(\chi_{t,\tau}(U_t)). \tag{4.9}$$

Lemma 4.1. *The functional F is non increasing,*

$$F(s) \ge F(t) \quad \text{for any } s,t \in (\tau, \tau+\varepsilon] \text{ with } s < t.$$

4.2.2. Area estimates. Under the above assumptions, we can prove the following Lemma, which relates the Laplacian of u with the area of the initial points of characteristics.

Lemma 4.2. *For ε small enough (depending only on the bound M for $\|H_{px}\|$), let $t \in (\tau, \tau+\varepsilon]$ and $A \subset \Omega_t$ be a Borel set. Then*

$$\mathcal{H}^n(X_{t,\tau}(A)) \ge C_1 \mathcal{H}^n(A) - C_2(t-\tau) \int_A d\Delta u(t,\cdot) + O((t-\tau)^2),$$

where C_1, C_2 are positive constants (depending on C, c_H). $\Delta u(t,\cdot)$ is the spatial-Laplacian of $u(t,\cdot)$.

Moreover, as in the scalar case, we have that

Lemma 4.3. *If $\varepsilon > 0$ is small enough, for any $t \in (\tau, \tau+\varepsilon]$, any $\delta \in [0, t-\tau]$ and any Borel set $A \subset \Omega_t$ we have*

$$\mathcal{H}^n(X_{t,\tau+\delta}(A)) \ge \left(\frac{1}{2}\right)^n \left(\frac{t-(\tau+\delta)}{t-\tau}\right)^n \mathcal{H}^n(X_{t,\tau}(A)).$$

One can next prove the following Lemma. In the following we will denote the Cantor part of $D^2_x u(t,\cdot)$ with $D^2_c u(t,\cdot)$.

Lemma 4.4. *For ε small enough, for any t in $(\tau, \tau+\varepsilon]$ such that $|D^2_c u(t,\cdot)|(\Omega_t) > 0$ and δ in $(0, \tau+\varepsilon-t]$, there exists a Borel set $A \subset \Omega_t$ such that*

i) *$\mathcal{H}^n(A) = 0$, $|D^2_c u(t,\cdot)|(A) > 0$ and $|D^2_c u(t,\cdot)|(\Omega_t \setminus A) = 0$;*

ii) *$X_{t,\tau}$ is single-valued on A;*

iii) *and*

$$\chi_{t,\tau}(A) \cap \chi_{t+\delta,\tau}(\Omega_{t+\delta}) = \varnothing.$$

At this point we can prove that the Cantor part appears only countably many times.

For $\varepsilon > 0$ sufficiently small such that Lemmas 4.1, 4.2, 4.3, and 4.4 hold, consider the functional F defined in (4.9) over the interval $[\tau, \tau+\varepsilon]$. F is bounded, and, from Lemma 4.1, F is a monotone function. Thus its points of discontinuity are at most countable.

We will prove that the presence of a Cantor part at a time t is related to a discontinuity of the functional F in t, hence there must be only a countable number of t's in $[\tau, \tau+\varepsilon]$ for which the Cantor part is negative.

Suppose there exists a t in $(\tau, \tau+\varepsilon)$ such that

$$|D^2_c u(t, \Omega_t)| > 0,$$

then for any $\delta > 0$ let A be the set of Lemma 4.4. Using Lemma 4.4-(iii) we get

$$F(t+\delta) \le F(t) - \mathcal{H}^n(X_{t,\tau}(A)) \tag{4.10}$$

To compute $\mathcal{H}^n(X_{t,\tau}(A))$ call $\omega := |D^2_c u(t,\cdot)|(A)$. As we saw in the previous lemma, if we choose $s \in [\tau, t)$ such that $t-s$ is small enough, we have

$$\mathcal{H}^n(X_{t,s}(A)) \ge \frac{C_2}{2}\omega^2.$$

Moreover for Lemma 4.3

$$\mathcal{H}^n(X_{t,\tau}(A)) \ge \left(\frac{1}{2}\right)^n \left(\frac{t-\tau}{t-s}\right)^n \mathcal{H}^n(X_{t,s}(A)).$$

Hence

$$\mathcal{H}^n(X_{t,\tau}(A)) \ge \left(\frac{1}{2}\right)^n \left(\frac{t-\tau}{t-s}\right)^n \frac{C_2}{2}\omega^2 \ge C\omega^2.$$

We can now use this estimate in (4.10) obtaining

$$F(t+\delta) \le F(t) - C\omega^2.$$

Letting $\delta \to 0$

$$\limsup_{\delta\to 0} F(t+\delta) < F(t).$$

Therefore t is a point of discontinuity for F, as we would like to prove.

5. Final remarks on some related cases

The SBV regularity can be proved for other kind of systems or equations. Here we list some interesting cases.

- SBV regularity for fluxes with countably many inflection points [23], or SBV regularity for $v_i(D\lambda_i r_i)$ [12]
- SBV regularity for Temple class systems with source terms

A very interesting open problem is the presence of Cantor part in the measure div$d$, where d is the direction of the optimal ray for the solution

$$u_t + H(\nabla u) = 0,$$

with H only smooth, convex. Some advances have been obtained in [10].

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