RENORMALIZATION FOR AUTONOMOUS NEARLY INCOMPRESSIBLE BV VECTOR FIELDS IN 2D

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Abstract. Given a bounded autonomous vector field $b\colon \mathbb{R}^2 \to \mathbb{R}^2$, we study the uniqueness of bounded solutions to the initial value problem for the related transport equation

$$\partial_t u + b \cdot \nabla u = 0.$$

Assuming that b is of class BV and it is nearly incompressible, we prove uniqueness of weak solutions to the transport equation. The starting point of the present work is the result which has been obtained in [10] (where the *steady nearly incompressible* case is treated). Our proof is based on splitting the equation onto a suitable partition of the plane: this technique was introduced in [3], using the results on the structure of level sets of Lipschitz maps obtained in [1]. Furthermore, in order to construct the partition, we use Ambrosio's superposition principle [4].

Keywords: transport equation, continuity equation, renormalization, disintegration of measures, Lipschitz functions, Superposition Principle.

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1. Introduction

Let Ω denote the torus $\mathbb{T}^d$ or the whole space $\mathbb{R}^d$, $d \in \mathbb{N}$. In this paper we consider the *continuity equation*

$$\partial_t u + \operatorname{div}(ub) = 0 \tag{1.1}$$

and the *transport equation*

$$\partial_t u + b \cdot \nabla u = 0, \tag{1.2}$$

for the unknown $u\colon I \times \Omega \to \mathbb{R}$ (where $I = (0, T)$, $T > 0$) with a given vector field $b\colon I \times \Omega \to \mathbb{R}^d$. We study the initial value problems for these equations with the same initial condition

$$u(0, \cdot) = u_0(\cdot), \tag{1.3}$$

where $u_0\colon \Omega \to \mathbb{R}$ is a given scalar field.

Our aim is to investigate uniqueness of weak solutions to (1.1), (1.3) (and to (1.2), (1.3)) under weak regularity assumptions on the vector field b.

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When $b \in L^\infty(I \times \Omega)$ then (1.1) is understood in the standard sense of distributions: $u \in L^\infty(I \times \Omega)$ is called a *weak solution* of the continuity equation if (1.1) holds in $\mathscr{D}'(I \times \Omega)$. One can prove (see e.g. [15]) that, if u is a weak solution of (1.1), then there exists a map $\widetilde{u} \in L^\infty([0,T] \times \Omega)$ such that $u(t,\cdot) = \widetilde{u}(t,\cdot)$ for a.e. $t \in I$ and $t \mapsto \widetilde{u}(t,\cdot)$ is weakly* continuous from $[0,T]$ into $L^\infty(\Omega)$. This allows us to prescribe an initial condition (1.3) for a weak solution u of the continuity equation in the following sense: we say that $u(0,\cdot) = u_0(\cdot)$ holds if $\widetilde{u}(0,\cdot) = u_0(\cdot)$.

Definition of weak solutions of the transport equation (1.2) is slightly more delicate. If the divergence of b is absolutely continuous with respect to the Lebesgue measure then (1.2) can be written as

$$\partial_t u + \operatorname{div}(ub) - u \operatorname{div} b = 0,$$

and the latter equation can be understood in the sense of distributions (see e.g. [16] for the details). We are interested in the case when $\operatorname{div} b$ is not absolutely continuous. In this case the notion of weak solution of (1.2) can be defined for the class of *nearly incompressible vector fields*.

Definition 1.1. A bounded, locally integrable vector field $b \colon I \times \Omega \to \mathbb{R}^d$ is called *nearly incompressible* if there exists a function $\rho \colon I \times \Omega \to \mathbb{R}$ (called *density* of b) such that $\ln(\rho) \in L^\infty(I \times \Omega)$ and

$$\partial_t \rho + \operatorname{div}(\rho b) = 0 \qquad \text{in } \mathscr{D}'(I \times \Omega). \tag{1.4}$$

Nearly incompressible vector fields were introduced in connection with the hyperbolic conservation laws, namely, the Keyfitz-Kranzer system [19]. See e.g. [15] for the details. Using mollification one can prove that if $\operatorname{div} b \in L^\infty(I \times \Omega)$ then b is nearly incompressible. The converse implication does not hold, so near incompressibility can be considered as a weaker version of the assumption $\operatorname{div} b \in L^\infty(I \times \Omega)$.

Definition 1.2. Let b be a nearly incompressible vector field with density ρ. We say that a function $u \in L^\infty(I \times \Omega)$ is a *(ρ–)weak solution* of (1.2) if

$$(\rho u)_t + \operatorname{div}(\rho u b) = 0 \quad \text{in } \mathscr{D}'(I \times \Omega).$$

Thanks to Definition 1.2 one can prescribe the initial condition for a ρ–weak solution of the transport equation similarly to the case of the continuity equation, which we mentioned above (see [15] for the details).

Existence of weak solutions to initial value problem for transport equation with a nearly incompressible vector field can be proved by a standard regularization argument [15]. The problem of *uniqueness* of weak solutions is much more delicate. The theory of uniqueness in the non-smooth framework has started with the seminal paper of R.J. DiPerna and P.-L. Lions [16] where uniqueness was obtained as a corollary of so-called *renormalization property* for the vector fields with Sobolev regularity. Thanks to Definition 1.2 the renormalization property can be defined also for nearly incompressible vector fields:

Definition 1.3. We say that a nearly incompressible vector field b with density ρ has the *renormalization property* if for every ρ–weak solution

$u \in L^\infty(I \times \Omega)$ of (1.2) and for any function $\beta \in C^1(\mathbb{R})$ the function $\beta(u)$ also is a ρ-weak solution of (1.2), i.e. it satisfies

$$\partial_t (\rho\beta(u)) + \operatorname{div} (\rho\beta(u)b) = 0 \quad \text{in } \mathscr{D}'(I \times \Omega).$$

Nearly incompressible vector fields are related to a conjecture, made by A. Bressan in [12]:

Conjecture 1.4 (Bressan's compactness conjecture)**.** *Let $\Omega = \mathbb{R}^d$ and let $b_n \colon \mathbb{R} \times \Omega \to \mathbb{R}^d$, $n \in \mathbb{N}$, be a sequence of smooth vector fields. Denote by Φ_n the solutions of the ODEs*

$$\begin{aligned} \frac{d}{dt}\Phi_n(t,x) &= b_n(t, \Phi_n(t,x)), \\ \Phi_n(0,x) &= x. \end{aligned}$$

Assume that $\|b_n\|_\infty + \|\nabla_{t,x} b_n\|_{L^1}$ is uniformly bounded and there exists a constant $C > 0$ such that

$$C^{-1} \leq \det(\nabla_x \Phi_n(t,x)) \leq C$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ and all $n \in \mathbb{N}$. Then the sequence Φ_n is strongly precompact in L^1_{loc}.

It has been proved in [5] that Bressan's conjecture would follow from the next one:

Conjecture 1.5 (Renormalization conjecture)**.** *Any bounded, nearly incompressible vector field $b \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R} \times \Omega)$ has the renormalization property (in the sense of Definition 1.3).*

The renormalization property can also be generalized for the systems of transport equations. Moreover, if η is another density of the nearly incompressible vector field b and b has the renormalization property with the density ρ, then any ρ–weak solution of (1.2) is also an η–weak solution and vice versa. In other words, the property of being a ρ–weak solution does not depend on the choice of the density ρ provided that renormalization holds. We refer to [15] for the details.

The problem of renormalization is also related to the chain rule problem (see also [6, 10]). In particular, if the functions ρ, u and b were smooth, renormalization property would be an easy corollary of the chain rule. Out of the smooth setting, the validity of this property is a key step to get uniqueness of weak solutions. Indeed, if we for simplicity consider $\Omega = \mathbb{T}^d$, then, integrating the equation above over the torus, we get

$$\partial_t \int_{\mathbb{T}^d} \rho\beta(u)\, dx = 0.$$

So if $u_0 = 0$ then for $\beta(y) = y^2$ we get

$$\int_{\mathbb{T}^d} \rho(t,x) u^2(t,x)\, dx = 0$$

for a.e. t which implies $u(t,\cdot) = 0$ for a.e. t.

Thus, uniqueness of weak solutions can be derived from the renormalization property for b. In [16] the authors proved that renormalization property holds under Sobolev regularity assumptions; some years later, L. Ambrosio

[8] improved this result, showing that renormalization holds for vector fields which are of class BV (locally in space) and have absolutely continuous divergence.
Another approach giving explicit compactness estimates has been introduced in [14], and further developed in [11, 18]: see also the references therein.

In the two dimensional autonomous case ($\Omega = \mathbb{R}^2$, b does not depend on time t) the problem of uniqueness is addressed in the papers [3], [1] and [10]. Indeed, in two dimensions and for divergence-free autonomous vector fields, renormalization theorems are available even under mild assumptions, because of the underlying Hamiltonian structure. In [3], the authors characterize the autonomous, divergence-free vector fields b on the plane such that the Cauchy problem for the continuity equation (1.1) admits a unique bounded weak solution for every bounded initial datum (1.3). The characterization they present relies on the so called *weak Sard property*, which is a (weaker) measure theoretic version of Sard's Lemma. Since the problem admits a Hamiltonian potential, uniqueness is proved following a strategy based on splitting the equation on the level sets of this function, reducing thus to a one-dimensional problem. This approach requires a preliminary study on the structure of level sets of Lipschitz maps defined on $\mathbb{R}^2$, which is carried out in the paper [1].

In [10] the *steady nearly incompressible* autonomous vector fields on $\Omega = \mathbb{R}^2$ were considered. Namely, an autonomous vector field $b\colon \mathbb{R}^2 \to \mathbb{R}^2$ is called steady nearly incompressible if it admits a steady density $\widetilde{\rho}$, i.e. there exists a function $\widetilde{\rho}$, uniformly bounded from below and above by some strictly positive constants, such that $\operatorname{div}(\widetilde{\rho} b) = 0$. It was proved in [10] that any steady nearly incompressible BV vector field on $\mathbb{R}^2$ has the renormalization property. In the present paper we extend this result to the non-steady case.

Any steady nearly incompressible vector field is nearly incompressible, but the inverse implication does not hold in general. For instance, consider a vector field $b\colon (0,2) \to \mathbb{R}$ given by $b(x) = |x-1| - 1$. If it was steady nearly incompressible, the function $\widetilde{\rho} \cdot b$ would be constant on $(0,2)$ and thus $\widetilde{\rho}$ could not be uniformly bounded from above by a positive constant. On the other hand this vector field b is nearly incompressible: the solution to the continuity equation $\partial_t \rho + \partial_x(\rho b) = 0$ with the initial condition $\rho|_{t=0} = 1$ satisfies $e^{-t} \leq \rho(t,x) \leq e^t$, as one can easily demostrate using the classical method of characteristics, since b is Lipschitz. This simple example can be generalized to higher dimensions.

The main result of this paper is a partial answer to the Conjecture 1.5:

Main Theorem. *Suppose that $\Omega = \mathbb{T}^2$ and $b\colon \Omega \to \mathbb{R}^2$ is a nearly incompressible* BV *vector field (with density ρ). Then*

(1) $\forall u_0 \in L^\infty(\Omega)$ there exists a unique (ρ-)weak solution $u \in L^\infty(I \times \Omega)$ to the transport equation (1.2) with the initial condition $u|_{t=0} = u_0$.
(2) $\forall u_0 \in L^\infty(\Omega)$ there exists a unique weak solution $u \in L^\infty(I \times \Omega)$ to the continuity equation (1.1) with the initial condition $u|_{t=0} = u_0$.
(3) the field b has the renormalization property.

We have formulated (and will prove) our result for $\Omega = \mathbb{T}^2$ for simplicity only. Minor adjustments of the proofs (in particular, taking into account the curves with endpoints at infinity in Lemma **??**, restricting to the curves starting from bounded sets in Lemma 6.5 in order to make the integrals finite, using more carefully that Radon measures are only *locally* finite on $\mathbb{R}^2$ etc.) allow to prove an analogous result for $\Omega = \mathbb{R}^2$.

We also show that if $b \neq 0$ a.e. then Main Theorem holds without assuming BV regularity of b.

However, our approach heavily relies on the assumption that $d = 2$ and b is *autonomous* (like the previous studies in [10]). In particular, we use representation of bounded divergence-free vector fields v in the form $v = \nabla^\perp H$ where H is a Lipschitz function. In this case the images of the integral curves of b can be locally represented by level sets of H and we use the properties such level sets, which are strictly two-dimensional (in higher dimensions, for example, the level sets $H^{-1}(y)$ can contain *triods* for an open set of values y; we refer to [1] for further details).

1.1. **Structure of the paper.** In Section 2 we collect the preliminary results, including Disintegration Theorem, Coarea Formula and the results from [1] concerning the structure of the level sets of Lipschitz functions on the plane. In particular, we introduce the notion of locally regular level set of a Lipschitz function $H: U \to \mathbb{R}$, where U is an open subset of $\mathbb{T}^2$ (or $\mathbb{R}^2$). Roughly saying, given $h \in \mathbb{R}$ the level set $H^{-1}(h)$ is *locally regular* if it is an (at most) countable union of disjoint simple Lipschitz curves.

In Section 2 we also recall Ambrosio's Superposition Principle. By this Principle the measure $\rho(t,\cdot)\mathscr{L}^2$ (where ρ is a nonnegative bounded solution of the continuity equation (1.4)) can be represented as an *image* of some measure η on the space of curves $C([0,T];\mathbb{T}^2)$ (which is concentrated on the solutions of the ODE $\gamma' = b(\gamma)$) under the *evaluation map* $e_t: \gamma \mapsto \gamma(t)$:

$$\rho(t,\cdot)\mathscr{L}^2 = e_{t\#}\eta.$$

In Section 3 study the *matching properties* of the level sets of two Lipschitz functions $H_1, H_2: U \to \mathbb{R}$. We prove that if $\nabla H_1 \parallel \nabla H_2$ a.e. then for a.e. x the connected components of the locally regular level sets of H_1 and H_2, containing x, coincide.

The proof of the Main Theorem is based on the characterization of the *divergence equation*

$$\operatorname{div}(ub) = \mu, \qquad u: \mathbb{T}^2 \to \mathbb{R}, \tag{1.5}$$

provided in Lemmas **??** and **??**. (Here μ is a Radon measure on $\mathbb{T}^2$.) Roughly saying, we prove that (1.5) is equivalent, to a family of equations along the images of the integral curves of b.

Such characterization is similar to the one obtained in [10] for *steady* nearly incompressible vector fields. In the latter case, since $\operatorname{div}(\rho b) = 0$, there exists a Lipschitz *Hamiltonian* $H: \mathbb{T}^2 \to \mathbb{R}$ such that

$$\rho b = \nabla^\perp H,$$

where $\nabla^\perp = (-\partial_2, \partial_1)$. This allows one to split (1.5) into an equivalent family of equations along the level sets of H (which are, essentially, the images of the integral curves of b).

In the general nearly incompressible case we are not able to construct a *global* Hamiltonian H directly as in the case of steady density. However, for any simply connected bounded open set $U \subset \mathbb{T}^2$ in Section 7.1 we construct a *local* Hamiltonian H_U. Let us present an outline of this construction.

Let η be a measure on $C([0,T];\mathbb{T}^2)$ given by Superposition Principle for the density ρ of the field b. In order to define H_U, we first restrict η to the set T_U of curves which stay in U for a positive amount of time and have the endpoints outside U. This provides us with a bounded nonnegative solution ρ_U of the continuity equation

$$\partial_t \rho_U + \operatorname{div}(\rho_U b) = 0 \quad \text{in } \mathscr{D}'((0,T)\times\mathbb{T}^2),$$

such that $\rho(0,x) = \rho(T,x) = 0$ for a.e. $x \in U$. Hence, integrating the continuity equation above over time t, we infer that the function

$$r_U(x) := \int_0^T \rho_U(t,x)\,dt$$

solves

$$\operatorname{div}(r_U b) = 0$$

in $\mathscr{D}'(U)$. Therefore in U one can construct a *local* Hamiltonian H_U such that

$$r_U b = \nabla^\perp H_U$$

a.e. in U.

Given the function r_U and the local Hamiltonian H_U, in Section 4 we characterize the divergence equation (4.1) in terms of the level sets of H_U. The results from [10] cannot be applied directly since r_U can vanish on a set of positive measure, in contrast to the steady nearly incompressible case. Therefore in Section 4 we adopt the methods from [3], [1] and [10] in order to treat this particular case.

A single local Hamiltonian may be insufficient to provide a *useful* characterization of the divergence equation, since r_U may vanish on a large part of U, or even the whole U. Therefore, in Section **??** we introduce a *countable family* of local Hamiltonians H_B, $B \in \mathscr{B}$, where $\mathscr{B}$ is the family of open balls with rational radii and centers having rational coordinates.

Using the matching properties, presented in Section 3, in Section **??** we "glue together" the regular level sets of H_B, $B \in \mathscr{B}$. We construct the *labelling function* f, such that the resulting curves are the level sets of f.

In Section **??** we "join" the "local" characterizations of the divergence equation, obtained for each ball $B \in \mathscr{B}$ using the results of Section 4. And ultimately, in Section **??** we reduce the continuity equation to the divergence equation.

The characterization of the divergence equation, obtained in Section **??**, is based on the fact that the endpoints of the level sets of f cannot be reached in finite time by the integral curves of b (see Lemma **??**). This property holds provided that $b \neq 0$ a.e. or $b \in \mathrm{BV}$. Indeed, in both cases the integral curves of b cannot "stick" to the set $\{b = 0\}$, i.e. cannot stay there for a positive amount of time (see Lemma 6.5).

In order to show that the set of "sticky" curves is η-negligible, in Section 5 we establish the locality of the divergence operator: if $\operatorname{div}(ub) = \mu$ for some measure μ, then μ vanishes on the set of points x such that $b(x) = 0$, x is a Lebesgue point of b and b is approximately differentiable in x.

Contents

1.2. **Notation.**

Throughout the paper, we use the following notation:

- (X, d) is a metric space;
- $\mathbb{1}_E$ is the characteristic function of the set $E \subset X$, defined as $\mathbb{1}_E(x) = 1$ if $x \in E$ and $\mathbb{1}_E(x) = 0$ otherwise;
- $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the two dimensional torus;
- Ω denotes in general a simply connected open set in $\mathbb{R}^2$;
- $\operatorname{dist}(x, E)$ is the distance of x from the set E, defined as the infimum of $d(x, y)$ as y varies in E;
- $B(x, r)$ or, equivalently, $B_r(x)$ is the open ball in $\mathbb{R}^d$ with radius r and centre x; $B(r)$ is the open ball in $\mathbb{R}^d$ with radius r and centre 0;
- $\fint_E f\, d\mu$ denotes the *average* of the function f over the set E with respect to the positive measure μ, that is

$$\fint_E f\, d\mu := \frac{1}{\mu(E)} \int_E f\, d\mu,$$

- $\mu \llcorner A$ denotes the restriction of a measure μ on a set A.
- $|\mu|$ is the total variation of a measure μ;
- μ^{sing} the singular component of μ with respect to the Lebesgue measure;

- $\mathscr{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$ and $\mathscr{H}^k$ is the k-dimensional Hausdorff measure;
- $\mathrm{Lip}(X)$ is the space of real-valued Lipschitz functions; $\mathrm{Lip}_c(X)$ is the space of real-valued compactly supported Lipschitz functions;
- $C_c^\infty(\Omega)$ is the space of smooth compactly supported functions, also called *test functions*;
- $\mathrm{BV}(\Omega)$ set of functions with bounded variation;
- $\mathscr{D}'(\Omega)$ is the space of distributions on the open set Ω;
- $\Gamma := C([0,T];\mathbb{T}^2)$ will denote the set of continuous curves in $\mathbb{T}^2$;
- $\dot\Gamma := \{\gamma \in \Gamma : \gamma(t) = \gamma(0),\ \forall t \in [0,T]\}$ denotes the set of constant curves (whose graphs are fixed points);
- $\widetilde\Gamma := \Gamma \setminus \dot\Gamma$ denotes the set of non-constant curves (whose graphs have positive length);
- $e_t \colon \Gamma \to \mathbb{T}^2$ is the *evaluation map* at time t, i.e. $e_t(\gamma) = \gamma(t)$.

Moreover, if $A \subset \mathbb{T}^2$ is a measurable set,

- $\Gamma_A := \{\gamma \in \Gamma : \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\}) > 0\}$ denotes the set of curves which stay in A for a positive amount of time;
- $\widetilde\Gamma_A := \widetilde\Gamma \cap \Gamma_A$ denotes the set of non-constant curves which stay in A for a positive amount of time;
- $\dot\Gamma_A := \dot\Gamma \cap \Gamma_A$ denotes the set of constant curves which stay in A for a positive amount of time.
- for every $s \in [0,T]$, we denote by

$$\begin{aligned}\Gamma_A^s &:= \{\gamma \in \Gamma : \gamma(s) \in A\},\\ \widetilde\Gamma_A^s &:= \left\{\gamma \in \widetilde\Gamma : \gamma(s) \in A\right\},\\ \dot\Gamma_A^s &:= \left\{\gamma \in \dot\Gamma :\ \gamma(s) \in A\right\}\end{aligned}$$

  accordingly the sets of all curves, non-constant curves and constant curves, which at time s belong to A;
- $\mathsf{T}_A := \{\gamma \in \Gamma_A :\ \gamma(0) \notin A, \gamma(T) \notin A\}$ denotes the set of curves which stay in A for a positive amount of time and have the endpoints outside A.

If $A \subseteq \mathbb{R}^2$ (or in $\mathbb{T}^2$), we denote by

$$\begin{aligned}\mathrm{Conn}(A) &:= \Big\{C \subset A : C \text{ is a connected component of } A\Big\},\\ \mathrm{Conn}^\star(A) &:= \Big\{C \in \mathrm{Conn}(A) : \mathscr{H}^1(C) > 0\Big\},\end{aligned}$$

and

$$A^\star := \bigcup_{C \in \mathrm{Conn}^\star(A)} C.$$

Given $x \in \mathbb{R}^2$ (or in $\mathbb{T}^2$) we denote by

$$\mathrm{conn}_x(A)$$

the connected component of A which contains x.

When the measure is not specified, it is assumed to be the Lebesgue measure, and we often write

$$\int f(x)\,dx$$

for the integral of f with respect to $\mathscr{L}^d$.

Let μ be a Radon measure on a metric space X. Let Y be a metric space and let $f\colon X \to Y$ be a Borel function. We denote by $f_\# \mu$ the *image measure* of μ under the map f. In particular, for any $\varphi \in C_c(Y)$ we have

$$\int_X \varphi(f(x))\,d\mu(x) = \int_Y \varphi(y)\,d(f_\# \mu)(y).$$

2. Preliminaries

2.1. **Disintegration of a measure.** The following well-known result can be considered as a curvilinear analog of Fubini's theorem (see Theorem 2.28 in [7] or for the most general statement Section 452 of [17]):

Theorem 2.1 (Disintegration)**.** *Let μ be a Radon measure on a metric space X. Let Y be a metric space and let $f\colon X \to Y$ be a Borel function. Let ν be a Radon measure on Y such that $f_\# |\mu| \ll \nu$. Then there exists a unique measurable family of Radon measures $\{\mu_y\}_{y\in Y}$ such that for ν-a.e. $y \in Y$ the measure μ_y is concentrated on the level set $f^{-1}(y)$ and*

$$\mu = \int_y \mu_y \,d\nu(y),$$

that is, for any $\phi \in C_c(X)$

$$\int_X \varphi(x)\,d\mu(x) = \int_Y \left(\int_X \varphi(x)\,d\mu_y(x) \right) d\nu(y).$$

Definition 2.2. The family $\{\mu_y\}_{y\in Y}$ given by Theorem 2.1 is called the *disintegration of μ with respect to f (and ν).*

2.2. **Coarea formula.** Suppose that H is a real function on $\mathbb{R}^2$ (or $\mathbb{T}^2$). A corollary of the coarea formula (see e.g. [7] for the general statement) provides an additional information on the structure of the disintegration of $|\nabla H| \mathscr{L}^2$ with respect to H:

Lemma 2.3. *Let $\{\varpi_h\}_{h\in\mathbb{R}}$ denote the disintegration of the measure $|\nabla H|\mathscr{L}^2$ with respect to H. Then for a.e. $h \in \mathbb{R}$ we have $\mathscr{H}^1(E_h) < \infty$ and $\varpi_h = \mathscr{H}^1 \llcorner E_h$, i.e. the disintegration of $|\nabla H|\mathscr{L}^2$ with respect to H is given by*

$$|\nabla H|\mathscr{L}^2 = \int_{\mathbb{R}} \mathscr{H}^1 \llcorner E_h \,dh.$$

2.3. **Structure of level sets of Lipschitz functions.** Let U be a bounded, open set in $\mathbb{T}^2$ (or in $\mathbb{R}^2$) and let $f\colon U \to \mathbb{R}$ be a Lipschitz function. For any $r \in \mathbb{R}$, we denote by $E_r := f^{-1}(r)$ the corresponding level set.

Theorem 2.4 ([1, Thm. 2.5], or [10, Thm. 5.4])**.** *Suppose that $f\colon U \to \mathbb{R}$ is a Lipschitz function. For any $r \in \mathbb{R}$, let $E_r := f^{-1}(r)$. Then the following statements hold for $\mathscr{L}^1$-a.e. $r \in f(U)$:*

(1) *$\mathscr{H}^1(E_r) < \infty$ and E_r is countable $\mathscr{H}^1$-rectifiable;*

(2) *for* $\mathscr{H}^1$*-a.e.* $x \in E_r$ *the function* f *is differentiable at* x *with* $\nabla f(x) \neq 0$*;*
(3) $\mathrm{Conn}^\star(E_r)$ *is countable and every* $C \in \mathrm{Conn}^\star(E_r)$ *is a simple (possibly closed) curve;*
(4) $\mathscr{H}^1(E_r \setminus E_r^\star) = 0$.

Definition 2.5. We will say that the level set E_r is *locally regular* if it satisfies conditions (1)-(2)-(3)-(4) (or it is empty).

In these terms, the theorem above states that for a.e. $r \in \mathbb{R}$ the level set E_r is locally regular.

Let $C \in \mathrm{Conn}^\star(E_r)$ be a connected component of some regular level set E_r. Let $\gamma \colon I \to \mathbb{T}^2$ be an injective Lipschitz parametrization of C, where $I = \mathbb{R}/\ell\mathbb{Z}$ or $I = (0, \ell)$ for some $\ell > 0$ is the *domain* of γ. In view of Remark 4.1) we can assume that the directions of b and $\nabla^\perp H$ agree $\mathscr{H}^1$-a.e. on C. So there exists a constant $\varpi \in \{+1, -1\}$ such that

$$\frac{b(\gamma(s))}{|b(\gamma(s))|} = \varpi \frac{\gamma'(s)}{|\gamma'(s)|} \tag{2.1}$$

for a.e. $s \in I$.

Definition 2.6. We will say that γ is an *admissible parametrization* of C if $\varpi = +1$. In the rest of the text we will consider only admissible parametrizations of the connected components C.

2.4. **Canonical disintegration of the Lebesgue measure with respect to a Lipschitz function.** Let U be a bounded open set in $\mathbb{R}^2$ (or $\mathbb{T}^2$). Suppose that $H \colon U \to \mathbb{R}$ is a Lipschitz function By Theorem 2.4 there exists a negligible set N_1 such that the level set $E_h := H^{-1}(h)$ is locally regular whenever $h \notin N_1$. Moreover, let N_2 denote the negligible set on which the measure $(H_\# \mathscr{L}^2)^{\mathrm{sing}}$ is concentrated, where $(H_\# \mathscr{L}^2)^{\mathrm{sing}}$ is the singular part of $H_\# \mathscr{L}^2$ with respect to $\mathscr{L}^1$. Then we set

$$N := N_1 \cup N_2 \qquad \text{and} \qquad E^\star := \cup_{h \notin N} E_h^\star \tag{2.2}$$

Note that the set $E^\star$ is measurable (see Appendix of [1] or [10]). Informally, $E^\star$ is the set of all "good parts" of the locally regular level sets of H.

Let S be the *critical set* of H, defined as the set of all $x \in U$ where H is not differentiable or $\nabla H(x) = 0$. In view of Rademacher's theorem we have $S = \{\nabla H = 0\} \mod \mathscr{L}^2$.

Let us decompose U into a disjoint union as follows:

$$U = (U \cap E^\star) \cup (U \setminus E^\star) = (U \cap E^\star \cap S) \cup (U \cap E^\star \setminus S) \cup (U \setminus E^\star)$$

By Lemma 2.3 we have $\nabla H = 0$ a.e. on $U \setminus E^\star$ and

$$\mathscr{L}^2 \llcorner (U \cap E^\star \setminus S) = \int c_h \mathscr{H}^2 \llcorner E_h \, dh$$

where $c_h := 1/|\nabla H| \in L^1(\mathscr{H}^1 \llcorner E_h)$.

By definition of the set $E^\star$ we have $H_\#(\mathscr{L}^2 \llcorner (U \cap E^\star \cap S)) \ll \mathscr{L}^1$ hence by Disintegration theorem

$$\mathscr{L}^2 \llcorner (U \cap E^\star \cap S) = \int \sigma_h \, dh,$$

where $\{\sigma_h\}_{h\in\mathbb{R}}$ is the disintegration of $\mathscr{L}^2 \llcorner (U \cap E^\star \cap S)$ with respect to H and $\mathscr{L}^1$.

Finally let

$$\zeta := H_\# \left(\mathscr{L}^2 \llcorner (U \setminus E^\star) \right). \tag{2.3}$$

By definition of the set N the measure ζ is concentrated on N hence by Disintegration theorem

$$\mathscr{L}^2 \llcorner (U \setminus E^\star) = \int \kappa_h \, d\zeta(h),$$

where $\{\kappa_h\}_{h\in\mathbb{R}}$ is the disintegration of $\mathscr{L}^2 \llcorner (U \setminus E^\star)$ with respect to H and ζ.

Bringing together the results above, we obtain

$$\mathscr{L}^2 \llcorner U = \int \left(c_h \mathscr{H}^1 \llcorner E_h + \sigma_h \right) dh + \int \kappa_h \, d\zeta(h), \tag{2.4}$$

Definition 2.7. The expression on the right-hand side of (2.4) is called the *canonical disintegration* of the Lebesgue measure $\mathscr{L}^2 \llcorner U$ *with respect to* H.

Let us summarize the properties of the canonical disintegration of the Lebesgue measure:

Lemma 2.8 ([3, Lemma 2.8])**.** *Suppose that the canonical disintegration of the Lebesgue measure is given by* (2.4)*. Then*

(1) $c_h \in L^1(\mathscr{H}^1 \llcorner E^\star_h)$, $c_h > 0$ *a.e.;*
(2) $c_h = 1/|\nabla H|$ *a.e. (w.r.t.* $\mathscr{H}^1 \llcorner E^\star_h$*);*
(3) σ_h *is concentrated on* $E^\star_h \cap \{\nabla H = 0\}$*;*
(4) κ_h *is concentrated on* $E^\star_h \cap \{\nabla H = 0\}$*;*
(5) ζ *is concentrated on* N *(hence* $\zeta \perp \mathscr{L}^1$*).*
(6) $\sigma_h \perp \mathscr{H}^1 \llcorner E_h$*.*

The proof follows directly from Definition 2.7 and Theorem 2.4.

Finally, let us mention a covering property of the set $E^\star$:

Lemma 2.9. *Let* $E^\star$ *be the set defined in* (2.2)*. Then*

$$E^\star \supset \{\nabla H \neq 0\} \mod \mathscr{L}^2.$$

Proof. Suppose that $P := \{\nabla H \neq 0\} \setminus E$ has positive measure. Then

$$0 < \int_P |\nabla H| \, dx = \int \int \mathbb{1}_P \, d\mathscr{H}^1 \llcorner E_h \, dh = 0$$

where the first equality is due to Coarea formula (Lemma 2.3) and the second equality holds since $\mathbb{1}_P$ is zero on E_h for a.e. h. □

Note that in general $E^\star$ can contain a subset of $\{\nabla H = 0\}$ with positive measure (see [1]). However, in the next section we show that, if H has the so-called *weak Sard property*, then in fact $E^\star = \{\nabla H \neq 0\} \mod \mathscr{L}^2$.

2.5. **The weak Sard property.** Let $H\colon U \to \mathbb{R}$ be a Lipschitz function and let S be the critical set of H, defined as in Section 2.4. We are interested in the following property:

the push-forward according to f of the restriction of $\mathscr{L}^2$ to S is singular with respect to $\mathscr{L}^1$, that is

$$H_\# \left(\mathscr{L}^2 \llcorner S\right) \perp \mathscr{L}^1.$$

This property clearly implies the following *weak Sard property*, which is used in [3, Section 2.13]:

$$H_\# \left(\mathscr{L}^2 \llcorner (S \cap E^\star)\right) \perp \mathscr{L}^1,$$

where the set $E^\star$ is the union of all connected components with positive length of all level sets of H (see (2.2)). We point out that the relevance of the weak Sard property in the framework of transport and continuity equation is explained is [3, Theorem 4.7].

Informally, the weak Sard property means that the "good" level sets of H do not intersect the critical set S, apart from a negligible set. In terms of the canonical disintegration of the Lebesgue measure, the definition of the weak Sard property can be reformulated as follows:

Lemma 2.10. *Suppose that the canonical disintegration of the Lebesgue measure is given by (2.4). Then H has the weak Sard property if and only if $\sigma_h = 0$ for a.e. h.*

We refer to [3] for an example of Lipschitz function such that σ_h is a Dirac delta on a non-negligible set of h. (In particular, this function does not posess the weak Sard property.)

We conclude this section with the following corollary of Lemma 2.10 concerning the covering properties of the set $E^\star$ defined in (2.2):

Lemma 2.11. *Suppose that H has the weak Sard property. Let $E^\star$ be the set defined in (2.2). Then*

$$E^\star = \{\nabla H \neq 0\} \mod \mathscr{L}^2.$$

Proof. The argument is similar to Lemma 2.9. Let $Q = E^\star \setminus \{\nabla H \neq 0\}$. By (2.4)

$$\mathscr{L}^2(Q) = \int \left(\int_Q d\sigma_h \right) dh = 0$$

since by Lemma 2.10 $\sigma_h = 0$ for a.e. h. □

2.6. **Ambrosio's Superposition Principle.** Consider the continuity equation in the form

$$\begin{cases} \partial_t \mu_t + \operatorname{div}(b\mu_t) = 0, \\ \mu_0 = \overline{\mu}, \end{cases} \tag{2.5}$$

where $[0, T] \ni t \mapsto \mu_t$ is a measure valued function and $b\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded, Borel vector field. A solution to (2.5) has to be understood in distributional sense.

In [8], L. Ambrosio proved the *Superposition Principle*. According to this principle, any nonnegative measure-valued solution to the continuity equation in fact is transported by a “probabilistic” flow of the ODE associated with b:

Theorem 2.12 (Superposition Principle)**.** *Let $b\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be a bounded, Borel vector field and let $[0,T]\ni t\mapsto \mu_t$ be a nonnegative, locally finite, measure-valued solution of the continuity equation (2.5). Then there exists a measurable family of probability measures $\{\eta_x\}_{x\in\mathbb{R}^d}$ on Γ such that*

$$\mu_t = \int e_{t\#}\eta_x \, d\bar{\mu}(x),$$

for any $t\in(0,T)$ and $(e_0)_\#\,\eta_x = \delta_x$. Moreover, η_x is concentrated on absolutely continuous integral solutions γ of the ODE $\gamma' = b(\gamma)$ starting from $\gamma(0)=x$, for $\overline{\mu}$-a.e. $x\in\mathbb{R}^d$.

In other words, any nonnegative measure-valued solution μ_t of the continuity equation (2.5) can be represented as

$$\mu_t = e_{t\#}\eta, \tag{2.6}$$

where η is some nonnegative measure on the space of continuous curves Γ, which is concentrated on the integral curves of the vector field b. In terms of Theorem 2.12 this measure η can be defined by

$$\eta = \int_{\mathbb{R}^d} \eta_x \, d\bar{\mu}(x).$$

(I.e. the family $\{\eta_x\}_{x\in\mathbb{R}^d}$ is the disintegration of η under the map e_0.)

Theorem 2.12 also holds if we replace $\mathbb{R}^d$ with the torus $\mathbb{T}^2$, since any vector field on $\mathbb{T}^2$ can be considered as a periodic vector field on $\mathbb{R}^2$.

3. Matching properties of Lipshitz functions

Suppose that in a bounded simply connected open set U in $\mathbb{R}^2$ (or in $\mathbb{T}^2$) we have two Lipshitz functions: $H_1\colon U\to\mathbb{R}$ and $H_2\colon U\to\mathbb{R}$. In this section we study the *matching properties* of the locally regular level sets of H_1 and H_2. Let $E_i^\star$ denote the set defined in (2.2) for H_i, $i=1,2$. Let $C_{i,x}$ denote the connected component of the level set $H_i^{-1}(H_i(x))$, containing x.

Definition 3.1. A point $x\in U$ is called a *matching point* of H_1 and H_2 in U, if the following property holds:

$$\text{if } x\in E_i^\star \text{ then } H_j \text{ is constant on } C_{i,x},\ i\neq j.$$

Remark 3.2. If $x\in E_1^\star\cap E_2^\star$ is a matching point, then $C_{1,x}=C_{2,x}$. Indeed, since H_j is constant on $C_{i,x}$ and $C_{i,x}$ is connected, it holds that $C_{j,x}\supset C_{i,x}$. Interchanging i and j we get $C_{i,x}=C_{j,x}$.

As usual, given two vectors a and b in $\mathbb{R}^2$ we write $a\parallel b$ if $a=\alpha b$ or $b=\alpha a$ for some real number α. The following lemma provides a sufficient condition for H_1 and H_2 to match in a.e. point of U:

Lemma 3.3 (Matching)**.** *Let H_1, H_2 be defined as above. If $\nabla H_1\parallel\nabla H_2$ a.e. on U, then there exists an $\mathscr{L}^2$-negligible set $X\subset U$ such that any $x\in U\setminus X$ is a matching point of H_1 and H_2 in U.*

Proof. Let X_i denote the set of $x \in U$ where H_i is not differentiable and let
$$X := \{x \in U : \nabla H_1(x) \nparallel \nabla H_2(x)\} \cup X_1 \cup X_2.$$

By disintegration (2.4) with respect to H_i we have
$$0 = \mathscr{L}^2(E_i^\star \cap X) = \int \left(\int_{E_{i,h} \cap X} c_{i,h} \, d\mathscr{H}^1 + \int_{E_{i,h} \cap X} d\sigma_{i,h} \right) dh,$$
where $E_{i,h} := H_i^{-1}(h)$. Hence there exists an $\mathscr{L}^1$-negligible set N_i such that for any $h \in H(E_i^\star) \setminus N_i$ it holds that
$$\mathscr{H}^1(E_{i,h} \cap X) = 0. \tag{3.1}$$

For any admissible parametrization $\gamma \colon I \to U$ of any nontrivial connected component C of $E_{i,h}$ we have $\mathscr{H}^1 \llcorner C = \gamma_\#(|\gamma'| \mathscr{L}^1)$. Then from (3.1) it follows that
$$\gamma' = 0 \text{ a.e. on } \gamma^{-1}(X).$$
Hence for a.e. $s \in I$ the map $s \mapsto H_j(\gamma(s))$ is differentiable and, moreover,
$$\partial_s H_j(\gamma(s)) = (\nabla H_j)(\gamma(s)) \cdot \gamma'(s) = 0,$$
since $\gamma'(s) \parallel (\nabla^\perp H_i)(\gamma(s)) \perp (\nabla H_j)(\gamma(s))$ for a.e. $s \in I$. Hence the map $s \mapsto H_j(\gamma(s))$ is constant on I. □

4. Characterisation of the divergence equation for steady nearly incompressible vector fields with vanishing density

Consider a bounded simply connected open set U in $\mathbb{R}^2$ (or $\mathbb{T}^2$). Suppose that $b \colon U \to \mathbb{R}^2$ is a bounded vector field on U. Let $u \colon U \to \mathbb{R}$ be a bounded function and suppose that μ is a Radon measure on U. In this section we study the *divergence equation*
$$\operatorname{div}(ub) = \mu \tag{4.1}$$
in $\mathscr{D}'(U)$ under assumption that there exists a bounded non-negative function $r \colon U \to \mathbb{R}$ such that
$$\operatorname{div}(rb) = 0 \tag{4.2}$$
in $\mathscr{D}'(U)$; since U is simply connected, there exists a Lipschitz *Hamiltonian* $H \colon U \to \mathbb{R}$ such that
$$\nabla^\perp H(x) = r(x) b(x), \qquad \text{for } \mathscr{L}^2\text{-a.e. } x \in U. \tag{4.3}$$

Remark 4.1. Thanks to (2.4) we always can add to the set N (defined in (2.2)) if necessary, an $\mathscr{L}^1$-negligible set so that for any $h \notin N$ for $\mathscr{H}^1$-a.e. $x \in E_h^\star$ we have $r(x) > 0$, $b(x) \neq 0$ and $r(x)b(x) = \nabla^\perp H(x)$.

Remark 4.2. We refer to H as the *local Hamiltonian*, since the function r may vanish on a set with positive measure. Therefore different choices of r provide, in general, different local Hamiltonians H and consequently different information on the equation (4.1).

The "global" case when $r \neq 0$ a.e. in U was treated in [10]. Therefore the goal of this section is to generalize the techniques from [10] in order to treat the "local" case, i.e. to allow r to vanish on a set with positive measure.

The main result of this section is characterisation of the equation (4.1) in terms of the level sets of the Hamiltonian H. For the convenience of the reader we present this characterisation in several lemmas.

First, we reduce the divergence equation to an equivalent family of equations along the level sets E_h of H:

Lemma 4.3. *Let $u, r\colon U \to \mathbb{R}$ and $b\colon U \to \mathbb{R}^2$ be bounded functions. Suppose that $r \geq 0$ a.e. in U. Let $H\colon U \to \mathbb{R}$ be a Lipschitz function such that (4.3) holds. Suppose that the canonical disintegration of $\mathscr{L}^2 \llcorner U$ with respect to H is given by (2.4). Then a Radon measure μ on U satisfies (4.1) in $\mathscr{D}'(U)$ if and only if*

- *the disintegration of μ with respect to H has the form*

$$\mu = \int \mu_h \, dh + \int \nu_h \, d\zeta(h), \tag{4.4}$$

  where ζ is defined in (2.3);
- *for $\mathscr{L}^1$-a.e. h*

$$\operatorname{div}\left(u c_h b \mathscr{H}^1 \llcorner E_h\right) + \operatorname{div}(u b \sigma_h) = \mu_h; \tag{4.5}$$

- *for ζ-a.e. h*

$$\operatorname{div}(u b \kappa_h) = \nu_h. \tag{4.6}$$

Then we show that for any locally regular level set E_h the equation (4.5) can be split into at most countable family of the equations along the nontrivial connected components of E_h:

Lemma 4.4. *The equation (4.5) holds iff*

- *for any nontrivial connected component C of E_h it holds*

$$\operatorname{div}\left(u c_h b \mathscr{H}^1 \llcorner C\right) + \operatorname{div}(u b \sigma_h \llcorner C) = \mu_h \llcorner C; \tag{4.7}$$

- *it holds*

$$\operatorname{div}(u b \sigma_h \llcorner (E_h \setminus E_h^\star)) = \mu_h \llcorner (E_h \setminus E_h^\star). \tag{4.8}$$

Then we show that (4.7) in fact can also be split into two different equations:

Lemma 4.5. *Equation (4.7) holds iff*

$$\operatorname{div}\left(u c_h b \mathscr{H}^1 \llcorner C\right) = \mu_h \llcorner C, \tag{4.9a}$$

$$\operatorname{div}(u b \sigma_h \llcorner C) = 0. \tag{4.9b}$$

And finally we derive an equivalent parametric version of the equation (4.9a):

Lemma 4.6. *Equation (4.9a) holds iff for any admissible parametrization γ of C*

$$\partial_s(\hat{u}\hat{c}_h|\hat{b}|) = \hat{\mu}_h \tag{4.10}$$

where $\gamma_\# \hat{\mu}_h = \mu_h \llcorner C$, $\hat{u} = u \circ \gamma$, $\hat{c}_h = c_h \circ \gamma$ and $\hat{b} = b \circ \gamma$.

We now proceed with the proofs.

4.1. **Reduction on the level sets.** First we reduce the divergence equation on the level sets of H using appropriate test functions.

Proof of Lemma 4.3. Let λ^s be a measure on $\mathbb{R}$ such that $H_\#|\mu| \ll \mathscr{L}^1 + \zeta + \lambda^s$, where ζ is defined as in Lemma 2.8 and $\lambda^s \perp \mathscr{L}^1 + \zeta$. Applying the Disintegration Theorem, we have that

$$\mu = \int \mu_h dh + \int \nu_h d\zeta(h) + \int \lambda_h d\lambda^s(h), \tag{4.11}$$

with μ_h, ν_h, λ_h concentrated on $\{H = h\}$. Writing equation (4.1) in distribution form we get

$$\int_{\mathbb{T}^2} u(b\cdot\nabla\phi)\,dx + \int \phi d\mu = 0, \qquad \forall \phi \in C^\infty_c(U).$$

By an elementary approximation argument, it is clear that we can use as test functions ϕ Lipschitz with compact support.

Using the disintegration of Lebesgue measure (2.4) and the disintegration (4.11) we thus obtain

$$\begin{aligned}\int \bigg[\int_{\mathbb{T}^2} uc_h(b\cdot\nabla\phi)\,d\mathscr{H}^1 \llcorner E_h + \int_{\mathbb{T}^2} u(b\cdot\nabla\phi)\,d\sigma_h\bigg]\,dh \\ + \int\int_{\mathbb{T}^2} u(b\cdot\nabla\phi)\,d\kappa_h\,d\zeta(h) + \int\int_{\mathbb{T}^2}\phi\,d\mu_h\,dh \\ + \int\int_{\mathbb{T}^2}\phi\,d\nu_h\,d\zeta(h) + \int\int_{\mathbb{T}^2}\phi\,d\lambda_h\,d\lambda^s(h) = 0,\end{aligned} \tag{4.12}$$

for every $\phi \in \mathrm{Lip}_c(U)$. In particular we can take

$$\phi = \psi(H(x))\varphi(x), \qquad \psi \in C^\infty(\mathbb{R}),\ \varphi \in C^\infty_c(U),$$

so that we can rewrite (4.12) as

$$\begin{aligned}\int \psi(h)\bigg[\int_{\mathbb{T}^2} uc_h(b\cdot\nabla\varphi)\,d\mathscr{H}^1 \llcorner E_h + \int_{\mathbb{T}^2} u(b\cdot\nabla\varphi)\,d\sigma_h\bigg)\,dh \\ + \int \psi(h)\int_{\mathbb{T}^2} u(b\cdot\nabla\varphi)\,d\kappa_h\,d\zeta(h) + \int\psi(h)\int_{\mathbb{T}^2}\varphi\,d\mu_h\,dh \\ + \int\psi(h)\int_{\mathbb{T}^2}\varphi\,d\nu_h\,d\zeta(h) + \int\psi(h)\int_{\mathbb{T}^2}\varphi\,d\lambda_h\,d\lambda^s(h) = 0,\end{aligned}$$

because

$$b\cdot\nabla\phi = \psi(H(x))b\cdot\nabla\varphi(x)$$

for $\mathscr{L}^2$-a.e. $x \in \mathbb{T}^2$.

Since the equalities above hold for all $\psi \in C^\infty(\mathbb{R})$ we have

$$\int \bigg[\int_{\mathbb{T}^2} uc_h(b\cdot\nabla\varphi)\,d\mathscr{H}^1 \llcorner E_h + \int_{\mathbb{T}^2} u(b\cdot\nabla\varphi)\,d\sigma_h\bigg]\,dh + \int\int_{\mathbb{T}^2}\varphi\,d\mu_h\,dh = 0,$$

$$\int\bigg[\int_{\mathbb{T}^2} u(b\cdot\nabla\varphi)\,d\kappa_h + \int_{\mathbb{T}^2}\varphi\,d\nu_h\bigg]\,d\zeta(h) = 0,$$

$$\int\int_{\mathbb{T}^2}\varphi\,d\lambda_h\,d\lambda^s(h) = 0,$$

which give, respectively, (4.5), (4.6) and (4.4). □

4.2. **Reduction on connected components of level sets.** If $K \subset \mathbb{R}^d$ is a compact then, in general, not any connected component C of K can be separated from $K \setminus C$ by a smooth function. However, it can be separated by a sequence of such functions:

Lemma 4.7 (Lemma 5.3 from [10]; see also [1, Section 2.8])**.** *If $K \subset \mathbb{R}^d$ is compact then for any connected component C of K there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ such that*

(1) $0 \le \phi_n \le 1$ on $\mathbb{R}^d$ and $\phi_n \in \{0,1\}$ on K for all $n \in \mathbb{N}$;
(2) for any $x \in C$, we have $\phi_n(x) = 1$ for every $n \in \mathbb{N}$;
(3) for any $x \in K \setminus C$, we have $\phi_n(x) \to 0$ as $n \to +\infty$;
(4) for any $n \in \mathbb{N}$, we have $\operatorname{supp} \nabla\phi_n \cap K = \emptyset$.

With the aid of this lemma we can now study the equation (4.5) on the nontrivial connected components of the level sets. In view of Lemma 4.3 in what follows we always assume that $h \notin N$ (see (2.2)).

Proof of Lemma 4.4. For any Borel set $A \subset \mathbb{T}^2$ we introduce the following functional

$$\Lambda_A(\psi) := \int_A u c_h (b \cdot \nabla\psi)\, d\mathscr{H}^1 \llcorner E_h + \int_A u(b\cdot\nabla\psi)\, d\sigma_h + \int_A \psi\, d\mu_h,$$

for all $\psi \in C_c^\infty(U)$.

Now fix a connected component C of E_h and take a sequence of functions $(\phi_n)_{n\in\mathbb{N}}$ given by Lemma 4.7 (applied with $K := E_h$). By assumption, we have that

$$\Lambda_{E_h}(\psi\phi_n) = 0 \tag{4.13}$$

for every $\psi \in C_c^\infty(U)$ and for every n. Let us pass to the limit as $n \to \infty$.

On one hand we have

$$\int \psi\phi_n\, d\mu_h = \int_C \psi\, d\mu + \int_{E_h\setminus C} \psi\phi_n\, d\mu \to \int_C \psi\, d\mu$$

because the second term converges to 0 since $\phi_n \to 0$ pointwise on $E_h \setminus C$.

On the other hand $\nabla(\psi\phi_n) = \psi\nabla\phi_n + \phi_n\nabla\psi$. In the terms with $\phi_n\nabla\psi$ we pass to the limit as above. The terms with the product $\psi\nabla\phi_n$ identically vanish thanks to the condition (4) on ϕ_n in Lemma 4.7. Therefore, we have that for every $\psi \in C_c^\infty(U)$

$$\Lambda_{E_h}(\psi\phi_n) \to \int_C u c_h(b\cdot\nabla\psi)\, d\mathscr{H}^1 + \int_C u(b\cdot\nabla\psi)\, d\sigma_h + \int_C \psi\, d\mu_h = \Lambda_C(\psi),$$

as $n \to +\infty$. Since (4.13) holds for every n, we deduce that $\Lambda_C(\psi) = 0$ and this gives us (4.7).

In order to get (4.8), it is enough to observe that $E_h^\star$ is a countable union of connected component C, therefore (from the previous step) we deduce that

$$\int_{E_h^\star} u c_h(b\cdot\nabla\psi)\, d\mathscr{H}^1 + \int_{E_h^\star} u(b\cdot\nabla\psi)\, d\sigma_h + \int_{E_h^\star} \psi\, d\mu_h = 0, \qquad \forall\psi \in C_c^\infty(U).$$

Hence

$$\Lambda_{E_h\setminus E_h^\star} = \int_{E_h^\star\setminus E_h} u c_h(b\cdot\nabla\psi)\, d\mathscr{H}^1 + \int_{E_h^\star\setminus E_h} u(b\cdot\nabla\psi)\, d\sigma_h + \int_{E_h^\star\setminus E_h} \psi\, d\mu_h = 0,$$

for every $\psi \in C_c^\infty(U)$. Remembering that $\mathscr{H}^1(E_h^\star \setminus E_h) = 0$ by Theorem 2.4 we get (4.8) and this concludes the proof.

The converse implication can be easily obtained by summing the equations (4.7) and (4.8). □

Before proving Lemma 4.5 we need to prove Lemma 4.6, i.e. establish the parametric version of the equation (4.9a). The proof of Lemma 4.6 we be based on the following result (we refer to [1, Section 7] for its proof):

Lemma 4.8 (Density lemma)**.** *Let $a \in L^1(I)$ and μ a Radon measure on I, where $I = \mathbb{R}/\ell\mathbb{Z}$ or $I = (0, \ell)$ for some $\ell > 0$. Suppose that $\gamma \colon I \to \Omega$ is an injective Lipschitz function such that $\gamma' \neq 0$ a.e. on I and $\gamma(0, \ell) \subset \Omega$. Consider the functional*

$$\Lambda(\phi) := \int_I \phi' a \, dt + \int_I \phi \, d\mu, \qquad \forall \phi \in \mathrm{Lip}_c(I).$$

If $\Lambda(\varphi \circ \gamma) = 0$ for any $\varphi \in C_c^\infty(\Omega)$ then $\Lambda(\phi) = 0$ for any $\phi \in \mathrm{Lip}_c(I)$.

Proof of Lemma 4.6. Let us recall a corollary from Area formula: if $\gamma \colon I \to \mathbb{T}^2$ is an injective Lipschitz parametrization of C then

$$\mathscr{H}^1 \llcorner C = \gamma_\# \left(|\gamma'| \mathscr{L}^1 \right).$$

Using this formula the distributional version of (4.9a),

$$\int_C u c_h b \cdot \nabla \phi d\mathscr{H}^1 \llcorner C + \int_C \phi d\mu_h = 0, \quad \forall \phi \in C_c^\infty(U),$$

can be written as

$$\int_I u(\gamma(s)) c_h(\gamma(s)) b(\gamma(s)) \cdot (\nabla \phi)(\gamma(s)) |\gamma'(s)| \, ds + \int_I \phi(\gamma(s)) d\hat{\mu}_h(s) = 0$$

where $\hat{\mu}_h$ is defined by $\hat{\mu}_h := \left(\gamma^{-1}\right)_\# \mu_h$.

Using (2.1) we can write the equation above as

$$\int_I u(\gamma(s)) c_h(\gamma(s)) \gamma'(s) (\nabla \phi)(\gamma(s)) |b(\gamma(s))| \, ds + \int_I \phi(\gamma(s)) d\hat{\mu}_h(s) = 0,$$

which reads as

$$\int_I \hat{u}(s) \hat{c}_h(s) |\hat{b}(s)| \partial_s(\phi(\gamma(s))) \, ds + \int_I \phi(\gamma(s)) d\hat{\mu}_h(s) = 0.$$

Since the equation above holds for any $\phi \in C_c^\infty(U)$ it remains to apply Lemma 4.8. □

Finally, let us turn to the proof of Lemma 4.5.

Proof of Lemma 4.5. This proof would be fairly easy in the case when γ is a straight line. Roughly saying, in this case (4.7) would read as

$$\int u(x) c_h(x) b(x) \psi'(x) \, dx + \int u(x) c_h(x) b(x) \psi'(x) \, d\sigma_h(x) + \int \psi(x) \, d\mu(x) = 0,$$

$\psi \in C_0^\infty(\mathbb{R})$. Since σ_h is concentrated on a $\mathscr{L}^1$-negligible set S, any $\phi \in C_0^1$ can be approximated in C^0-norm with a sequence of C^1-functions ϕ_n having

0-derivative on S. Consequently, ϕ_n' converge to ϕ' weak* in L^∞ as $n \to \infty$. Then, substituting $\psi = \phi_n$ and passing to the limit as $n \to \infty$ we get

$$\int u(x)c_h(x)b(x)\phi'(x)\,dx + \int \phi(x)\,d\mu(x) = 0.$$

Hence the only technicality here is to repeat this argument on a curve.

Let $\Lambda(\phi) := M(\phi) + N(\phi)$, where

$$M(\phi) := \int_C uc_h(b \cdot \nabla\phi)\,d\mathscr{H}^1 + \int_C \phi\,d\mu_h$$

and

$$N(\phi) := \int_C ub \cdot \nabla\phi d\sigma_h$$

for every $\phi \in C_c^\infty(U)$.

Fix a test function ϕ: we are going to "perturb" ϕ in such a way that $N(\phi)$ becomes arbitrarily small and $M(\phi)$ remains almost unchanged. Since $\Lambda(\phi) = 0$ we will obtain that $|M(\phi)| < \varepsilon$ and this will imply that $M(\phi) = N(\phi) = 0$.

By Lemma 2.8, we have $\sigma_h \perp \mathscr{H}^1 \llcorner C$ therefore there exists a $\mathscr{H}^1$-negligible set $S \subset C$ such that σ_h is concentrated on S. Moreover, by inner regularity, for every $n \in \mathbb{N}$, we can find a compact $K \subset S$ such that

$$\sigma_h(S \setminus K) < \frac{1}{n}.$$

Using the fact that $\mathscr{H}^1(K) = 0$, for every $n \in \mathbb{N}$, we can find countably many open balls $\{B_{r_j}(z_j)\}_{j\in\mathbb{N}}$ which cover K and whose radii r_j satisfy

$$\sum_{j\in\mathbb{N}} r_j < \frac{1}{n}.$$

Furthermore, by compactness, we can extract from $\{B_{r_j}(z_j)\}_{l\in\mathbb{N}}$ a finite subcovering, $\{B_{r_j}(z_j)\}$ with $j = 1, \dots, \nu$ where $\nu = \nu(n) \in \mathbb{N}$ (we stress that ν depends on n).

For every $j \in \{1, \dots, \nu\}$, let

$$P_i^{j,n} := (z_{j,i} - r_j, z_{j,i} + r_j)$$

denote the projection of $B_{r_j}(z_j)$ onto the x_i-axis, with $i = 1, 2$. Since $P_i^{j,n}$ is an open interval we can find a smooth function $\psi_i^{j,n} \colon \mathbb{R} \to \mathbb{R}$ such that

$$\psi_i^{j,n}(\xi) = \begin{cases} 0 & \xi \in P_i^{j,n}, \\ 1 & \operatorname{dist}(\xi, \partial P_i^{j,n}) > 2r_i, \end{cases}$$

and $0 \le \psi_i^{j,n} \le 1$ for every $\xi \in \mathbb{R}$. Now we consider the product $\psi_i^n := \psi_i^{1,n}\psi_i^{2,n} \cdots \psi_i^{\nu,n}$ and we define the functions $\chi_i^n \colon \mathbb{R} \to \mathbb{R}$ as

$$\chi_i^n(\xi) := \int_0^\xi \psi_i^n(w)\,dw$$

for $i = 1, 2$ and $n \in \mathbb{N}$. Now we set $\chi^n(x) := (\chi_1^n(x), \chi_2^n(x))$ and $\phi_n := \phi \circ \chi^n$. Since $\|\chi^n - \mathrm{id}\|_\infty \le 4\sum_i r_i \le \frac{4}{n}$ we deduce that $\phi_n \to \phi$ uniformly in C because

$$|\phi_n(x) - \phi(x)| \le \|\nabla\phi\|_\infty \|\chi^n - \mathrm{id}\|_\infty \to 0$$

as $n \to +\infty$.

Let us now take an admissible parametrization of C, $\gamma\colon I \to \mathbb{R}$, and let us introduce the functions $\hat{\phi}_n := \phi_n \circ \gamma$. Using for instance the density of C^1 functions in $L^1(I)$, we can actually show that $\partial_s \hat{\phi}_n \rightharpoonup^\star \partial_s \hat{\phi}$ in weak* topology of L^∞. Passing to the parametrization as in the proof of Lemma 4.6 we get

$$\int_C u c_h (b \cdot \nabla \phi_n) \, d\mathscr{H}^1 = \int_I \hat{u} \hat{c}_h \hat{b} \, \partial_s \hat{\phi}_n \, ds,$$

where we denote by $\hat{\cdot}$ the composition with γ.

Using weak* convergence, we obtain that

$$\int_C u c_h (b \cdot \nabla \phi_n) \, d\mathscr{H}^1 \to \int_C u c_h (b \cdot \nabla \phi) \, d\mathscr{H}^1.$$

On the other hand, by uniform convergence, we immediately get

$$\int \phi_n \, d\mu_h \to \int \phi \, d\mu_h,$$

as $n \to +\infty$. In particular, we have that $M(\phi_n) \to M(\phi)$.

Now observe that $\nabla \phi_n = 0$ on K by construction, hence we get

$$N(\phi_n) \leq \int_{S \setminus K} |ub| |\nabla \phi_n| d\sigma_h \leq \|ub\|_\infty \|\nabla \phi\|_\infty \frac{1}{n} \to 0$$

and this implies that $N(\phi) = 0$. Therefore, $0 = \Lambda(\phi) = M(\phi)$, which concludes the proof. □

5. Locality of the divergence

In this section we prove that the if $\operatorname{div}(ub)$ is a measure, then it is 0 on the set

$$M := \Big\{ x \in \mathbb{T}^2 : b(x) = 0, \, x \in \mathcal{D}_b \text{ and } \nabla^{\mathrm{appr}} b(x) = 0 \Big\}, \tag{5.1}$$

where $\mathcal{D}_b$ is the set of approximate differentiability points and $\nabla^{\mathrm{appr}} b$ is the approximate differential, according to Definition [7, Def. 3.70]. For shortness, we will call this property *locality of the divergence*.

Let U be an open set in $\mathbb{R}^d$ (or in $\mathbb{T}^d$), $d \in \mathbb{N}$. The main result of this section is the following

Proposition 5.1. *$u \in L^\infty(U)$ and suppose that $\operatorname{div}(ub) = \lambda$ in the sense of distributions, where λ is a Radon measure on U. Then $|\lambda| \llcorner M = 0$.*

Note that we do not assume any weak differentiability of u or ub, so the conclusion of Proposition 5.1 does not follow immediately from the standard locality properties of the approximate derivative (see e.g. [7], Proposition 3.73).

The proof is based on Besicovitch-Vitali covering Lemma ([7, Thm. 2.19]) and uses some basic facts about the trace properties of L^∞ vector fields whose divergence is a measure (we refer to [13, 9] or [15]). In particular, we recall the following Theorem (for the proof, see [15, Prop 7.10]):

Theorem 5.2 (Fubini's Theorem for traces)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set and $B \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^d)$ be a vector field whose distributional divergence* $\operatorname{div} B =: \mu$ *is a Radon measure with locally finite variation in Ω. Let $F \in C^1(\Omega)$. Then for a.e. $t \in \mathbb{R}$ we have*

$$Tr(B, \partial\{F > t\}) = B \cdot \nu \qquad \mathscr{H}^{d-1}\text{-a.e. on } \Omega \cap \partial\{F > t\}, \tag{5.2}$$

where ν denotes the exterior unit normal to $\partial\{F > t\}$ and the distribution $Tr(B, \partial\Omega')$ is defined by

$$\langle Tr(B, \partial\Omega'), \phi\rangle := \int_{\Omega'} \phi \, d\mu + \int_{\Omega'} \nabla\phi \cdot B \, dx, \qquad \forall \phi \in C^\infty_c(\Omega).$$

for every open subset $\Omega' \subset \Omega$ with C^1 boundary.

Furthermore, we will use the following elementary

Lemma 5.3. *Let $G \colon \mathbb{R}^d \to \mathbb{R}$ be a bounded, Borel function. For every $r > 0$ there exists a set of positive measure of real numbers $s = s(r) \in [r, 2r]$ such that*

$$\int_{\partial B_{s(r)}} |G(x)| \, d\mathscr{H}^{d-1}(x) \le \frac{1}{r} \int_{B_{2r}} |G(y)| \, dy.$$

Proof of Proposition 5.1. Fix an arbitrary $x \in M$. For brevity let $B_r := B_r(x)$. By (5.2) with $F(y) := |x - y|^2$, there exists an $\mathscr{L}^1$-negligible set N_x such that for any positive number $r \notin N_x$ we have

$$|\lambda(B_r)| = \left| \int_{\partial B_r} u b \cdot \nu \, d\mathscr{H}^{d-1} \right| \le C \int_{\partial B_r} |b| \, d\mathscr{H}^{d-1},$$

where ν denotes the exterior unit normal to ∂B_r. By Lemma 5.3

$$C \int_{\partial B_r} |b| \, d\mathscr{H}^{d-1} \le \frac{C}{r} \int_{B_{2r}} |b(x)| \, dx = o(r^d)$$

because, by definition of M, we have $\fint_{B_r} |b| \, dx = o(r)$. Therefore

$$|\lambda(B_r)| = o(r^d). \tag{5.3}$$

Fix $\varepsilon > 0$. By (5.3) for any $x \in M$ there exists $\delta_x > 0$ such that for any positive number $r < \delta_x$ such that $r \notin N_x$ we have

$$|\lambda(B_r(x))| \le \varepsilon r^d. \tag{5.4}$$

Let $S \subset M$ be an arbitrary bounded subset.

By regularity of λ, there exists a bounded open set $O \supset S$ such that $|\lambda|(O \setminus S) < \varepsilon$. Hence, for any $x \in S$ there exists $\rho_x > 0$ such that $B(x, r) \subset O$ for any positive number $r < \rho_x$. Consequently

$$\mathscr{F} := \{B(x, r) : x \in S, r < \min(\rho_x, \delta_x), r \notin N_x\}$$

is a fine covering of S.

Hence we can apply Besicovitch-Vitali covering Lemma ([7, Thm. 2.19]): there exists a countable disjoint subfamily $\{B_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ such that

$$|\lambda| \left(S \setminus \bigcup_i B_i \right) = 0.$$

On the other hand, since $\bigcup_i B_i \subset O$ by construction, we have

$$|\lambda| \left(\bigcup_i B_i \setminus S \right) < \varepsilon.$$

Using (5.4), since the balls B_i are disjoint, we have

$$\lambda \left(\bigcup_i B_i \right) = \sum_i \lambda(B_i) \leq \varepsilon \mathscr{L}^2 \left(\bigcup_i B_i \right).$$

Hence

$$\lambda(S) = \lambda \left(\bigcup_i B_i \right) - \lambda \left(\bigcup_i B_i \setminus S \right) \to 0$$

as $\varepsilon \downarrow 0$. Hence $\lambda \llcorner S = 0$ and, by arbitrariness of $S \subset M$, $\lambda \llcorner M = 0$. □

6. Global properties of nearly incompressible vector fields

Suppose that $b \colon \mathbb{T}^2 \to \mathbb{R}^2$ is bounded nearly incompressible vector field with density $\rho \colon [0, T] \times \mathbb{T}^2 \to \mathbb{R}$. By Definition 1.1 and Theorem 2.12 there exists a measure η on Γ such that

$$\rho(t, \cdot) \mathscr{L}^2 = e_{t\#} \eta. \tag{6.1}$$

6.1. **Comparison between $\mathscr{L}^2$ and η.** In this section we study the relation between η and $\mathscr{L}^2$. Namely, we prove that a set $A \subset \mathbb{T}^2$ is $\mathscr{L}^2$-negligible if and only if the set of curves which stay in A for a positive amount of time is η-negligible. We also show that A is $\mathscr{L}^2$-negligible if and only if for any $s \in [0, T]$ the set of curves which pass through A at time s is η-negligible.

Lemma 6.1. *Let $A \subset \mathbb{T}^2$ be a measurable set. Then $\mathscr{L}^2(A) = 0$ if and only if $\eta(\Gamma_A) = 0$ where*

$$\Gamma_A := \left\{ \gamma \in \Gamma : \mathscr{L}^1(\{t \in [0, T] : \gamma(t) \in A\}) > 0 \right\}.$$

Proof. Let us prove first that $\mathscr{L}^2(A) = 0$ implies $\eta(\Gamma_A) = 0$. We denote by ρ_A the density such that $\rho_A(t, \cdot) \mathscr{L}^2 = e_{t\#} (\eta \llcorner \Gamma_A)$ and $r_A(x) := \int_0^T \rho_A(t, x) \, dt$. We have, using Fubini,

$$\begin{aligned}
0 = \mathscr{L}^2(A) = r_A \mathscr{L}^2(A) &= \int_0^T \int_\Gamma \mathbb{1}_A(x) \rho_A(t, x) \, dx \, dt \\
&= \int_0^T \int_\Gamma \mathbb{1}_A(\gamma(t)) \, d\eta(\gamma) \, dt \\
&= \int_\Gamma \int_0^T \mathbb{1}_A(\gamma(t)) \, dt \, d\eta(\gamma) \\
&= \int_{\Gamma_A} \int_0^T \mathbb{1}_A(\gamma(t)) \, dt \, d\eta(\gamma) \\
&= \int_{\Gamma_A} \mathscr{L}^1(\{t \in [0, T] : \gamma(t) \in A\}) \, d\eta(\gamma),
\end{aligned}$$

hence, $\mathscr{L}^1(\{t \in [0, T] : \gamma(t) \in A\}) = 0$ for η-a.e. $\gamma \in \Gamma_A$.

For the opposite direction, using that ρ is uniformly bounded from below by $1/C$, we get

$$\begin{aligned}
\frac{T}{C}\mathscr{L}^2(A) = \frac{T}{C}\int \mathbb{1}_A(x)\,dx = \frac{1}{C}\int_0^T \int \mathbb{1}_A(x)\,dx\,dt \\
&\leq \int_0^T \int \mathbb{1}_A(x)\rho(t,x)\,dx\,dt \\
&= \int_0^T \int_\Gamma \mathbb{1}_A(\gamma(t))\,d\eta(\gamma)\,dt \\
&= \int_\Gamma \int_0^T \mathbb{1}_A(\gamma(t))\,dt\,d\eta(\gamma) \\
&= \int_{\Gamma_A} \int_0^T \mathbb{1}_A(\gamma(t))\,dt\,d\eta(\gamma) \\
&= \int_{\Gamma_A} \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\})\,d\eta(\gamma) = 0. \qquad \square
\end{aligned}$$

Lemma 6.2. *We have* $\mathscr{L}^2(A) = 0$ *if and only if* $\eta(\Gamma_A^s) = 0$ *for every* $s \in [0,T]$.

Proof. For direct implication

$$\begin{aligned}
0 = \mathscr{L}^2(A) &= \int \mathbb{1}_A(x)\rho(s,x)\,dx \\
&= \int_\Gamma \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) \\
&= \int_{\Gamma_A^s} \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) = \eta(\Gamma_A^s).
\end{aligned}$$

For the opposite direction,

$$\begin{aligned}
\frac{1}{C}\mathscr{L}^2(A) &= \frac{1}{C}\int \mathbb{1}_A(x)\,dx \\
&\leq \int \mathbb{1}_A(x)\rho(s,x)\,dx \\
&= \int_\Gamma \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) \\
&= \int_{\Gamma_A^s} \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) = \eta(\Gamma_A^s) = 0. \qquad \square
\end{aligned}$$

6.2. **Properties of "sticky" integral curves.** In general the integral curves of b may "stick" to the set

$$Z := \{b = 0\}, \tag{6.2}$$

i.e. stay in Z for a positive amount of time. In this section show that if

$$Z = M \mod \mathscr{L}^2, \tag{6.3}$$

where the set M is defined in (5.1), then the set of nonconstant integral curves which "stick" to Z is η-negligible.

The condition (6.3) is satisfied for some weakly differentiable vector fields. In particular,

Lemma 6.3. *If* $b \in \mathrm{BV}(\mathbb{T}^2)$ *then* (6.3) *holds.*

Proof. Since $b \in \mathrm{BV}(\mathbb{T}^2)$, b is approximately differentiable a.e. on $\mathbb{T}^2$ (see e.g. Theorem 3.83 from [7]), $\nabla^{\mathrm{appr}} b(x)$ exists for a.e. $x \in Z$. Moreover, by locality of the derivative of b (see e.g. Proposition 3.92(a) from [7]) it holds that $\nabla^{\mathrm{appr}} b(x) = 0$ for $\mathscr{L}^2$-a.e. $x \in Z$. □

Remark 6.4. Equality (6.3) also holds for any bounded b whose divergence and curl are measures. Indeed, any such b is approximately differentiable a.e., see [2]. Furthermore, using locality property stated in [7, Prop. 3.73 - Rem. 3.93], we have that at every Lebesgue point x of the set $\{b = 0\}$ at which b is approximate differentiable we also have $\nabla^{\mathrm{appr}} b(x) = 0$.

Recall the notation introduced in Section 1.2: consider the set of non-constant curves which stay in Z for a positive amount of time

$$\widetilde{\Gamma}_Z := \widetilde{\Gamma} \cap \Gamma_Z$$

and the set of non-constant curves intersecting Z (at least) at time s

$$\widetilde{\Gamma}_Z^s := \left\{ \gamma \in \widetilde{\Gamma} : \gamma(s) \in Z \right\}.$$

Using Proposition 5.1, we are able to prove that these sets are η-negligible, i.e. the non-constant integral curves of b do not stay in Z for a positive amount of time:

Lemma 6.5. *Suppose that* (6.3) *holds. Then:*

- $\eta(\widetilde{\Gamma}_Z^s) = 0$ *for a.e.* $s \in [0, T]$*;*
- $\eta(\widetilde{\Gamma}_Z) = 0$.

Proof. Let us define $g \colon \Gamma \to \mathbb{R}$ by

$$g(\gamma) := \int_0^T \mathbb{1}_Z(\gamma(s))\, ds. \tag{6.4}$$

By the definition of $\widetilde{\Gamma}_Z$ we have $g(\gamma) > 0$ for η-a.e. $\gamma \in \widetilde{\Gamma}_Z$. Hence

$$\eta(\widetilde{\Gamma}_Z) = 0 \quad \Leftrightarrow \quad I_g := \int_{\widetilde{\Gamma}_Z} g(\gamma)\, d\eta(\gamma) = 0 \tag{6.5}$$

By Fubini's theorem

$$I_g = \int_0^T \int_{\widetilde{\Gamma}_Z} \mathbb{1}_Z(\gamma(s))\, d\eta(\gamma)\, ds$$

Since $\int_{\widetilde{\Gamma}_Z} \mathbb{1}_Z(\gamma(s))\, ds \le \int_{\widetilde{\Gamma}} \mathbb{1}_Z(\gamma(s))\, ds = \eta(\widetilde{\Gamma}_Z^s)$ it holds that

$$I_g \le \int_0^T \eta(\widetilde{\Gamma}_Z^s)\, ds.$$

Therefore it remains to prove that $\eta(\widetilde{\Gamma}_Z^s) = 0$ for a.e. $s \in [0, T]$.

Since $e_{t\#}(\eta \llcorner \widetilde{\Gamma}_Z^s) \ll e_{t\#}\eta \ll \mathscr{L}^2$ there exists a non-negative Borel function $\rho_Z^s \colon [0, T] \times \mathbb{T}^2 \to \mathbb{R}$ such that

$$\rho_Z^s(t, \cdot) \mathscr{L}^2 = e_{t\#}(\eta \llcorner \widetilde{\Gamma}_Z^s).$$

(In fact $0 \le \rho_Z^s(t, x) \le \rho(t, x)$ for a.e. $(t, x) \in [0, T] \times \mathbb{T}^2$.)

It is easy to see that ρ^s_Z solves the continuity equation

$$\partial_t \rho^s_Z + \operatorname{div}(\rho^s_Z b) = 0.$$

For a.e. $t \in [0,T]$ and a.e. $s \in [0,T]$, integrating in time on $[s,t]$, we get

$$\operatorname{div}\left(b \int_s^t \rho_Z(\tau,\cdot)\,d\tau \right) = (\rho_Z(t,\cdot) - \rho_Z(s,\cdot))\mathscr{L}^2. \tag{6.6}$$

By Proposition 5.1, we have

$$(\rho_Z(t,\cdot) - \rho_Z(s,\cdot))\mathscr{L}^2 \llcorner M = 0, \tag{6.7}$$

hence, in view of (6.3), it holds that $\rho_Z(t,x) = \rho_Z(s,x)$ for a.e. $x \in Z$.

Let us fix $s \in [0,T]$ such that (6.6) and (6.7) hold for a.e. $t \in [0,T]$.

Integrating (6.6) over $\mathbb{T}^2$ we obtain conservation of the total mass:

$$\int_{\mathbb{T}^2} \rho^s_Z(t,x)\,dx = \int_{\mathbb{T}^2} \rho^s_Z(s,x)\,dx.$$

Hence by (6.7) for a.e. $t \in [0,T]$ we have

$$\int_{\mathbb{T}^2} \mathbb{1}_{\mathbb{T}^2\setminus Z}(x)\rho^s_Z(t,x)\,dx = \int_{\mathbb{T}^2} \mathbb{1}_{\mathbb{T}^2\setminus Z}(x)\rho^s_Z(s,x)\,dx. \tag{6.8}$$

By definition of ρ^s_Z, since $\mathbb{1}_{\widetilde{\Gamma}^s_Z}(\gamma) = \mathbb{1}_Z(\gamma(s))$, we have

$$\int_{\mathbb{T}^2} \mathbb{1}_{\mathbb{T}^2\setminus Z}(x)\rho^s_Z(s,x)\,dx = \int \mathbb{1}_{\mathbb{T}^2\setminus Z}(\gamma(s))\mathbb{1}_Z(\gamma(s))\,d\eta(\gamma) = 0.$$

Consequently, by (6.8), for a.e. $t \in [0,T]$ it holds that

$$\int_{\widetilde{\Gamma}^s_Z} \mathbb{1}_{\mathbb{T}^2\setminus Z}(\gamma(t))\,d\eta(\gamma) = 0$$

We have thus proved that $\mathbb{1}_Z(\gamma(t)) = 1$ for a.e. $t \in [0,T]$ for η-a.e. $\gamma \in \widetilde{\Gamma}^s_Z$. If $\gamma(t) \in Z$ for a.e. $t \in [0,T]$ then $b(\gamma(t)) = 0$ for a.e. t and hence the graph of γ is a fixed point, i.e. $\gamma \in \dot{\Gamma}$. But $\widetilde{\Gamma}^s_Z \subset \widetilde{\Gamma} = \Gamma \setminus \dot{\Gamma}$. Therefore $\eta(\Gamma^s_Z) = 0$. □

7. Local properties of nearly incompressible vector fields

As in the previous Section, let $b\colon \mathbb{T}^2 \to \mathbb{R}^2$ be an bounded nearly incompressible vector field with density $\rho\colon [0,T]\times\mathbb{T}^2 \to \mathbb{R}$. Let η be a measure on Γ given by (2.6). Consider a simply connected open set $U \subset \mathbb{T}^2$.

7.1. Construction of the local Hamiltonian.

In this section we use η to construct in U the *local steady density* r_U such that $\operatorname{div}(r_U b) = 0$ in $\mathscr{D}'(U)$ and the *local Hamiltonian* $H_U\colon U \to \mathbb{R}$ such that $\nabla^\perp H_U = r_U b$ a.e. in U. Recall the notation introduced in Section 1.2:

$$\mathsf{T}_U := \left\{ \gamma \in \Gamma\colon\ \mathscr{L}^1(\gamma^{-1}(U)) > 0,\ \gamma(0) \notin U, \gamma(T) \notin U \right\}$$

is the set of the curves which stay in U for a positive amount of time and have the endpoints outside U.

Consider the measure

$$\eta_U := \eta \llcorner \mathsf{T}_U \tag{7.1}$$

and define the *local density* ρ_U by

$$\rho_U(t,\cdot)\mathscr{L}^2 = (e_t)_\# \eta_U. \tag{7.2}$$

Indeed, since $\eta_U \ll \eta$ we also have $e_{t\#}\eta_U \ll e_{t\#}\eta \ll \mathscr{L}^2$.

Finally, we define the *local steady density*

$$r_U(x) := \int_0^T \rho_U(t,x)dt, \qquad x \in U. \tag{7.3}$$

Lemma 7.1. *It holds* $\operatorname{div}(r_U b) = 0$ *in* $\mathscr{D}'(U)$.

Proof. For any $\phi \in C_c^\infty(U)$ we have

$$\begin{aligned}
\int_U r_U b(x) \cdot \nabla\phi(x) dx &= \int_U \int_0^T \rho_U(t,x) b(x) \cdot \nabla\phi(x)\, dt\, dx \\
&= \int_0^T \int_{\mathsf{T}_U} b(\gamma(t)) \cdot (\nabla\phi)(\gamma(t))\, d\eta_U\, dt \\
&= \int_0^T \int_{\mathsf{T}_U} \dot\gamma(t) \cdot (\nabla\phi)(\gamma(t))\, d\eta_U\, dt \\
&= \int_0^T \int_{\mathsf{T}_U} \frac{d}{dt}\phi(\gamma(t))\, d\eta_U\, dt \\
&= \int_{\mathsf{T}_U} \Big[\phi(\gamma(T)) - \phi(\gamma(0))\Big]\, d\eta_U = 0.
\end{aligned}$$

because for η_U-a.e. $\gamma \in \mathsf{T}_U$, $\gamma(0) \notin U$, $\gamma(T) \notin U$. □

In the remaining part of this section we are going to show that the images of the integral curves of b, having the endpoints outside U, are contained in the locally regular level sets of H_U, up to an η-negligible set.

Let $H\colon \mathbb{T}^2 \to \mathbb{R}$ be an extension of H_U to the whole $\mathbb{T}^2$ (using standard theorems for the extension of Lipschitz maps).

Lemma 7.2. *Let* $t_1, t_2 \in [0,T]$ *and set* $\mathsf{T} := \{\gamma : \gamma((t_1,t_2)) \subset U\}$. *Then* η*-a.e.* $\gamma \in \mathsf{T}$ *the map* $(t_1,t_2) \ni t \mapsto H(\gamma(t))$ *is a constant function.*

Proof. Let $(\varrho_\varepsilon)_\varepsilon$ be the standard family of convolution kernels in $\mathbb{R}^2$. We set $H_\varepsilon(x) := H \star \varrho_\varepsilon(x)$ for any $x \in U$.

For every $t \in [t_1,t_2]$ define

$$I(t) := \int_{\mathsf{T}} |H(\gamma(t)) - H(\gamma(0))| d\eta(\gamma)$$

and we will prove $I \equiv 0$.

First note that I is positive because the integrand is non-negative and η is positive. On the other hand,

$$\begin{aligned}
I(t) \le \underbrace{\int_{\mathsf{T}} |H(\gamma(t)) - H_\varepsilon(\gamma(t))| d\eta(\gamma)}_{I_1^\varepsilon} + \underbrace{\int_{\mathsf{T}} |H_\varepsilon(\gamma(t)) - H_\varepsilon(\gamma(0))| d\eta(\gamma)}_{I_2^\varepsilon} \\
+ \underbrace{\int_{\mathsf{T}} |H_\varepsilon(\gamma(0)) - H(\gamma(0))| d\eta(\gamma)}_{I_3^\varepsilon}.
\end{aligned}$$

Now for a.e. $x \in \mathbb{T}^2$ we have $H_\varepsilon(x) \to H(x)$: hence

$$\int_{\mathsf{T}} |H_\varepsilon(\gamma(t)) - H(\gamma(t))|\, d\eta(\gamma) \le \int_U |H_\varepsilon(x) - H(x)| \rho(t,x) dx \to 0$$

as $\varepsilon \to 0$. Therefore, we can infer that

$$I_1^\varepsilon \to 0, \qquad I_3^\varepsilon \to 0$$

as $\varepsilon \downarrow 0$.

Let us study I_2^ε. We have

$$\begin{aligned} I_2^\varepsilon(t) &\le \int_{\mathsf{T}} \int_{t_1}^t |\partial_s H_\varepsilon(\gamma(s))| \, ds \, d\eta(\gamma) \\ &= \int_{\mathsf{T}} \int_{t_1}^t |\nabla H_\varepsilon(\gamma(s)) \cdot b(\gamma(s))| \, ds \, d\eta(\gamma) \\ &= \int_{t_1}^t \int |\nabla H_\varepsilon(x) \cdot b(x)| \, d(e_{t\#}\eta \llcorner \mathsf{T})(x) \, ds \\ &\le \int_0^T \int |\nabla H_\varepsilon(x) \cdot b(x)| \rho_{\mathsf{T}}(t,x) \, dx \, ds \\ &= \int |\nabla H_\varepsilon(x) \cdot b(x)| r_{\mathsf{T}}(x) \, dx \to \int |\nabla H(x) \cdot b(x)| r_{\mathsf{T}}(x) \, dx = 0 \end{aligned}$$

where we have used $\nabla H_\varepsilon(x) \to \nabla H(x)$ for a.e. x. In the end, we have that $I_2^\varepsilon \to 0$ as $\varepsilon \downarrow 0$ and this concludes the proof. □

We now improve Lemma 7.2, showing that η_U-a.e. γ is contained in *locally regular* level sets of H_U. Actually, we prove a slightly stronger statement:

Lemma 7.3. *Suppose that $N \subset \mathbb{R}$ is $\mathscr{L}^1$-negligible. Then there exists an η_U-negligible set $\mathsf{N} \subset \Gamma$ such that for any $\gamma \in \mathsf{T}_U \setminus \mathsf{N}$ it holds that*

$$\gamma([0,T]) \cap U \ \subset \ E_U^\star \setminus H^{-1}(N).$$

Proof. Using Lemma 7.2, we can remove an η_U-negligible set of trajectories along which H_U is not constant.

Let $E := E_U^\star \setminus H^{-1}(N)$ and $E^c := U \setminus E$. We claim that η_U-a.e. $\gamma \in \mathsf{T}_U$ has the following property:

$$\text{for a.e. } t \in \gamma^{-1}(U) \text{ if } \gamma(t) \notin E \text{ then } b(\gamma(t)) = 0. \tag{7.4}$$

Assuming this claim is proved, consider any connected component C of $\gamma([0,T]) \cap U$. Since $\mathscr{H}^1(C) > 0$ there exists $I \subset \gamma^{-1}(C)$ with $\mathscr{L}^1(I) > 0$ such that $b(\gamma(t)) \neq 0$ for a.e. $t \in I$. Hence by (7.4) for a.e. $t \in I$ we have $\gamma(t) \in E$. Consequently there exists a point $x \in C$ such that $x \in E$. By definition of $E_U^\star \supset E$ the level set $H_U^{-1}(H_U(x))$ is locally regular. But H_U is constant on C, hence $C \subset H_U^{-1}(H_U(x)) \subset E$.

It remains to prove the claim (7.4).

By Coarea Formula (see Lemma 2.3), $|\nabla H| \mathscr{L}^2 \llcorner E^c = 0$, i.e.

$$\int \mathbb{1}_{E^c}(x) |\nabla H(x)| \, dx = 0.$$

Since $\nabla H = r_U b^\perp$ in U and $r_U \geq 0$ (since $\rho_U > 0$), we have

$$\begin{aligned} 0 &= \int \mathbb{1}_{E^c}(x)|r_U(x)b(x)|\,dx \\ &= \int\int_0^T \mathbb{1}_{E^c}(x)\rho_U(t,x)|b(x)|\,dx\,dt \\ &= \int_0^T \int \mathbb{1}_{E^c}(\gamma(t))|b(\gamma(t))|\,d\eta(\gamma)\,dt, \end{aligned}$$

where in the last equality we used (6.1). Hence, by Fubini's theorem,

$$\int_0^T \mathbb{1}_{E^c}(\gamma(t))|b(\gamma(t))|\,dt = 0$$

for η-a.e. γ. This implies (7.4). □

7.2. **Weak Sard property of a local Hamiltonian.** In this section, in the context of Section 4, we study the local Hamiltonian H under the following additional assumptions:

(1) b is nearly incompressible;
(2) either $b \in BV(U)$ or $b \neq 0$ a.e. in U.

We show that under these assumptions the Hamiltonian H has the weak Sard property.

Since $S = \{\nabla H = 0\} \mod \mathscr{L}^2$ and $\nabla H = rb \mod \mathscr{L}^2$, for a.e. $x \in S$ either $b(x) = 0$ or $r(x) = 0$. But if b is nearly incompressible, we are able to prove that for a.e. $x \in E^\star \cap S$ it holds that $b(x) = 0$. Consequently r can vanish only on a negligible subset of $E^\star \cap S \cap \{b \neq 0\}$. This can be viewed as a *partial* weak Sard property:

Lemma 7.4. *Let $r\colon U \to \mathbb{R}$ and $b\colon U \to \mathbb{R}^2$ be bounded functions. Suppose that $r \geq 0$ a.e. in U. Let $H\colon U \to \mathbb{R}$ be a Lipschitz function such that (4.3) holds. Suppose that the canonical disintegration of $\mathscr{L}^2 \llcorner U$ with respect to H is given by (2.4). If b is nearly incompressible then for a.e. h the measure σ_h is actually concentrated on $E_h \cap \{b = 0\}$.*

In other words, for nearly incompressible vector fields for a.e. h the measure σ_h is concentrated not only on $S \cap E_h$, but on $S \cap E_h \cap \{b = 0\}$.

Proof. Since b is nearly incompressible the function $m(t,x) := \int_0^T \rho(\tau,x)\,d\tau$, where ρ is the density of b, solves

$$\operatorname{div}(mb) = \rho(T,\cdot) - \rho(0,\cdot) \tag{7.5}$$

in $\mathscr{D}'(U)$. (Here $\rho(T,\cdot)$ and $\rho(0,\cdot)$, strictly saying, are weak* limits in L^∞ of $\rho(t,\cdot)$ as $t \to T$ and $t \to 0$ respectively.)

Applying Lemmas 4.3, 4.4, 4.5 with $u = m$, from (4.9b) we obtain

$$\operatorname{div}(mb\sigma_h \llcorner C) = 0. \tag{7.6}$$

By the definition of near incompressibility $m > 0$ a.e. in U. Let

$$V := \{b \neq 0, r = 0\} \cap E^\star. \tag{7.7}$$

denote the subset of $\{b \neq 0\} \cap E^\star$ where r vanishes.

In view of (7.6), applying Lemmas 4.5, 4.4 and 4.3 for $u = \mathbb{1}_V \cdot m$, we get

$$\operatorname{div}(\mathbb{1}_V m) = 0$$

in $\mathscr{D}'(U)$. Consequently

$$\operatorname{div}(\widetilde{r}b) = 0 \tag{7.8}$$

in $\mathscr{D}'(U)$, where

$$\widetilde{r} := r + m\mathbb{1}_V.$$

In view of (7.8) we can construct a Lipshitz function $\widetilde{H} \colon U \to \mathbb{R}$ such that

$$\nabla^\perp \widetilde{H}(x) = \widetilde{r}(x)b(x), \qquad \text{for } \mathscr{L}^2\text{-a.e. } x \in U. \tag{7.9}$$

Since $\nabla H \parallel \nabla \widetilde{H}$ a.e. in U, by Matching Lemma 3.3 for a.e. $\mathscr{L}^2$-a.e. $x \in E^\star \cap \widetilde{E}^\star$ we have $C_x = \widetilde{C}_x$, where C_x and $\widetilde{C}_x$ are respectively the connected components of $H^{-1}(H(x))$ and $\widetilde{H}^{-1}(\widetilde{H}(x))$, containing x. Let M denote the set of all $x \in E^\star \cap \widetilde{E}^\star$ such that $C_x = \widetilde{C}_x$.

By definition $M \subset E^\star$. Let us prove that $E^\star \subset M \mod \mathscr{L}^2$.

Consider the canonical disintegrations of the Lebesgue measure (on $E^\star$ and $\widetilde{E}^\star$) with respect to H and $\widetilde{H}$:

$$\begin{aligned} \mathscr{L}^2 \llcorner E^\star &= \int (c_h \mathscr{H}^1 \llcorner E_h^\star + \sigma_h)\, dh, \\ \mathscr{L}^2 \llcorner \widetilde{E}^\star &= \int (\widetilde{c}_h \mathscr{H}^1 \llcorner \widetilde{E}_h^\star + \widetilde{\sigma}_h)\, dh, \end{aligned} \tag{7.10}$$

where σ_h is concentrated on S and $\widetilde{\sigma}_h$ is concentrated on $\widetilde{S}$.

Suppose that $\mathscr{L}^2(P) > 0$, where $P := E^\star \setminus M$. Then there exist a subset $I \subset \mathbb{R}$ with positive measure such that

$$\mathscr{H}^1(E_h^\star \cap P) + \sigma_h(E_h^\star \cap P) > 0 \tag{7.11}$$

for all $h \in I$.

Fix $h \in I$ and consider $C \in \operatorname{Conn}^\star(E_h)$. Suppose that $P \cap C \neq C$. Then some point of C belongs to M. But then $C \subset \widetilde{E}^\star$ and therefore $C \subset M$.

Hence for any $C \in \operatorname{Conn}^\star(E_h)$ either $C \subset P$ or $C \cap P = \emptyset$. Then (7.11) implies that there exists a connected component $C \in \operatorname{Conn}^\star(E_h)$ such that $C \subset P$ and $\mathscr{H}^1(C) > 0$. Hence $\mathscr{H}^1(E_h^\star \cap P) > 0$ for all $h \in I$ and, consequently,

$$\int_P |\nabla \widetilde{H}|\, dx \geq \int_P |\nabla H|\, dx > 0$$

where the first inequality holds since $|\nabla \widetilde{H}| \geq |\nabla H|$ and the second inequality follows from disintegration (7.10).

Therefore we can find a set $Q \subset P$ such that $\nabla \widetilde{H} \neq 0$ a.e. on Q. But then by coarea formula (Lemma 2.3) a.e. $x \in Q$ belongs to $\widetilde{E}^\star$. Hence $\mathscr{L}^2(Q \cap M) > 0$. Then by contradiction we have $\mathscr{L}^2(E^\star \setminus M) = 0$.

To complete the proof of the lemma it remains to restrict the disintegration of $\mathscr{L}^2 \llcorner \widetilde{E}^\star$ to M and compare it with the disintegration of $\mathscr{L}^2 \llcorner E$. For any $x \in M$ we have $C_x = \widetilde{C}_x = C$, and $c_h > 0$ and $\widetilde{c}_h > 0$ on C a.e. w.r.t $\mathscr{H}^1$. Hence by definition of V we have $c_h = \widetilde{c}_h$ a.e. on C. Therefore from (7.10), for a.e. h we obtain that $c_h \mathscr{H}^1 \llcorner (E_h^\star \cap M) = \widetilde{c}_h \mathscr{H}^1 \llcorner (\widetilde{E}_h^\star \cap M)$ and consequently

$$\sigma_h \llcorner E^\star = \widetilde{\sigma}_h \llcorner E^\star.$$

Since $\widetilde{\sigma}_h$ is concentrated on $\{b=0\}$ (for a.e. h) we conclude that σ_h also is concentrated on $\{b=0\}$ for a.e. h.

By (7.7) and (7.10) this actually means that $\mathscr{L}^2(V)=0$, so eventually $\widetilde{r}=r$ a.e. and therefore $H=\widetilde{H}$ (up to an additive constant). □

Lemma 7.5. *Suppose that the assumptions of Lemma 7.4 hold and, in addition, either $b\neq 0$ a.e. in U, or $b\in BV(U)$. Then the Hamiltonian H has the weak Sard property.*

Proof. First of all, σ_h in the canonical disintegration (2.4) is concentrated on $\{b=0\}$ for a.e. h by Lemma 7.4.

Hence, when $b\neq 0$ a.e., there is nothing to prove: $H_\#\mathscr{L}^2 \llcorner \{b=0\}$ will be zero.

When b vanishes on a set with positive measure, but has BV regularity, this lemma can be proved using minor modifications of the proof of [10, Theorem 8.4]. Indeed, since $b\in BV(U)$, it is approximately differentiable a.e. and then $H_\#\mathscr{L}^2\llcorner\{b=0\}\perp\mathscr{L}^1$. By comparing two disintegrations of $\mathscr{L}^2\llcorner\{b=0\}$ we conclude that σ_h is concentrated on $\{b\neq 0\}$ for a.e. h. □

8. New stuff

Let us consider the countable covering $\mathscr{B}$ of $\mathbb{T}^2$ given by

$$\mathscr{B} := \left\{B(x,r) : x\in\mathbb{Q}^2, r\in\mathbb{Q}^+\right\}.$$

For each ball $B\in\mathscr{B}$ and rational numbers $s,t\in\mathbb{Q}\cap(0,T)$ such that $s<t$ let

$$\mathsf{T}_{B,s,t} := \left\{\gamma\in\Gamma_B : \mathscr{L}^1(\gamma^{-1}(B))>0,\ \gamma(s)\notin B, \gamma(t)\notin B\right\}.$$

For each $B\in\mathscr{B}$, $s\in\mathbb{Q}\cap(0,T)$, $t\in\mathbb{Q}\cap(s,T)$ restricting η to $\mathsf{T}_{B,s,t}$, we construct the local Hamiltonian $H_{B,s,t}$ as in Section 7.1.

Lemma 8.1. *There exists an η-negligible set $N\subset\Gamma$ such that any integral curve $\gamma\in\widetilde{\Gamma}\setminus N$ of the vector field b has the following properties:*

(1) for any $B\in\mathscr{B}$
   - *the Hamiltonian H_B is constant along each connected component of $\gamma([0,T])\cap B$;*
   - *if $\gamma\in\mathsf{T}_{B,s,t}$ then each connected component of $\gamma([s,t])\cap B$ is contained in a regular level set of H_B;*

(2) for any $\tau\in(0,T)$ there exist a ball $B\in\mathscr{B}$, $s\in\mathbb{Q}\cap(0,T)$ and $t\in\mathbb{Q}\cap(\tau,T)$ such that $\gamma\in\mathsf{T}_{B,s,t}$;

(3) there exists $T_\gamma\in(0,T]$ such that $[0,T_\gamma]\ni t\mapsto\gamma(t)$ is a simple (possibly closed) curve.

Proof. First of all, using Lemma 6.5 we can remove a negligible set of integral curves of b which stay in the set $\{b=0\}$ for a positive amount of time. Applying Lemmas 7.2 and 7.3 countably many times (for each ball $B\in\mathscr{B}$ and all rationals $s\in\mathbb{Q}\cap(0,T)$ and $t\in\mathbb{Q}\cap(s,T)$) we obtain the set $N\subset\Gamma$ such that the first property holds.

Next, for any $\tau\in(0,T)$ there exists $s\in\mathbb{Q}\cap(0,\tau)$ such that $\gamma(s)\neq\gamma(\tau)$. (Otherwise, since γ is an integral curve of b, it would have to stay in $\{b=0\}$ for a positive amount of time). Similarly there exists $t\in(s,T)$ such that $\gamma(t)\neq\gamma(\tau)$. Then for any ball $B\in\mathscr{B}$ with sufficiently small radius, containing $\gamma(\tau)$ and not containing $\gamma(s)$ and $\gamma(t)$ it clearly holds that $\gamma \in T_{B,s,t}$.

Let us prove that $\gamma([0,T])$ cannot contain a triod. By contradiction, suppose that x is a vertex of some triod contained in $\gamma([0,T])$. Then there exists $\tau \in (0,T)$ such that $x = \gamma(\tau)$. By the second property we can find rational numbers $s, t \in \mathbb{Q} \cap (0,T)$ such that $\tau \in (s,t)$ and hence $\gamma \in \mathsf{T}_{B,s,t}$. Hence x is contained in some connected component C of $H_{s,t,B}$. But $H_{s,t,B}$ is constant along each connected component of $B \cap \gamma([0,T])$ and $\gamma([0,T])$ contains a triod with the vertex $x \in B$. Hence C contains a triod. This is not possible since C is a regular level set.

Let $T_1 > 0$ be such that $\gamma(T_1) = \gamma(0)$. Since $\gamma([0,T])$ does not contain triods it follows that $\gamma([T_1,T]) \subset \gamma([0,T_1])$ and hence $\mathscr{H}^1(\gamma([0,T])) \le \mathscr{H}^1(\gamma([0,T_1]))$. Let $T_\gamma := \inf\{T_1 > 0 \colon \gamma(T_1) = \gamma(0)\} \cup \{T\}$. Since $\gamma([0,T])$ has positive length, it holds that $T_\gamma > 0$. □

Lemma 8.2. *If u solves the transport equation*

$$\partial_t(\rho u) + \operatorname{div}(\rho u b) = 0$$

with the initial condition $u(0,\cdot)$ then $[0,T] \ni t \mapsto u(t,\gamma(t))$ is constant for η-a.e. γ.

Proof. It is clear that for any $\gamma \in \dot\Gamma$ it holds that $[0,T] \ni t \mapsto u(t,\gamma(t))$ is constant. Hence it is sufficient to consider only $\widetilde\Gamma$.

Let N be the set given by Lemma 8.1. Let $\gamma \in \widetilde\Gamma \setminus N$.

Consider some $\sigma \in \mathbb{Q} \cap (0,T)$ and $\theta \in \mathbb{Q} \cap (\sigma, T)$. Let C be a connected component of some regular level set of $H_{B,\sigma,\theta}$.

Disintegrating the transport equation along the level sets of $H_{B,\sigma,\theta}$ we get

$$\partial_t(\hat u \hat c_h |\hat b|) + \partial_s(\hat u \hat c_h |\hat b|) = 0.$$

(we can remove an η-negligible set of curves which pass through the regular level sets along which this equation does not hold.)

Hence for any $s \in [0,T]$ such that $\gamma(s) \in C$ there exists $\delta > 0$ such that

$$u(t,\gamma(t)) = u(s,\gamma(s))$$

for any $t \in (s-\delta, s+\delta) \cap [0,T]$.

(γ without loss of generality starts from the set $E_{\sigma,\theta,B}$ for some σ, θ, B)

Covering the compact $[0,T]$ with finitely many open intervals we conclude. □

Lemma 8.3. *If $[0,T] \ni t \mapsto u(t,\gamma(t))$ is constant for η-a.e. γ, then u solves the transport equation with the initial condition $u(0,\cdot)$.*

Proof. If $\varphi = \varphi(t,x)$ is a smooth test function which vanishes at T then

$$\begin{aligned}
&\int (\rho u \varphi_t + \rho u b \nabla\varphi)\, dx\, dt + \int \rho(0,x) u(0,x) \varphi(0,x)\, dx \\
&= \int u(t,\gamma(t)) \partial_t \varphi(t,\gamma(t))\, d\eta(\gamma)\, dt + \int u(0,\gamma(0)) \varphi(0,\gamma(0))\, d\eta(\gamma) \\
&= \int u(0,\gamma(0)) \partial_t \varphi(t,\gamma(t))\, d\eta(\gamma)\, dt + \int u(0,\gamma(0)) \varphi(0,\gamma(0))\, d\eta(\gamma) \\
&= -\int u(0,\gamma(0)) \varphi(0,\gamma(0))\, d\eta(\gamma) + \int u(0,\gamma(0)) \varphi(0,\gamma(0))\, d\eta(\gamma) = 0.
\end{aligned}$$

□

Proposition 8.4. *if u solves transport then renormalization.*

Proof. Since u solve transport, by Lemma 8.2 the function $t \mapsto u(t, \gamma(t))$ is constant for η-a.e. γ.

Then for any $\beta \in C^1(\mathbb{R})$ the function $t \mapsto \beta(u(t, \gamma(t)))$ is constant for η-a.e. γ. Hence by Lemma 8.3 the function $\beta(u)$ solves transport. □

8.1. **Covering property of the regular level sets.** Let

$$\hat{E} := \cup_{B,s,t} E^{\star}_{B,s,t}. \tag{8.1}$$

The following covering property is a global analog of Lemma 2.9:

Lemma 8.5. *It holds that $E \supset \{b \neq 0\} \mod \mathscr{L}^2$.*

Proof. Since $E \supset \hat{E} \mod \mathscr{L}^2$ it is sufficient to prove that $\hat{E} \supset \{b \neq 0\}$ mod $\mathscr{L}^2$. Let $P := \{b \neq 0\} \setminus \hat{E}$. Then for any $B \in \mathscr{B}$ it holds that $P \subset \{\nabla H_B = 0\} \mod \mathscr{L}^2$. Since $b \neq 0$ on P and $\nabla H^\perp = r_B b$ it holds that $r_B = 0$ a.e. on P for all $B \in \mathscr{B}$. Then for any $B \in \mathscr{B}$

$$\begin{aligned}
0 &= \int_{P \cap B} r_B \, dx \\
&= \int_0^T \int \mathbb{1}_{P \cap B}(x) \rho_B(t,x) \, dx \, dt \\
&= \int_{\widetilde{\Gamma}} \int_0^T \mathbb{1}_{P \cap B}(\gamma(t)) \, d\eta(\gamma) \, dt,
\end{aligned}$$

hence η-a.e. $\gamma \in \widetilde{\Gamma}$ spends zero amount of time in $P \cap B$. Since B is arbitrary and $\mathscr{B}$ is countable, we can generalize this claim to the whole set P:

$$\int_{\widetilde{\Gamma}} \int_0^T \mathbb{1}_P(\gamma(t)) \, dt \, d\eta(\gamma) = 0. \tag{8.2}$$

By near incompressibility

$$\begin{aligned}
\mathscr{L}^2(P) &\leq C \int_0^T \int \mathbb{1}_P(x) \rho(t,x) \, dx \, dt \\
&= C \int_0^T \int_{\dot{\Gamma} \cup \widetilde{\Gamma}} \mathbb{1}_P(\gamma(t)) \, d\eta(\gamma) \, dt \\
&\overset{(*)}{=} C \int_0^T \int_{\dot{\Gamma}} \mathbb{1}_P(\gamma(t)) \, d\eta(\gamma) \, dt \\
&\overset{(**)}{=} C \int_0^T \int_{\dot{\Gamma}} \mathbb{1}_P(\gamma(t)) \mathbb{1}_{\{b=0\}}(\gamma(t)) \, d\eta(\gamma) \, dt \\
&\leq C \int_0^T \int \mathbb{1}_P(\gamma(t)) \mathbb{1}_{\{b=0\}}(\gamma(t)) \, d\eta(\gamma) \, dt \\
&\leq C \int_0^T \int \mathbb{1}_P(x) \mathbb{1}_{\{b=0\}}(x) \rho(t,x) \, dx \, dt \\
&\overset{(***)}{=} 0,
\end{aligned}$$

where

- (*) holds by (8.2),
- (**) holds because $1_{\{b=0\}}(\gamma(t)) = 0$ for any $t \in [0, T]$ and any $\gamma \in \dot{\Gamma}$,
- (***) holds because P and $\{b = 0\}$ are disjoint. □

Remark 8.6. If we always have WSP then actually $E = \{b \neq 0\}$ since on each $E_{B,s,t}$ it holds that $b \neq 0$ a.e.

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