EXTREMAL FACES OF THE RANGE OF A VECTOR MEASURE AND A THEOREM OF LYAPUNOV

Stefano Bianchini

S.I.S.S.A (Trieste)

December 1998

Abstract. A Theorem of Lyapunov states that the range $\mathcal{R}(\mu)$ of a non–atomic vector measure μ is compact and convex. In this paper we give a condition to detect the dimension of the extremal faces of $\mathcal{R}(\mu)$ in terms of the Radon–Nikodym derivative of μ with respect to its total variation $|\mu|$: namely $\mathcal{R}(\mu)$ has an extremal face of dimension less or equal to k if and only if the set $(x_1, \dots, x_{k+1})$ such that $f(x_1), \dots, f(x_{k+1})$ are linear dependent has positive $|\mu|^{\otimes(k+1)}$–measure. Decomposing the set X in a suitable way, we obtain $\mathcal{R}(\mu)$ as vector sum of sets which are strictly convex. This result allows us to study the problem of the description of the range of μ if μ has atoms, achieving an extension of Lyapunov's Theorem.

1991 *Mathematics Subject Classification.* Primary: 46G10, 28B05. Secondary: 52A20.

Key words and phrases. Exposed points, strictly convex, Lyapunov, range of a vector measure, atomic measures.

I wish to thank Carlo Mariconda for his very precious help. I wish also to thank Arrigo Cellina for his warm encouragement.

Introduction

A well known Theorem of Lyapunov states that the range $\mathcal{R}(\mu)$ of a non–atomic vector measure μ on a measurable space $(X, \mathcal{M})$ with values in $\mathbb{R}^n$ is compact and convex. We recall that by definition $\mathcal{R}(\mu)$ is the set $\{\mu(E) : \ E \in \mathcal{M}\}$.
In [6] Halkin proved a Lyapunov's type Theorem introducing the measurable sets $D^+(p) = \{x \in X : \ f(x) \cdot p > 0\}$, $D^0(p) = \{x \in X : \ f(x) \cdot p = 0\}$ where p in a vector in $\mathbb{R}^n \setminus \{0\}$, $f(\cdot)$ is the Radon–Nikodym derivative of μ with respect to its total variation $|\mu|$ and " $\cdot$ " is the scalar product in $\mathbb{R}^n$. These sets were used in [3] to obtain the following Theorem:

Theorem A. *The following equivalence holds:*

1) $\mathcal{R}(\mu)$ *is a n–dimensional strictly convex set;*
2) *For every p in $\mathbb{R}^n \setminus \{0\}$ the set $\{x \in X : \ f(x) \cdot p = 0\}$ is negligible;*
3) $\det[f(x_1), \cdots, f(x_n)] \neq 0$ $|\mu|^{\otimes n}$*–a.e. on* X^n.

Generalizing some results of [3] and [7], we extend the previous Theorem to the case of extremal faces of dimension k, $1 \leq k \leq n-1$ (see section 1 for the definition of extremal face). In fact we have:

Theorem B. *These conditions are equivalent:*

1) $\mathcal{R}(\mu)$ *has an extremal face of dimension less than or equal to k, $1 \leq k \leq n-1$;*
2) *There exists in $\mathbb{R}^n$ a k–dimensional subspace S_k such that $|\mu|(f^{-1}(S_k)) > 0$;*
3) $f(x_1) \wedge \cdots \wedge f(x_{k+1})$ *is* 0 *on a set of* $|\mu|^{\otimes(k+1)}$*–positive measure in* X^{k+1}*, where* " $\wedge$ " *denotes the external product in* $\mathbb{R}^n$.

The last point means that the set $(x_1, \ldots, x_{k+1})$ in X^{k+1} such that $f(x_1), \ldots, f(x_{k+1})$ are linear dependent has positive $|\mu|^{\otimes(k+1)}$–measure. This result is achieved in section 1.
In section 2 we present an application of these results to the decomposition of X into a sum of measurable sets. It is well know that a zonoid, i.e. the range of a vector measure, is decomposable. This means that it can be written as the vector sum of sets which are not homothetic to itself: in fact, if $\{Y_j\}_{j \in \mathbb{N}}$ is any measurable partition of X, $X = \bigcup_{j=1}^{\infty} Y_j$, then $\mathcal{R}(\mu)$ is the vector sum of $\mathcal{R}(\mu|_{Y_j})$. We decompose the space X into a sum of measurable sets,

$$X \ = \ X_n \cup \big(\bigcup_{i \in I_{n-1}} X^i_{n-1} \big) \cup \cdots \cup \big(\bigcup_{i \in I_1} X^i_1 \big),$$

such that the range of μ restricted to X^i_k is a compact k–dimensional strictly convex set: this means that its extremal faces are either points or the whole set $\mathcal{R}(\mu_{X^i_k})$.
We recall that a set $H \in \mathcal{M}$ is an atom of μ if $|\mu|(H) > 0$ and for all $A \in \mathcal{M}$ such that $A \subseteq H$, $|\mu|(A)$ is either 0 or $|\mu|(H)$. We call a measure non–atomic if it has no atoms, atomic if there is at least an atom, purely atomic if every non–negligible set E is a union of atoms of μ.

In section 3 we study the range of an atomic real measure. It is well known that the set $\{H \in \mathcal{M} : H \text{ is an atom of } \mu\}$ is countable. In the first part of this section we prove that even for an atomic vector measure $\mathcal{R}(\mu)$ is compact. However, in general $\mathcal{R}(\mu)$ is not convex if μ is atomic: it is sufficient to consider a purely atomic positive measure with just one atom H_1. In this case $\mathcal{R}(\mu) = \{0, \mu(H_1)\}$. Denoting with a_i the measure of the atom H_i, we show that the range of a purely atomic real measure is the set $K = \big\{x = \sum_{i=1}^{\infty} a_i s_i,\ s_i \in \{0,1\}\big\}$, and that the maximal gap in K is $\sup\big\{|a_i| - \sum_{|a_j|<|a_i|} |a_j|,\ i \in \mathbb{N}\big\}$. With these results we obtain a first extension of Lyapunov's Theorem to atomic real measures, simply filling the gaps in K with the non–atomic part of the measure μ, namely $\mu|_{X\setminus(\cup H_i)}$:

$$|\mu|(X \setminus (\cup H_i)) \geq \sup_{i\geq 1}\Big(|\mu|(H_i) - \sum_{j:|\mu|(H_j)<|\mu|(H_i)} |\mu|(H_j)\Big) \qquad (*).$$

In section 4 we apply the previous results to an atomic vector measure. We prove that $\mathcal{R}(\mu)$ is convex if and only if $\mathcal{R}(\mu)$ contains the 1–dimensional extremal faces of its convex envelope. Since we also prove that the decomposition of X given in section 2 is still valid, even if μ is atomic, we just need to check condition $(*)$ on each set X_1^i, achieving an extension of Lyapunov's Theorem.

1. Extremal faces of the range of a vector measure

We recall some properties of extremal faces of a convex set $D \subseteq \mathbb{R}^n$. These results can be found in [10].
A subset $D_1 \subseteq D$ is called extremal face if we have:

$$\forall y \in D_1,\ y_1,\ y_2 \in D \text{ and } y = \frac{(y_1+y_2)}{2} \quad \Longrightarrow \quad y_1,\ y_2 \in D_1.$$

By a cone we mean a convex subset C of $\mathbb{R}^n$ such that if $x \in C$ implies $\lambda x \in C$ for every positive λ; we call C a generating cone if $C \cup (-C) = \mathbb{R}^n$. Let $\Pi_C = C \cap (-C)$. In the following by cone we will mean a generating cone.
Any cone C induces a pseudo-order relation in $\mathbb{R}^n$: we say $x \leq_C y$ if $y - x \in C$. Note that it is possible to have x and $-x$ in C: this happens if and only if $x \in \Pi_C$.
The following result is well known:

Proposition 1.1. *Suppose that D is a convex and compact subset of $\mathbb{R}^n$. Then $D_1 \subseteq D$ is an extremal face if and only if there exists a generating cone C such that*

$$D_1 = \big\{y_1 \in D : y_1 \geq_C y \ \forall y \in D\big\}.$$

We say that D_1 is defined by the order relation induced by C. The properties of extremal faces of convex sets were used by Olech in [7] to characterize extremal subsets of

decomposable families of functions.
Let X be a set, $\mathcal{M}$ a σ–algebra of subsets of X, ν a positive non–atomic measure on $(X,\mathcal{M})$. A family K of functions in $L^1(X;\mathbb{R}^n)$ is called decomposable if for each pair u, $v \in K$ and any measurable $\chi : X \to \{0,1\}$ we have $\chi u + (1-\chi)v \in K$. We say that a subset K_1 of K is an extremal face in K if there exists a generating cone C such that

$$K_1 = \big\{u \in K : u(x) - v(x) \in C \ \nu\text{–a.e.} \ \ \forall v \in K, \ x \in X\big\}.$$

As it is shown in [7], the set $D = \{\int_X u(x)\, d\nu : u \in K\}$ is convex and the following result holds:

Proposition 1.2. *K_1 is an extremal face in K if and only if the set $D_1 = \{\int_X u(x)\, d\nu : u \in K_1\}$ is extremal in $D = \{\int_X v(x)\, d\nu : v \in K\}$. Moreover D_1 is the extremal face of D defined by the order relation induced by C.*

In section 1 and 2 we consider a non–atomic vector measure μ on $(X,\mathcal{M})$ with values in $\mathbb{R}^n$ and let $f = \frac{d\mu}{d|\mu|}$ be the density of μ with respect to its total variation $|\mu|$; we will denote by $|\mu|^{\otimes n}$ the n–product measure of $|\mu|$ on X^n. Unless the contrary is expressly stated, for A, B in X by $A \subseteq B$ we mean that $B \setminus A$ is $|\mu|$–negligible. The application of the previous results to the decomposable set $K = \{f\chi_A : A \in \mathcal{M}\}$ and $\nu = |\mu|$ gives the following result:

Proposition 1.3. *Let $\mathcal{M}_1$ a subset of $\mathcal{M}$. The set F defined as $F = \big\{\mu(B) : B \in \mathcal{M}_1 \subseteq \mathcal{M}\big\}$ is an extremal face of $\mathcal{R}(\mu) = \big\{\mu(A) : A \in \mathcal{M}\big\}$ if and only if there exists a cone C such that:*

$$\mathcal{M}_1 = \big\{E \in \mathcal{M} : f\chi_E - f\chi_A \in C \ |\mu|\text{–a.e.} \ \forall A \in \mathcal{M}, \ \text{on } X\big\}.$$

Moreover, F is the extremal face of $\mathcal{R}(\mu)$ defined by the order relation induced by C.

We give now a new characterization of the above set $\mathcal{M}_1$. By $f^{-1}(D)$ we denote the inverse image through f of a subset D of $\mathbb{R}^n$.

Theorem 1.4. *Let F be an extremal face for $\mathcal{R}(\mu)$ and C the corresponding cone so that $F = \big\{y_1 \in \mathcal{R}(\mu) : y_1 \geq_C y, \ \forall y \in \mathcal{R}(\mu)\big\}$. Then the following conditions are equivalent:*

a) *$\mu(E)$ belongs to F;*
b) *E satisfies the condition:*

$$f^{-1}(C \setminus \Pi_C) \subseteq E \subseteq f^{-1}(C). \tag{1}$$

Proof. Assume that $\mu(E) \in F$ for some $E \in \mathcal{M}$. By Proposition 1.3 for every $A \in \mathcal{M}$ we have $f\chi_E - f\chi_A \in C \quad |\mu|$–a.e.. If we choose $A = \emptyset$, then $f\chi_E \in C$ so that $E \subseteq f^{-1}(C)$. If A is the whole set X, then $f\chi_E - f = -f\chi_{(X\setminus E)} \in C$. Thus $X \setminus E \subseteq f^{-1}(-C)$, or equivalently $X\setminus f^{-1}(-C) = f^{-1}(C\setminus\Pi_C) \subseteq E$, since $X = f^{-1}(\mathbb{R}^n) = f^{-1}((-C)\cup(C\setminus\Pi_C))$. Conversely assume that (1) holds and consider $\varphi(x) = f(x)\chi_E(x) - f(x)\chi_A(x)$ for $A \in \mathcal{M}$. If x is in E, then $\varphi(x)$ may be 0 or $f(x)$, thus $\varphi(x) \in C$. If x is not in E, then $\varphi(x)$ may equal $-f(x)$ or 0. But by definition $f(x)$ is not in $C \setminus \Pi_C$, implying that $f(x)$ is in $-C$ because C is generating. Thus in any case $\varphi(x) \in C$. $\square$

Remark. With the notation of Proposition 1.3, Theorem 1.4 yields that

$$\mathcal{M}_1 = \big\{E \in \mathcal{M} : f^{-1}(C \setminus \Pi_C) \subseteq E \subseteq f^{-1}(C)\big\}.$$

We denote by $S_k \subseteq \mathbb{R}^n$ a linear subspace of dimension k and by Ω_k the family of these subspaces. We say that an extremal face F of a convex set K has dimension k if the smallest affine set that contains it has linear dimension k: in particular the dimension of a convex set K is the linear dimension of the smallest affine set containing it. It is easy to see that if C is the cone related to F as in Theorem 1.4, then Π_C has at least dimension k. We say that the cone C_F, corresponding to F, is minimal if Π_{C_F} has the same dimension of F and C_F is generating. We recall that if C is a generating cone and "$\cdot$" is the usual scalar product in $\mathbb{R}^n$, then there exists an orthonormal family of vectors $p_1, \ldots, p_k$ such that

$$\begin{aligned} C \ &= \big\{y \in \mathbb{R}^n : y \cdot p_1 > 0\big\} \cup \\ &\cup \big\{y \in \mathbb{R}^n : y \cdot p_1 = 0,\ y \cdot p_2 > 0\big\} \cup \cdots \cup \\ &\cup \big\{y \in \mathbb{R}^n : y \cdot p_1 = 0, \ldots, y \cdot p_{k-1} = 0, y \cdot p_k > 0\big\} \cup \Pi_C, \end{aligned} \tag{2}$$

where Π_C is defined as before:

$$\Pi_C = C \cap (-C) = \big\{y \in \mathbb{R}^n : y \cdot p_i = 0, i = 1, \ldots, k\big\}.$$

Conversely if a cone is defined as in (2), it is clearly generating.

Lemma 1.5. *The minimal cone C_F corresponding to an extremal face F of K exists.*

Proof. Let C be the cone corresponding to F as in Proposition 1.1; we can assume that C has the form (2). The only case to study is when the dimension of Π_C is greater than the dimension of F: let us suppose that the dimension of F is k and the dimension of Π_C is $k + k_1$ for some $k_1 > 0$. Let $S_k(F) \subseteq \Pi_C$ be the minimal subspace of $\mathbb{R}^n$ such that its translate contains F. Consider an orthonormal base $\{q_i\}_{i=1}^{k_1}$ in the orthogonal complement of $S_k(F)$ in Π_C, and define

$$\begin{aligned} C_F \ &= \big\{y \in \mathbb{R}^n : y \cdot p_1 > 0\big\} \cup \cdots \cup \\ &\cup \big\{y \in \mathbb{R}^n : y \cdot p_1 = 0, \ldots, y \cdot p_{n-k-k_1} > 0\big\} \cup \\ &\cup \big\{y \in \mathbb{R}^n : y \cdot p_1 = 0, \ldots, y \cdot p_{n-k-k_1} = 0, y \cdot q_1 > 0\big\} \cup \cdots \cup \\ &\cup \big\{y \in \mathbb{R}^n : y \cdot p_1 = 0, \ldots, y \cdot p_{n-k-k_1} = 0, y \cdot q_1 = 0, \ldots, y \cdot q_{k_1} > 0\big\} \cup \Pi_{C_F}, \end{aligned}$$

where

$$\begin{aligned}\Pi_{C_F} &= C_F \cap (-C_F) \\ &= \big\{ y \in \mathbb{R}^n : y \cdot p_i = 0,\ y \cdot q_j = 0,\ i = 1, \dots, n-k-k_1,\ j = 1, \dots, k_1 \big\}.\end{aligned}$$

It is obvious that C_F is generating and by construction

$$\Pi_{C_F} = S_k(F).$$

Finally it is easy to show that $F = \big\{ y_1 \in \mathcal{R}(\mu) : y_1 \geq_{C_F} y,\ \forall y \in \mathcal{R}(\mu) \big\}$. $\square$

In what follows a closed convex subset C of $\mathbb{R}^n$ is said to be strictly convex if it has no non–trivial (e.g. different from a point and from $\mathcal{R}(\mu)$) extremal faces, i.e. if its only proper extremal faces are points.
Let $k \in \{1, \dots, n-1\}$. The following corollary is an easy consequence of the previous results.

Corollary 1.6. *$\mathcal{R}(\mu)$ has a non–trivial extremal face of dimension less than or equal to k if and only if there exists a k–dimensional subspace S_k in Ω_k such that $|\mu|(f^{-1}(S_k)) > 0$.*

Remark. By Lyapunov's Theorem $\mathcal{R}(\mu)$ is compact and convex, so that it does have at least an extreme point. However $|\mu|(f^{-1}(0)) = 0$ since $|f| = 1$ and 0 is the linear dimension of a point.

Proof. Assume that $|\mu|(f^{-1}(S_k)) = 0$ for every k–dimensional subspace $S_k \in \Omega_k$. Consider an extremal face F with dimension l, $0 < l \leq k$, and let C_F be its corresponding minimal cone. Lemma 1.5 shows that C_F exists and Π_{C_F} has dimension less or equal than k. By Theorem 1.4, $\mu(E)$ belongs to F if and only if $f^{-1}(C_F \setminus \Pi_{C_F}) \subseteq E \subseteq f^{-1}(C_F)$. Since $|\mu|\big(f^{-1}(\Pi_{C_F})\big) = 0$, it follows that $E = f^{-1}(C_F)$, so that F is reduced to one point.
Conversely, if there exists an $S_k \in \Omega_k$ such that $|\mu|(f^{-1}(S_k)) > 0$, there are two subsets A_1, A_2 in $f^{-1}(S_k)$ such that $\mu(A_1) \neq \mu(A_2)$. Consider a orthonormal base $\{p_i\}_{i=1}^{n-k}$ in $S_k^{\perp}$, the orthogonal complement of S_k, and let C be the generating cone defined by the family $\{p_i\}_{i=1}^{n-k}$ as in (2):

$$\begin{aligned} C &= \big\{ y \in \mathbb{R}^n : y \cdot p_1 > 0 \big\} \cup \\ &\cup \big\{ y \in \mathbb{R}^n : y \cdot p_1 = 0,\, y \cdot p_2 > 0 \big\} \cup \cdots \cup \\ &\cup \big\{ y \in \mathbb{R}^n : y \cdot p_1 = 0, \dots, y \cdot p_{k-1} = 0, y \cdot p_k > 0 \big\} \cup S_k. \end{aligned}$$

It is easy to show that the two sets

$$E_1 = f^{-1}(C \setminus S_k) \cup A_1 \quad \text{and} \quad E_2 = f^{-1}(C \setminus S_k) \cup A_2$$

correspond to different points on the face determined by C_{h_k}. $\square$

Remark. Corollary 1.6 is a generalization of Proposition 3.1 in [3], stating that $\mathcal{R}(\mu)$ is a n–dimensional strictly convex set if and only if $|\mu|(\{x : p \cdot f(x) = 0\}) = 0$ for all $p \in (\mathbb{R}^n \backslash 0)$; in fact the orthonormal space of a non–zero vector p is a $(n-1)$–dimensional subspace.

Notations. For $u_1, \dots, u_m$ in $\mathbb{R}^n$ we denote by $<u_1, \dots, u_m>$ the vector space spanned by $u_1, \dots, u_m$ and by $<u_1, \dots, u_m>^{\perp}$ its orthogonal space.

We denote by $u_1 \wedge \cdots \wedge u_m$ the external product on $\mathcal{R}^n$ and by $|u_1 \wedge \cdots \wedge u_m|$ its norm, i.e. the square root of the Gramian of $u_1, \dots, u_m$ $(m \leq n)$; we recall that the latter is the sum of the squares of the minors of order m of the matrix $(e_i \cdot u_j)_{i,j}$ (where $(e_i)_i$ is the standard basis in $\mathbb{R}^n$) and that $u_1, \dots, u_m$ are linearly dependent if and only if their Gramian vanishes. We write $\{u_1, \dots, \hat{u}_i, \dots, u_m\} = \{u_j : 1 \leq j \leq m,\ j \neq i\}$.

We introduce the subset Δ_k of X^k defined by

$$\Delta_k = \{(x_1, \dots, x_k) \in X^k :\ f(x_1) \wedge \cdots \wedge f(x_k) = 0\}.$$

Theorem 1.7. *$\mathcal{R}(\mu)$ has a non–trivial extremal faces of dimension less than or equal to k, $1 \leq k \leq (n-1)$, if and only if $|\mu|^{\otimes(k+1)}(\Delta_{k+1}) > 0$.*

Remark. Theorem 1.7 generalizes Theorem 3.2 of [3] stating that $\mathcal{R}(\mu)$ is a n–dimensional strictly convex set if and only if $|\mu|^{\otimes n}(\Delta_n) = 0$.

Proof. If there exists a l–dimensional face F on $\mathcal{R}(\mu)$, with $l \in \{1, \dots, k\}$, then by Corollary 1.6 we can find a k-dimensional subspace S_k in Ω_k such that $f^{-1}(S_k)$ is non–negligible; since $(f^{-1}(S_k))^{k+1} \subseteq \Delta_{k+1}$ then we obtain

$$|\mu|^{\otimes(k+1)}(\Delta_{k+1}) \geq \left[|\mu|(f^{-1}(S_k))\right]^{k+1} > 0.$$

We will prove now the opposite implication. For each set $S \subseteq X^{k+1}$ and $(x_2, \dots, x_{k+1})$ in X^k we set $S(x_2, \dots, x_{k+1}) = \{x_1 \in X : (x_1, \dots, x_{k+1}) \in S\}$. The measurability of the set

$$B = \{(x_1, \dots, x_{k+1}) \in X^{k+1} : f(x_1) \in <f(x_2), \dots, f(x_{k+1})>\}$$

is shown in [3]. Fubini's theorem gives

$$|\mu|^{\otimes(k+1)}(\Delta_{k+1}) = \int_{X^k} \left\{ \int_{\Delta_{k+1}(x_2, \dots, x_n)} d|\mu|(x_1) \right\} d(|\mu|(x_2) \otimes \cdots \otimes |\mu|(x_{k+1})).$$

Assume that $\mathcal{R}(\mu)$ has no l–dimensional faces, with $l \in \{1, \dots, k\}$; then Corollary 1.6 yields

$$|\mu|(\{x_1 \in X : f(x_1) \in <f(x_2), \dots, f(x_{k+1})>\}) = 0$$

so that if Δ^1_{k+1} is the subset of Δ_{k+1} defined by

$$\Delta^1_{k+1} = \{(x_1, \dots, x_{k+1}) \in \Delta_{k+1} : f(x_1) \notin <f(x_2), \dots, f(x_{k+1})>\}$$

from the above formula we obtain

$$|\mu|^{\otimes(k+1)}(\Delta)=\int_{X^k}\left\{\int_{\Delta^1_{k+1}(x_2,\dots,x_{k+1})}d|\mu|(x_1)\right\}d(|\mu|(x_2)\otimes\cdots\otimes|\mu|(x_{k+1})).$$

The set Δ^1_{k+1} being measurable, Tonelli's Theorem yields

$$|\mu|^{\otimes(k+1)}(\Delta_{k+1})=|\mu|^{\otimes(k+1)}(\Delta^1_{k+1}).$$

Similarly if for i in $\{2,\dots,k+1\}$ we put

$$\Delta^i_{k+1}=\{(x_1,\dots,x_{k+1})\in\Delta_{k+1}: f(x_i)\notin<f(x_1),\dots,\widehat{f(x_i)}\dots,f(x_{k+1})>\}$$

the same arguments give $|\mu|^{\otimes(k+1)}(\Delta_{k+1})=|\mu|^{\otimes(k+1)}(\Delta^i_{k+1})$. As a consequence we have

$$|\mu|^{\otimes(k+1)}(\Delta_{k+1})=|\mu|^{\otimes(k+1)}\left(\bigcap_{i=1}^{k+1}\Delta^i_{k+1}\right).$$

Obviously the set $\bigcap_{i=1}^{k+1}\Delta^i_{k+1}$ is empty; the conclusion follows. □

2. A decomposition of X

In [4] it is shown that a zonoid, i.e. the range of a vector measure, is decomposable. This means that $\mathcal{R}(\mu)$ can be decomposed into the vector sum of convex sets D_i, $i=1,2,\dots$, where D_i is not homothetic to $\mathcal{R}(\mu)$ for all i; in [4] this criterion was used to decide whether a convex set can be a zonoid or not. In this section we give a decomposition of X,

$$X \;=\; X_n\cup\Big(\bigcup_{i\in I_{n-1}}X^i_{n-1}\Big)\cup\cdots\cup\Big(\bigcup_{i\in I_1}X^i_1\Big),$$

that reflects the structure of the extremal faces of $\mathcal{R}(\mu)$. Actually we will decompose $\mathcal{R}(\mu)$ as the sum of the ranges of the measure μ restricted to the sets X^i_k in such a way that $\mathcal{R}(\mu|_{X^i_k})$ is a strictly convex set of dimension k. The following lemma is the base of our construction.

Lemma 2.1. *Suppose that there exists a $k\in\{0,\dots,n-1\}$ such that $|\mu|\big(f^{-1}(S_k)\big)=0$ for all subspace S_k in Ω_k. Then:*

a) *The set $N_{k+1}=\big\{S_{k+1}\in\Omega_{k+1}:\ |\mu|\big(f^{-1}(S_k)\big)>0\big\}$ is at most countable;*

b) *If X^{k+1} is defined as*

$$X^{k+1}=X\backslash\Big(\bigcup_{S_{k+1}\in N_{k+1}}f^{-1}(S_{k+1})\Big)$$

then $\mathcal{R}\big(\mu|_{X^{k+1}}\big)$ has no extremal faces of dimension l, $1\le l\le k+1$;

c) *$\mathcal{R}\big(\mu|_{f^{-1}(S_{k+1})}\big)$ is a strictly convex set of dimension k for all S_{k+1} in N_{k+1}.*

Proof. If we consider two distinct $(k+1)$–dimensional subspaces S^1_{k+1}, S^2_{k+1} of $\mathbb{R}^n$, the dimension of $S^1_{k+1} \cap S^2_{k+1}$ is strictly less than $k+1$; it follows that

$$|\mu|\big(f^{-1}(S^1_{k+1}) \cap f^{-1}(S^2_{k+1})\big) = |\mu|\big(f^{-1}(S^1_{k+1} \cap S^2_{k+1})\big) = 0.$$

The sets $\big\{f^{-1}(S_{k+1})\big\}_{S_{k+1} \subseteq \Omega_{k+1}}$ are then disjoint $|\mu|$–a.e. and this implies that the number of different linear subspaces S_{k+1} with $|\mu|(f^{-1}(S_{k+1})) > \epsilon$ is less than $\frac{|\mu|(X)}{\epsilon}$. As a consequence the set N_{k+1} is at most countable. By the σ–additivity of $\mathcal{M}$, the set

$$X^{k+1} = X \setminus \Big(\bigcup_{S_{k+1} \in N_{k+1}} f^{-1}(S_{k+1}) \Big)$$

is measurable and the application of Corollary 1.6 to $\mu|_{X^{k+1}}$ and $\mu|_{f^{-1}(S_{k+1})}$, with $S_{k+1} \in N_{k+1}$, completes the proof. $\square$

Remark. We note that for $k=0$ the fact that $|\mu|(f^{-1}(0)) = 0$ is always true: it follows that the number of one dimensional subspace S_1 in $\mathbb{R}^n$ such that $|\mu|(f^{-1}(S_1)) > 0$ is at most countable. Moreover we observe that we have used the fact that μ is non–atomic only in the application of Corollary 1.6. Thus part a) of Lemma 2.1 is valid even if μ is atomic.

Theorem 2.2. *Suppose that $(X, \mathcal{M})$ is a measurable space and μ is a non–atomic vector measure with values in $\mathbb{R}^n$. There exists a decomposition of the space X,*

$$\begin{aligned} X &= \big(\bigcup_{i \in I_l} X^i_l\big) \cup \cdots \cup \big(\bigcup_{i \in I_1} X^i_1\big) \qquad \textit{or} \\ X &= X_n \cup \big(\bigcup_{i \in I_{n-1}} X^i_{n-1}\big) \cup \cdots \cup \big(\bigcup_{i \in I_1} X^i_1\big), \end{aligned} \tag{3}$$

with $l \leq n-1$, such that:

a) *the sets I_k are at most countable;*
b) *the sets X^i_k are disjoint and $\mathcal{R}(\mu|_{X^i_k})$ is contained in a k–dimensional linear space S^i_k;*
c) *$\mathcal{R}(\mu|_{X^i_k})$ is a strictly convex k–dimensional zonoid;*
d) *If $|\mu|(X_n) > 0$, then $\mathcal{R}(\mu|_{X_n})$ is a strictly convex n–dimensional zonoid.*

Proof. By Lemma 2.1 and the subsequent remark, the set N_1 is at most countable. Let us write $N_1 = \{S^i_1\}_{i \in I_1}$ and define $X^i_1 = f^{-1}(S^i_1)$ for all i in I_1. Lemma 2.1 ensures that $\mathcal{R}(\mu|_{X^i_1})$ is a strictly convex 1–dimensional zonoid. Let $X^2 = X \setminus \big(\bigcup_{i \in I_1} X^i_1\big)$. If X^2 is empty, then the Theorem is proved; otherwise Lemma 2.1 can be applied to X^2 with

$k = 1$, $\mu_2 = \mu|_{X^2}$. It is obvious that if $f_2(\cdot)$ is the Radon–Nikodym derivative of μ_2 with respect to its total variation, then $f_2(x) = f(x)$ for all $x \in X^2$. Thus by construction for all subspaces S_1 in Ω_1 we have $|\mu_2|(f_2^{-1}(S_1)) = |\mu|(f^{-1}(S_1) \cap X_2) = 0$, and by Lemma 2.1 we obtain that the set $N_2 = \{S_2 \in \Omega_2 : |\mu_2|(f_2^{-1}(S_2)) > 0\}$ is at most countable: let as denote this family by $N_2 = \{S_2^i\}_{i \in I_2}$. If $X^3 = X \setminus \left[\left(\bigcup_{i \in I_2} X_2^i\right) \cup \left(\bigcup_{i \in I_1} X_1^i\right)\right]$ is empty, then the theorem is proved. Otherwise we proceed with this construction until either we stop at an index $k = l < n$, so that

$$X \;=\; \big(\bigcup_{i \in I_l} X_l^i\big) \cup \cdots \cup \big(\bigcup_{i \in I_1} X_1^i\big),$$

or, if we define

$$X_n \;=\; X \setminus \Big[\big(\bigcup_{i \in I_{n-1}} X_{n-1}^i\big) \cup \cdots \cup \big(\bigcup_{i \in I_1} X_1^i\big)\Big],$$

we have the following decomposition of X:

$$X \;=\; X_n \cup \big(\bigcup_{i \in I_{n-1}} X_{n-1}^i\big) \cup \cdots \cup \big(\bigcup_{i \in I_1} X_1^i\big).$$

Lemma 2.1 ensures that each $\mathcal{R}(\mu|_{X_k^i})$ is a strictly convex k–dimensional zonoid and that $\mathcal{R}(\mu|_{X_n})$ is a strictly convex n–dimensional zonoid in $\mathbb{R}^n$, if $|\mu|(X_n) > 0$. $\square$

Remark. A consequence of the decomposition of Theorem 2.2 is that if F is a k–dimensional strictly convex extremal face of $\mathcal{R}(\mu)$, with $k \geq 1$, then F is the translate of some $\mathcal{R}(\mu_{X_k^i})$. In fact Theorem 1.4 shows that if C_F is the minimal cone corresponding to F, then

$$F = \mu(C_F \setminus \Pi_{C_F}) + \mathcal{R}(\mu|_{f^{-1}(\Pi_{C_F})})$$

and that $|\mu|(f^{-1}(\Pi_{C_F})) > 0$. Since F is strictly convex, Corollary 1.6 implies that $|\mu|(f^{-1}(\Pi_{C_F}) \cap X_l^i) = 0$ for all $i \in I_l$, $l = 1, \ldots, k-1$. As a consequence $f^{-1}(\Pi_{C_F}) = X_k^i$ for some X_k^i.

3. On atomic real measures

In this section we study the range of an atomic real measure. We recall that a set $H \in \mathcal{M}$ is an atom of μ if $|\mu|(H) > 0$ and for all $A \in \mathcal{M}$ such that $A \subseteq H$, $|\mu|(A)$ is either 0 or $|\mu|(H)$. The main result of this section is an extension of Lyapunov's Theorem to atomic real measures.

As a preliminary step, consider a sequence of real vectors $a = (a_i)_{i \in \mathbb{N}}$, $a_i \in \mathbb{R}^n$, such that

$\sum_{i\in\mathbb{N}} |a_i| < \infty$, where $|a_i|$ is the Euclidean norm of the vector a_i. To every sequence $a = (a_i)_{i\in\mathbb{N}}$ we associate the following map from the Cantor set $\{0,1\}^{\mathbb{N}}$ in $\mathbb{R}^n$:

$$\begin{aligned} h_a: \quad \{0,1\}^{\mathbb{N}} \quad &\to \quad \mathbb{R}^n \\ s = \{s_i\}_{i\in\mathbb{N}} \quad &\to \; h(s) = \sum_{i=1}^{\infty} a_i s_i. \end{aligned} \tag{4}$$

If the sequence a is finite, we suppose $a_i = 0$ from an index onwards. Define $K_a = h_a(\{0,1\}^{\mathbb{N}})$, i.e. the image through h_a of the Cantor set. We prove the following Lemma.

Lemma 3.1. *If $\sum_{i\in\mathbb{N}} |a_i| < \infty$, where $|a_i|$ is the Euclidean norm of the vector a_i, then K_a is a compact set in $\mathbb{R}^n$.*

Proof. If $\{0,1\}^{\mathbb{N}}$ is equipped with the usual product topology, then $\{0,1\}^{\mathbb{N}}$ is a compact space. Since $\sum_{i=1}^{\infty} |a_i| < \infty$, it is easy to prove that h_a is continuous. As a conseguence, $h_a(\{0,1\}^{\mathbb{N}})$ is compact. $\square$

In the following, we will use some result on atomic measures; these results can be found in [1]. The first Theorem of this section is the application of Corollary 5.2.13 of [1] to $|\mu|$.

Theorem 3.2. *Let μ an atomic vector measure on $\mathcal{M}$. Then we can write*

$$X = \bigcup_{i=0}^{\infty} H_i \text{ or } X = \bigcup_{i=0}^{N} H_i \tag{5}$$

for some $N \geq 0$ with the following properties:

a) *the H_i are pairwise disjoint;*
b) *H_i is an atom of μ for every $i \geq 1$;*
c) *$\mu|_{H_0}$ is a non–atomic vector measure.*

Now we can extend Theorem 11.4.4 of [1] to vector measures.

Theorem 3.3. *Let μ any vector measure. Then $\mathcal{R}(\mu)$ is compact.*

Proof. Consider the decomposition of X given in Theorem 3.2. Since $\mu|_{H_0}$ is a non–atomic vector measure, its range is compact by Lyapunov's Theorem, and since the vector sum of two compact sets in $\mathbb{R}^n$ is compact, we just have to prove that $\mathcal{R}(\mu|_{X\setminus H_0})$ is compact.
Let $a_i = \mu(H_i)$ for every H_i with $i \geq 1$, and consider the sequence $a = (a_i)_{i\in\mathbb{N}}$. (If the atoms H_i are finite, we suppose $a_i = 0$ from an index onwards.) Since

$$\sum_{i=1}^{\infty} |a_n| = \sum_{i=1}^{\infty} |\mu(H_n)| = \sum_{i=1}^{\infty} |\mu|(H_n) = |\mu|(X \setminus H_0),$$

Lemma 3.1 assures that K_a is compact. Thus we are left to prove that $\mathcal{R}(\mu_{X\setminus H_0})$ is equal to the set K_a. If $s = \{s_i\}_{i\in\mathbb{N}}$ is in $\{0,1\}^{\mathbb{N}}$, then

$$h_a(s) = \sum_{i=1}^{\infty} a_i s_i = \sum_{i=1}^{\infty} \mu(H_i)s_i = \mu\Big(\bigcup_{i:s_i=1} H_i\Big),$$

so that $h_a(\{0,1\}^{\mathbb{N}})$ is contained in $\mathcal{R}(\mu|_{X\setminus H_0})$. Conversely, if A is a measurable set in $X \setminus H_0$, then by σ–additivity

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A \cap H_i) = \sum_{i=1}^{\infty} \mu(H_i)s_i(A),$$

where $s_i(A) = 1$ if and only if $\mu(A\cap H_i) = \mu(H_i)$. (We recall that $\mu(A\cap H_i)$ can be either 0 or $\mu(H_i)$.) Thus, defining $s(A) = (s_i(A))_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$, we have $\mu(A) = h_a(s(A))$. It follows that $K_a = h_a(\{0,1\}^{\mathbb{N}}) = \mathcal{R}(\mu|_{X\setminus H_0})$. $\square$

Let $\{a_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers such that $\sum_{i=1}^{\infty}|a_i| < \infty$. We wonder what is the maximal gap in the set $K_a = h_a\big(\{0,1\}^{\mathbb{N}}\big)$, i.e. the number $\inf\{\delta \geq 0 : K_a + [0,\delta]$ is connected $\}$. Lemma 3.1 shows that K_a is compact. Let m be the minimum of K_a: $m = \min\{x : x \in K_a\}$. Consider the function $\delta(x)$ on K_a defined as

$$\delta(x) = \begin{cases} 0 & x = m \\ \inf\big\{(x-y), y < x, y \in K_a\big\} & x > m \end{cases}$$

This function is upper semi-continuous, because if we consider the family of upper semi-continuous functions

$$\delta_y(x) = \begin{cases} 2(x-m) & x \leq y \\ x - y & x > y \end{cases}$$

then

$$\delta(x) = \inf\big\{\delta_y(x), y \in K_a\big\}.$$

We conclude that $\max\{\delta(x), x \in K_a\}$ exists and this value is the maximal gap in the set K_a. It is obvious that if $\delta(x)$ is 0 on K_a, then K_a is an interval.

Theorem 3.4. *If $\delta(x)$ and $\{a_i\}_{i\in\mathbb{N}}$ are defined as above, then*

$$\max\big\{\delta(x), x \in K_a\big\} = \max\Big\{|a_i| - \sum_{|a_j|<|a_i|} |a_j|, i \in \mathbb{N}\Big\}$$

if there exists an index i such that $|a_i| - \sum_{|a_j|<|a_i|}|a_j| > 0$; otherwise $\max\{\delta(x), x \in K_a\} = 0$.

Remark. This Theorem is a slight generalization of Exercise 131 in [9]. We note also that in both cases

$$\max\Big\{\delta(x), x \in K\Big\} = \sup\Big\{|a_i| - \sum_{|a_j|<|a_i|} |a_j|, i \in \mathbb{N}\Big\}. \tag{6}$$

In fact, if the numbers $|a_i| - \sum_{|a_j|<|a_i|} |a_j|$ are negative for all $i \in \mathbb{N}$, then, by the convergence of the series $\sum_{i\in\mathbb{N}} |a_i|$, for all $\epsilon > 0$ there exists an index i_0 such that $|a_{i_0}| < \epsilon$ and $\sum_{|a_j|<|a_{i_0}|} |a_j| < \epsilon$. For this index i_0 we have

$$-\epsilon < |a_{i_0}| - \sum_{|a_j|<|a_{i_0}|} |a_j| \leq 0,$$

so that $\sup\Big\{|a_i| - \sum_{|a_j|<|a_i|} |a_j|, i \in \mathbb{N}\Big\} = 0$. If there exists an index i_0 such that $|a_{i_0}| - \sum_{|a_j|<|a_{i_0}|} |a_j| > 0$, then (6) is a consequence of the statement of the Theorem.

Proof. For every real sequence $a = (a_i)_{i\in\mathbb{N}}$ such that $\sum_{i=1}^{\infty} |a_i| < \infty$, we define the sequence $\tilde{a} = (|a_i|)_{i\in\mathbb{N}}$. We start showing that $K_{\tilde{a}}$ is a translate of K_a. If the series is positive, the claim is trivial. Otherwise let m be the minimum of K_a: $m = \sum_{a_i<0} a_i$. To every x in K_a, $x = \sum_{i=1}^{\infty} a_i s_i(x)$, $s_i(x) \in \{0, 1\}$, we associate the point $\tilde{x}$ in $K_{\tilde{a}}$ defined as:

$$\begin{aligned}\tilde{x} &= \sum_{a_i>0} a_i s_i(x) - \sum_{a_i<0} a_i(1 - s_i(x)) \\ &= \sum_{i=0}^{\infty} a_i s_i(x) - \sum_{a_i<0} a_i = x - m.\end{aligned}$$

Conversely, if $\tilde{y}$ is in $K_{\tilde{a}}$, $\tilde{y} = \sum_{i=1}^{\infty} |a_i| s_i(\tilde{y})$, $s_i(\tilde{y}) \in \{0, 1\}$, consider the point in K_a:

$$\begin{aligned}y &= \sum_{a_i>0} |a_i| s_i(\tilde{y}) - \sum_{a_i<0} |a_i|(1 - s_i(\tilde{y})) \\ &= \sum_{a_i>0} |a_i| s_i(\tilde{y}) + \sum_{a_i<0} |a_i| s_i(\tilde{y}) - \sum_{a_i<0} |a_i| = \tilde{y} + m.\end{aligned}$$

These two formulae imply that $K_{\tilde{a}} = K_a - m$, so that the maximal gap in these two sets is the same. Without loss of generality, we can then suppose that the sequence is strictly positive and, by the absolute convergence, decreasing. In this case the numbers $|a_i| - \sum_{|a_j|<|a_i|} |a_j|$ become $a_i - \sum_{a_j<a_i} a_j$ and $j > i$, and since equation (6) is unaffected by terms less than zero, we have

$$\max\big\{\delta(x), x \in K\big\} = \sup\big\{a_i - \sum_{a_j<a_1} a_j, i \in \mathbb{N}\big\} = \sup\big\{a_i - \sum_{j=i+1}^{\infty} a_j, i \in \mathbb{N}\big\}. \tag{6'}$$

Define the series $c_i = a_i - \sum_{j=i+1}^{\infty} a_j$. If $c_i \leq 0$ for all $i \in \mathbb{N}$, then we can apply the result of Exercise 131 in [9], obtaining $\delta(x) = 0$.
Suppose now that there exists at least one c_i greater than 0. We begin showing that $\max_{i\in\mathbb{N}} c_i$ exists: this follows from the fact that the series is convergent. If $c_i > 2\epsilon$ for some ii, then there exists an index k such that $a_l < \epsilon$, $\sum_{j=l+1}^{\infty} a_j < \epsilon$ for all $l > k$; then

$$|c_l| = \left| a_l - \sum_{j=l+1}^{\infty} a_j \right| < 2\epsilon.$$

Thus the maximum is taken over a finite set of c_i 's.
Now we have to prove that there exists a point x in K_a such that $\delta(x)$ is equal to $c_{i_0} = \max_{i\in\mathbb{N}} c_i > 0$. Consider the point a_{i_0}: the nearest point y lower than a_{i_0} is obviously $y = \sum_{a_i < a_{i_0}} a_i$, and then $x - y = c_{i_0}$. $\square$

At this point we can prove the following extension of Lyapunov's Theorem to atomic real measures. We recall that a measure μ is said to admit a Hahn set if there exists a set $D \in \mathcal{M}$ such that $\mu(A \cap D) \leq 0$ and $\mu(A \cap (X \setminus D)) \geq 0$ for all $A \in \mathcal{M}$.

Theorem 3.5. *Let μ a real–valued measure on $(X, \mathcal{M})$ such that $|\mu|(X) < \infty$ and let $\{H_i\}_{i\geq 0}$ be the decomposition of X as in (5). Then $\mathcal{R}(\mu)$ is convex in $\mathbb{R}$ if and only if the following condition holds:*

$$|\mu|(H_0) \geq \sup_{i\geq 1} \Big(|\mu|(H_i) - \sum_{j:|\mu|(H_j)<|\mu|(H_i)} |\mu|(H_j) \Big). \tag{7}$$

Moreover $\mathcal{R}(\mu) = [-\mu^-(X), \mu^+(X)]$, where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ.

Proof. If we define $a_i = \mu(H_i)$, then we can apply Theorem 3.3 and the result is that the maximal gap c_0 in $\mathcal{R}(\mu|_{X\setminus H_0})$ is equal to

$$c_0 = \sup_{i\geq 1} \Big(|\mu|(H_i) - \sum_{j:|\mu|(H_j)<|\mu|(H_i)} |\mu|(H_j) \Big).$$

It is obvious that $\mathcal{R}(\mu) = \mathcal{R}(\mu|_{H_0}) + \mathcal{R}(\mu|_{X\setminus H_0})$ and by Lyapunov's Theorem $\mathcal{R}(\mu|_{H_0})$ is an interval of length $|\mu|(H_0)$. Thus $\mathcal{R}(\mu)$ is an interval if and only if $|\mu|(H_0)$ is greater of the maximal gap in $\mathcal{R}(\mu|_{X\setminus H_0})$, i.e.

$$|\mu|(H_0) \geq \sup_{i\geq 1} \Big(|\mu|(H_i) - \sum_{j:|\mu|(H_j)<|\mu|(H_i)} |\mu|(H_j) \Big).$$

Since μ admits a Hahn set, obviously choosing the set $D = \{x \in X : f(x) \leq 0\}$, by Proposition 11.4.6 of [1] we have

$$\alpha = \sup_{A \in \mathcal{M}} \mu(A) = \ \mu^+(X) \ \in \ \mathcal{R}(\mu)$$
$$\beta = \inf_{A \in \mathcal{M}} \mu(A) = -\mu^-(X) \ \in \ \mathcal{R}(\mu),$$

and thus $\mathcal{R}(\mu) = [-\mu^-(X), \mu^+(X)]$. $\square$

4. An extension of Lyapunov's Theorem

In this last section we study the range of an atomic vector measure μ and we extend Theorem 3.4 to vector measures. The first step is the following Lemma, that relates the decomposition (3) of Theorem 2.2 with (5) of Theorem 3.2.

Lemma 4.1. *Let μ be an atomic vector measure. Then there exists a decomposition of X,*

$$\begin{aligned} X &= \Big(\bigcup_{i\in I_l} X_l^i\Big) \cup \cdots \cup \Big(\bigcup_{i \in I_1} X_1^i\Big) \qquad or \\ X &= X_n \cup \Big(\bigcup_{i\in I_{n-1}} X_{n-1}^i\Big) \cup \cdots \cup \Big(\bigcup_{i\in I_1} X_1^i\Big), \end{aligned} \tag{3'}$$

with $l \leq n-1$, such that:

a) *the sets I_k are at most countable;*
b) *the sets X_k^i are disjoint and $\mathcal{R}(\mu|_{X_k^i})$ is contained in a k–dimensional linear space S_k^i.*

Moreover if

$$X = \bigcup_{j=0}^{\infty} H_j \ or \ X = \bigcup_{j=0}^{N} H_j \tag{5'}$$

for some $N \geq 0$ is the decomposition of Theorem 3.2, then every atom H_j, $j \geq 1$, is contained in some X_1^i, $i \geq 0$.

Proof. By the remark following Lemma 2.1, the set $N_1 = \{S_1 \in \Omega_1 : |\mu|(f^{-1}(S_1) > 0\}$ is at most countable. Let us write $N_1 = \{S_1^i\}_{i\in I_1}$ and define $X_1^i = f^{-1}(S_1^i)$ for all i in I_1. We note that if H_i is an atom of μ, then $|\mu|\big(f^{-1}(<\mu(H_i)>)\big)$ is greater or equal to $|\mu|(H_i)$: then every atom of μ is contained in some X_1^i. (We recall that with $< u >$ we denote the span of the non–zero vector u.) Let $X^2 = X \setminus \big(\bigcup_{i\in I_1} X_1^i\big)$. If X^2 is empty, then the Lemma is proved: in fact $\mathcal{R}(\mu|_{X_1^i})$ is contained in S_1^i. Otherwise part a) of Lemma 2.1 can be applied to X^2 with $k = 1$, $\mu_2 = \mu|_{X^2}$. As in the proof of Theorem 2.2, we obtain that the set $N_2 = \{S_2 \in \Omega_2 : |\mu_2|(f_2^{-1}(S_2)) > 0\}$ is at most countable: let as denote this family by $N_2 = \{S_2^i\}_{i\in I_2}$. If $X^3 = X \setminus \left[\left(\bigcup_{i\in I_2} X_2^i\right) \cup \left(\bigcup_{i\in I_1} X_1^i\right)\right]$ is empty, then the Lemma is proved. Otherwise we proceed with this construction until either we stop at an index $k = l < n$, so that

$$X \;=\; \big(\bigcup_{i\in I_l} X_l^i\big) \cup \cdots \cup \big(\bigcup_{i\in I_1} X_1^i\big),$$

or, if we define

$$X_n \;=\; X \setminus \Big[\big(\bigcup_{i\in I_{n-1}} X_{n-1}^i\big) \cup \cdots \cup \big(\bigcup_{i\in I_1} X_1^i\big)\Big],$$

we have the following decomposition of X:

$$X \;=\; X_n \cup \big(\bigcup_{i\in I_{n-1}} X_{n-1}^i\big) \cup \cdots \cup \big(\bigcup_{i\in I_1} X_1^i\big).$$

Since by construction $X_k^i \subseteq f^{-1}(S_k^i)$ for some S_k^i, the conclusion follows. $\square$

Now we show that the convexity of $\mathcal{R}(\mu)$ follows form the convexity of $\mathcal{R}(\mu|_{X_1^i})$, $i \in I_1$. We recall that with X_1^i we denote a measurable subset of X such that $|\mu|(X_1^i) > 0$ and $X_1^i = f^{-1}(S_1^i)$, S_1^i being a 1–dimensional space in $\mathbb{R}^n$ and $f = \frac{d\mu}{d|\mu|}$. By formula (3') of Lemma 4.1 these subsets are at most countable: let us denote them by $\{S_1^i : i \in I_1\}$.

Theorem 4.2. *The range of a vector measure μ is convex if and only if the range of the restriction of μ to the sets X_1^i is convex for all $i \in I_1$.*

Proof. If $\mathcal{R}(\mu)$ is convex, choose a convex cone C^i such that $f^{-1}(\Pi_{C^i})$ is the set X_1^i, $i \in I_1$. The face corresponding to C^i is obviously a 1–dimensional strictly convex set, and, by the remark following Theorem 2.2, if we define $y = \mu(f^{-1}(C^i \setminus \Pi_{C^i}))$, then x belongs to F if and only if $x = y + \mu(E)$, $E \subseteq X_1^i$. If F is convex, it is obvious that $F - y = \mathcal{R}(\mu|_{X_1^i})$ is convex.
The converse follows immediately since the vector sum of convex sets is convex. Lemma 4.1 shows that every atom H_i, $i \geq 1$, is contained in some X_1^i, $i \in I_1$. As a consequence, if we define, as in the proof of Lemma 4.1, $X^2 = X \setminus (\cup_{i\in I_1} X_1^i)$, Lyapunov's Theorem yields that $\mathcal{R}(\mu|_{X^2})$ is convex. Since

$$\mathcal{R}(\mu) = \mathcal{R}(\mu|_{X^2}) + \sum_{i\in I_1} \mathcal{R}(\mu|_{X_1^i}),$$

and by assumption $\mathcal{R}(\mu|_{X_1^i})$ are convex, the conclusion follows. $\square$

At this point we can prove the main result.

Theorem 4.3. *Given a vector measure μ on the measurable space $(X,\mathcal{M})$, with values in $\mathbb{R}^n$, let $\{X_1^i\}_{i\in I_1}$ be the family of subsets of X such that $|\mu|(X_1^i)>0$ and $X_1^i=f^{-1}(S_1^i)$, S_1^i being a 1–dimensional space in $\mathbb{R}^n$ and $f=\frac{d\mu}{d|\mu|}$. Let $\{H_j^i\}_{j\geq 0}$ be the decomposition of X_1^i such that H_j^i, $j\geq 1$, are the atoms of $\mu|_{X_1^i}$ and $\mu|_{H_0^i}$ is a non–atomic vector measure. The range of μ is convex if and only if*

$$|\mu|(H_0^i)\geq \sup_{j\geq 1}\Big(|\mu|(H_j^i)-\sum_{k:|\mu|(H_k^i)<|\mu|(H_j^i)}|\mu|(H_k^i)\Big)$$

for all $i\in I_1$.

Proof. Let us consider a generic vector measure on the space $(X,\mathcal{M})$. By Theorem 4.2, the measure has a convex range if and only if $\mathcal{R}(\mu|_{X_1^i})$ is convex for all i in I_1. By Theorem 3.5, $\mathcal{R}(\mu|_{X_1^i})$ is convex if and only if

$$|\mu|(H_0^i)\geq \sup_{j\geq 1}\Big(|\mu|(H_j^i)-\sum_{k:|\mu|(H_k^i)<|\mu|(H_j^i)}|\mu|(H_k^i)\Big),$$

where the H_j^i are the decomposition of X_1^i as in Theorem 3.2. The Theorem is proved. □

References

1. K. P. Bhaskara Rao, M. Bhaskara Rao, *Theory of Charges*, Academic Press, London, 1983.
2. S. Bianchini, R. Cerf, C. Mariconda, *Two dimensional zonoids and Chebyshev measures*, J. Math. Anal. Appl. **211–2** (1997), 512–526.
3. S. Bianchini, C. Mariconda, *The vector measures whose range is strictly convex* (to appear).
4. E.D. Bolker, *A class of convex bodies*, Trans. Amer. Math. Soc. **145** (1969), 323–345.
5. R. Cerf, C. Mariconda, *Chebyshev measures*, Proc. Amer. Math. Soc. **125** (1997), 3321–3329.
6. H. Halkin, *On a generalization of a Theorem of Lyapunov*, J. Math. Anal. Appl. **10** (1965), 325–329.
7. C. Olech, *The Lyapunov Theorem: its extensions and applications*, Methods of Nonconvex Analysis (A. Cellina, eds.), vol. 1446, Springer–Verlag, Berlin, 1989, pp. 84–103.
8. C. Olech, *Lectures on the integration of set–valued functions*, preprint 44/87/M–S.I.S.S.A., (1987).
9. G. Pólya and G. Szegö, *Problems and Theorems in Analysis 1*, Springer Verlag, Berlin, German, 1978.
10. R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.
11. W. Rudin, *Functional Analysis*, Mc Graw-Hill, New York, 1991.
12. R. Schneider, *Measures in convex geometry*, Rend. Istit. Mat. Univ. Trieste (1996).
13. R.L. Wheeden and A. Zygmund, *Measure and Integral*, Pure and Applied Mathematics, Dekken, New York, 1977.

Stefano Bianchini, S.I.S.S.A. (I.S.A.S.), Via Beirut 2/4, 34013 Trieste, Italy
E-mail address: bianchin@sissa.it