INTERNATIONAL SCHOOL

FOR ADVANCED STUDIES

Trieste

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INTRODUCTION TO ALGEBRAIC TOPOLOGY AND ALGEBRAIC GEOMETRY

Notes of a course delivered during the academic year 2002/2003

La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si può intendere se prima non si impara a intender la lingua, e conoscer i caratteri, ne' quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro laberinto.

GALILEO GALILEI (from "Il Saggiatore")

Preface

These notes assemble the contents of the introductory courses I have been giving at SISSA since 1995/96. Originally the course was intended as introduction to (complex) algebraic geometry for students with an education in theoretical physics, to help them to master the basic algebraic geometric tools necessary for doing research in algebraically integrable systems and in the geometry of quantum field theory and string theory. This motivation still transpires from the chapters in the second part of these notes.

The first part on the contrary is a brief but rather systematic introduction to two topics, singular homology (Chapter 2) and sheaf theory, including their cohomology (Chapter 3). Chapter 1 assembles some basics fact in homological algebra and develops the first rudiments of de Rham cohomology, with the aim of providing an example to the various abstract constructions.

Chapter 5 is an introduction to spectral sequences, a rather intricate but very powerful computation tool. The examples provided here are from sheaf theory but this computational techniques is also very useful in algebraic topology.

I thank all my colleagues and students, in Trieste and Genova and other locations, who have helped me to clarify some issues related to these notes, or have pointed out mistakes. In this connection special thanks are due to Fabio Pioli. Most of Chapter 3 is an adaptation of material taken from [**2**]. I thank my friends and collaborators Claudio Bartocci and Daniel Hernández Ruipérez for granting permission to use that material. I thank Lothar Göttsche for useful suggestions and for pointing out an error and the students of the 2002/2003 course for their interest and constant feedback.

Genova, 4 December 2004

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Part 1

Algebraic Topology

CHAPTER 1

Introductory material

The aim of the first part of these notes is to introduce the student to the basics of algebraic topology, especially the singular homology of topological spaces. The future developments we have in mind are the applications to algebraic geometry, but also students interested in modern theoretical physics may find here useful material (e.g., the theory of spectral sequences).

As its name suggests, the basic idea in algebraic topology is to translate problems in topology into algebraic ones, hopefully easier to deal with.

In this chapter we give some very basic notions in homological algebra and then introduce the fundamental group of a topological space. De Rham cohomology is introduced as a first example of a cohomology theory, and is homotopic invariance is proved.

1. Elements of homological algebra

1.1. Exact sequences of modules. Let R be a ring, and let M, M', M'' be R-modules. We say that two R-module morphisms $i\colon M' \to M$, $p\colon M \to M''$ form an *exact sequence* of R-modules, and write

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0\,,$$

if i is injective, p is surjective, and $\ker p = \operatorname{Im} i$.

Example 1.1. Set $R = \mathbb{Z}$, the ring of integers (recall that $\mathbb{Z}$-modules are just abelian groups), and consider the sequence

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to 1 \tag{1.1}$$

where i is the inclusion of the integers into the complex numbers $\mathbb{C}$, while $\mathbb{C}^* = \mathbb{C} - \{0\}$ is the multiplicative group of nonzero complex numbers. The morphism exp is defined as $\exp(z) = e^{2\pi i z}$. The reader may check that this sequence is exact.

A morphism of exact sequences is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0
\end{array}$$

of R-module morphisms whose rows are exact.

1.2. Differential complexes. Let R be a ring, and M an R-module.

DEFINITION 1.2. *A differential on M is a morphism $d\colon M \to M$ of R-modules such that $d^2 \equiv d \circ d = 0$. The pair (M, d) is called a differential module.*

The elements of the spaces M, $Z(M, d) \equiv \ker d$ and $B(M, d) \equiv \operatorname{Im} d$ are called *cochains*, *cocycles* and *coboundaries* of (M, d), respectively. The condition $d^2 = 0$ implies that $B(M, d) \subset Z(M, d)$, and the R-module

$$H(M, d) = Z(M, d)/B(M, d)$$

is called the *cohomology group* of the differential module (M, d). We shall often write $Z(M)$, $B(M)$ and $H(M)$, omitting the differential d when there is no risk of confusion.

Let (M, d) and (M', d') be differential R-modules.

DEFINITION 1.3. *A morphism of differential modules is a morphism $f\colon M \to M'$ of R-modules which commutes with the differentials, $f \circ d' = d \circ f$.*

A morphism of differential modules maps cocycles to cocycles and coboundaries to coboundaries, thus inducing a morphism $H(f)\colon H(M) \to H(M')$.

PROPOSITION 1.4. *Let $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ be an exact sequence of differential R-modules. There exists a morphism $\delta\colon H(M'') \to H(M')$ (called connecting morphism) and an exact triangle of cohomology*

$$\begin{array}{ccc} H(M) & \xrightarrow{H(p)} & H(M'') \\ {\scriptstyle H(i)}\uparrow & \swarrow{\scriptstyle \delta} & \\ H(M') & & \end{array}$$

PROOF. The construction of δ is as follows: let $\xi'' \in H(M'')$ and let m'' be a cocycle whose class is ξ''. If m is an element of M such that $p(m) = m''$, we have $p(d(m)) = d(m'') = 0$ and then $d(m) = i(m')$ for some $m' \in M'$ which is a cocycle. Now, the cocycle m' defines a cohomology class $\delta(\xi'')$ in $H(M')$, which is independent of the choices we have made, thus defining a morphism $\delta\colon H(M'') \to H(M')$. One proves by direct computation that the triangle is exact. □

The above results can be translated to the setting of complexes of R-modules.

DEFINITION 1.5. *A complex of R-modules is a differential R-module $(M^\bullet, d)$ which is $\mathbb{Z}$-graded, $M^\bullet = \bigoplus_{n\in\mathbb{Z}} M^n$, and whose differential fulfills $d(M^n) \subset M^{n+1}$ for every $n \in \mathbb{Z}$.*

We shall usually write a complex of R-modules in the more pictorial form

$$\dots \xrightarrow{d_{n-2}} M^{n-1} \xrightarrow{d_{n-1}} M^n \xrightarrow{d_n} M^{n+1} \xrightarrow{d_{n+1}} \dots$$

For a complex $M^\bullet$ the cocycle and coboundary modules and the cohomology group split as direct sums of terms $Z^n(M^\bullet) = \ker d_n$, $B^n(M^\bullet) = \operatorname{Im} d_{n-1}$ and $H^n(M^\bullet) = Z^n(M^\bullet)/B^n(M^\bullet)$ respectively. The groups $H^n(M^\bullet)$ are called the *cohomology groups* of the complex $M^\bullet$.

DEFINITION 1.6. *A morphism of complexes of R-modules $f\colon N^\bullet \to M^\bullet$ is a collection of morphisms $\{f_n\colon N^n \to M^n \,|\, n \in \mathbb{Z}\}$, such that the following diagram commutes:*

$$\begin{array}{ccc} M^n & \xrightarrow{f_n} & N^n \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ M^{n+1} & \xrightarrow{f_{n+1}} & N^{n+1} \end{array} .$$

For complexes, Proposition 1.4 takes the following form:

PROPOSITION 1.7. *Let $0 \to N^\bullet \xrightarrow{i} M^\bullet \xrightarrow{p} P^\bullet \to 0$ be an exact sequence of complexes of R-modules. There exist connecting morphisms $\delta_n\colon H^n(P^\bullet) \to H^{n+1}(N^\bullet)$ and a long exact sequence of cohomology*

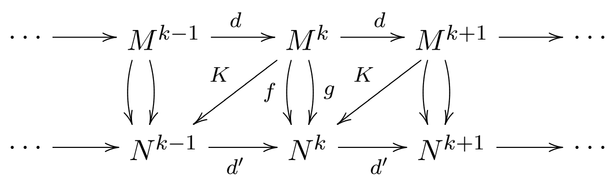
$$\dots \xrightarrow{\delta_{n-1}} H^n(N^\bullet) \xrightarrow{H(i)} H^n(M^\bullet) \xrightarrow{H(p)} H^n(P^\bullet) \xrightarrow{\delta_n}$$
$$\xrightarrow{\delta_n} H^{n+1}(N^\bullet) \xrightarrow{H(i)} H^{n+1}(M^\bullet) \xrightarrow{H(p)} H^{n+1}(P^\bullet) \xrightarrow{\delta_{n+1}} \dots$$

PROOF. The connecting morphism $\delta\colon H^\bullet(P^\bullet) \to H^\bullet(N^\bullet)$ defined in Proposition 1.4 splits into morphisms $\delta_n\colon H^n(P^\bullet) \to H^{n+1}(N^\bullet)$ (indeed the connecting morphism increases the degree by one) and the long exact sequence of the statement is obtained by developing the exact triangle of cohomology introduced in Proposition 1.4. □

1.3. Homotopies. Different (i.e., nonisomorphic) complexes may nevertheless have isomorphic cohomologies. A sufficient conditions for this to hold is that the two complexes are *homotopic*. While this condition is not necessary, in practice the (by far) commonest way to prove the isomorphism between two cohomologies is to exhibit a homototopy between the corresponding complexes.

DEFINITION 1.8. *Given two complexes of R-modules, $(M^\bullet, d)$ and $(N^\bullet, d')$, and two morphisms of complexes, $f, g\colon M^\bullet \to N^\bullet$, a homotopy between f and g is a morphism $K\colon N^\bullet \to M^{\bullet-1}$ (i.e., for every k, a morphism $K\colon N^k \to M^{k-1}$) such that $d' \circ K + K \circ d = f - g$.*

The situation is depicted in the following commutative diagram.



Proposition 1.9. *If there is a homotopy between f and g, then $H(f) = H(g)$, namely, homotopic morphisms induce the same morphism in cohomology.*

Proof. Let $\xi = [m] \in H^k(M^\bullet, d)$. Then

$$H(f)(\xi) = [f(m)] = [g(m)] + [d'(K(m))] + [K(dm)] = [g(m)] = H(g)(\xi)$$

since $dm = 0$, $[d'(K(m))] = 0$. □

Definition 1.10. *Two complexes of R-modules, $(M^\bullet, d)$ and $(N^\bullet, d')$, are said to be* homotopically equivalent *(or* homotopic*) if there exist morphisms $f\colon M^\bullet \to N^\bullet$, $g\colon N^\bullet \to M^\bullet$, such that:*

$f \circ g\colon N^\bullet \to N^\bullet$ is homotopic to the identity map id_N;

$g \circ f\colon M^\bullet \to M^\bullet$ is homotopic to the identity map id_M.

Corollary 1.11. *Two homotopic complexes have isomorphic cohomologies.*

Proof. We use the notation of the previous Definition. One has

$$H(f) \circ H(g) = H(f \circ g) = H(\mathrm{id}_N) = \mathrm{id}_{H(N)}$$

$$H(g) \circ H(f) = H(g \circ f) = H(\mathrm{id}_M) = \mathrm{id}_{H(M)}$$

so that both $H(f)$ and $H(g)$ are isomorphism. □

Definition 1.12. *A homotopy of a complex of R-modules $(M^\bullet, d)$ is a homotopy between the identity morphism on M, and the zero morphism; more explicitly, it is a morphism $K\colon M^\bullet \to M^{\bullet -1}$ such that $d \circ K + K \circ d = \mathrm{id}_M$.*

Proposition 1.13. *If a complex of R-modules $(M^\bullet, d)$ admits a homotopy, then it is exact (i.e., all its cohomology groups vanish; one also says that the complex is* acyclic*).*

Proof. One could use the previous definitions and results to yield a proof, but it is easier to note that if $m \in M^k$ is a cocycle (so that $dm = 0$), then

$$d(K(m)) = m - K(dm) = m$$

so that m is also a coboundary. □

Remark 1.14. More generally, one can state that if a homotopy $K\colon M^k \to M^{k-1}$ exists for $k \geq k_0$, then $H^k(M, d) = 0$ for $k \geq k_0$. In the case of complexes bounded below zero (i.e., $M = \oplus_{k \in \mathbb{N}} M^k$) often a homotopy is defined only for $k \geq 1$, and it may happen that $H^0(M,d) \neq 0$. Examples of such situations will be given later in this chapter.

REMARK 1.15. One might as well define a homotopy by requiring $d' \circ K - K \circ d = \dots$; the reader may easily check that this change of sign is immaterial.

2. De Rham cohomology

As an example of a cohomology theory we may consider the *de Rham* cohomology of a differentiable manifold X. Let $\Omega^k(X)$ be the vector space of differential k-forms on X, and let $d\colon \Omega^k(X) \to \Omega^{k+1}(X)$ be the exterior differential. Then $(\Omega^\bullet(X), d)$ is a differential complex of $\mathbb{R}$-vector spaces (the *de Rham* complex), whose cohomology groups are denoted $H^k_{dR}(X)$ and are called the *de Rham cohomology groups of* X. Since $\Omega^k(X) = 0$ for $k > n$ and $k < 0$, the groups $H^k_{dR}(X)$ vanish for $k > n$ and $k < 0$. Moreover, since $\ker[d\colon \Omega^0(X) \to \Omega^1(X)]$ is formed by the locally constant functions on X, we have $H^0_{dR}(X) = \mathbb{R}^C$, where C is the number of connected components of X.

If $f\colon X \to Y$ is a smooth morphism of differentiable manifolds, the pullback morphism $f^*\colon \Omega^k(Y) \to \Omega^k(X)$ commutes with the exterior differential, thus giving rise to a morphism of differential complexes $(\Omega^\bullet(Y), d) \to (\Omega^\bullet(X), d))$; the corresponding morphism $H(f)\colon H^\bullet_{dR}(Y) \to H^\bullet_{dR}(X)$ is usually denoted $f^\sharp$.

We may easily compute the cohomology of the Euclidean spaces $\mathbb{R}^n$. Of course one has $H^0_{dR}(\mathbb{R}^n) = \ker[d : C^\infty(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)] = \mathbb{R}$.

PROPOSITION 1.1. (Poincaré lemma) $H^k_{dR}(\mathbb{R}^n) = 0$ *for* $k > 0$.

PROOF. We define a linear operator $K\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$ by letting, for any k-form $\omega \in \Omega^k(\mathbb{R}^n)$, $k \geq 1$, and all $x \in \mathbb{R}^n$,

$$(K\omega)(x) = k \left[\int_0^1 t^{k-1} \omega_{i_1 i_2 \dots i_k}(tx)\, dt\right] x^{i_1}\, dx^{i_2} \wedge \dots \wedge dx^{i_k}\,.$$

One easily shows that $dK + Kd = \mathrm{Id}$; this means that K is a homotopy of the de Rham complex of $\mathbb{R}^n$ defined for $k \geq 1$, so that, according to Proposition 1.13 and Remark 1.14, all cohomology groups vanish in positive degree. Explicitly, if ω is closed, we have $\omega = dK\omega$, so that ω is exact. □

EXERCISE 1.2. Realize the circle S^1 as the unit circle in $\mathbb{R}^2$. Show that the integration of 1-forms on S^1 yields an isomorphism $H^1_{dR}(S^1) \simeq \mathbb{R}$. This argument can be quite easily generalized to show that, if X is a connected, compact and orientable n-dimensional manifold, then $H^n_{dR}(X) \simeq \mathbb{R}$.

If a manifold is a cartesian product, $X = X_1 \times X_2$, there is a way to compute the de Rham cohomology of X out of the de Rham cohomology of X_1 and X_2 (Künneth theorem, cf. [**3**]). For later use, we prove here a very particular case. This will serve also as an example of the notion of homotopy between complexes.

PROPOSITION 1.3. *If X is a differentiable manifold, then $H^k_{dR}(X \times \mathbb{R}) \simeq H^k_{dR}(X)$ for all $k \geq 0$.*

PROOF. Let t a coordinate on $\mathbb{R}$. Denoting by p_1, p_2 the projections of $X \times \mathbb{R}$ onto its two factors, every k-form ω on $X \times \mathbb{R}$ can be written as

$$\omega = f\, p_1^*\omega_1 + g\, p_1^*\omega_2 \wedge p_2^* dt \tag{1.2}$$

where $\omega_1 \in \Omega^k(X)$, $\omega_2 \in \Omega^{k-1}(X)$, and f, g are functions on $X \times \mathbb{R}$.[1] Let $s\colon X \to X \times \mathbb{R}$ be the section $s(x) = (x, 0)$. One has $p_1 \circ s = \mathrm{id}_X$ (i.e., s is indeed a section of p_1), hence $s^* \circ p_1^*\colon \Omega^\bullet(X) \to \Omega^\bullet(X)$ is the identity. We also have a morphism $p_1^* \circ s^*\colon \Omega^\bullet(X \times \mathbb{R}) \to \Omega^\bullet(X \times \mathbb{R})$. This is not the identity (as a matter of fact one, has $p_1^* \circ s^*(\omega) = f(x,0)\, p_1^*\omega_1$). However, this morphism is homotopic to $\mathrm{id}_{\Omega^\bullet(X\times\mathbb{R})}$, while $\mathrm{id}_{\Omega^\bullet(X)}$ is definitely homotopic to itself, so that the complexes $\Omega^\bullet(X)$ and $\Omega^\bullet(X \times \mathbb{R})$ are homotopic, thus proving our claim as a consequence of Corollary 1.11. So we only need to exhibit a homotopy between $p_1^* \circ s^*$ and $\mathrm{id}_{\Omega^\bullet(X\times\mathbb{R})}$.

This homotopy $K\colon \Omega^\bullet(X \times \mathbb{R}) \to \Omega^{\bullet-1}(X \times \mathbb{R})$ is defined as (with reference to equation (1.2))

$$K(\omega) = (-1)^k \left[\int_0^t g(x,s)\, ds\right] p_2^*\, \omega_2.$$

The proof that K is a homotopy is an elementary direct computation,[2] after which one gets

$$d \circ K + K \circ d = \mathrm{id}_{\Omega^\bullet(X\times\mathbb{R})} - p_1^* \circ s^*.$$

□

In particular we obtain that the morphisms

$$p_1^\sharp\colon H^\bullet_{dR}(X) \to H^\bullet_{dR}(X \times \mathbb{R}), \qquad s^\sharp\colon H^\bullet_{dR}(X \times \mathbb{R}) \to H^\bullet_{dR}(X \times)$$

are isomorphisms.

REMARK 1.4. If we take $X = \mathbb{R}^n$ and make induction on n we get another proof of Poincaré lemma.

EXERCISE 1.5. By a similar argument one proves that for all $k > 0$

$$H^k_{dR}(X \times S^1) \simeq H^k_{dR}(X) \oplus H^{k-1}_{dR}(X).$$ □

Now we give an example of a long cohomology exact sequence within de Rham's theory. Let X be a differentiable manifold, and Y a closed submanifold. Let $r_k\colon \Omega^k(X) \to$

[1] In intrinsic notation this means that

$$\Omega^k(X \times \mathbb{R}) \simeq C^\infty(X \times \mathbb{R}) \otimes_{C^\infty(X)} [\Omega^k(X) \oplus \Omega^{k-1}(X)].$$

[2] The reader may consult e.g. [**3**], §I.4.

$\Omega^k(Y)$ be the restriction morphism; this is surjective. Since the exterior differential commutes with the restriction, after letting $\Omega^k(X,Y) = \ker r_k$ a differential $d'\colon \Omega^k(X,Y) \to \Omega^{k+1}(X,Y)$ is defined. We have therefore an exact sequence of differential modules, in a such a way that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{k-1}(X,Y) & \longrightarrow & \Omega^{k-1}(X) & \xrightarrow{r_{k-1}} & \Omega^{k-1}(Y) & \longrightarrow & 0 \\ & & \downarrow d' & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & \Omega^{k}(X,Y) & \longrightarrow & \Omega^{k}(X) & \xrightarrow{r_{k}} & \Omega^{k}(Y) & \longrightarrow & 0 \end{array}$$

commutes. The complex $(\Omega^\bullet(X,Y), d')$ is called the relative de Rham complex, [3] and its cohomology groups by $H^k_{dR}(X,Y)$ are called the relative de Rham cohomology groups. One has a long cohomology exact sequence

$$\begin{aligned} 0 \quad \to \quad & H^0_{dR}(X,Y) \to H^0_{dR}(X) \to H^0_{dR}(Y) \xrightarrow{\delta} H^1_{dR}(X,Y) \\ \to \quad & H^1_{dR}(X) \to H^1_{dR}(Y) \xrightarrow{\delta} H^2_{dR}(X,Y) \to \dots \end{aligned}$$

EXERCISE 1.6. 1. Prove that the space $\ker d'\colon \Omega^k(X,Y) \to \Omega^{k+1}(X,Y)$ is for all $k \geq 0$ the kernel of r_k restricted to $Z^k(X)$, i.e., is the space of closed k-forms on X which vanish on Y. As a consequence $H^0_{dR}(X,Y) = 0$ whenever X and Y are connected.

2. Let $n = \dim X$ and $\dim Y \leq n-1$. Prove that $H^n_{dR}(X,Y) \to H^n_{dR}(X)$ surjects, and that $H^k_{dR}(X,Y) = 0$ for $k \geq n+1$. Make an example where $\dim X = \dim Y$ and check if the previous facts still hold true.

EXAMPLE 1.7. Given the standard embedding of S^1 into $\mathbb{R}^2$, we compute the relative cohomology $H^\bullet_{dR}(\mathbb{R}^2, S^1)$. We have the long exact sequence

$$\begin{aligned} 0 \quad \to \quad & H^0_{dR}(\mathbb{R}^2,S^1) \to H^0_{dR}(\mathbb{R}^2) \to H^0_{dR}(S^1) \xrightarrow{\delta} H^1_{dR}(\mathbb{R}^2,S^1) \\ \to \quad & H^1_{dR}(\mathbb{R}^2) \to H^1_{dR}(S^1) \xrightarrow{\delta} H^2_{dR}(\mathbb{R}^2,S^1) \to H^2_{dR}(\mathbb{R}^2) \to 0\,. \end{aligned}$$

As in the previous exercise, we have $H^k_{dR}(\mathbb{R}^2,S^1) = 0$ for $k \geq 3$. Since $H^0_{dR}(\mathbb{R}^2) \simeq \mathbb{R}$, $H^1_{dR}(\mathbb{R}^2) = H^2_{dR}(\mathbb{R}^2) = 0$, $H^0_{dR}(S^1) \simeq H^1_{dR}(S^1) \simeq \mathbb{R}$, we obtain the exact sequences

$$0 \to H^0_{dR}(\mathbb{R}^2,S^1) \to \mathbb{R} \xrightarrow{r} \mathbb{R} \to H^1_{dR}(\mathbb{R}^2,S^1) \to 0$$

$$0 \to \mathbb{R} \to H^2_{dR}(\mathbb{R}^2,S^1) \to 0$$

where the morphism r is an isomorphism. Therefore from the first sequence we get $H^0_{dR}(\mathbb{R}^2,S^1) = 0$ (as we already noticed) and $H^1_{dR}(\mathbb{R}^2,S^1) = 0$. From the second we obtain $H^2_{dR}(\mathbb{R}^2,S^1) \simeq \mathbb{R}$. □

From this example we may abstract the fact that whenever X and Y are connected, then $H^0_{dR}(X,Y) = 0$.

[3] Sometimes this term is used for another cohomology complex, cf. [**3**].

Exercise 1.8. Consider a submanifold Y of $\mathbb{R}^2$ formed by two disjoint embedded copies of S^1. Compute $H^\bullet_{dR}(\mathbb{R}^2, Y)$.

3. Mayer-Vietoris sequence in de Rham cohomology

The Mayer-Vietoris sequence is another example of long cohomology exact sequence associated with de Rham cohomology, and is very useful for making computations. Assume that a differentiable manifold X is the union of two open subset U, V. For every k, $0 \leq k \leq n = \dim X$ we have the sequence of morphisms

$$0 \to \Omega^k(X) \xrightarrow{i} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{p} \Omega^k(U \cap V) \to 0 \tag{1.3}$$

where

$$i(\omega) = (\omega_{|U}, \omega_{|V}), \qquad p((\omega_1, \omega_2)) = \omega_{1|U\cap V} - \omega_{2|U\cap V}.$$

One easily checks that i is injective and that $\ker p = \operatorname{Im} i$. The surjectivity of p is somehow less trivial, and to prove it we need a partition of unity argument. From elementary differential geometry we recall that a *partition of unity* subordinated to the cover $\{U, V\}$ of X is a pair of smooth functions $f_1, f_2 \colon X \to \mathbb{R}$ such that

$$\operatorname{supp}(f_1) \subset U, \qquad \operatorname{supp}(f_2) \subset V, \qquad f_1 + f_2 = 1.$$

Given $\tau \in \Omega^k(U \cap V)$, let

$$\omega_1 = f_2\,\tau, \qquad \omega_2 = -f_1\,\tau.$$

These k-form are defined on U and V, respectively. Then $p((\omega_1, \omega_2)) = \tau$. Thus the sequence (1.3) is exact. Since the exterior differential d commutes with restrictions, we obtain a long cohomology exact sequence

$$\begin{aligned} 0 \to H^0_{dR}(X) \to H^0_{dR}(U) \oplus H^0_{dR}(V) \to H^0_{dR}(U \cap V) \xrightarrow{\delta} H^1_{dR}(X) \to \\ \to H^1_{dR}(U) \oplus H^1_{dR}(V) \to H^1_{dR}(U \cap V) \xrightarrow{\delta} H^2_{dR}(X) \to \dots \end{aligned} \tag{1.4}$$

This is the Mayer-Vietoris sequence. The argument may be generalized to a union of several open sets.[4]

Exercise 1.1. Use the Mayer-Vietoris sequence (1.4) to compute the de Rham cohomology of the circle S^1.

Example 1.2. We use the Mayer-Vietoris sequence (1.4) to compute the de Rham cohomology of the sphere S^2 (as a matter of fact we already know the 0th and 2nd group, but not the first). Using suitable stereographic projections, we can assume that U and V are diffeomorphic to $\mathbb{R}^2$, while $U \cap V \simeq S^1 \times \mathbb{R}$. Since $S^1 \times \mathbb{R}$ is homotopic to S^1, it has the same de Rham cohomology. Hence the sequence (1.4) becomes

$$0 \to H^0_{dR}(S^2) \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to H^1_{dR}(S^2) \to 0$$

$$0 \to \mathbb{R} \to H^2_{dR}(S^2) \to 0.$$

[4]The Mayer-Vietoris sequence foreshadows the Čech cohomology we shall study in Chapter 3.

From the first sequence, since $H^0_{dR}(S^2) \simeq \mathbb{R}$, the map $H^0_{dR}(S^2) \to \mathbb{R} \oplus \mathbb{R}$ is injective, and then we get $H^1_{dR}(S^2) = 0$; from the second sequence, $H^2_{dR}(S^2) \simeq \mathbb{R}$.

EXERCISE 1.3. Use induction to show that if $n \geq 3$, then $H^k_{dR}(S^n) \simeq \mathbb{R}$ for $k = 0, n$, $H^k_{dR}(S^n) = 0$ otherwise.

EXERCISE 1.4. Consider $X = S^2$ and $Y = S^1$, embedded as an equator in S^2. Compute the relative de Rham cohomology $H^\bullet_{dR}(S^2, S^1)$.

4. Elementary homotopy theory

4.1. Homotopy of paths. Let X be a topological space. We denote by I the closed interval $[0, 1]$. A *path* in X is a continuous map $\gamma \colon I \to X$. We say that X is *pathwise connected* if given any two points x_1, $x_2 \in X$ there is a path γ such that $\gamma(0) = x_1$, $\gamma(1) = x_2$.

A *homotopy* Γ between two paths γ_1, γ_2 is a continuous map $\Gamma \colon I \times I \to X$ such that

$$\Gamma(t, 0) = \gamma_1(t), \qquad \Gamma(t, 1) = \gamma_2(t).$$

If the two paths have the same end points (i.e. $\gamma_1(0) = \gamma_2(0) = x_1$, $\gamma_1(1) = \gamma_2(1) = x_2$), we may introduce the stronger notion of *homotopy with fixed end points* by requiring additionally that $\Gamma(0, s) = x_1$, $\Gamma(1, s) = x_2$ for all $s \in I$.

Let us fix a base point $x_0 \in X$. A *loop based at* x_0 is a path such that $\gamma(0) = \gamma(1) = x_0$. Let us denote $\mathcal{L}(x_0)$ th set of loops based at x_0. One can define a composition between elements of $\mathcal{L}(x_0)$ by letting

$$(\gamma_2 \cdot \gamma_1)(t) = \begin{cases} \gamma_1(2t), & 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

This does not make $\mathcal{L}(x_0)$ into a group, since the composition is not associative (composing in a different order yields different parametrizations).

PROPOSITION 1.1. *If $x_1, x_2 \in X$ and there is a path connecting x_1 with x_2, then $\mathcal{L}(x_1) \simeq \mathcal{L}(x_2)$.*

PROOF. Let c be such a path, and let $\gamma_1 \in \mathcal{L}(x_1)$. We define $\gamma_2 \in \mathcal{L}(x_2)$ by letting

$$\gamma_2(t) = \begin{cases} c(1 - 3t), & 0 \leq t \leq \frac{1}{3} \\ \gamma_1(3t - 1), & \frac{1}{3} \leq t \leq \frac{2}{3} \\ c(3t - 2), & \frac{2}{3} \leq t \leq 1. \end{cases}$$

This establishes the isomorphism. □

4.2. The fundamental group. Again with reference with a base point x_0, we consider in $\mathcal{L}(x_0)$ an equivalence relation by decreeing that $\gamma_1 \sim \gamma_2$ if there is a homotopy with fixed end points between γ_1 and γ_2. The composition law in $\mathcal{L}_{x_0}$ descends to a group structure in the quotient

$$\pi_1(X, x_0) = \mathcal{L}(x_0)/\sim .$$

$\pi_1(X, x_0)$ is the *fundamental group of X with base point x_0;* in general it is nonabelian, as we shall see in examples. As a consequence of Proposition 1.1, if $x_1, x_2 \in X$ and there is a path connecting x_1 with x_2, then $\pi_1(X, x_1) \simeq \pi_1(X, x_2)$. In particular, if X is pathwise connected the fundamental group $\pi_1(X, x_0)$ is independent of x_0 up to isomorphism; in this situation, one uses the notation $\pi_1(X)$.

Definition 1.2. *X is said to be simply connected if it is pathwise connected and $\pi_1(X) = \{e\}$.*

The simplest example of a simply connected space is the one-point space $\{*\}$.

Since the definition of the fundamental group involves the choice of a base point, to describe the behaviour of the fundamental group we need to introduce a notion of map which takes the base point into account. Thus, we say that a *pointed space* (X, x_0) is a pair formed by a topological space X with a chosen point x_0. A map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ is a continuous map $f\colon X \to Y$ such that $f(x_0) = y_0$. It is easy to show that a map of pointed spaces induces a group homomorphism $f_*\colon \pi(X, x_0) \to \pi_1(Y, y_0)$.

4.3. Homotopy of maps. Given two topological spaces X, Y, a homotopy between two continuous maps f, $g\colon X \to Y$ is a map $F\colon X \times I \to Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$ for all $x \in X$. One then says that f and g are homotopic.

Definition 1.3. *One says that two topological spaces X, Y are homotopically equivalent if there are continuous maps $f\colon X \to Y$, $g\colon Y \to X$ such that $g \circ f$ is homotopic to id_X, and $f \circ g$ is homotopic to id_Y. The map f, g are said to be homotopical equivalences,.*

Of course, homeomorphic spaces are homotopically equivalent.

Example 1.4. For any manifold X, take $Y = X \times \mathbb{R}$, $f(x) = (x, 0)$, g the projection onto X. Then $F\colon X \times I \to X$, $F(x, t) = x$ is a homotopy between $g \circ f$ and id_X, while $G\colon X \times \mathbb{R} \times I \to X \times \mathbb{R}$, $G(x, s, t) = (x, st)$ is a homotopy between $f \circ g$ and id_Y. So X and $X \times \mathbb{R}$ are homotopically equivalent. The reader should be able to concoct many similar examples.

Given two pointed spaces (X, x_0), (Y, y_0), we say they are homotopically equivalent if there exist maps of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$, $g\colon (Y, y_0) \to (X, x_0)$ that make the topological spaces X, Y homotopically equivalent.

PROPOSITION 1.5. *Let* $f\colon (X,x_0)\to(Y,y_0)$ *be a homotopical equivalence. Then* $f_*\colon \pi_*(X,x_0)\to(Y,y_0)$ *is an isomorphism.*

PROOF. Let $g\colon (Y,y_0)\to(X,x_0)$ be a map that realizes the homotopical equivalence, and denote by F a homotopy between $g\circ f$ and id_X. Let γ be a loop based at x_0. Then $g\circ f\circ\gamma$ is again a loop based at x_0, and the map

$$\Gamma\colon I\times I\to X,\qquad \Gamma(s,t)=F(\gamma(s),t)$$

is a homotopy between γ and $g\circ f\circ\gamma$, so that $\gamma=g\circ f\circ\gamma$ in $\pi_1(X,x_0)$. Hence, $g_*\circ f_*=\mathrm{id}_{\pi_1(X,x_0)}$. In the same way one proves that $f_*\circ g_*=\mathrm{id}_{\pi_1(Y,y_0)}$, so that the claim follows. □

COROLLARY 1.6. *If two pathwise connected spaces* X *and* Y *are homotopic, then their fundamental groups are isomorphic.*

DEFINITION 1.7. *A topological space is said to be contractible if it is homotopically equivalent to the one-point space* $\{*\}$.

A contractible space is simply connected.

EXERCISE 1.8. 1. Show that $\mathbb{R}^n$ is contractible, hence simply connected. 2. Compute the fundamental groups of the following spaces: the punctured plane ($\mathbb{R}^2$ minus a point); $\mathbb{R}^3$ minus a line; $\mathbb{R}^n$ minus a $(n-2)$-plane (for $n\geq 3$).

4.4. Homotopic invariance of de Rham cohomology. We may now prove the invariance of de Rham cohomology under homotopy.

LEMMA 1.9. *Let* X*,* Y *be differentiable manifolds, and let* $f,g\colon X\to Y$ *be two homotopic smooth maps. Then the morphisms they induce in cohomology coincide,* $f^\sharp=g^\sharp$.

PROOF. We choose a homotopy between f and g in the form of a *smooth*[5] map $F\colon X\times\mathbb{R}\to Y$ such that

$$F(x,t)=f(x)\quad\text{if}\quad t\leq 0,\qquad F(x,t)=g(x)\quad\text{if}\quad t\geq 1\,.$$

We define sections $s_0,s_1\colon X\to X\times\mathbb{R}$ by letting $s_0(x)=(x,0)$, $s_1(x)=(x,1)$. Then $f=F\circ s_0$, $g=F\circ s_1$, so $f^\sharp=s_0^\sharp\circ F^\sharp$ and $g^\sharp=s_1^\sharp\circ F^\sharp$. Let $p_1\colon X\times\mathbb{R}\to X$, $p_2\colon X\times\mathbb{R}\to\mathbb{R}$ be the projections. Then $s_0^\sharp\circ p_1^\sharp=s_1^\sharp\circ p_1^\sharp=\mathrm{Id}$. By Proposition 1.3 $p_1^\sharp$ is an isomorphism. Then $s_0^\sharp=s_1^\sharp$, and $f^\sharp=F^\sharp=g^\sharp$. □

PROPOSITION 1.10. *Let* X *and* Y *be homotopic differentiable manifolds. Then* $H^k_{dR}(X)\simeq H^k_{dR}(Y)$ *for all* $k\geq 0$.

PROOF. If f, g are two smooth maps realizing the homotopy, then $f^\sharp\circ g^\sharp=g^\sharp\circ f^\sharp=\mathrm{Id}$, so that both $f^\sharp$ and $g^\sharp$ are isomorphisms. □

[5]For the fact that F can be taken smooth cf. [**3**].

4.5. The van Kampen theorem. The computation of the fundamental group of a topological space is often unsuspectedly complicated. An important tool for such computations is the van Kampen theorem, which we state without proof. This theorem allows one, under some conditions, to compute the fundamental group of an union $U \cup V$ if one knows the fundamental groups of U, V and $U \cap V$. As a prerequisite we need the notion of *amalgamated product of two groups.* Let G, G_1, G_2 be groups, with fixed morphisms $f_1 \colon G \to G_1$, $f_2 \colon G \to G_2$. Let F the free group generated by $G_1 \amalg G_2$ and denote by $\cdot$ the product in this group.[6] Let R be the normal subgroup generated by elements of the form[7]

$$(xy) \cdot y^{-1} \cdot x^{-1} \quad \text{with } x,\, y \text{ both in } G_1 \text{ or } G_2$$

$$f_1(g) \cdot f_2(g)^{-1} \quad \text{for } g \in G.$$

Then one defines the amalgamated product $G_1 *_G G_2$ as F/R. There are natural maps $g_1 \colon G_1 \to G_1 *_G G_2$, $g_2 \colon G_2 \to G_1 *_G G_2$ obtained by composing the inclusions with the projection $F \to F/R$, and one has $g_1 \circ f_1 = g_2 \circ f_2$. Intuitively, one could say that $G_1 *_G G_2$ is the smallest subgroup generated by G_1 and G_2 with the identification of $f_1(g)$ and $f_2(g)$ for all $g \in G$.

Exercise 1.11. (1) Prove that if $G_1 = G_2 = \{e\}$, and G is any group, then $G_1 *_G G_2 = \{e\}$.
(2) Let G be the group with three generators a, b, c, satisfying the relation $ab = cba$. Let $\mathbb{Z} \to G$ be the homomorphism induced by $1 \mapsto c$. Prove that $G *_{\mathbb{Z}} G$ is isomorphic to a group with four generators m, n, p, q, satisfying the relation $m\,n\,m^{-1}\,n^{-1}\,p\,q\,p^{-1}\,q^{-1} = e$. □

Suppose now that a pathwise connected space X is the union of two pathwise connected open subsets U, V, and that $U \cap V$ is pathwise connected. There are morphisms $\pi_1(U \cap V) \to \pi_1(U)$, $\pi_1(U \cap V) \to \pi_1(V)$ induced by the inclusions.

Proposition 1.12. $\pi_1(X) \simeq \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$.

This is a simplified form of van Kampen's theorem, for a full statement see [**7**].

Example 1.13. We compute the fundamental group of a figure 8. Think of the figure 8 as the union of two circles X in $\mathbb{R}^2$ which touch in one pount. Let p_1, p_2 be points in the two respective circles, different from the common point, and take $U = X - \{p_1\}$, $V = X - \{p_2\}$. Then $\pi_1(U) \simeq \pi_1(V) \simeq \mathbb{Z}$, while $U \cap V$ is simply connected. It follows that $\pi_1(X)$ is a free group with two generators. The two generators do not commute; this can also be checked "experimentally" if you think of winding a string along the

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[6] F is the group whose elements are words $x_1^{\epsilon_1} x_2 \dots x_n^{\epsilon}$ or the empty word, where the letters x_i are either in G_1 or G_2, $\epsilon_i = \pm 1$, and the product is given by juxtaposition.

[7] The first relation tells that the product of letters in the words of F are the product either in G_1 or G_2, when this makes sense. The second relation kind of "glues" G_1 and G_2 along the images of G.

figure 8 in a proper way... More generally, the fundamental group of the corolla with n petals (n copies of S^1 all touching in a single point) is a free group with n generators.

EXERCISE 1.14. Prove that for any $n \geq 2$ the sphere S^n is simply connected. Deduce that for $n \geq 3$, $\mathbb{R}^n$ minus a point is simply connected.

EXERCISE 1.15. Compute the fundamental group of $\mathbb{R}^2$ with n punctures.

4.6. Other ways to compute fundamental groups. Again, we state some results without proof.

PROPOSITION 1.16. *If G is a simply connected topological group, and H is a normal discrete subgroup, then $\pi_1(G/H) \simeq H$.*

Since $S^1 \simeq \mathbb{R}/\mathbb{Z}$, we have thus proved that

$$\pi_1(S^1) \simeq \mathbb{Z}.$$

In the same way we compute the fundamental group of the n-dimensional torus

$$T^n = S^1 \times \cdots \times S^1 \ (n \text{ times}) \ \simeq \mathbb{R}^n/\mathbb{Z}^n,$$

obtaining $\pi_1(T^n) \simeq \mathbb{Z}^n$.

EXERCISE 1.17. Compute the fundamental group of a 2-dimensional punctured torus (a torus minus a point). Use van Kampen's theorem to compute the fundamental group of a Riemann surface of genus 2 (a compact, orientable, connected 2-dimensional differentiable manifold of genus 2, i.e., "with two handles"). Generalize your result to any genus.

EXERCISE 1.18. Prove that, given two pointed topological spaces (X, x_0), (Y, y_0), then

$$\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0). \qquad \square$$

This gives us another way to compute the fundamental group of the n-dimensional torus T^n (once we know $\pi_1(S^1)$).

EXERCISE 1.19. Prove that the manifolds S^3 and $S^2 \times S^1$ are not homeomorphic.

EXERCISE 1.20. Let X be the space obtained by removing a line from $\mathbb{R}^2$, and a circle linking the line. Prove that $\pi_1(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Prove the stronger result that X is homotopic to the 2-torus.

CHAPTER 2

Singular homology theory

1. Singular homology

In this Chapter we develop some elements of the homology theory of topological spaces. There are many different homology theories (simplicial, cellular, singular, Čech-Alexander, ...) even though these theories coincide when the topological space they are applied to is reasonably well-behaved. Singular homology has the disadvantage of appearing quite abstract at a first contact, but in exchange of this we have the fact that it applies to any topological space, its functorial properties are evident, it requires very little combinatorial arguments, it relates to homotopy in a clear way, and once the basic properties of the theory have been proved, the computation of the homology groups is not difficult.

1.1. Definitions. The basic blocks of singular homology are the continuous maps from standard subspaces of Euclidean spaces to the topological space one considers. We shall denote by $P_0, P_1, \dots, P_n$ the points in $\mathbb{R}^n$

$$P_0 = 0, \qquad P_i = (0, 0, \dots, 0, 1, 0, \dots, 0) \quad \text{(with just one 1 in the } i\text{th position).}$$

The convex hull of these points is denoted by Δ_n and is called the *standard n-simplex*. Alternatively, one can describe Δ_k as the set of points in $\mathbb{R}^n$ such that

$$x_i \geq 0, \quad i = 1, \dots, n, \qquad \sum_{i=1}^{n} x_i \leq 1.$$

The boundary of Δ_n is formed by $n+1$ faces F_n^i ($i = 0, 1, \dots, n$) which are images of the standard $(n-1)$-simplex by affine maps $\mathbb{R}^{n-1} \to \mathbb{R}^n$. These faces may be labelled by the vertex of the simplex which is opposite to them: so, F_n^i is the face opposite to P_i.

Given a topological space X, a singular n-simplex in X is a continuous map $\sigma\colon \Delta_n \to X$. The restriction of σ to any of the faces of Δ_n defines a singular $(n-1)$-simplex $\sigma_i = \sigma_{|F_n^i}$ (or $\sigma \circ F_n^i$ if we regard F_n^i as a singular $(n-1)$-simplex).

If $Q_0, \dots, Q_k$ are $k+1$ points in $\mathbb{R}^n$, there is a unique affine map $\mathbb{R}^k \to \mathbb{R}^n$ mapping $P_0, \dots, P_k$ to the Q's. This affine map yields a singular k-simplex in $\mathbb{R}^n$ that we denote $< Q_0, \dots, Q_k >$. If $Q_i = P_i$ for $0 \leq i \leq k$, then the affine map is the identity on $\mathbb{R}^k$, and we denote the resulting singular k-simplex by δ_k. The standard n-simplex Δ_n may so also be denoted $< P_0, \dots, P_n >$, and the face F_n^i of Δ_n is the singular $(n-1)$-simplex $< P_0, \dots, \hat{P}_i, \dots, P_n >$, where the hat denotes omission.

Choose now a commutative unital ring R. We denote by $S_k(X, R)$ the free group generated over R by the singular k-simplexes in X. So an element in $S_k(X, R)$ is a "formal" finite linear combination (called a *singular chain*)

$$\sigma = \sum_j a_j \, \sigma_j$$

with $a_j \in R$, and the σ_j are singular k-simplexes. Thus, $S_k(X, R)$ is an R-module, and, via the inclusion $\mathbb{Z} \to R$ given by the identity in R, an abelian group. For $k \geq 1$ we define a morphism $\partial \colon S_k(X, R) \to S_{k-1}(X, R)$ by letting

$$\partial \sigma = \sum_{i=0}^{k} (-1)^i \, \sigma \circ F_k^i$$

for a singular k-simplex σ and exteding by R-linearity. For $k = 0$ we define $\partial \sigma = 0$.

EXAMPLE 2.1. If $Q_0, \dots, Q_k$ are $k + 1$ points in $\mathbb{R}^n$, one has

$$\partial < Q_0, \dots, Q_k > = \sum_{i=0}^{k} (-1)^i < Q_0, \dots, \hat{Q}_i, \dots, Q_k > \ .$$

PROPOSITION 2.2. $\partial^2 = 0$.

PROOF. Let σ be a singular k-simplex.

$$\begin{aligned} \partial^2 \sigma &= \sum_{i=0}^{k} (-1)^i \, \partial(\sigma \circ F_k^i) = \sum_{i=0}^{k} (-1)^i \sum_{j=0}^{k-1} (-1)^j \ \sigma \circ F_k^i \circ F_{k-1}^j \\ &= \sum_{j<i=1}^{k} (-1)^{i+j} \ \sigma \circ F_k^j \circ F_{k-1}^{i-1} + \sum_{0=i\leq j}^{k-1} (-1)^{i+j} \ \sigma \circ F_k^i \circ F_{k-1}^j \end{aligned}$$

Resumming the first sum by letting $i = j$, $j = i - 1$ the last two terms cancel. □

So $(S_\bullet(X, R), \partial)$ is a (homology) graded differential module. Its homology groups $H_k(X, R)$ are the *singular homology groups of X with coefficients in R.* We shall use the following notation and terminology:

$Z_k(X, R) = \ker \partial \colon S_k(X, R) \to S_{k-1}(X, R)$ (the module of k-cycles);

$B_k(X, R) = \operatorname{Im} \partial \colon S_{k+1}(X, R) \to S_k(X, R)$ (the module of k-boundaries);

therefore, $H_k(X, R) = Z_k(X, R)/B_k(X, R)$. Notice that $Z_0(X, R) \equiv S_0(X, R)$.

1.2. Basic properties.

PROPOSITION 2.3. *If X is the union of pathwise connected components X_j, then $H_k(X, R) \simeq \oplus_j H_k(X_j, R)$ for all $k \geq 0$.*

PROOF. Any singular k-simplex must map Δ_k inside a pathwise connected components (if two points of Δ_k would map to points lying in different components, that would yield path connecting the two points). □

PROPOSITION 2.4. *If X is pathwise connected, then $H_0(X,R) \simeq R$.*

PROOF. This follows from the fact that a 0-cycle $c = \sum_j a_j\, x_j$ is a boundary if and only if $\sum_j a_j = 0$. Indeed, if c is a boundary, then $c = \partial(\sum_j b_j \gamma_j)$ for some paths γ_j, so that $c = \sum_j b_j(\gamma_j(1) - \gamma_j(0))$, and the coefficients sum up to zero. On the other hand, if $\sum_j a_j = 0$, choose a base point $x_0 \in X$. Then one can write

$$c = \sum_j a_j\, x_j = \sum_j a_j\, x_j - (\sum_j a_j) x_0 = \sum_j a_j(x_j - x_0) = \partial \sum_j a_j \gamma_j$$

if γ_j is a path joining x_0 to x_j.

This means that $B_0(X,R)$ is the kernel of the surjective map $Z_0(X,R) = S_0(X,R) \to R$ given by $\sum_j a_j\, x_j \mapsto \sum_j a_j$, so that $H_0(X,R) = Z_0(X,R)/B_0(X,R) \simeq R$. □

Let $f\colon X \to Y$ be a continuous map of topological spaces. If σ is a singular k-simplex in X, then $f\circ\sigma$ is a singular k-simplex in Y. This yields a morphism $S_k(f)\colon S_k(X,R) \to S_k(Y,R)$ for every $k \geq 0$. It is immediate to prove that $S_k(f) \circ \partial = \partial \circ S_{k+1}(f)$:

$$S_k(f)(\partial\sigma) = f \circ \sum_{i=0}^{k+1} (-1)^i \sigma \circ F^i_{k+1} = \partial(f \circ \sigma) = \partial(S_k(f)(\sigma))\,.$$

This implies that f induces a morphism $H_k(X,R) \to H_k(Y,R)$, that we denote $f_\flat$. It is also easy to check that, if $g\colon Y \to W$ is another continous map, then $S_k(g\circ f) = S_k(g) \circ S_k(f)$, and $(g\circ f)_\flat = g_\flat \circ f_\flat$.

1.3. Homotopic invariance.

PROPOSITION 2.5. *If $f, g\colon X \to Y$ are homotopic map, the induced maps in homology coincide.*

It should be by now clear that this yields as an immediate consequence the homotopic invariance of the singular homology.

COROLLARY 2.6. *If two topological spaces are homotopically equivalent, their singular homologies are isomorphic.*

To prove Proposition 2.5 we build, for every $k \geq 0$ and any topological space X, a morphism (called the *prism operator*) $P\colon S_k(X) \to S_{k+1}(X \times I)$ (here I denotes again the unit closed interval in $\mathbb{R}$). We define the morphism P in two steps.

Step 1. The first step consists in definining a singular $(k+1)$-chain π_{k+1} in the topological space $\Delta_k \times I$ by subdiving the polyhedron $\Delta_k \times I \subset \mathbb{R}^{k+1}$ (a "prysm"

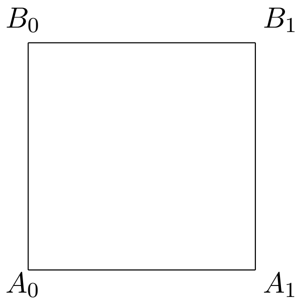


FIGURE 1. The prism π_2 over Δ_1

over the standard symplesx Δ_k) into a number of singular $(k+1)$-simplexes, and summing them with suitable signs. The polyhedron $\Delta_k \times I \subset \mathbb{R}^{k+1}$ has $2(k+1)$ vertices $A_0, \ldots, A_k, B_0, \ldots, B_k$, given by $A_i = (P_i, 0)$, $B_i = (P_i, 1)$. We define

$$\pi_{k+1} = \sum_{i=0}^{k} (-1)^i < A_0, \ldots, A_i, B_i, \ldots, B_k > .$$

For instance, for $k = 1$ we have

$$\pi_2 =< A_0, B_0, B_1 > - < A_0, A_1, B_1 > .$$

Step 2. If σ is a singular k-simplex in a topological space X, then $\sigma \times \mathrm{id}$ is a continous map $\Delta_k \times I \to X \times I$. Therefore it makes sense to define the singular $(k+1)$-chain $P(\sigma)$ in X as

$$P(\sigma) = S_{k+1}(\sigma \times \mathrm{id})(\pi_{k+1}). \tag{2.1}$$

The definition of the prism operator implies its functoriality:

PROPOSITION 2.7. *If $f\colon X \to Y$ is a continuous map, the diagram*

$$\begin{array}{ccc} S_k(X) & \xrightarrow{P} & S_{k+1}(X \times I) \\ {\scriptstyle S_k(f)} \downarrow & & \downarrow {\scriptstyle S_{k+1}(f \times id)} \\ S_k(Y) & \xrightarrow{P} & S_{k+1}(Y \times I) \end{array}$$

commutes.

PROOF. It is just a matter of computation.

$$\begin{aligned} S_{k+1}(f \times \mathrm{id}) \circ P(\sigma) &= S_{k+1}(f \times \mathrm{id}) \circ S_{k+1}(\sigma \times \mathrm{id})(\pi_{k+1}) \\ &= S_{k+1}(f \circ \sigma \times \mathrm{id})(\pi_{k+1}) = P(S_k(f)) . \end{aligned}$$

$\square$

The relevant property of the prism operator is proved in the next Lemma.

LEMMA 2.8. *Let* $\lambda_0, \lambda_i \colon X \to X \times I$ *be the maps* $\lambda_0(x) = (x, 0)$, $\lambda_1(x) = (x, 1)$. *Then*

$$\partial \circ P + P \circ \partial = S_k(\lambda_1) - S_k(\lambda_0) \tag{2.2}$$

as maps $S_k(X) \to S_k(X \times I)$.

PROOF. Let $\delta_k \colon \Delta_k \to \Delta_k$ be the identity map regarded as singular k-simplex in Δ_k. Notice that $P(\delta_k) = \pi_{k+1}$.

We first check the identity (2.2) for $X = \Delta_k$, applying both sides of (2.2) to δ_k. The right side yields

$$< B_0, \dots, B_k > - < A_0, \dots A_k > .$$

We compute now the action of the left side of (2.2) on δ_k.

$$\begin{aligned} \partial P(\delta_k) &= \sum_{i=0}^{k} (-1)^i \partial < A_0, \dots, A_i, B_i, \dots, B_k > \\ &= \sum_{j \le i = 0}^{k} (-1)^{i+j} < A_0, \dots, \hat{A}_j, \dots A_i, B_i, \dots, B_k > \\ &+ \sum_{i \le j = 0}^{k} (-1)^{i+j+1} < A_0, \dots A_i, B_i, \dots, \hat{B}_j, \dots B_k > . \end{aligned}$$

All terms with $i = j$ cancel with the exception of $< B_0, \dots, B_k > - < A_0, \dots A_k >$. So we have

$$\begin{aligned} \partial P(\delta_k) &= < B_0, \dots, B_k > - < A_0, \dots A_k > \\ &+ \sum_{j < i = 1}^{k} (-1)^{i+j} < A_0, \dots, \hat{A}_j, \dots A_i, B_i, \dots, B_k > \\ &- \sum_{i < j = 1}^{k} (-1)^{i+j} < A_0, \dots A_i, B_i, \dots, \hat{B}_j, \dots B_k > . \end{aligned}$$

On the other hand, one has

$$\partial \delta_k = \sum_{j=0}^{k} (-1)^j < P_0, \dots, \hat{P}_j, \dots, P_k > .$$

Since

$$\begin{aligned} P(< P_0, \dots, \hat{P}_j, \dots, P_k >) &= \sum_{i<j} (-1)^i < A_0, \dots, A_i, B_i, \dots, \hat{B}_j, \dots, B_k > \\ &- \sum_{i>j} (-1)^i < A_0, \dots, \hat{A}_j, \dots, A_i, B_i, \dots, B_k > \end{aligned}$$

we obtain the equation (2.2) (note that exchanging the indices i, j changes the sign).

We must now prove that if equation (2.2) holds when both sides are applied to δ_k, then it holds in general. One has indeed

$$\partial P(\sigma) = \partial S_{k+1}(\sigma \times \mathrm{id})(P(\delta_k)) = S_k(\sigma \times \mathrm{id})(\partial P(\delta_k))$$

$$\begin{aligned} P(\partial\sigma) &= P\partial(S_k(\sigma)(\delta_k)) \\ &= P(S_{k-1}(\sigma)(\partial\delta_k)) = S_k(\sigma \times \mathrm{id})(P(\partial\delta_k)) \end{aligned}$$

so that

$$\begin{aligned} \partial P(\sigma) + P(\partial\sigma) &= S_{k+1}(\sigma \times \mathrm{id})(\partial P(\delta_k)) + P(\partial\delta_k)) \\ &= S_{k+1}(\sigma \times \mathrm{id})(S_k(\bar\lambda_1) - S_k(\bar\lambda_0)) = S_k(\lambda_1) - S_k(\lambda_0) \end{aligned}$$

where $\bar\lambda_0$, $\bar\lambda_1$ are the obvious maps $\Delta_k \to \Delta_k \times I$. □

Equation (2.2) states that P is a hotomopy (in the sense of homological algebra) between the maps λ_0 and λ_1, so that one has $(\lambda_1)_\flat = (\lambda_2)_\flat$ in homology.

Proof of Proposition 2.5. Let F be a hotomopy between the maps f and g. Then, $f = F \circ \lambda_0$, $g = F \circ \lambda_1$, so that

$$f_\flat = (F \circ \lambda_0)_\flat = F_\flat \circ (\lambda_0)_\flat = F_\flat \circ (\lambda_1)_\flat = (F \circ \lambda_1)_\flat = g_\flat.$$

□

Corollary 2.9. *If X is a contractible space then*

$$H_0(X, R) \simeq R, \qquad H_k(X, R) = 0 \quad \textit{for} \quad k > 0.$$

1.4. Relation between the first fundamental group and homology. A loop γ in X may be regarded as a closed singular 1-simplex. If we fix a point $x_0 \in X$, we have a set-theoretic map $\chi\colon \mathcal{L}(x_0) \to S_1(X, \mathbb{Z})$. The following result tells us that χ descends to a group homomorphism $\chi\colon \pi_1(X, x_0) \to H_1(X, \mathbb{Z})$.

Proposition 2.10. *If two loops γ_1, γ_2 are homotopic, then they are homologous as singular 1-simplexes. Moreover, given two loops at x_0, γ_1, γ_2, then $\chi(\gamma_2 \circ \gamma_1) = \chi(\gamma_1) + \chi(\gamma_2)$ in $H_1(X, \mathbb{Z})$.*

Proof. Choose a homotopy with fixed endpoints between γ_1 and γ_2, i.e., a map $\Gamma\colon I \times I \to X$ such that

$$\Gamma(t, 0) = \gamma_1(t), \qquad \Gamma(t, 1) = \gamma_2(t), \qquad \Gamma(0, s) = \Gamma(1, s) = x_0 \text{ for all } s \in I.$$

Define the loops $\gamma_3(t) = \Gamma(1, t)$, $\gamma_4(t) = \Gamma(0, t)$, $\gamma_5(t) = \Gamma(t, t)$. Both loops γ_3 and γ_4 are actually the constant loop at x_0. Consider the points P_0, P_1, P_2, $Q = (1, 1)$ in $\mathbb{R}^2$, and define the singular 2-simplex

$$\sigma = \Gamma \circ < P_0, P_1, Q > - \Gamma \circ < P_0, P_2, Q >$$

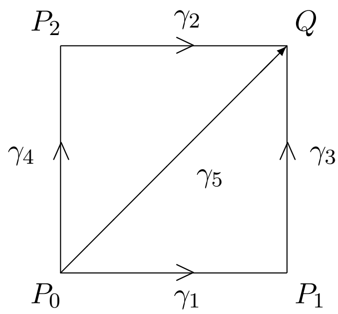


FIGURE 2

(cf. Figure 2). We then have

$$\begin{aligned}\partial\sigma &= \Gamma\circ < P_1, Q > -\Gamma\circ < P_0, Q > +\Gamma\circ < P_0, P_1 > \\ &- \Gamma\circ < P_2, Q > +\Gamma\circ < P_0, Q > -\Gamma\circ < P_0, P_2 > \\ &= \gamma_3 - \gamma_5 + \gamma_1 - \gamma_2 + \gamma_5 + \gamma_4 = \gamma_1 - \gamma_2.\end{aligned}$$

This proves that $\chi(\gamma_1)$ and $\chi(\gamma_2)$ are homologous. To prove the second claim we need to define a singular 2-simplex σ such that

$$\partial\sigma = \gamma_1 + \gamma_2 - \gamma_2 \cdot \gamma_1.$$

Consider the point $T = (0, \frac{1}{2})$ in the standard 2-simplex Δ_2 and the segment Σ joining T with P_1 (cf. Figure 3). If $Q \in \Delta_2$ lies on or below Σ, consider the line joining P_0 with Q, parametrize it with a parameter t such that $t = 0$ in P_0 and $t = 1$ in the intersection of the line with Σ, and set $\sigma(Q) = \gamma_1(t)$. Analogously, if Q lies above or on Σ, consider the line joining P_2 with Q, parametrize it with a parameter t such that $t = 1$ in P_2 and $t = 0$ in the intersection of the line with Σ, and set $\sigma(Q) = \gamma_2(t)$. This defines a singular 2-simplex $\sigma \colon \Delta_2 \to$X, and one has

$$\begin{aligned}\partial\sigma &= \sigma\circ < P_1, P_2 > -\sigma\circ < P_0, P_2 > +\sigma\circ < P_0, P_1 > \\ &= \gamma_2 - \gamma_2 \cdot \gamma_1 + \gamma_1.\end{aligned}$$

□

We recall from basic group theory the notion of *commutator subgroup*. Let G be any group, and let $C(G)$ be the subgroup generated by elements of the form $ghg^{-1}h^{-1}$, $g, h \in G$. The subgroup $C(G)$ is obviously normal in G; the quotient group $G/C(G)$ is abelian. We call it the *abelianization* of G. It turns out that the first homology group of a space with integer coefficients is the abelianization of the fundamental group.

PROPOSITION 2.11. *If X is pathwise connected, the morphism $\chi \colon \pi_1(X, x_0) \to H_1(X, \mathbb{Z})$ is surjective, and its kernel is the commutator subgroup of $\pi_1(X, x_0)$.*

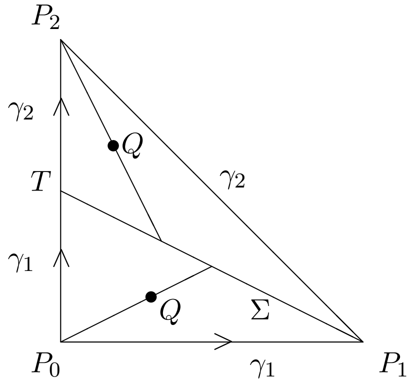


FIGURE 3

PROOF. Let $c = \sum_j a_j \sigma_j$ be a 1-cycle. So we have

$$0 = \partial c = \sum_j a_i(\sigma_j(1) - \sigma_j(0)).$$

In this linear combination of points with coefficients in $\mathbb{Z}$ some of the points may coincide; the sum of the coefficients corresponding to the same point must vanish. Choose a base point $x_0 \in X$ and for every j choose a path α_j from x_0 to $\sigma_j(0)$ and a path β_j from x_0 to $\sigma_j(1)$, in such a way that they depend on the endpoints and not on the indexing (e.g, if $\sigma_j(0) = \sigma_k(0)$, choose $\alpha_j = \alpha_k$). Then we have

$$\sum_j a_j(\beta_j - \alpha_j) = 0.$$

Now if we set $\bar{\sigma}_j = \alpha_j + \sigma_j - \beta_j$ we have $c = \sum_j a_j \bar{\sigma}_j$. Let γ_j be the loop $\beta^{-1} \cdot \sigma_j \cdot \alpha$; then,

$$\chi(\left[\Pi_j \gamma_j^{a_j}\right]) = [c]$$

so that χ is surjective.

To prove the second claim we need to show that the commutator subgroup of $\pi_1(X, x_0)$ coincides with $\ker \chi$. We first notice that since $H_1(X, \mathbb{Z})$ is abelian, the commutator subgroup is necessarily contained in $\ker \chi$. To prove the opposite inclusion, let γ be a loop that in homology is a 1-boundary, i.e., $\gamma = \partial \sum_j a_j \sigma_j$. So we may write

$$\sigma_j = \gamma_{0j} - \gamma_{1j} + \gamma_{2j} \tag{2.3}$$

for some paths γ_{kj}, $k = 0, 1, 2$. Choose paths (cf. Figure 4)

$$\begin{array}{llll} \alpha_{0j} & \text{from} & x_0 & \text{to} \quad \gamma_{1j}(0) = \gamma_{2j}(0) = P_0 \\ \alpha_{1j} & \text{from} & x_0 & \text{to} \quad \gamma_{2j}(1) = \gamma_{0j}(0) = P_1 \\ \alpha_{2j} & \text{from} & x_0 & \text{to} \quad \gamma_{1j}(1) = \gamma_{0j}(1) = P_2 \end{array}$$

and consider the loops

$$\beta_{0j} = \alpha_{0j}^{-1} \cdot \gamma_{1j}^{-1} \cdot \alpha_{2j}, \quad \beta_{1j} = \alpha_{2j}^{-1} \cdot \gamma_{0j} \cdot \alpha_{1j}, \quad \beta_{2j} = \alpha_{1j}^{-1} \cdot \gamma_{2j} \cdot \alpha_{0j}.$$

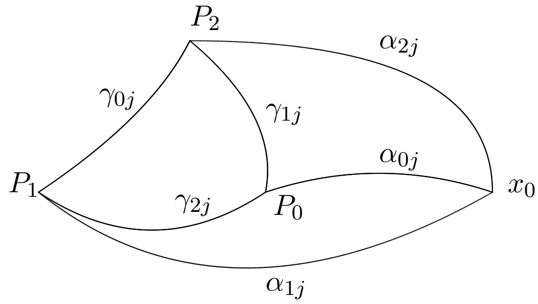


FIGURE 4

Note that the loops

$$\beta_j = \beta_{0j} \cdot \beta_{1j} \cdot \beta_{2j} = \alpha_{0j}^{-1} \cdot \gamma_{1j}^{-1} \cdot \gamma_{0j} \cdot \gamma_{2j} \cdot \alpha_{0j}$$

are homotopic to the constant loop at x_0 (since the image of a singular 2-simplex is contractible). As a consequence one has the equality in $\pi_1(X, x_0)$

$$\Pi_j [\beta_j]^{a_j} = e.$$

This implies that the image of $\Pi_j [\beta_j]^{a_j}$ in $\pi_1(X, x_0)/C(\pi_1(X, x_0))$ is the identity. On the other hand from (2.3) we see that γ coincides, up to reordering of terms, with $\Pi_j \beta_j^{a_j}$, so that the image of the class of γ in $\pi_1(X, x_0)/C(\pi_1(X, x_0))$ is the identity as well. This means that γ lies in the commutator subgroup. □

So whenever in the examples in Chapter 1 the fundamental groups we computed turned out to be abelian, we were also computing the group $H_1(X, \mathbb{Z})$. In particular,

COROLLARY 2.12. $H_1(X, \mathbb{Z}) = 0$ *if* X *is simply connected.*

EXERCISE 2.13. Compute $H_1(X, \mathbb{Z})$ when X is: 1. the corolla with n petals, 2. $\mathbb{R}^n$ minus a point, 3. the circle S^1, 4. the torus T^2, 5. a punctured torus, 6. a Riemann surface of genus g.

2. Relative homology

2.1. The relative homology complex. Given a topological space X, let A be any subspace (that we consider with the relative topology). We fix a coefficient ring R which for the sake of conciseness shall be dropped from the notation. For every $k \geq 0$ there is a natural inclusion (injective morphism of R-modules) $S_k(A) \subset S_k(X)$; the homology operators of the complexes $S_\bullet(A)$, $S_\bullet(X)$ define a morphism $\delta \colon S_k(X)/S_k(A) \to S_{k-1}(X)/S_{k-1}(A)$ which squares to zero. If we define

$$Z'_k(X, A) = \ker \partial \colon \frac{S_k(X)}{S_k(A)} \to \frac{S_{k-1}(X)}{S_{k-1}(A)}$$

$$B'_k(X,A) = \operatorname{Im} \partial \colon \frac{S_{k+1}(X)}{S_{k+1}(A)} \to \frac{S_k(X)}{S_k(A)}$$

we have $B'_k(X,A) \subset Z'_k(X,A)$.

Definition 2.1. *The homology groups of X relative to A are the R-modules $H_k(X,A) = Z'_k(X,A)/B'_k(X,A)$. When we want to emphasize the choice of the ring R we write $S_k(X,A;R)$.*

The relative homology is more conveniently defined in a slightly different way, which makes clearer its geometrical meaning. It will be useful to consider the following diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & Z_k(X) & \xrightarrow{q_k} & Z'_k(X,A) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & S_k(A) & \longrightarrow & S_k(X) & \xrightarrow{q_k} & S_k(X)/S_k(A) & \longrightarrow & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \longrightarrow & B_{k-1}(A) & \longrightarrow & B_{k-1}(X) & \xrightarrow[q_{k-1}]{} & B'_{k-1}(X,A) & \longrightarrow & 0
\end{array}$$

Let

$$Z_k(X,A) = \{c \in S_k(X) \,|\, \partial c \in S_{k-1}(A)\}$$

$$B_k(X,A) = \{c \in S_k(X) \,|\, c = \partial b + c' \text{ with } b \in S_{k+1}(X),\ c' \in S_k(A)\}\,.$$

Thus, $Z_k(X,A)$ is formed by the chains whose boundary is in A, and $B_k(A)$ by the chains that are boundaries up to chains in A.

Lemma 2.2. *$Z_k(X,A)$ is the pre-image of $Z'_k(X,A)$ under the quotient homomorphism q_k; that is, an element $c \in S_k(X)$ is in $Z_k(X,A)$ if and only if $q_k(c) \in Z'_k(X,A)$.*

Proof. If $q_k(c) \in Z'_k(X,A)$ then $0 = \partial \circ q_k(c) = q_{k-1} \circ \partial(c)$ so that $c \in Z_k(X,A)$. If $c \in Z_k(X,A)$ then $q_{k-1} \circ \partial(c) = 0$ so that $q_k(c) \in Z'_k(X,A)$. □

Lemma 2.3. *$c \in S_k(X)$ is in $B_k(X,A)$ if and only if $q_k(c) \in B'_k(X,A)$.*

Proof. If $c = \partial b + c'$ with $b \in S_{k+1}(X)$ and $c' \in S_k(A)$ then $q_k(c) = q_k \circ \partial b = \partial \circ q_{k+1}(b) \in B'_k(X,A)$. Conversely, if $q_k(c) \in B'_k(X,A)$ then $q_k(c) = \partial \circ q_{k+1}(b)$ for some $b \in S_{k+1}(X)$, then $c - \partial b \in \ker q_{k-1}$ so that $c = \partial b + c'$ with $c' \in S_k(A)$. □

Proposition 2.4. *For all $k \geq 0$, $H_k(X,A) \simeq Z_k(X,A)/B_k(X,A)$.*

PROOF. What we should do is to prove the commutativity and the exactness of the rows of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_k(A) & \longrightarrow & B_k(X,A) & \xrightarrow{q_k} & B'_k(X,A) & \longrightarrow & 0 \\ & & \big\downarrow\wr & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & S_k(A) & \longrightarrow & Z_k(X,A) & \xrightarrow{q_k} & Z'_k(X,A) & \longrightarrow & 0 \end{array}$$

Commutativity is obvious. For the exactness of the first row, it is obvious that $S_k(A) \subset B_k(X,A)$ and that $q_k(c) = 0$ if $c \in S_k(A)$. On the other hand if $c \in B_k(X,A)$ we have $c = \partial b + c'$ with $b \in S_{k+1}(X)$ and $c' \in S_k(A)$, so that $q_k(c) = 0$ implies $0 = q_k \circ \partial b = \partial \circ q_{k+1}(b)$, which in turn implies $c \in S_k(A)$. To prove the surjectivity of q_k, just notice that by definition an element in $B'_k(X,A)$ may be represented as ∂b with $b \in S_{k+1}(X)$.

As for the second row, we have $S_k(A) \subset Z_k(X,A)$ from the definition of $Z_k(X,A)$. If $c \in S_k(A)$ then $q_k(c) = 0$. If $c \in Z_k(X,A)$ and $q_k(c) = 0$ then $c \in S_k(A)$ by the definition of $Z'_k(X,A)$. Moreover q_k is surjective by Lemma 2.2. □

2.2. Main properties of relative homology. We list here the main properties of the cohomology groups $H_k(X,A)$. If a proof is not given the reader should provide one by her/himself.

• If A is empty, $H_k(X,A) \simeq H_k(X)$.

• The relative cohomology groups are functorial in the following sense. Given topological spaces X, Y with subsets $A \subset X$, $B \subset Y$, a continous map of pairs is a continuous map $f\colon X \to Y$ such that $f(A) \subset B$. Such a map induces in natural way a morphisms of R-modules $f_\flat\colon H_\bullet(X,A) \to H_\bullet(Y,B)$. If we consider the inclusion of pairs $(X,\emptyset) \hookrightarrow (X,A)$ we obtain a morphism $H_\bullet(X) \to_\bullet H(X,A)$.

• The inclusion map $i\colon A \hookrightarrow X$ induces a morphism $H_\bullet(A) \to H_\bullet(X)$ and the composition $H_\bullet(A) \to H_\bullet(X) \to H_\bullet(X,A)$ vanishes (since $Z_k(A) \subset B_k(X,A)$).

• If $X = \cup_j X_j$ is a union of pathwise connected components, then $H_k(X,A) \simeq \oplus_j H_k(X_j,A_j)$ where $A_j = A \cap X_j$.

PROPOSITION 2.5. *If X is pathwise connected and A is nonempty, then $H_0(X,A)$* $= 0$.

PROOF. If $c = \sum_j a_j x_j \in S_0(X)$ and γ_j is a path from $x_0 \in A$ to x_j, then $\partial(\sum_j a_j x_j) = c - (\sum_j a_j)x_0$ so that $c \in B_0(X,A)$. □

COROLLARY 2.6. *$H_0(X,A)$ is a free R-module generated by the components of X that do not meet A.*

Indeed $H_j(X_j,A_j) = 0$ if A_j is empty.

PROPOSITION 2.7. *If $A = \{x_0\}$ is a point, $H_k(X,A) \simeq H_k(X)$ for $k > 0$.*

PROOF.

$$\begin{aligned} Z_k(X,A) &= \{c \in S_k(X) \,|\, \partial c \in S_{k-1}(A)\} = Z_k(X) \text{ when } k > 0 \\ B_k(X,A) &= \{c \in S_k(X) \,|\, c = \partial b + c' \text{ with } b \in S_{k+1}(X),\, c' \in S_k(A)\} \\ &= B_k(X) \text{ when } k > 0. \end{aligned}$$

□

2.3. The long exact sequence of relative homology. By definition the relative homology of X with respect to A is the homology of the quotient complex $S_\bullet(X)/S_\bullet(A)$. By Proposition 1.7, adapted to homology by reversing the arrows, one obtains a long exact cohomology sequence

$$\begin{aligned} \cdots \to H_2(A) \to H_2(X) \to H_2(X,A) \\ \to H_1(A) \to H_1(X) \to H_1(X,A) \\ \to H_0(A) \to H_0(X) \to H_0(X,A) \to 0 \end{aligned}$$

EXERCISE 2.8. Assume to know that $H_1(S^1,R) \simeq R$ and $H_k(S^1,R) = 0$ for $k > 1$. Use the long relative homology sequence to compute the relative homology groups $H_\bullet(\mathbb{R}^2, S_1; R)$.

3. The Mayer-Vietoris sequence

The Mayer-Vietoris sequence (in its simplest form, that we are going to consider here) allows one to compute the homology of a union $X = U \cup V$ from the knowledge of the homology of U, V and $U \cap V$. This is quite similar to what happens in de Rham cohomology, but in the case of homology there is a subtlety. Let us denote $A = U \cap V$. One would think that there is an exact sequence

$$0 \to S_k(A) \xrightarrow{i} S_k(U) \oplus S_k(V) \xrightarrow{p} S_k(X) \to 0$$

where i is the morphism induced by the inclusions $A \hookrightarrow U$, $A \hookrightarrow V$, and p is given by $p(\sigma_1, \sigma_2) = \sigma_1 - \sigma_2$ (again using the inclusions $U \hookrightarrow X$, $V \hookrightarrow X$). However, it is not possible to prove that p is surjective (if σ is a singular k-simplex whose image is not contained in U or V, it is not in general possible to write it as a difference of standard k-simplexes in U, V). The trick to circumvent this difficulty consists in replacing $S_\bullet(X)$ with a different complex that however has the same homology.

Let $\mathfrak{U} = \{U_\alpha\}$ be an open cover of X.

DEFINITION 2.1. *A singular k-chain $\sigma = \sum_j a_j \sigma_j$ is $\mathfrak{U}$-small if every singular k-simplex σ_j maps into an open set $U_\alpha \in \mathfrak{U}$ for some α. Moreover we define $S^{\mathfrak{U}}_\bullet(X)$ as the subcomplex of $S_\bullet(X)$ formed by $\mathfrak{U}$-small chains.*[1]

The homology differential ∂ restricts to $S^{\mathfrak{U}}_\bullet(X)$, so that one has a homology $H^{\mathfrak{U}}_\bullet(X)$.

[1] Again, we understand the choice of a coefficient ring R.

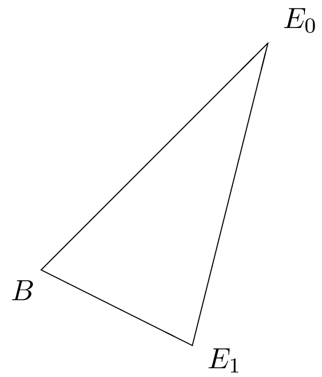


FIGURE 5. The join $B(< E_0, E_1 >)$

PROPOSITION 2.2. *$H_\bullet^{\mathfrak{U}}(X) \simeq H_\bullet(X)$.*

To prove this isomorphism we shall build a homotopy between the complexes $S_\bullet^{\mathfrak{U}}(X)$ and $S_\bullet(X)$. This will be done in several steps.

Given a singular k-simplex $< Q_0, \dots, Q_k >$ in $\mathbb{R}^n$ and a point $B \in \mathbb{R}^n$ we consider the singular simplex $< B, Q_0, \dots, Q_k >$, called the *join* of B with $< Q_0, \dots, Q_k >$. This operator B is then extended to singular chains in $\mathbb{R}^n$ by linearity. The following Lemma is easily proved.

LEMMA 2.3. *$\partial \circ B + B \circ \partial = \mathrm{Id}$ on $S_k(\mathbb{R}^n)$ if $k > 0$, while $\partial \circ B(\sigma) = \sigma - (\sum_j a_j)B$ if $\sigma = \sum_j a_j x_j \in S_0(\mathbb{R}^n)$.*

Next we define operators $\Sigma\colon S_k(X) \to S_k(X)$ and $T\colon S_k(X) \to S_{k+1}(X)$. The operator Σ is called the *subdivision operator* and its effect is that of subdividing a singular simplex into a linear combination of "smaller" simplexes. The operators Σ and T, analogously to what we did for the prism operator, will be defined for $X = \Delta_k$ (the space consisting of the standard k-simplex) and for the "identity" singular simplex $\delta_k\colon \Delta_k \to \Delta_k$, and then extended by functoriality. This should be done for all k. One defines

$$\Sigma(\delta_0) = \delta_0, \qquad T(\delta_0) = 0.$$

and then extends recursively to positive k:

$$\Sigma(\delta_k) = B_k(\Sigma(\partial\delta_k)), \qquad T(\delta_k) = B_k(\delta_k - \Sigma(\delta_k) - T(\partial\delta_k))$$

where the point B_k is the *barycenter* of the standard k-simplex Δ_k,

$$B_k = \frac{1}{k+1}\sum_{j=0}^{k} P_j\,.$$

EXAMPLE 2.4. For $k = 1$ one gets $\Sigma(\delta_1) =< B_1P_1 > - < B_1P_0 >$; for $k = 2$, the action of Σ splits Δ_2 into smaller simplexes as shown in Figure 6. $\square$

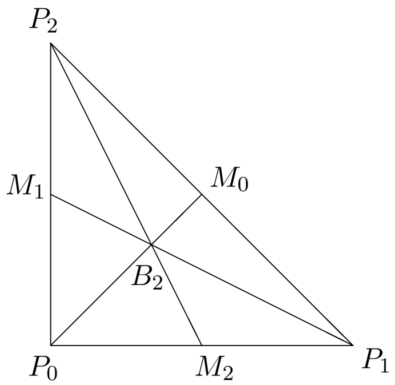


FIGURE 6. The subdivision operator Σ splits Δ_2 into the chain $< B_2, M_0, P_2 > - < B_2, M_0, P_1 > - < B_2, M_1, P_2 > + < B_2, M_1, P_0 >$ $+ < B_2, M_2, P_1 > - < B_2, M_2, P_0 >$

The definition of Σ and T for every topological space and every singular k-simplex σ in X is

$$\Sigma(\sigma) = S_k(\sigma)(\Sigma(\delta_k)), \qquad T(\sigma) = S_{k+1}(\sigma)(T(\delta_k)).$$

LEMMA 2.5. *One has the identities*

$$\partial \circ \Sigma = \Sigma \circ \partial, \qquad \partial \circ T + T \circ \partial = \mathrm{Id} - \Sigma.$$

PROOF. These identities are proved by direct computation (it is enough to consider the case $X = \Delta_k$). $\square$

The first identity tells us that Σ is a morphism of differential complexes, and the second that T is a homotopy between Σ and Id, so that the morphism $\Sigma_\flat$ induced in homology by Σ is an isomorphism.

The *diameter* of a singular k-simplex σ in $\mathbb{R}^n$ is the maximum of the lengths of the segments contained in σ. The proof of the following Lemma is an elementary computation.

LEMMA 2.6. *Let* $\sigma =< E_0, \dots, E_k >$*, with* $E_0, \dots, E_k \in \mathbb{R}^n$*. The diameter of every simplex in the singular chain* $\Sigma(\sigma) \in S_k(\mathbb{R}^n)$ *is at most* $k/k+1$ *times the diameter of* σ*.*

PROPOSITION 2.7. *Let* X *be a topological space,* $\mathfrak{U} = \{U_\alpha\}$ *an open cover, and* σ *a singular* k*-simplex in* X*. There is a natural number* $r > 0$ *such that every singular simplex in* $\Sigma^r(\sigma)$ *is contained in a open set* U_α*.*

PROOF. As Δ_k is compact there is a real positive number ϵ such that σ maps a neighbourhood of radius ϵ of every point of Δ_k into some U_α. Since

$$\lim_{r \to +\infty} \frac{k^r}{(k+1)^r} = 0$$

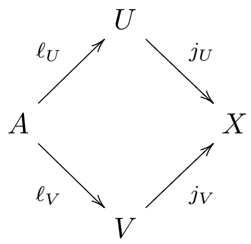
there is an $r > 0$ such that $\Sigma^r(\delta_k)$ is a linear combination of simplexes whose diameter is less than ϵ. But as $\Sigma^r(\sigma) = S_k(\sigma)(\Sigma^r(\delta_k))$ we are done. $\square$

This completes the proof of Proposition 2.2. We may now prove the exactness of the Mayer-Vietoris sequence in the following sense. If $X = U \cup V$ (union of two open subsets), let $\mathfrak{U} = \{U, V\}$ and $A = U \cap V$.

Proposition 2.8. *For every k there is an exact sequence of R-modules*

$$0 \to S_k(A) \xrightarrow{i} S_k(U) \oplus S_k(V) \xrightarrow{p} S_k^{\mathfrak{U}}(X) \to 0\,.$$

Proof. One has a diagram of inclusions



Defining $i(\sigma) = (\ell_U \circ \sigma, -\ell_V \circ \sigma)$ and $p(\sigma_1, \sigma_2) = j_U \circ \sigma_1 + j_V \circ \sigma_2$, the exactness of the Mayer-Vietoris sequence is easily proved. $\square$

The morphisms i and p commute with the homology operator ∂, so that one obtains a long homology exact sequence involving the homologies $H_\bullet(A)$, $H_\bullet(V) \oplus H_\bullet(V)$ and $H_\bullet^{\mathfrak{U}}(X)$. But in view of Proposition 2.2 we may replace $H_\bullet^{\mathfrak{U}}(X)$ with the homology $H_\bullet(X)$, so that we obtain the exact sequence

$$\begin{aligned} \cdots \to H_2(A) \to H_2(U) \oplus H_2(V) \to H_2(X) & \\ \to H_1(A) \to H_1(U) \oplus H_1(V) \to H_1(X) & \\ \to H_0(A) \to H_0(U) \oplus H_0(V) \to H_0(X) \to 0 & \end{aligned}$$

Exercise 2.9. Prove that for any ring R the homology of the sphere S^n with coefficients in R, $n \geq 2$, is

$$H_k(S^n, R) = \begin{cases} R & \text{for } \quad k = 0 \text{ and } k = n \\ 0 & \text{for } \quad 0 < k < n \text{ and } k > n\,. \end{cases}$$

Exercise 2.10. Show that the relative homology of S^2 mod S^1 with coefficients in $\mathbb{Z}$ is concentrated in degree 2, and $H_2(S^2, S^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Exercise 2.11. Use the Mayer-Vietoris sequence to compute the homology of a cylinder $S^1 \times \mathbb{R}$ minus a point with coefficients in $\mathbb{Z}$. (Hint: since the cylinder is homotopic to S^1, it has the same homology). The result is (calling X the space)

$$H_0(X, Z) \simeq \mathbb{Z}, \quad H_1(X, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}, \quad H_2(X, \mathbb{Z}) = 0\,.$$

Compare this with the homology of S^2 minus three points.

4. Excision

If a space X is the union of subspaces, the Mayer-Vietoris suquence allows one to compute the homology of X from the homology of the subspaces and of their intersections. The operation of *excision* in some sense gives us information about the reverse operation, i.e., it tells us what happen to the homology of a space if we "excise" a subpace out of it. Let us recall that given a map $f\colon (X,A)\to(Y,B)$ (i.e., a map $f\colon X\to Y$ such that $f(A)\subset B$) there is natural morphism $f_\flat\colon H_\bullet(X,A)\to H_\bullet(Y,B)$.

Definition 2.1. *Given nested subspaces $U\subset A\subset X$, the inclusion map $(X-U,A-U)\to(X,A)$ is said to be an excision if the induced morphism $H_k(X-U,A-U)\to H_k(X,A)$ is an isomorphism for all k.*

If $(X-U,A-U)\to(X,A)$ is an excision, we say that U "can be excised."

To state the main theorem about excision we need some definitions from topology.

Definition 2.2. *1. Let $i\colon A\to X$ be an inclusion of topological spaces. A map $r\colon X\to A$ is a retraction of i if $r\circ i=\mathrm{Id}_A$.*

2. A subspace $A\subset X$ is a deformation retract of X if Id_X is homotopically equivalent to $i\circ r$, where $r\colon X\to A$ is a retraction.

If $r\colon X\to A$ is a retraction of $i\colon A\to X$, then $r_\flat\circ i_\flat=\mathrm{Id}_{H_\bullet(A)}$, so that $i_\flat\colon H_\bullet(A)\to H_\bullet(X)$ is injective. Moreover, if A is a deformation retract of X, then $H_\bullet(A)\simeq H_\bullet(X)$. The same notion can be given for inclusions of pairs, $(A,B)\hookrightarrow(X,Y)$; if such a map is a deformation retract, then $H_\bullet(A,B)\simeq H_\bullet(X,Y)$.

Exercise 2.3. Show that no retraction $S^n\to S^{n-1}$ can exist.

Theorem 2.4. *If the closure $\overline{U}$ of U lies in the interior* $\mathrm{int}(A)$ *of A, then U can be excised.*

Proof. We consider the cover $\mathfrak{U}=\{X-\overline{U},\mathrm{int}(A)\}$ of X. Let $c=\sum_j a_j\sigma_j\in Z_k(X,A)$, so that $\partial c\in S_{k-1}(A)$. In view of Proposition 2.2 we may assume that c is $\mathfrak{U}$-small. If we cancel from σ those singular simplexes σ_j taking values in $\mathrm{int}(A)$, the class $[c]\in H_k(X,A)$ is unchanged. Therefore, after the removal, we can regard c as a relative cycle in $X-U$ mod $A-U$; this implies that the morphism $H_k(X-U,A-U)\to H_k(X,A)$ is surjective.

To prove that it is injective, let $[c]\in H_k(X-U,A-U)$ be such that, regarding c as a cycle in X mod A, it is a boundary, i.e., $c\in B_k(X,A)$. This means that

$$c=\partial b+c' \qquad \text{with} \quad b\in S_{k+1}(X),\ c'\in S_k(A)\,.$$

We apply the operator Σ^r to both sides of this inequality, and split $\Sigma^r(b)$ into b_1+b_2, where b_1 maps into $X-\bar{U}$ and b_2 into $\mathrm{int}(A)$. We have

$$\Sigma^r(c)-\partial b_1=\Sigma^r(c')+\partial b_2\,.$$

The chain in the left side is in $X-U$ while the chain in the right side is in A; therefore, both chains are in $(X-U)\cap A = A-U$. Now we have

$$\Sigma^r(c) = \Sigma^r(c') + \partial b_2 + \partial b_1$$

with $\Sigma^r(c')+\partial b_2 \in S_k(A-U)$ and $\partial b_1 \in S_{k+1}(X-U)$ so that $\Sigma^r(c) \in B_k(X-U, A-U)$, which implies $[c]=0$ (in $H_k(X-U,A-U)$). □

EXERCISE 2.5. Let B an open band around the equator of S^2, and $x_0 \in B$. Compute the relative homology $H_\bullet(S^2-x_0, B-x_0;\mathbb{Z})$.

To describe some more applications of excision we need the notion of *augmented homology modules.* Given a topological space X and a ring R, let us define

$$\begin{aligned} \partial^\sharp \colon S_0(X,R) &\to R \\ \sum_j a_j\sigma_j &\mapsto \sum_j a_j\,. \end{aligned}$$

We define the augmented homology modules

$$H_0^\sharp(X,R) = \ker\partial^\sharp / B_0(X,R)\,, \qquad H_k^\sharp(X,R) = H_k(X,R) \text{ for } k>0\,.$$

If $A\subset X$, one defines the augmented relative homology modules $H_k^\sharp(X,A;R)$ in a similar way, i.e.,

$$H_k^\sharp(X,A;R) = H_k(X,A;R) \text{ if } A\neq\emptyset, \qquad H_k^\sharp(X,A;R) = H_k(X,R) \text{ if } A=\emptyset\,.$$

EXERCISE 2.6. Prove that there is a long exact sequence for the augmented relative homology modules.

EXERCISE 2.7. Let B^n be the closed unit ball in $\mathbb{R}^{n+1}$, S^n its boundary, and let $E_n^\pm$ be the two closed (northern, southern) emispheres in S^n.

1. Use the long exact sequence for the augmented relative homology modules to prove that $H_k^\sharp(S^n) \simeq H_k^\sharp(S^n, E_n^-)$ and $H_{k-1}^\sharp(S^{n-1}) \simeq H_k^\sharp(B^n, S^{n-1})$. So we have $H_k^\sharp(B^n,S^{n-1}) = 0$ for $k<n$, $H_n^\sharp(B^n,S^{n-1})\simeq R$
2. Use excision to show that $H_k^\sharp(S^n,E_n^-) \simeq H_k^\sharp(B^n,S^{n-1})$.
3. Deduce that $H_k^\sharp(S^n) \simeq H_{k-1}^\sharp(S^{n-1})$.

EXERCISE 2.8. Let S^n be the sphere realized as the unit sphere in $\mathbb{R}^{n+1}$, and let $r\colon S^n \to S^n \to S^n$ be the reflection

$$r(x^0,x^1,\dots,x^n) = (-x^0,x^1,\dots,x^n).$$

Prove that $r_\flat \colon H_n(S^n) \to H_n(S^n)$ is the multiplication by -1. (Hint: this is trivial for $n=0$, and can be extended by induction using the commutativity of the diagram

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{\sim} & H^\sharp_{n-1}(S^{n-1}) \\ \downarrow{\scriptstyle r_\flat} & & \downarrow{\scriptstyle r_\flat} \\ H_n(S^n) & \xrightarrow{\sim} & H^\sharp_{n-1}(S^{n-1}) \end{array}$$

which follows from Exercise 2.7.

Exercise 2.9. 1. The rotation group $O(n+1)$ acts on S^n. Show that for any $M \in O(n+1)$ the induced morphism $M_\flat \colon H_n(S^n) \to H_n(S^n)$ is the multiplication by $\det M = \pm 1$.

2. Let $a \colon S^n \to S^n$ be the antipodal map, $a(x) = -x$. Show that $a_\flat \colon H_n(S^n) \to H_n(S^n)$ is the multiplication by $(-1)^{n+1}$.

Example 2.10. We show that the inclusion map $(E^+_n, S^{n-1}) \to (S^n, E^-_n)$ is an excision. (Here we are excising the open southern emisphere, i.e., with reference to the general theory, $X = S^n$, $U =$ the open southern emisphere, $A = E^-_n$.)

The hypotheses of Theorem 2.4 are not satisfied. However it is enough to consider the subspace

$$V = \left\{ x \in S^n \,|\, x^0 > -\tfrac{1}{2} \right\} .$$

V can be excised from (S^n, E^-_n). But (E^+_n, S^{n-1}) is a deformation retract of $(S^n - V, E^-_n - V)$ so that we are done. $\square$

We end with a standard application of algebraic topology. Let us define a *vector field* on S^n as a continous map $v \colon S^n \to \mathbb{R}^{n+1}$ such that $v(x) \cdot x = 0$ for all $x \in S^n$ (the product is the standard scalar product in $\mathbb{R}^{n+1}$).

Proposition 2.11. *A nowhere vanishing vector field v on S^n exists if and only if n is odd.*

Proof. If $n = 2m+1$ a nowhere vanishing vector field is given by

$$v(x_0, \dots, x_{2m+1}) = (-x_1, x_0, -x_3, x_2, \dots, -x_{2m+1}, x_{2m}) .$$

Conversely, assume that such a vector field exists. Define

$$w(x) = \frac{v(x)}{\|v(x)\|};$$

this is a map $S^n \to S^n$, with $w(x) \cdot x = 0$ for all $x \in S^n$. Define

$$\begin{aligned} F \colon S^n \times I &\to S^n \\ F(x,t) &= x \cos t\pi + w(x) \sin t\pi. \end{aligned}$$

Since

$$F(x,0) = x, \qquad F(x, \tfrac{1}{2}) = w(x), \qquad F(x,1) = -x$$

the three maps Id, w, a are homotopic. But as a consequence of Exercise 2.9, n must be odd. $\square$

CHAPTER 3

Introduction to sheaves and their cohomology

1. Presheaves and sheaves

Let X be a topological space.

Definition 3.1. *A presheaf of Abelian groups on X is a rule*[1] *$\mathcal{P}$ which assigns an Abelian group $\mathcal{P}(U)$ to each open subset U of X and a morphism (called restriction map) $\varphi_{U,V} \colon \mathcal{P}(U) \to \mathcal{P}(V)$ to each pair $V \subset U$ of open subsets, so as to verify the following requirements:*

(1) $\mathcal{P}(\emptyset) = \{0\}$;

(2) $\varphi_{U,U}$ is the identity map;

(3) if $W \subset V \subset U$ are open sets, then $\varphi_{U,W} = \varphi_{V,W} \circ \varphi_{U,V}$.

The elements $s \in \mathcal{P}(U)$ are called *sections* of the presheaf $\mathcal{P}$ on U. If $s \in \mathcal{P}(U)$ is a section of $\mathcal{P}$ on U and $V \subset U$, we shall write $s_{|V}$ instead of $\varphi_{U,V}(s)$. The restriction $\mathcal{P}_{|U}$ of $\mathcal{P}$ to an open subset U is defined in the obvious way.

Presheaves of rings are defined in the same way, by requiring that the restriction maps are ring morphisms. If $\mathcal{R}$ is a presheaf of rings on X, a presheaf $\mathcal{M}$ of Abelian groups on X is called a *presheaf of modules* over $\mathcal{R}$ (or an $\mathcal{R}$-module) if, for each open subset U, $\mathcal{M}(U)$ is an $\mathcal{R}(U)$-module and for each pair $V \subset U$ the restriction map $\varphi_{U,V} \colon \mathcal{M}(U) \to \mathcal{M}(V)$ is a morphism of $\mathcal{R}(U)$-modules (where $\mathcal{M}(V)$ is regarded as an $\mathcal{R}(U)$-module via the restriction morphism $\mathcal{R}(U) \to \mathcal{R}(V)$). The definitions in this Section are stated for the case of presheaves of Abelian groups, but analogous definitions and properties hold for presheaves of rings and modules.

Definition 3.2. *A morphism $f \colon \mathcal{P} \to \mathcal{Q}$ of presheaves over X is a family of morphisms of Abelian groups $f_U \colon \mathcal{P}(U) \to \mathcal{Q}(U)$ for each open $U \subset X$, commuting with the*

[1] This rather naive terminology can be made more precise by saying that a presheaf on X is a contravariant functor from the category $\mathfrak{O}_X$ of open subsets of X to the category of Abelian groups. $\mathfrak{O}_X$ is defined as the category whose objects are the open subsets of X while the morphisms are the inclusions of open sets.

restriction morphisms; i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}(U) & \xrightarrow{f_U} & \mathcal{Q}(U) \\ {\scriptstyle \varphi_{U,V}}\downarrow & & \downarrow{\scriptstyle \varphi_{U,V}} \\ \mathcal{P}(V) & \xrightarrow{f_V} & \mathcal{Q}(V) \end{array}$$

DEFINITION 3.3. *The stalk of a presheaf $\mathcal{P}$ at a point $x \in X$ is the Abelian group*

$$\mathcal{P}_x = \varinjlim_{U} \mathcal{P}(U)$$

where U ranges over all open neighbourhoods of x, directed by inclusion.

REMARK 3.4. We recall here the notion of direct limit. A *directed set I* is a partially ordered set such that for each pair of elements i, $j \in I$ there is a third element k such that $i < k$ and $j < k$. If I is a directed set, a directed system of Abelian groups is a family $\{G_i\}_{i \in I}$ of Abelian groups, such that for all $i < j$ there is a group morphism $f_{ij} \colon G_i \to G_j$, with $f_{ii} = id$ and $f_{ij} \circ f_{jk} = f_{ik}$. On the set $\mathfrak{G} = \coprod_{i \in I} G_i$, where $\coprod$ denotes disjoint union, we put the following equivalence relation: $g \sim h$, with $g \in G_i$ and $h \in G_j$, if there exists a $k \in I$ such that $f_{ik}(g) = f_{jk}(h)$. The direct limit $\mathfrak{l}$ of the system $\{G_i\}_{i \in I}$, denoted $\mathfrak{l} = \varinjlim_{i \in I} G_i$, is the quotient $\mathfrak{G}/\sim$. Heuristically, two elements in $\mathfrak{G}$ represent the same element in the direct limit if they are 'eventually equal.' From this definition one naturally obtains the existence of canonical morphisms $G_i \to \mathfrak{l}$. The following discussion should make this notion clearer; for more detail, the reader may consult [**13**]. □

If $x \in U$ and $s \in \mathcal{P}(U)$, the image s_x of s in $\mathcal{P}_x$ via the canonical projection $\mathcal{P}(U) \to \mathcal{P}_x$ (see footnote) is called the germ of s at x. From the very definition of direct limit we see that two elements $s \in \mathcal{P}(U)$, $s' \in \mathcal{P}(V)$, U, V being open neighbourhoods of x, define the same germ at x, i.e. $s_x = s'_x$, if and only if there exists an open neighbourhood $W \subset U \cap V$ of x such that s and s' coincide on W, $s_{|W} = s'_{|W}$.

DEFINITION 3.5. *A sheaf on a topological space X is a presheaf $\mathcal{F}$ on X which fulfills the following axioms for any open subset U of X and any cover $\{U_i\}$ of U.*

- S1) *If two sections $s \in \mathcal{F}(U)$, $\bar{s} \in \mathcal{F}(U)$ coincide when restricted to any U_i, $s_{|U_i} = \bar{s}_{|U_i}$, they are equal, $s = \bar{s}$.*
- S2) *Given sections $s_i \in \mathcal{F}(U_i)$ which coincide on the intersections, $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for every i, j, there exists a section $s \in \mathcal{F}(U)$ whose restriction to each U_i equals s_i, i.e. $s_{|U_i} = s_i$.*

Thus, roughly speaking, sheaves are presheaves defined by local conditions. The *stalk of a sheaf* is defined as in the case of a presheaf.

EXAMPLE 3.6. If $\mathcal{F}$ is a sheaf, and $\mathcal{F}_x = \{0\}$ for all $x \in X$, then $\mathcal{F}$ is the zero sheaf, $\mathcal{F}(U) = \{0\}$ for all open sets $U \subset X$. Indeed, if $s \in \mathcal{F}(U)$, since $s_x = 0$ for all $x \in U$, there is for each $x \in U$ an open neighbourhood U_x such that $s_{|U_x} = 0$. The first sheaf axiom then implies $s = 0$. This is not true for a presheaf, cf. Example 3.15 below. □

A morphism of sheaves is just a morphism of presheaves. If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X, for every $x \in X$ the morphism f induces a morphism between the stalks, $f_x : \mathcal{F}_x \to \mathcal{G}_x$, in the following way: since the stalk $\mathcal{F}_x$ is the direct limit of the groups $\mathcal{F}(U)$ over all open U containing x, any $g \in \mathcal{F}_x$ is of the form $g = s_x$ for some open $U \ni x$ and some $s \in \mathcal{F}(U)$; then set $f_x(g) = (f_U(s))_x$.

A sequence of morphisms of sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is *exact* if for every point $x \in X$, the sequence of morphisms between the stalks $0 \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to 0$ is exact. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, for every open subset $U \subset X$ the sequence of groups $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$ is exact, but the last arrow may fail to be surjective. Instances of this situation are shown in Examples 3.11 and 3.12 below.

EXERCISE 3.7. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of sheaves. Show that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is an exact sequence of presheaves.

EXAMPLE 3.8. Let G be an Abelian group. Defining $\mathcal{P}(U) \equiv G$ for every open subset U and taking the identity maps as restriction morphisms, we obtain a presheaf, called the *constant presheaf* $\tilde{G}_X$. All stalks $(\tilde{G}_X)_x$ of $\tilde{G}_X$ are isomorphic to the group G. The presheaf $\tilde{G}_X$ is not a sheaf: if V_1 and V_2 are disjoint open subsets of X, and $U = V_1 \cup V_2$, the sections $g_1 \in \tilde{G}_X(V_1) = G$, $g_2 \in \tilde{G}_X(V_2) = G$, with $g_1 \neq g_2$, satisfy the hypothesis of the second sheaf axiom S2) (since $V_1 \cap V_2 = \emptyset$ there is nothing to satisfy), but there is no section $g \in \tilde{G}_X(U) = G$ which restricts to g_1 on V_1 and to g_2 on V_2.

EXAMPLE 3.9. Let $\mathcal{C}_X(U)$ be the ring of real-valued continuous functions on an open set U of X. Then $\mathcal{C}_X$ is a sheaf (with the obvious restriction morphisms), the sheaf of continuous functions on X. The stalk $\mathcal{C}_x \equiv (\mathcal{C}_X)_x$ at x is the ring of germs of continuous functions at x.

EXAMPLE 3.10. In the same way one can define the following sheaves:

The sheaf $\mathcal{C}^\infty_X$ of differentiable functions on a differentiable manifold X.

The sheaves Ω^p_X of differential p-forms, and all the sheaves of tensor fields on a differentiable manifold X.

The sheaf of holomorphic functions on a complex manifold and the sheaves of holomorphic p-forms on it.

The sheaves of forms of type (p, q) on a complex manifold X.

EXAMPLE 3.11. Let X be a differentiable manifold, and let $d\colon \Omega^\bullet_X \to \Omega^\bullet_X$ be the exterior differential. We can define the presheaves $\mathcal{Z}^p_X$ of closed differential p-forms, and

$\mathcal{B}^p_X$ of exact p-differential forms,

$$\mathcal{Z}^p_X(U) = \{\omega \in \Omega^p_X(U) \,|\, d\omega = 0\},$$

$$\mathcal{B}^p_X(U) = \{\omega \in \Omega^p_X(U) \,|\, \omega = d\tau \quad \text{for some} \quad \tau \in \Omega^{p-1}_X(U)\}.$$

$\mathcal{Z}^p_X$ is a sheaf, since the condition of being closed is local: a differential form is closed if and only if it is closed in a neighbourhood of each point of X. On the contrary, $\mathcal{B}^p_X$ is not a sheaf. In fact, if $X = \mathbb{R}^2$, the presheaf $\mathcal{B}^1_X$ of exact differential 1-forms does not fulfill the second sheaf axiom: consider the form

$$\omega = \frac{xdy - ydx}{x^2 + y^2}$$

defined on the open subset $U = X - \{(0,0)\}$. Since ω is closed on U, there is an open cover $\{U_i\}$ of U by open subsets where ω is an exact form, $\omega_{|U_i} \in \mathcal{B}^1_X(U_i)$ (this is Poincaré's lemma). But ω is not an exact form on U because its integral along the unit circle is different from 0.

This means that, while the sequence of sheaf morphisms $0 \to \mathbb{R} \to \mathcal{C}^\infty_X \xrightarrow{d} \mathcal{Z}^1_X \to 0$ is exact (Poincaré lemma), the morphism $\mathcal{C}^\infty_X(U) \xrightarrow{d} \mathcal{Z}^1_X(U)$ may fail to be surjective.

Example 3.12. Let X be a complex manifold, $\mathbb{Z}$ the constant sheaf with stalk the integers, $\mathcal{O}$ the sheaf of holomorphic functions on X, and $\mathcal{O}^*$ the sheaf of nowhere vanishing holomorphic functions. In analogy with the exact sequence (1.1) we may consider the sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 1 \tag{3.1}$$

This is an exact sequence of sheaves, in particular $\exp\colon \mathbb{C} \to \mathbb{C}^*$ is surjective as a map of sheaves, since in a neighbourhood of every point, an inverse my be found by applying the logarithm function. However, since the latter is multi-valued, surjectivity fails on non-simply connected open sets. See Example 3.11.

1.1. Étalé space. We wish now to describe how, given a presheaf, one can naturally associate with it a sheaf having the same stalks. As a first step we consider the case of a constant presheaf $\tilde{G}_X$ on a topological space X, where G is an Abelian group. We can define another presheaf G_X on X by putting $G_X(U) = \{$locally constant functions $f\colon U \to G\}$, [2] where $\tilde{G}_X(U) = G$ is included as the constant functions. It is clear that $(G_X)_x = G_x = G$ at each point $x \in X$ and that G_X is a sheaf, called the *constant sheaf with stalk* G. Notice that the functions $f\colon U \to G$ are the sections of the projection $\pi\colon \coprod_{x\in X} G_x \to X$ and the locally constant functions correspond to those sections which locally coincide with the sections produced by the elements of G.

Now, let $\mathcal{P}$ be an arbitrary presheaf on X. Consider the disjoint union of the stalks $\underline{\mathcal{P}} = \coprod_{x\in X} \mathcal{P}_x$ and the natural projection $\pi\colon \underline{\mathcal{P}} \to X$. The sections $s \in \mathcal{P}(U)$ of the

[2]A function is locally constant on U if it is constant on any connected component of U.

presheaf $\mathcal{P}$ on an open subset U produce sections $\underline{s}\colon U \hookrightarrow \underline{\mathcal{P}}$ of π, defined by $\underline{s}(x) = s_x$, and we can define a new presheaf $\mathcal{P}^\natural$ by taking $\mathcal{P}^\natural(U)$ as the group of those sections $\sigma\colon U \hookrightarrow \underline{\mathcal{P}}$ of π such that for every point $x \in U$ there is an open neighbourhood $V \subset U$ of x which satisfies $\sigma_{|V} = \underline{s}$ for some $s \in \mathcal{P}(V)$.

That is, $\mathcal{P}^\natural$ is the presheaf of all sections that locally coincide with sections of $\mathcal{P}$. It can be described in another way by the following construction.

Definition 3.13. *The set $\underline{\mathcal{P}}$, endowed with the topology whose base of open subsets consists of the sets $s(U)$ for U open in X and $s \in \mathcal{P}(U)$, is called the étalé space of the presheaf $\mathcal{P}$.*

Exercise 3.14.
(1) Show that $\pi\colon \underline{\mathcal{P}} \to X$ is a local homeomorphism, i.e., every point $u \in \underline{\mathcal{P}}$ has an open neighbourhood U such that $\pi\colon U \to \pi(U)$ is a homeomorphism.
(2) Show that for every open set $U \subset X$ and every $s \in \mathcal{P}(U)$, the section $\underline{s}\colon U \to \underline{\mathcal{P}}$ is continuous.
(3) Prove that $\mathcal{P}^\natural$ is the sheaf of continuous sections of $\pi\colon \underline{\mathcal{P}} \to X$.
(4) Prove that for all $x \in X$ the stalks of $\mathcal{P}$ and $\mathcal{P}^\natural$ at x are isomorphic.
(5) Show that there is a presheaf morphism $\phi\colon \mathcal{P} \to \mathcal{P}^\natural$.
(6) Show that ϕ is an isomorphism if and only if $\mathcal{P}$ is a sheaf. □

$\mathcal{P}^\natural$ is called the *sheaf associated with the presheaf* $\mathcal{P}$. In general, the morphism $\phi\colon \mathcal{P} \to \mathcal{P}^\natural$ is neither injective nor surjective: for instance, the morphism between the constant presheaf $\tilde{G}_X$ and its associated sheaf G_X is injective but nor surjective.

Example 3.15. As a second example we study the sheaf associated with the presheaf $\mathcal{B}^k_X$ of exact k-forms on a differentiable manifold X. For any open set U we have an exact sequence of Abelian groups (actually of $\mathbb{R}$-vector spaces)

$$0 \to \mathcal{B}^k_X(U) \to \mathcal{Z}^k_X(U) \to \mathcal{H}^k_X(U) \to 0$$

where $\mathcal{H}^k_X$ is the presheaf that with any open set U associates its k-th de Rham cohomology group, $\mathcal{H}^k_X(U) = H^k_{DR}(U)$. Now, the open neighbourhoods of any point $x \in X$ which are diffeomorphic to $\mathbb{R}^n$ (where $n = \dim X$) are cofinal[3] in the family of all open neighbourhoods of x, so that $(\mathcal{H}^k_X)_x = 0$ by the Poincaré lemma. In accordance with Example 3.6 this means that $(\mathcal{H}^k_X)^\natural = 0$, which is tantamount to $(\mathcal{B}^k_X)^\natural \simeq \mathcal{Z}^k_X$.

In this case the natural morhism $\mathcal{H}^k_X \to (\mathcal{H}^k_X)^\natural$ is of course surjective but not injective. On the other hand, $\mathcal{B}^k_X \to (\mathcal{B}^k_X)^\natural = \mathcal{Z}^k_X$ is injective but not surjective. □

[3]Let I be a directed set. A subset J of I is said to be *cofinal* if for any $i \in I$ there is a $j \in J$ such that $i < j$. By the definition of direct limit we see that, given a directed family of Abelian groups $\{G_i\}_{i\in I}$, if $\{G_j\}_{j\in J}$ is the subfamily indexed by J, then

$$\varinjlim_{i\in I} G_i \simeq \varinjlim_{j\in J} G_j\,;$$

that is, direct limits can be taken over cofinal subsets of the index set.

Definition 3.16. *Given a sheaf $\mathcal{F}$ on a topological space X and a subset (not necessarily open) $S \subset X$, the sections of the sheaf $\mathcal{F}$ on S are the continuous sections $\sigma\colon S \hookrightarrow \underline{\mathcal{F}}$ of $\pi\colon \underline{\mathcal{F}} \to X$. The group of such sections is denoted by $\Gamma(S, \mathcal{F})$.*

Definition 3.17. *Let $\mathcal{P}$, $\mathcal{Q}$ be presheaves on a topological space X.* [4]

(1) The direct sum of $\mathcal{P}$ and $\mathcal{Q}$ is the presheaf $\mathcal{P} \oplus \mathcal{Q}$ given, for every open subset $U \subset X$, by $(\mathcal{P} \oplus \mathcal{Q})(U) = \mathcal{P}(U) \oplus \mathcal{Q}(U)$ with the obvious restriction morphisms.

(2) For any open set $U \subset X$, let us denote by $\operatorname{Hom}(\mathcal{P}_{|U}, \mathcal{Q}_{|U})$ the space of morphisms between the restricted presheaves $\mathcal{P}_{|U}$ and $\mathcal{Q}_{|U}$; this is an Abelian group in a natural manner. The presheaf of homomorphisms is the presheaf $\mathcal{H}om(\mathcal{P}, \mathcal{Q})$ given by $\mathcal{H}om(\mathcal{P}, \mathcal{Q})(U) = \operatorname{Hom}(\mathcal{P}_{|U}, \mathcal{Q}_{|U})$ with the natural restriction morphisms.

(3) The tensor product of $\mathcal{P}$ and $\mathcal{Q}$ is the presheaf $(\mathcal{P} \otimes \mathcal{Q})(U) = \mathcal{P}(U) \otimes \mathcal{Q}(U)$.

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then the presheaves $\mathcal{F} \oplus \mathcal{G}$ and $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ are sheaves. On the contrary, the tensor product of $\mathcal{F}$ and $\mathcal{G}$ previously defined may not be a sheaf. Indeed one defines the tensor product of the *sheaves* $\mathcal{F}$ and $\mathcal{G}$ as the sheaf associated with the presheaf $U \to \mathcal{F}(U) \otimes \mathcal{G}(U)$.

It should be noticed that in general $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) \not\simeq \operatorname{Hom}(\mathcal{F}(U), \mathcal{G}(U))$ and $\mathcal{H}om(\mathcal{F}, \mathcal{G})_x \not\simeq \operatorname{Hom}(\mathcal{F}_x, \mathcal{G}_x)$.

1.2. Direct and inverse images of presheaves and sheaves. Here we study the behaviour of presheaves and sheaves under change of base space. Let $f\colon X \to Y$ be a continuous map.

Definition 3.18. *The direct image by f of a presheaf $\mathcal{P}$ on X is the presheaf $f_*\mathcal{P}$ on Y defined by $(f_*\mathcal{P})(V) = \mathcal{P}(f^{-1}(V))$ for every open subset $V \subset Y$. If $\mathcal{F}$ is a sheaf on X, then $f_*\mathcal{F}$ turns out to be a sheaf.*

Let $\mathcal{P}$ be a presheaf on Y.

Definition 3.19. *The inverse image of $\mathcal{P}$ by f is the presheaf on X defined by*

$$U \to \varinjlim_{U \subset f^{-1}(V)} \mathcal{P}(V).$$

The inverse image sheaf of a sheaf $\mathcal{F}$ on Y is the sheaf $f^{-1}\mathcal{F}$ associated with the inverse image presheaf of $\mathcal{F}$.

The stalk of the inverse image presheaf at a point $x \in X$ is isomorphic to $\mathcal{P}_{f(x)}$. It follows that if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves on Y, the induced sequence

$$0 \to f^{-1}\mathcal{F}' \to f^{-1}\mathcal{F} \to f^{-1}\mathcal{F}'' \to 0$$

[4] Since we are dealing with Abelian groups, i.e. with $\mathbb{Z}$-modules, the Hom modules and tensor products are taken over $\mathbb{Z}$.

of sheaves on X, is also exact (that is, the inverse image functor for sheaves of Abelian groups is exact).

The étalé space $\underline{f^{-1}\mathcal{F}}$ of the inverse image sheaf is the fibred product [5] $Y \times_X \underline{\mathcal{F}}$. It follows easily that the inverse image of the constant sheaf G_X on X with stalk G is the constant sheaf G_Y with stalk G, $f^{-1}G_X = G_Y$.

2. Cohomology of sheaves

We wish now to describe a cohomology theory which associates cohomology groups to a sheaf on a topological space X.

2.1. Čech cohomology. We start by considering a presheaf $\mathcal{P}$ on X and an open cover $\mathfrak{U}$ of X. We assume that $\mathfrak{U}$ is labelled by a totally ordered set I, and define

$$U_{i_0\dots i_p} = U_{i_0} \cap \dots \cap U_{i_p} .$$

We define the *Čech complex of* $\mathfrak{U}$ *with coefficients in* $\mathcal{P}$ as the complex whose p-th term is the Abelian group

$$\check{C}^p(\mathfrak{U},\mathcal{P}) = \prod_{i_0<\dots<i_p} \mathcal{P}(U_{i_0\dots i_p}) .$$

Thus a p-cochain α is a collection $\{\alpha_{i_0\dots i_p}\}$ of sections of $\mathcal{P}$, each one belonging to the space of sections over the intersection of $p+1$ open sets in $\mathfrak{U}$. Since the indexes of the open sets are taken in strictly increasing order, each intersection is counted only once.

The Čech differential $\delta\colon \check{C}^p(\mathfrak{U},\mathcal{P}) \to \check{C}^{p+1}(\mathfrak{U},\mathcal{P})$ is defined as follows: if $\alpha = \{\alpha_{i_0\dots i_p}\} \in \check{C}^p(\mathfrak{U},\mathcal{P})$, then

$$\{(\delta\alpha)_{i_0\dots i_{p+1}}\} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0\dots\widehat{\imath}_k\dots i_{p+1}|U_{i_0\dots i_{p+1}}} .$$

Here a caret denotes omission of the index. For instance, if $p=0$ we have $\alpha = \{\alpha_i\}$ and

$$(\delta\alpha)_{ik} = \alpha_{k|U_i\cap U_k} - \alpha_{i|U_i\cap U_k} . \tag{3.2}$$

It is an easy exercise to check that $\delta^2 = 0$. Thus we obtain a cohomology theory. We denote the corresponding cohomology groups by $\check{H}^k(\mathfrak{U},\mathcal{P})$.

Lemma 3.1. *If $\mathcal{F}$ is a sheaf, one has an isomorphism $\check{H}^0(\mathfrak{U},\mathcal{F}) \simeq \mathcal{F}(X)$*

Proof. We have $\check{H}^0(\mathfrak{U},\mathcal{F}) = \ker\delta\colon \check{C}^0(\mathfrak{U},\mathcal{P}) \to \check{C}^1(\mathfrak{U},\mathcal{P})$. So if $\alpha \in \check{H}^0(\mathfrak{U},\mathcal{F})$ by (3.2) we see that

$$\alpha_{k|U_i\cap U_k} = \alpha_{i|U_i\cap U_k} .$$

By the second sheaf axiom this implies that there is a global section $\tilde{\alpha} \in \mathcal{F}(X)$ such that $\tilde{\alpha}_{|U_i} = \alpha_i$. This yields a morphism $\check{H}^0(\mathfrak{U},\mathcal{F}) \to \mathcal{F}(X)$, which is evidently surjective and is injective because of the first sheaf axiom. □

[5] For a definition of fibred product see e.g. [**16**].

EXAMPLE 3.2. We consider an open cover $\mathfrak{U}$ of the circle S^1 formed by three sets which intersect only pairwise. We compute the Čech cohomology of $\mathfrak{U}$ with coefficients in the constant sheaf $\mathbb{R}$. We have $C^0(\mathfrak{U},\mathbb{R}) = C^1(\mathfrak{U},\mathbb{R}) = \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$, $C^k(\mathfrak{U},\mathbb{R}) = 0$ for $k>1$ because there are no triple intersections. The only nonzero differential $d_0\colon C^0(\mathfrak{U},\mathbb{R}) \to C^1(\mathfrak{U},\mathbb{R})$ is given by

$$d_0(x_0,x_1,x_2) = (x_1-x_2, x_2-x_0, x_0-x_1).$$

Hence

$$\check{H}^0(\mathfrak{U},\mathbb{R}) = \ker d_0 \simeq \mathbb{R}$$

$$\check{H}^1(\mathfrak{U},\mathbb{R}) = C^1(\mathfrak{U},\mathbb{R})/\operatorname{Im} d_0 \simeq \mathbb{R}.$$

□

It is possible to define Čech cohomology groups depending only on the pair $(X,\mathcal{F})$, and not on a cover, by letting

$$\check{H}^k(X,\,\mathcal{F}) = \varinjlim_{\mathfrak{U}} \check{H}^k(\mathfrak{U},\mathcal{F}).$$

The direct limit is taken over a cofinal subset of the directed set of all covers of X (the order is of course the refinement of covers: a cover $\mathfrak{V}=\{V_j\}_{j\in J}$ is a refinement of $\mathfrak{U}$ if there is a map $f\colon I\to J$ such that $V_{f(i)}\subset U_i$ for every $i\in I$). The order must be fixed at the outset, since a cover may be regarded as a refinement of another in many ways. As different cofinal families give rise to the same inductive limit, the groups $\check{H}^k(X,\,\mathcal{F})$ are well defined.

2.2. Fine sheaves. Čech cohomology is well-behaved when the base space X is paracompact. (It is indeed the bad behaviour of Čech cohomology on non-paracompact spaces which motivated the introduction of another cohomology theory for sheaves, usually called *sheaf cohomology;* cf. [**6**].) In this and in the following sections we consider some properties of Čech cohomology that hold in that case.

DEFINITION 3.3. *A sheaf of rings $\mathcal{R}$ on a topological space X is fine if, for any locally finite oper cover $\mathfrak{U}=\{U_i\}_{i\in I}$ of X,*[6] *there is a family $\{s_i\}_{i\in I}$ of global sections of $\mathcal{R}$ such that:*

(1) $\sum_{i\in I} s_i = 1$;

(2) for every $i\in I$ there is a closed subset $S_i\subset U_i$ such that $(s_i)_x=0$ whenever $x\notin S_i$.

[6]We recall that an oper cover $\mathfrak{U}$ is *locally finite* if every point in X has an open neighbourhood which intersects only a finite number of elements of $\mathfrak{U}$. It is possible to show that whenever X is paracompact, any open cover has a locally finite refinement [**17**].

The family $\{s_i\}$ is called a *partition of unity* subordinated to the cover $\mathfrak{U}$. For instance, the sheaf of continuous functions on a paracompact topological space as well as the sheaf of smooth functions on a differentiable manifold are fine, while sheaves of complex or real analytic functions are not.

Definition 3.4. *A sheaf $\mathcal{F}$ of Abelian groups on a topological space X is said to be acyclic if $\check{H}^k(X, \mathcal{F}) = 0$ for $k > 0$.*

Proposition 3.5. *Let $\mathcal{R}$ be a fine sheaf of rings on a paracompact space X. Every sheaf $\mathcal{M}$ of $\mathcal{R}$-modules is acyclic.*

Proof. Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be a locally finite open cover of X, and let $\{\rho_i\}$ be a partition of unity of $\mathcal{R}$ subordinated to $\mathfrak{U}$. For any $\alpha \in \check{C}^q(\mathfrak{U}, \mathcal{M})$ with $q > 0$ we set

$$
\begin{aligned}
(K\alpha)_{i_0\ldots i_{q-1}} &= \sum_{\substack{j\in I\\ j<i_0}} \rho_j\, a_{ji_0\ldots i_{q-1}} - \sum_{\substack{j\in I\\ i_0<j<i_1}} \rho_j\, a_{i_0 j i_1\ldots i_{q-1}} + \ldots \\
&= \sum_{k=0}^{q} (-1)^k \sum_{\substack{j\in I\\ i_{k-1}<j<i_k}} \rho_j\, a_{i_0\ldots i_{k-1} j i_k \ldots i_{q-1}}\,.
\end{aligned}
$$

This defines a morphism $K\colon \check{C}^k(\mathfrak{U}, \mathcal{M}) \to \check{C}^{k-1}(\mathfrak{U}, \mathcal{M})$ such that $\delta K + K\delta = \mathrm{id}$ (i.e., K is a homotopy operator); then $\alpha = \delta K\alpha$ if $\delta\alpha = 0$, so that $\check{H}^k(\mathfrak{U}, \mathcal{M}) = 0$ for $k > 0$. Since on a paracompact space the locally finite open covers are cofinal in the family of all covers, we can take direct limit on such covers, thus getting $\check{H}^k(X, \mathcal{M}) = 0$ for $k > 0$. □

Example 3.6. Using this result we may recast the proof of the exactness of the Mayer-Vietoris sequence for de Rham cohomology in a slightly different form. Given a differentiable manifold X, let $\mathfrak{U}$ be the open cover formed by two sets U and V. Since $\check{C}^2(\mathfrak{U}, \Omega^k) = 0$ (there are no triple intersections) we have an exact sequence

$$0 \to \check{H}^0(\mathfrak{U}, \Omega^k) \to \check{C}^0(\mathfrak{U}, \Omega^k) \xrightarrow{\delta} \check{C}^1(\mathfrak{U}, \Omega^k) \to 0\,.$$

which in principle is exact everywhere but at $C^1(\mathfrak{U}, \Omega^k)$. However since the sheaves Ω^k are acyclic by Proposition 3.5, one has $\check{H}^1(\mathfrak{U}, \Omega^k) = 0$, which means that δ is surjective, and the sequence is exact at that place as well. We have the identifications

$$\check{H}^0(\mathfrak{U}, \Omega^k) = \Omega^k(X), \qquad C^0(\mathfrak{U}, \Omega^k) = \Omega^k(U) \oplus \Omega^k(V), \qquad C^1(\mathfrak{U}, \Omega^k) = \Omega^k(U \cap V)$$

so that we obtain the exactness of the Mayer-Vietoris sequence.

2.3. Long exact sequences in Čech cohomology. We wish to show that when X is paracompact, any exact sequence of sheaves induces a corresponding long exact sequence in Čech cohomology.

LEMMA 3.7. *Let X be any topological space, and let*

$$0 \to \mathcal{P}' \to \mathcal{P} \to \mathcal{P}'' \to 0 \tag{3.3}$$

be an exact sequence of presheaves on X. Then one has a long exact sequence

$$\begin{aligned} 0 \to \check{H}^0(X,\mathcal{P}') \to \check{H}^0(X,\mathcal{P}) \to \check{H}^0(X,\mathcal{P}'') \to \check{H}^1(X,\mathcal{P}') \to \dots \\ \to \check{H}^k(X,\mathcal{P}') \to \check{H}^k(X,\mathcal{P}) \to \check{H}^k(X,\mathcal{P}'') \to \check{H}^{k+1}(X,\mathcal{P}') \to \dots \end{aligned}$$

PROOF. For any open cover $\mathfrak{U}$ the exact sequence (3.3) induces an exact sequence of differential complexes

$$0 \to \check{C}^\bullet(\mathfrak{U},\mathcal{P}') \to \check{C}^\bullet(\mathfrak{U},\mathcal{P}) \to \check{C}^\bullet(\mathfrak{U},\mathcal{P}'') \to 0$$

which induces the long cohomology sequence

$$\begin{aligned} 0 \to \check{H}^0(\mathfrak{U},\mathcal{P}') \to \check{H}^0(\mathfrak{U},\mathcal{P}) \to \check{H}^0(\mathfrak{U},\mathcal{P}'') \to \check{H}^1(\mathfrak{U},\mathcal{P}') \to \dots \\ \to \check{H}^k(\mathfrak{U},\mathcal{P}') \to \check{H}^k(\mathfrak{U},\mathcal{P}) \to \check{H}^k(\mathfrak{U},\mathcal{P}'') \to \check{H}^{k+1}(\mathfrak{U},\mathcal{P}') \to \dots \end{aligned}$$

Since the direct limit of a family of exact sequences yields an exact sequence, by taking the direct limit over the open covers of X one obtains the required exact sequence. □

LEMMA 3.8. *Let X be a paracompact topological space, $\mathcal{P}$ a presheaf on X whose associated sheaf is the zero sheaf, let $\mathfrak{U}$ be an open cover of X, and let $\alpha \in \check{C}^k(\mathfrak{U},\mathcal{P})$. There is a refinement $\mathfrak{W}$ of $\mathfrak{U}$ such that $\tau(\alpha) = 0$, where $\tau\colon \check{C}^k(\mathfrak{U},\mathcal{P}) \to \check{C}^k(\mathfrak{W},\mathcal{P})$ is the morphism induced by restriction.*

PROOF. We shall need to use the following fact [**5, ?**]: given an open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of a paracompact space X,[7] there is an open cover $\mathfrak{V} = \{V_i\}_{i\in I}$ having the same cardinality of $\mathfrak{U}$, such that $\bar{V}_i \subset U_i$.

□

PROPOSITION 3.9. *Let $\mathcal{P}$ be a presheaf on a paracompact space X, and let $\mathcal{P}^\natural$ be the associated sheaf. For all $k \geq 0$, the natural morphism $\check{H}^k(X,\mathcal{P}) \to \check{H}^k(X,\mathcal{P}^\natural)$ is an isomorphism.*

PROOF. One has an exact sequence of presheaves

$$0 \to \mathcal{Q}_1 \to \mathcal{P} \to \mathcal{P}^\natural \to \mathcal{Q}_2 \to 0$$

with

$$\mathcal{Q}_1^\natural = \mathcal{Q}_2^\natural = 0\,. \tag{3.4}$$

This gives rise to

$$0 \to \mathcal{Q}_1 \to \mathcal{P} \to \mathcal{T} \to 0\,, \qquad 0 \to \mathcal{T} \to \mathcal{P}^\natural \to \mathcal{Q}_2 \to 0 \tag{3.5}$$

[7] It is enough that X is normal, however, any paracompact space is normal [**17**].

where $\mathcal{T}$ is the quotient presheaf $\mathcal{P}/\mathcal{Q}_1$, i.e. the presheaf $U \to \mathcal{P}(U)/\mathcal{Q}_1(U)$. By Lemma 3.8 the isomorphisms (3.4) yield $\check{H}^k(X,\mathcal{Q}_1) = \check{H}^k(X,\mathcal{Q}_2) = 0$. Then by taking the long exact sequences of cohomology from the exact sequences (3.5) we obtain the desired isomorphism. □

Using these results we may eventually prove that on paracompact spaces one has long exact sequences in Čech cohomology.

THEOREM 3.10. *Let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be an exact sequence of sheaves on a paracompact space X. There is a long exact sequence of Čech cohomology groups

$$0 \to \check{H}^0(X,\mathcal{F}') \to \check{H}^0(X,\mathcal{F}) \to \check{H}^0(X,\mathcal{F}'') \to \check{H}^1(X,\mathcal{F}') \to \dots$$
$$\to \check{H}^k(X,\mathcal{F}') \to \check{H}^k(X,\mathcal{F}) \to \check{H}^k(X,\mathcal{F}'') \to \check{H}^{k+1}(X,\mathcal{F}') \to \dots$$

PROOF. Let $\mathcal{P}$ be the quotient *presheaf* $\mathcal{F}/\mathcal{F}'$; then $\mathcal{P}^\natural \simeq \mathcal{F}''$. One has an exact sequence of presheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{P} \to 0\,.$$

By taking the associated long exact sequence in cohomology (cf. Lemma 3.7) and using the isomorphism $\check{H}^k(X,\mathcal{P}) = \check{H}^k(X,\mathcal{F}'')$ one obtains the required exact sequence. □

EXAMPLE 3.11. The long exact sequence in cohomology associated with the exact sequence (3.1) starts with

$$0 \to H^0(U,\mathbb{Z}) \to H^0(U,\mathcal{O}) \to H^0(U,\mathcal{O}^*) \to H^1(U,\mathbb{Z}) \to \dots$$

This shows that the obstruction to the sequence (3.1) to be exact as a sequence of presheaves is the first cohomology group with coefficients in Z. Since X, being a manifold, is paracompact and locally Euclidean , the Čech cohomology of $\mathbb{Z}$ coincides with the singular cohomology (see Proposition 3.29); therefore the above mentioned obstruction is the non-simply connectedness of U.

2.4. Abstract de Rham theorem. We describe now a very useful way of computing cohomology groups; this result is sometimes called “abstract de Rham theorem.” As a particular case it yields one form of the so-called de Rham theorem, which states that the de Rham cohomology of a differentiable manifold and the Čech cohomology of the constant sheaf $\mathbb{R}$ are isomorphic.

DEFINITION 3.12. *Let $\mathcal{F}$ be a sheaf of abelian groups on X. A resolution of $\mathcal{F}$ is a collection of sheaves of abelian groups $\{\mathcal{L}^k\}_{k\in\mathbb{N}}$ with morphisms $i\colon \mathcal{F} \to \mathcal{L}^0$, $d_k\colon \mathcal{L}^k \to \mathcal{L}^{k+1}$ such that the sequence*

$$0 \to \mathcal{F} \xrightarrow{i} \mathcal{L}^0 \xrightarrow{d_0} \mathcal{L}^1 \xrightarrow{d_1} \dots$$

is exact. If the sheaves $\mathcal{L}^\bullet$ are acyclic (fine) the resolution is said to be acyclic (fine).

LEMMA 3.13. *If* $0 \to \mathcal{F} \to \mathcal{L}^\bullet$ *is a resolution, the morphism* $i_X \colon \mathcal{F}(X) \to \mathcal{L}^0(X)$ *is injective.*

PROOF. Let $\mathcal{Q}$ be the quotient $\mathcal{L}^0/\mathcal{F}$. Then the sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{L}^0 \to \mathcal{Q} \to 0$$

is exact. By Exercise 3.7, the sequence of abelian groups

$$0 \to \mathcal{F}(X) \to L^0(X) \to \mathcal{Q}(X)$$

is exact. This implies the claim. □

However the sequence of abelian groups

$$0 \to \mathcal{L}^0(X) \xrightarrow{d_0} \mathcal{L}^1(X) \xrightarrow{d_1} \dots$$

is not exact. We shall consider its cohomology $H^\bullet(\mathcal{L}^\bullet(X), d)$. By the previous Lemma we have $H^0(\mathcal{L}^\bullet(X), d) \simeq H^0(X, \mathcal{F})$.

THEOREM 3.14. *If* $0 \to \mathcal{F} \to \mathcal{L}^\bullet$ *is an acyclic resolution there is an isomorphism* $\check{H}^k(X, \mathcal{F}) \simeq H^k(\mathcal{L}^\bullet(X), d)$ *for all* $k \geq 0$.

PROOF. Define $\mathcal{Q}^k = \ker d_k \colon \mathcal{L}^k \to \mathcal{L}^{k+1}$. The resolution may be split into

$$0 \to \mathcal{F} \to \mathcal{L}^0_X \to \mathcal{Q}^1 \to 0\,, \qquad 0 \to \mathcal{Q}^k \to \mathcal{L}^k \to \mathcal{Q}^{k+1} \to 0\,. \quad k \geq 1$$

Since the sheaves $\mathcal{L}^k$ are acyclic by taking the long exact sequences of cohomology we obtain a chain of isomorphisms

$$\check{H}^k(X, \mathcal{F}) \simeq \check{H}^{k-1}(X, \mathcal{Q}^1) \simeq \cdots \simeq \check{H}^1(X, \mathcal{Q}^{k-1}) \simeq \frac{\check{H}^0(X, \mathcal{Q}^k)}{\operatorname{Im} \check{H}^0(X, \mathcal{L}^{k-1})}$$

By Exercise 3.7 $\check{H}^0(X, \mathcal{Q}^k) = \mathcal{Q}^k(X)$ is the kernel of $d_k \colon \mathcal{L}^k(X) \to \mathcal{L}^{k+1}$ so that the claim is proved. □

COROLLARY 3.15. *(de Rham theorem.) Let* X *be a differentiable manifold. For all* $k \geq 0$ *the cohomology groups* $H^k_{DR}(X)$ *and* $\check{H}^k(X, \mathbb{R})$ *are isomorphic.*

PROOF. Let $n = \dim X$. The sequence

$$0 \to \mathbb{R} \to \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \to \Omega^n_X \to 0 \tag{3.6}$$

(where $\Omega^0_X \equiv \mathcal{C}^\infty_X$) is exact (this is Poincaré's lemma). Moreover the sheaves $\Omega^\bullet_X$ are modules over the fine sheaf of rings $\mathcal{C}^\infty_X$, hence are acyclic. The claim then follows for the previous theorem. □

COROLLARY 3.16. *Let* U *be a subset of a differentiable manifold* X *which is diffeomorphic to* $\mathbb{R}^n$. *Then* $H^k(U, \mathbb{R}) = 0$ *for* $k > 0$. □

2.5. Soft sheaves. For later use we also introduce and study the notion of *soft sheaf.* However, we do not give the proofs of most claims, for which the reader is referred to [**2, 6, 25**]. The contents of this subsection will only be used in Section 5.5.

DEFINITION 3.17. *Let $\mathcal{F}$ be a sheaf a on a topological space X, and let $U \subset X$ be a closed subset of X. The space $\mathcal{F}(U)$ (called "the space of sections of $\mathcal{F}$ over U") is defined as*

$$\mathcal{F}(U) = \varinjlim_{V \supset U} \mathcal{F}(V)$$

where the direct limit is taken over all open neighbourhoods V of U.

A consequence of this definition is the existence of a natural restriction morphism $\mathcal{F}(X) \to \mathcal{F}(U)$.

DEFINITION 3.18. *A sheaf $\mathcal{F}$ is said to be soft if for every closed subset $U \subset X$ the restriction morphism $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.*

PROPOSITION 3.19. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of soft sheaves on a paracompact space X, for every closed subset $V \subset X$ the sequence of groups*

$$0 \to \mathcal{F}'(V) \to \mathcal{F}(V) \to \mathcal{F}''(V) \to 0$$

is exact.

PROOF. The proof of Proposition 3.2 can be easily adapted to this situation. □

COROLLARY 3.20. *The quotient of two soft sheaves on a paracompact space is soft.*

PROOF. If $\mathcal{F}'' = \mathcal{F}/\mathcal{F}'$ is the quotient of two soft sheaves, by Proposition 3.19 the restriction morphism $\mathcal{F}(X) \to \mathcal{F}(V)$ is surjective (where V is any closed subset of X), so that $\mathcal{F}''(X) \to \mathcal{F}''(V)$ is surjective as well. □

PROPOSITION 3.21. *Any soft sheaf of rings $\mathcal{R}$ on a paracompact space is fine.*

PROOF. Cf. Lemma II.3.4 in [**2**]. □

PROPOSITION 3.22. *Every sheaf $\mathcal{F}$ on a paracompact space admits soft resolutions.*

PROOF. Let $\mathcal{S}^0(\mathcal{F})$ be the sheaf of discontinuous sections of $\mathcal{F}$ (i.e., the sheaf of all sections of the sheaf space $\underline{\mathcal{F}}$). The sheaf $\mathcal{S}^0(\mathcal{F})$ is obviously soft. Now we have an exact sequence $0 \to \mathcal{F} \to \mathcal{S}^0(\mathcal{F}) \to \mathcal{F}_1 \to 0$. The sheaf $\mathcal{F}_1$ is not soft in general, but it may embedded into the soft sheaf $\mathcal{S}^0(\mathcal{F}_1)$, and we have an exact sequence $0 \to \mathcal{F}_1 \to \mathcal{S}^0(\mathcal{F}_1) \to \mathcal{F}_2 \to 0$. Upon iteration we have exact sequences

$$0 \to \mathcal{F}_k \xrightarrow{i_k} \mathcal{S}^k(\mathcal{F}) \xrightarrow{p_k} \mathcal{F}_{k+1} \to 0$$

where $\mathcal{S}^k(\mathcal{F}) = \mathcal{S}^0(\mathcal{F}_k)$. One can check that the sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{S}^0(\mathcal{F}) \xrightarrow{f_0} \mathcal{S}^1(\mathcal{F}) \xrightarrow{f_1} \dots$$

(where $f_k = i_{k+1} \circ p_k$) is exact. □

PROPOSITION 3.23. *If $\mathcal{F}$ is a sheaf on a paracompact space, the sheaf $\mathcal{S}^0(\mathcal{F})$ is acyclic.*

PROOF. The endomorphism sheaf $\mathcal{E}nd(\mathcal{S}^0(\mathcal{F}))$ is soft, hence fine by Proposition 3.21. Since $\mathcal{S}^0(\mathcal{F})$ is an $\mathcal{E}nd(\mathcal{S}^0(\mathcal{F}))$-module, it is acyclic.[8] □

PROPOSITION 3.24. *On a paracompact space soft sheaves are acyclic.*

PROOF. If $\mathcal{F}$ is a soft sheaf, the sequence $0 \to \mathcal{F}(X) \to \mathcal{S}^0\mathcal{F}(X) \to \mathcal{F}_1(X) \to 0$ obtained from $0 \to \mathcal{F} \to \mathcal{S}^0\mathcal{F} \to \mathcal{F}_1 \to 0$ is exact (Proposition 3.19). Since $\mathcal{F}$ and $\mathcal{S}^0\mathcal{F}$ are soft, so is $\mathcal{F}_1$ by Corollary 3.20, and the sequence $0 \to \mathcal{F}_1(X) \to \mathcal{S}^1\mathcal{F}(X) \to \mathcal{F}_2(X) \to 0$ is also exact. With this procedure we can show that the complex $\mathcal{S}^\bullet(\mathcal{F})(X)$ is exact. But since all sheaves $\mathcal{S}^\bullet(\mathcal{F})$ are acyclic by the previous Proposition, by the abstract de Rham theorem the claim is proved. □

Note that in this way we have shown that for any sheaf $\mathcal{F}$ on a paracompact space there is a *canonical soft resolution.*

2.6. Leray's theorem for Čech cohomology. If an open cover $\mathfrak{U}$ of a topological space X is suitably chosen, the Čech cohomologies $\check{H}^\bullet(\mathfrak{U}, \mathcal{F})$ and $\check{H}^\bullet(X, \mathcal{F})$ are isomorphic. Leray's theorem establishes a sufficient condition for such an isomorphism to hold. Since the cohomology $\check{H}^\bullet(\mathfrak{U}, \mathcal{F})$ is in generally much easier to compute, this turns out to be a very useful tool in the computation of Čech cohomology groups.

We say that an open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of a topological space X is acyclic for a sheaf $\mathcal{F}$ if $\check{H}^k(U_{i_0\dots i_p}, \mathcal{F}) = 0$ for all $k > 0$ and all nonvoid intersections $U_{i_0\dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$, $i_0 \dots i_p \in I$.

THEOREM 3.25. (Leray's theorem) *Let $\mathcal{F}$ be a sheaf on a paracompact space X, and let $\mathfrak{U}$ be an open cover of X which is acyclic for $\mathcal{F}$ and is indexed by an ordered set. Then, for all $k \geq 0$, the cohomology groups $\check{H}^k(\mathfrak{U}, \mathcal{F})$ and $\check{H}^k(X, \mathcal{F})$ are isomorphic.*

To prove this theorem we need to construct the so-called *Čech sheaf complex.* For every nonvoid intersection $U_{i_0\dots i_p}$ let $j_{i_0\dots i_p} \colon U_{i_0\dots i_p} \to X$ be the inclusion. For every p define the sheaf

$$\check{\mathcal{C}}^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} (j_{i_0\dots i_p})_* \mathcal{F}_{|U_{i_0\dots i_p}} \tag{3.7}$$

(every factor $(j_{i_0\dots i_p})_* \mathcal{F}_{|U_{i_0\dots i_p}}$ is the sheaf $\mathcal{F}$ first restricted to $U_{i_0\dots i_p}$ and the extended by zero to the whole of X). The Čech differential induces sheaf morphisms $\delta \colon \check{\mathcal{C}}^p(\mathfrak{U}, \mathcal{F}) \to \check{\mathcal{C}}^{p+1}(\mathfrak{U}, \mathcal{F})$. From the definition, we get isomorphisms

$$\check{\mathcal{C}}^p(\mathfrak{U}, \mathcal{F})(X) \simeq \check{C}^p(\mathfrak{U}, \mathcal{F}), \tag{3.8}$$

[8] We are cheating a little bit, since the sheaf of rings $\mathcal{E}nd(\mathcal{S}^0(\mathcal{F}))$ is not commutative. However a closer inspection of the proof would show that it works anyways.

i.e., by taking global sections of the Čech sheaf complex we get the Čech cochain group complex. Moreover we have:

LEMMA 3.26. *For all p and k,*

$$\check{H}^k(X, \check{\mathcal{C}}^p(\mathfrak{U}, \mathcal{F})) \simeq \prod_{i_0 < \cdots < i_p} \check{H}^k(U_{i_0 \dots i_p}, \mathcal{F}) \,.$$

PROOF. By the definition of the Čech cohomology groups we have

$$\check{H}^k(X, \check{\mathcal{C}}^p(\mathfrak{U}, \mathcal{F})) = \varinjlim_{\mathfrak{V}} \check{H}^k(\mathfrak{V}, \check{\mathcal{C}}^p(\mathfrak{U}, \mathcal{F}))$$

where $\mathfrak{V}$ runs over all open covers of X. The groups $\check{H}^k(\mathfrak{V}, \check{\mathcal{C}}^p(\mathfrak{U}, \mathcal{F}))$ are the cohomology of the complex $\check{C}^\bullet(\mathfrak{V}, \check{\mathcal{C}}^p(\mathfrak{U}, \mathcal{F}))$, which may be written as

$$\begin{aligned} \check{C}^k(\mathfrak{V}, \check{\mathcal{C}}^p(\mathfrak{U}, \mathcal{F})) &= \prod_{\ell_0 < \cdots < \ell_k} \check{\mathcal{C}}^p(\mathfrak{U}, \mathcal{F})(V_{\ell_0 \dots \ell_k})) \\ &\simeq \prod_{\ell_0 < \cdots < \ell_k} \prod_{i_0 < \cdots < i_p} \mathcal{F}(V_{\ell_0 \dots \ell_k} \cap U_{i_0 \dots i_p}) \\ &\simeq \check{C}^k(\mathfrak{V}_{i_0 \dots i_k}, \mathcal{F}_{|U_{i_0 \dots i_p}}) \end{aligned}$$

where $\mathfrak{V}_{i_0 \dots i_p}$ is the restriction of the cover $\mathfrak{V}$ to $U_{i_0 \dots i_p}$. This implies the claim. □

We may now prove Leray's theorem. As an immediate consequence of the fact that $\mathcal{F}$ fulfils the sheaf axioms, the complex $\check{\mathcal{C}}^\bullet(\mathfrak{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$. Under the hypothesis of Leray's theorem, by Lemma 3.26 this resolution is acyclic. By the abstract de Rham theorem, the cohomology of the global sections of the resolution is isomorphic to the cohomology of $\mathcal{F}$. But, due to the isomorphisms (3.8), the cohomology of the global sections of the resolution is the cohomology $\check{H}^\bullet(\mathfrak{U}, \mathcal{F})$.

2.7. Good covers. By means of Leray's theorem we may reduce the problem of computing the Čech cohomology of a differentiable manifold with coefficients in the constant sheaf $\mathbb{R}$ (which, via de Rham theorem, amounts to computing its de Rham cohomology) to the computation of the cohomology of a cover with coefficients in $\mathbb{R}$; thus a problem which in principle would need the solution of differential equations on topologically nontrivial manifolds is reduced to a simpler problem which only involves the intersection pattern of the open sets of a cover.

DEFINITION 3.27. *A locally finite open cover $\mathfrak{U}$ of a differentiable manifold is good if all nonempty intersections of its members are diffeomorphic to $\mathbb{R}^n$.*

Good covers exist on any differentiable manifold (cf. [**19**]). Due to Corollary 3.16, good covers are acyclic for the constant sheaf $\mathbb{R}$. We have therefore

PROPOSITION 3.28. *For any good cover* $\mathfrak{U}$ *of a differentiable manifold* X *one has isomorphisms*

$$\check{H}^k(\mathfrak{U},\mathbb{R}) \simeq \check{H}^k(X,\mathbb{R}), \qquad k \geq 0\,.$$

□

The cover of Example 3.2 was good, so we computed there the de Rham cohomology of the circle S^1.

2.8. Comparison with other cohomologies. In algebraic topology one attaches to a topological space X several cohomologies with coefficients in an abelian group G. Loosely speaking, whenever X is paracompact and locally Euclidean, all these cohomologies coincide with the Čech cohomology of X with coefficients in the constant sheaf G. In particular, we have the following result:

PROPOSITION 3.29. *Let* X *be a paracompact locally Euclidean topological space, and let* G *be an abelian group. The singular cohomology of* X *with coefficients in* G *is isomorphic to the Čech cohomology of* X *with coefficients in the constant sheaf* G. □

3. Sheaf cohomology

Another kind of sheaves which can be introduced is that of *flabby sheaves* (also called "flasque"). A sheaf $\mathcal{F}$ on a topological space X is said to be flabby if for every open subset $U \subset X$ the restriction morphism $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. It is easy to prove that flabby sheaves are soft: if $U \subset X$ is a closed subset, by definition of direct limit, for every $s \in \mathcal{F}(U)$ there is an open neighbourhood V of U and a section $s' \in \mathcal{F}(V)$ which restricts to s. Since $\mathcal{F}$ is flabby, s' can be extended to the whole of X. So on a paracompact space, flabby sheaves are acyclic, and by the abstract de Rham theorem flabby resolution can be used to compute cohomology. We should also notice that the canonical soft resolution $\mathcal{S}^\bullet(\mathcal{F})$ we constructed in Section 2.5 is flabby, as one can easily check by the definition itself. We shall then call $\mathcal{S}^\bullet(\mathcal{F})$ the *canonical flabby resolution* of the sheaf $\mathcal{F}$ (this is also called the *Godement resolution* of $\mathcal{F}$).

One can further pursue this line and use flabby resolutions (for instance, the canonical flabby resolution) to *define* cohomology. That is, for every sheaf $\mathcal{F}$, its cohomology is by definition the cohomology of the global sections of its canonical flabby resolution (it then turns out that cohomology can be computed with any acyclic resolution). This has the advantage of producing a cohomology theory (called *sheaf cohomology*) which is bell-behaved (e.g., it has long exact sequences in cohomology) on every topological space, not just on paracompact ones. In this section we briefly outline the basics of this theory; for a more comprehensive treatment the reader may refer to [**6, 4, 2**], or to [**23**] where a different and more general approach to sheaf cohomology (using *injective resolutions*) is pursued; also the original paper by Grothendieck [**9**] can be fruitfully read. It follows from our treatment that on a paracompact topological space the sheaf and Čech cohomology coincide, but in general they do not. In the next chapter we shall establish the relation between the two cohomologies in terms of a spectral sequence (cf. also [**12**], especially the exercise section, for a discussion of the comparison between the two cohomologies).

Definition 3.1. *If $\mathcal{F}$ is a sheaf on a topological space X, its* sheaf cohomology groups *are defined as*

$$H^i(X,\mathcal{F}) = H^i(\mathcal{S}^\bullet(\mathcal{F})(X))$$

for $i \geq 0$.

The following two results are basic for this construction. Here X is any topological space.

Proposition 3.2. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, with $\mathcal{F}'$ flabby, for any open set $U \subset X$ the sequence of abelian groups*

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$$

is exact (namely, the sequence is exact as a sequence of presheaves).

Proof. Let $U \subset X$ and $s'' \in \mathcal{F}''(U)$. We need to show the existence of $s \in \mathcal{F}(U)$ such that $p(s) = s''$ under the map $p\colon \mathcal{F} \to \mathcal{F}''$. Let I be the set of all pairs (W, s), where $W \subset U$ is open, and $s \in \mathcal{F}(W)$ represents s'' on W (i.e., $p(s) = s''_{|W}$). The set I is nonempty since the morphism p is surjective in the sense of sheaves. The set I may be given a partial ordering "by extension", i.e., $(W, s) < (W', t)$ if $W \subset W'$ and $s = t_{|W}$. The set has an upper bound (the union of all its elements) and then by Zorn's lemma it has a maximal element $(\overline{W}, \bar{s})$. If $x \in U \setminus \overline{W}$ there is a neighbourhood V of x and a section $t \in \mathcal{F}(V)$ which represents s'' in V. Over the intersection $V \cap \overline{W}$ the section $\bar{s} - t$ lies in $\mathcal{F}'$ and since $\mathcal{F}'$ is flabby it may be extended to V. We can then modify t so that $\bar{s} = t$ in $V \cap \overline{W}$, which contradicts the fact that $(\overline{W}, \bar{s})$ is maximal. Then such a x cannot exist, and $\overline{W} = U$. □

Corollary 3.3. *The quotient of two flabby sheaves is flabby.*

Proof. If we have the sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, with $\mathcal{F}'$ and $\mathcal{F}$ flabby, we may apply the previous Lemma. If $s'' \in \mathcal{F}''(U)$ there exists $s \in \mathcal{F}(U)$ such that $p(s) = s''$. Since $\mathcal{F}$ is flabby, s extends to a section t of $\mathcal{F}$ on X, and then $p(t)$ extends s''. □

Corollary 3.4. *If*

$$0 \to \mathcal{L}^0 \to \mathcal{L}^1 \to \dots$$

is an exact sequence of flabby sheaves, for every open set $U \subset X$ the sequence of abelian groups

$$0 \to \mathcal{L}^0(U) \to \mathcal{L}^1(U) \to \dots$$

is exact.

Corollary 3.5. *Flabby sheaves are acyclic with respect to sheaf cohomology, i.e.,* $H^p(X,\mathcal{F})=0$ *for all* $p>0$ *if* $\mathcal{F}$ *is a flabby sheaf.*

Proof. Apply the previous corollary to the canonical flabby resolution of $\mathcal{F}$. □

Corollary 3.6. *Flabby sheaves are acyclic with respect to Čech cohomology, i.e.,* $\check{H}^p(\mathfrak{U},\mathcal{F})=0$ *for every open cover* $\mathfrak{U}$ *of* X *and for all* $p>0$ *if* $\mathcal{F}$ *is a flabby sheaf.*

Proof. Since $\mathcal{F}$ is flabby, the sheaves $\check{\mathcal{C}}^p(\mathfrak{U},\mathcal{F})$ defined in Eq. (3.7) are flabby as well. By Corollary 3.4 the sequence

$$\check{\mathcal{C}}^0(\mathfrak{U},\mathcal{F})(X)\xrightarrow{\delta}\check{\mathcal{C}}^1(\mathfrak{U},\mathcal{F})(X)\xrightarrow{\delta}\dots$$

is exact. Since $\check{\mathcal{C}}^p(\mathfrak{U},\mathcal{F})(X)=\check{C}^p(\mathfrak{U},\mathcal{F})$, this implies that the Čech complex $\check{C}^\bullet(\mathfrak{U},\mathcal{F})$ is exact. □

As a further consequence, we have the isomorphism between Čech and sheaf cohomology on a paracompact space.

Corollary 3.7. *For any sheaf* $\mathcal{F}$ *on a paracompact space* X*, the Čech cohomology* $\check{H}^\bullet(X,\mathcal{F})$ *and the sheaf cohomology* $H^\bullet(X,\mathcal{F})$ *are isomorphic.*

Proof. By the previous Corollary, the canonical flabby resolution of $\mathcal{F}$ is acyclic for the Cech cohomology, so that the abstract de Rham theorem implies the claim. □

We want to show that sheaf cohomology is well behaved with respect to exact sequences of sheaves on any topological space. Let us denote by $\mathrm{Sh}_{/X}$, $K(\mathrm{Sh}_{/X})$ and $K(\mathrm{Ab})$ the categories of sheaves (of abelian groups) on X, of complexes of sheaves on X, and of complexes of abelian groups, respectively. The canonical flabby resolution allows one to define two functors:

$$\begin{aligned} F_1\colon \mathrm{Sh}_{/X} &\to K(\mathrm{Sh}_{/X}) \\ \mathcal{F} &\mapsto \mathcal{S}^\bullet(\mathcal{F}) \\[1ex] F_2\colon \mathrm{Sh}_{/X} &\to K(\mathrm{Ab}) \\ \mathcal{F} &\mapsto \mathcal{S}^\bullet(\mathcal{F})(X) \end{aligned}$$

Proposition 3.8. *The two functors* F_1, F_2 *are exact (i.e., they map exact sequences to exact sequences).*

Proof. If

$$0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0 \tag{3.9}$$

is an exact sequence of sheaves we have for every $x\in X$ an exact sequence

$$0\to\prod_{x\in X}\mathcal{F}'_x\to\prod_{x\in X}\mathcal{F}_x\to\prod_{x\in X}\mathcal{F}''_x\to 0$$

so that the sequence of complexes of sheaves

$$0 \to \mathcal{S}^\bullet(\mathcal{F}') \to \mathcal{S}^\bullet(\mathcal{F}) \to \mathcal{S}^\bullet(\mathcal{F}'') \to 0$$

induced by (3.9) is exact. This proves that F_1 is exact. Moreover, by Proposition 3.2 the sequence

$$0 \to \mathcal{S}^\bullet(\mathcal{F}')(X) \to \mathcal{S}^\bullet(\mathcal{F})(X) \to \mathcal{S}^\bullet(\mathcal{F}'')(X) \to 0 \tag{3.10}$$

is exact as well, so that F_2 is exact. □

COROLLARY 3.9. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, there is a long exact sequence of cohomology*

$$\begin{aligned} &0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \to \dots \\ &\to H^k(X, \mathcal{F}') \to H^k(X, \mathcal{F}) \to H^k(X, \mathcal{F}'') \to H^{k+1}(X, \mathcal{F}') \to \dots \end{aligned} \tag{3.11}$$

PROOF. The long exact sequence of cohomology associated with the exact sequence of complexes of abelian groups (3.10) is just the sequence (3.11). □

An immediate consequence of this result is that the proof of the abstract de Rham theorem for the Čech cohomology on a paracompact space may be applied to provide a proof of the same theorem for sheaf cohomology on any space; thus, on any topological space, the sheaf cohomology of a sheaf $\mathcal{F}$ is isomorphic to the cohomology of the complex of global sections of a resolution of $\mathcal{F}$ which is acyclic for the sheaf cohomology.

CHAPTER 4

More homological algebra

CHAPTER 5

Spectral sequences

Spectral sequences are a powerful tool for computing homology, cohomology and homotopy groups. Often they allow one to trade a difficult computation for an easier one. Examples that we shall consider are another proof of the Čech-de Rham theorem, the Leray spectral sequence, and the Künneth theorem.

Spectral sequences are a difficult topic, basically because the theory is quite intricate and the notation is correspondingly cumbersome. Therefore we have chosen what seems to us to be the simplest approach, due to Massey [**20**]. Our treatment basically follows [**3**].

1. Filtered complexes

Let (K, d) be a graded differential module, i.e.,

$$K = \bigoplus_{n\in\mathbb{Z}} K^n\,, \qquad d\colon K^n \to K^{n+1}\,, \qquad d^2 = 0\,.$$

A graded submodule of (K, d) is a graded subgroup $K' \subset K$ such that $dK' \subset K'$, i.e.,

$$K' = \bigoplus_{n\in\mathbb{Z}} K'^n\,, \qquad K'^n \subset K^n\,, \qquad d\colon K'^n \to K'^{n+1}\,.$$

A sequence of nested graded submodules

$$K = K_0 \supset K_1 \supset K_2 \supset \dots$$

is a *filtration* of (K, d). We then say that (K, d) is filtered, and associate with it the graded complex[1]

$$\mathrm{Gr}(K) = \bigoplus_{p\in\mathbb{Z}} K_p/K_{p+1}, \qquad K_p = K \text{ if } p \leq 0\,.$$

Note that by assumption (since every K_{p+1} is a graded subgroup of K_p) the filtration is compatible with the grading, i.e., if we define $K_p^i = K^i \cap K_p$, then

$$K^n = K_0^n \supset K_1^n \supset K_2^n \supset \dots \tag{5.1}$$

is a filtration of K^i, and moreover $dK_p^n \subset K_p^{n+1}$.

[1]The choice of having $K_p = K$ for $p \leq 0$ is due to notational convenience.

EXAMPLE 5.1. A *double complex* is a collection of abelian groups $K^{p,q}$, with $p, q \geq 0$,[2] and morphisms $\delta_1 \colon K^{p,q} \to K^{p+1,q}$, $\delta_2 \colon K^{p,q} \to K^{p,q+1}$ such that

$$\delta_1{}^2 = \delta_2{}^2 = 0\,, \qquad \delta_1\delta_2 + \delta_2\delta_1 = 0\,.$$

Let (T, d) be the associated *total complex*:

$$T^i = \bigoplus_{p+q=i} K^{p,q}\,, \qquad d \colon T^i \to T^{i+1} \text{ defined by } d = \delta_1 + \delta_2$$

(note that the definition of d implies $d^2 = 0$). Then letting

$$T_p = \bigoplus_{i \geq p,\ q \geq 0} K^{i,q}$$

we obtain a filtration of (T, d). This satifies $T_p \simeq T$ for $p \leq 0$. The successive quotients of the filtration are

$$T_p/T_{p+1} = \bigoplus_{q \in \mathbb{N}} K^{p,q}\,.$$

□

DEFINITION 5.2. *A filtration $K_\bullet$ of (K, d) is said to be regular if for every $i \geq 0$ the filtration (5.1) is finite; in other words, for every i there is a number $\ell(i)$ such that $K^i_p = 0$ for $p > \ell(i)$.*

For instance, the filtration in Example 5.1 is regular since $T^i_p = 0$ for $p > i$, and indeed

$$T^i_p = T^i \cap T_p = \bigoplus_{j=0}^{i-p} K^{i-j,j}.$$

2. The spectral sequence of a filtered complex

At first we shall not consider the grading. Let $K_\bullet$ be a filtration of a differential module (K, d), and let

$$G = \bigoplus_{p \in \mathbb{Z}} K_p\,.$$

The inclusions $K_{p+1} \to K_p$ induce a morphism $i \colon G \to G$ ("the shift by the filtering degree"), and one has an exact sequence

$$0 \to G \xrightarrow{\ i\ } G \xrightarrow{\ j\ } E \to 0 \tag{5.2}$$

[2] This assumption is made here for simplicity but one could let p, q range over the integers; however some of the results we are going to give would be no longer valid.

with $E \simeq \mathrm{Gr}(K)$. The differential d induces differentials in G and E, so that from (5.2) one gets an exact triangle in cohomology (cf. Section 1.1)

$$
\begin{array}{ccc}
H(G) & \xrightarrow{\quad i \quad} & H(G) \\
{\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\
& H(E) &
\end{array}
\tag{5.3}
$$

where k is the connecting morphism.

Let us now assume that the filtration $K_\bullet$ has *finite length*, i.e., $K_p = 0$ for p greater than some ℓ (called the *length* of the filtration).

Since $dK_p \subset K_p$ for every p, we may consider the cohomology groups $H(K_p)$. The morphism i induces morphisms $i\colon H(K_{p+1}) \to H(K_p)$. Define G_1 to be the direct sum of the terms on the sequence (which is not exact)

$$
\begin{aligned}
0 \to H(K_\ell) \xrightarrow{\ i\ } H(K_{\ell-1}) \xrightarrow{\ i\ } \dots \\
\xrightarrow{\ i\ } H(K_1) \xrightarrow{\ i\ } H(K) \xrightarrow{\ \sim\ } H(K_{-1}) \xrightarrow{\ \sim\ } \dots\,,
\end{aligned}
$$

i.e., $G_1 = \bigoplus_{p\in\mathbb{Z}} H(K_p) \simeq H(G)$. Next we define G_2 as the sum of the terms of the sequence

$$
\begin{aligned}
0 \to i(H(K_\ell))) \to i(H(K_{\ell-1})) \to \dots \\
\to i(H(K_1)) \to H(K) \xrightarrow{\ \sim\ } H(K_{-1}) \xrightarrow{\ \sim\ } \dots
\end{aligned}
$$

Note that the morphism $i(H(K_1)) \to H(K)$ is injective, since it is the inclusion of the image of $i\colon H(K_1) \to H(K)$ into $H(K)$. This procedure is then iterated: G_3 is the sum of the terms in the sequence

$$
\begin{aligned}
0 \to i(i(H(K_\ell)))) \to i(i(H(K_{\ell-1}))) \to i(i(H(K_2)) \\
\to i(H(K_1)) \to H(K) \xrightarrow{\ \sim\ } H(K_{-1}) \xrightarrow{\ \sim\ } \dots
\end{aligned}
$$

and now the morphisms $i(i(H(K_2)) \to i(H(K_1))$ and $i(H(K_1)) \to H(K)$ are injective. When we reach the step ℓ, all the morphisms in the sequence

$$
\begin{aligned}
0 \to i^\ell(H(K_\ell))) \to i^{\ell-1}(H(K_{\ell-1})) \to \dots \\
\to i(H(K_1)) \to H(K) \xrightarrow{\ \sim\ } H(K_{-1}) \xrightarrow{\ \sim\ } \dots
\end{aligned}
$$

are injective, so that $G_{\ell+2} \simeq G_{\ell+1}$, and the procedure stabilizes: $G_r \simeq G_{r+1}$ for $r \geq \ell+1$. We define $G_\infty = G_{\ell+1}$; we have

$$
G_\infty \simeq \bigoplus_{p\in\mathbb{Z}} F_p
$$

where $F_p = i^p(H(K_p))$, i.e., F_p is the image of $H(K_p)$ into $H(K)$. The groups F_p provide a filtration of $H(K)$,

$$
H(K) = F_0 \supset F_1 \supset \dots \supset F_\ell \supset F_{\ell+1} = 0\,. \tag{5.4}
$$

We come now to the construction of the spectral sequence. Recall that since $dK_p \subset K_p$, and $E = \bigoplus_p K_p/K_{p+1}$, the differential d acts on E, and one has a cohomology group $H(E)$ wich splits into a direct sum

$$H(E) \simeq \bigoplus_{p\in\mathbb{Z}} H(K_p/K_{p+1}, d)\,.$$

The cohomology group $H(E)$ fits into the exact triangle (5.3), that we rewrite as

(5.5)
$$\begin{array}{ccc} G_1 & \xrightarrow{\;i_1\;} & G_1 \\ {\scriptstyle k_1}\nwarrow & & \swarrow{\scriptstyle j_1} \\ & E_1 & \end{array}$$

where $E_1 = H(E)$. We define $d_1 \colon E_1 \to E_1$ by letting $d_1 = j_1 \circ k_1$; then $d_1^2 = 0$ since the triangle is exact. Let $E_2 = H(E_1, d_1)$ and recall that G_2 is the image of G_1 under $i \colon G_1 \to G_1$. We have morphisms

$$i_2 \colon G_2 \to G_2, \,, \qquad j_2 \colon G_2 \to E_2\,, \qquad k_2 \colon E_2 \to G_2$$

where

(i) i_2 is induced by i_1 by letting $i_2(i_1(x)) = i_1(i_1(x))$ for $x \in G_1$;

(ii) j_2 is induced by j_1 by letting $j_2(i_1(x)) = [j_1(x)]$ for $x \in G_1$, where $[\,]$ denotes taking the homology class in $E_2 = H(E_1, d_1)$.

(iii) k_2 is induced by k_1 by letting $k_2([e]) = i_1(k_1(e))$.

EXERCISE 5.1. Show that the morphisms j_2 and k_2 are well defined, and that the triangle

(5.6)
$$\begin{array}{ccc} G_2 & \xrightarrow{\;i_2\;} & G_2 \\ {\scriptstyle k_2}\nwarrow & & \swarrow{\scriptstyle j_2} \\ & E_2 & \end{array}$$

is exact. □

We call (5.6) the *derived triangle* of (5.5). The procedure leading from (5.5) to the triangle (5.6) can be iterated, and we get a sequence of exact triangles

$$\begin{array}{ccc} G_r & \xrightarrow{\;i_r\;} & G_r \\ {\scriptstyle k_r}\nwarrow & & \swarrow{\scriptstyle j_r} \\ & E_r & \end{array}$$

where each group E_r is the cohomology group of the differential module (E_{r-1}, d_{r-1}), with $d_{r-1} = j_{r-1} \circ k_{r-1}$.

As we have already noticed, due to the assumption that the filtration $K_\bullet$ has finite length ℓ, the groups G_r stabilize when $r \geq \ell + 1$, and the morphisms $i_r \colon G_r \to G_r$

become injective. Thus all morphisms $k_r\colon E_r \to G_r$ vanish in that range, which implies $d_r = 0$, so that the groups E_r stabilize as well: $E_{r+1} \simeq E_r$ for $r \geq \ell + 1$. We denote by $E_\infty = E_{\ell+1}$ the stable value.

Thus, the sequence

$$0 \to G_\infty \xrightarrow{i_\infty} G_\infty \to E_\infty \to 0$$

is exact, which implies that E_∞ is the associated graded module of the filtration (5.4) of $H(K)$:

$$E_\infty \simeq \bigoplus_{p\leq \ell} F_p/F_{p+1}\,.$$

DEFINITION 5.2. *A sequence of differential modules $\{(E_r, d_r)\}$ such that $H(E_r, d_r) \simeq E_{r+1}$ is said to be a spectral sequence. If the groups E_r eventually become stationary, we denote the stationary value by E_∞. If E_∞ is isomorphic to the associated graded module of some filtered group H, we say that the spectral sequence converges to H.*

So what we have seen so far in this section is that if (K, d) is a differential module with a filtration of finite length, one can build a spectral sequence which converges to $H(K)$.

REMARK 5.3. It may happen in special cases that the groups E_r stabilize before getting the value $r = \ell + 1$. That happens if and only if $d_r = 0$ for some value $r = r_0$. This implies that $d_r = 0$ also for $r > r_0$, and $E_{r+1} \simeq E_r$ for all $r \geq r_0$. When this happens we say that the spectral sequence *degenerates* at step r_0. □

Now we consider the presence of a grading.

THEOREM 5.4. *Let (K, d) be a graded differential module, and $K_\bullet$ a regular filtration. There is a spectral sequence $\{(E_r, d_r)\}$, where each E_r is graded, which converges to the graded group $H^\bullet(K, d)$.*

Note that the filtration need not be of finite length: the length $\ell(i)$ of the filtration of K^i is finite for every i, but may increase with i.

PROOF. For every n and p we have $d(K_p^n) \subset K_p^{n+1}$, therefore we have cohomology groups $H^n(K_p)$. As a consequence, the groups G_r are graded:

$$G_r \simeq \bigoplus_{n\in\mathbb{Z}} F_r^n = \bigoplus_{n,p\in\mathbb{Z}} i^{r-1}(H^n(K_p))$$

and the groups E_r are accordingly graded. We may construct the derived triangles as before, but now we should pay attention to the grading: the morphisms i and j have degree zero, but k has degree one (just check the definition: k is basically a connecting morphism).

Fix a natural number n, and let $r \geq \ell(n + 1) + 1$; for every p the morphisms

$$i_r\colon F_r^{n+1} \to F_r^{n+1}$$

are injective, and the morphisms

$$k_r\colon E_r^n \to F_r^{n+1}$$

are zero. These are the same statements as in the ungraded case. Therefore, as it happened in the ungraded case, the groups E_r^n become stationary for r big enough. Note that $G_\infty^n = \oplus_{p\in\mathbb{Z}} F_p^n$, where $F_{p+1}^{n+1} = i^{\ell(n+1)}(H^{n+1}(K_{p+1}))$, and that the morphism i_∞ sends F_{p+1}^n injectively into F_p^n for every n, and there is an exact sequence

$$0 \to G_\infty^n \xrightarrow{i_\infty} G_\infty^n \to E_\infty^n \to 0\,.$$

This implies that E_r is the graded module associated with the graded complex $H^\bullet(K,d)$. $\square$

The last statement in the proof means that for each n, $F_\bullet^n$ is a filtration of $H^n(K,d)$, and $E_\infty^n \simeq \bigoplus_{p\in\mathbb{Z}} F_p^n/F_{p+1}^n$.

3. The bidegree and the five-term sequence

The terms E_r of the spectral sequence are actually bigraded; for instance, since the filtration and the degree of K are compatible, we have

$$K_p/K_{p+1} \simeq \bigoplus_{q\in\mathbb{Z}} K_p^q/K_{p+1}^q \simeq \bigoplus_{q\in\mathbb{Z}} K_p^{p+q}/K_{p+1}^{p+q}$$

and $E_0 = E$ is bigraded by

$$E_0 = \bigoplus_{p,q\in\mathbb{Z}} E_0^{p,q} \qquad \text{with} \quad E_0^{p,q} = K_p^{p+q}/K_{p+1}^{p+q}\,.$$

Note that the total complex associated with this bidegree yields the gradation of E.

Let us go to next step. Since $d\colon K_p^{p+q} \to K_p^{p+q+1}$, i.e., $d\colon E_0^{p,q} \to E_0^{p,q+1}$, and $E_1 = H(E,d)$, if we set

$$E_1^{p,q} = H^q(E_0^{p,\bullet},d) \simeq H^{p+q}(K_p/K_{p+1})$$

we have $E_1 \simeq \bigoplus_{p,q\in\mathbb{Z}} E_1^{p,q}$.

If we go one step further we can show that

$$d_1\colon E_1^{p,q} \to E_1^{p+1,q}\,.$$

Indeed if $x \in E_1^{p,q} \simeq H^{p+q}(K_p/K_{p+1})$ we write x as $x = [e]$ where $e \in K_p^{p+q}/K_{p+1}^{p+q}$ so that $k_1(x) = i_1(k(e)) \in H^{p+q+1}(K_{p+1})$ and

$$d_1(x) = j_1(k_1(x)) = j_1(k(e)) \in H^{p+q+1}(K_{p+1}/K_{p+2}) \simeq E_1^{p+1,q}\,.$$

As a result we have $E_2 \simeq \bigoplus_{p,q\in\mathbb{Z}} E_2^{p,q}$ with

$$E_2^{p,q} \simeq H^p(E_1^{\bullet,q},d_1)\,.$$

The same analysis shows that in general $E_r \simeq \bigoplus_{p,q\in\mathbb{Z}} E_r^{p,q}$ with

$$d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$$

and moreover we have

$$E_\infty^{p,q} \simeq F_p^{p+q}/F_{p+1}^{p+q}\,.$$

The next two Lemmas establish the existence of the morphisms that we shall use to introduce the so-called *five-term sequence,* and will anyway be useful in the following.

LEMMA 5.1. *There are canonical morphisms $H^q(K) \to E_r^{0,q}$.*

PROOF. Since $K_p \simeq K$ for $p \leq 0$ we have $F_p^n \simeq H^n(K_p) = H^n(K)$ for $p \leq 0$, hence $E_\infty^{p,q} = 0$ for $p < 0$ and $E_\infty^{0,q} \simeq F_0^q/F_1^q \simeq H^q(K)/F_1^q$, so that there is a surjective morphism $H^q(K) \to E_\infty^{0,q}$.

Note now that a nonzero class in $E_r^{0,q}$ cannot be a boundary, since then it should come from $E_r^{-r,q+r-1} = 0$. So cohomology classes are cycles. Since cohomology classes are elements in $E_{r+1}^{0,q}$, we have inclusions $E_{r+1}^{0,q} \subset E_r^{0,q}$ ($E_{r+1}^{0,q}$ is the subgroup of cycles in $E_r^{0,q}$). This yields an inclusion $E_\infty^{0,q} \subset E_r^{0,q}$ for all r.

Combining the two arguments we obtain morphisms $H^q(K) \to E_r^{0,q}$. □

LEMMA 5.2. *Assume that $K_p^n = 0$ if $p > n$ (so, in particular, the filtration is regular). Then for every $r \geq 2$ there is a morphism $E_r^{p,0} \to H^p(K)$.*

PROOF. The hypothesis of the Lemma implies that $E_r^{p,q} = 0$ for $q < 0$ (indeed, $F_p^{p+q} = i^r(H^{p+q}(K_p))$ for r big enough, so that $F_q^{p+q} = 0$ if $q < 0$ since then $K_p^{p+1} = 0$). As a consequence, for $r \geq 2$ the differential $d_r\colon E_r^{p,0} \to E_r^{p+r,1-r}$ maps to zero, i.e., all elements in $E_r^{p,0}$ are cycles, and determine cohomology classes in $E_{r+1}^{p,0}$. This means we have a morphism $E_r^{p,0} \to E_{r+1}^{p,0}$, and composing, morphisms $E_r^{p,0} \to E_\infty^{p,0}$.

Since $F_p^n = 0$ for $p > n$ we have $E_\infty^{p,0} \simeq F_p^p/F_{p+1}^p \simeq F_p^p$ so that one has an injective morphism $E_\infty^{p,0} \to H^p(K)$. Composing we have a morphism $E_r^{p,0} \to H^p(K)$. □

PROPOSITION 5.3. *(The five-term sequence). Assume that $K_p^n = 0$ if $p > n$. There is an exact sequence*

$$0 \to E_2^{1,0} \to H^1(K) \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to H^2(K)\,.$$

PROOF. The morphisms involved in the sequence in addition to d_2 have been defined in the previous two Lemmas. We shall not prove the exactness of the sequence here, for a proof cf. e.g. [**6**]. □

4. The spectral sequences associated with a double complex

In this Section we consider a double complex as we have defined in Example 5.1. Due to the presence of the bidegree, the result in Theorem 5.4 may be somehow refined.

We shall use the notation in Example 5.1. The group

$$G = \bigoplus_{p\in\mathbb{Z}} T_p = \bigoplus_{p\in\mathbb{Z}} \; \bigoplus_{n\geq p,\; q\in\mathbb{N}} K^{i,q}$$

has natural gradation $G = \oplus_{n\in\mathbb{Z}} G^n$ given by

$$G^n = \bigoplus_{p\in\mathbb{Z}} T_p^n \simeq \bigoplus_{p\in\mathbb{Z}} \bigoplus_{j=0}^{n-p} K^{n-j,j} \tag{5.7}$$

but it also bigraded, with bidegree

$$G^{p,q} = T_q^{p+q}\,.$$

Notice that if we form the total complex $\bigoplus_{p+q=n} G^{p,q}$ we obtain the complex (5.7) back:

$$\bigoplus_{p+q=n} G^{p,q} \simeq \bigoplus_{p+q=n} \bigoplus_{j=0}^{q} K^{p+q-j,j} = \bigoplus_{j=0}^{n-p} K^{n-j,j} = G^n\,.$$

The operators δ_1, δ_2 and $d = \delta_1 + \delta_2$ act on G:

$$\delta_1\colon G^{n,q} \to G^{n+1,q}\,, \qquad \delta_2 = G^{n,q} \to G^{n,q+1}\,, \qquad d\colon G^k \to G^{k+1}\,.$$

We analyze the spectral sequence associated with these data. The first three terms are easily described. One has

$$E_0^{p,q} \simeq T_p^{p+q}/T_{p+1}^{p+q} \simeq K^{p,q}$$

so that the differential $d_0\colon E_0^{p,q} \to E_0^{p,q+1}$ coincides with $\delta_2\colon K^{p,q} \to K^{p,q+1}$, and one has

$$E_1^{p,q} \simeq H^q(K^{p,\bullet}, \delta_2)\,. \tag{5.8}$$

At next step we have $d_1\colon E_1^{p,q} \to E_1^{p+1,q}$ with $E_1^{p,q} \simeq H^{p+q}(T_p/T_{p+1})$ and $T_p/T_{p+1} \simeq \bigoplus_{q\in Z} K^{p,q}$. Hence the differential

$$d_1\colon H^{p+q}(\bigoplus_{n\in\mathbb{Z}} K^{p,n}) \to H^{p+q+1}(\bigoplus_{n\in Z} K^{p+1,n})$$

is identified with δ_1, and

$$E_2^{p,q} \simeq H^p(E_1^{\bullet,q}, \delta_1)\,. \tag{5.9}$$

One should notice that by exchanging the two degrees in K (i.e., considering another double complex $'K$ such that $'K^{p,q} = K^{q,p}$), we obtain another spectral sequence, that we denote by $\{'E_r, 'd_r\}$. Both sequences converge to the same graded group, i.e., the cohomology of the total complex (but the corresponding filtrations are in general different), and this often provides interesting information. For the second spectral sequence we get

$$'E_1^{q,p} \simeq H^p(K^{\bullet,q}, \delta_1) \tag{5.10}$$

$$'E_2^{q,p} \simeq H^q('E_1^{p,\bullet}, \delta_2)\,. \tag{5.11}$$

EXAMPLE 5.1. A simple application of the two spectral sequences associated with a double complex provides another proof of the Čech-de Rham theorem, i.e., the isomorphism $H^\bullet(X,\mathbb{R}) \simeq H^\bullet_{DR}(X)$ for a differentiable manifold X. Let $\mathfrak{U} = \{U_i\}$ be a good cover of X, and define the double complex

$$K^{p,q} = \check{C}^p(\mathfrak{U}, \Omega^q)\,,$$

i.e., $K^{\bullet,q}$ is the complex of Čech cochains of $\mathfrak{U}$ with coefficients in the sheaf of differential q-forms. The first differential δ_1 is basically the Čech differential δ, while δ_2 is the exterior differential d.[3] Actually δ and d commute rather than anticommute, but this is easily settled by defining the action of δ_1 on $K^{p,q}$ as $\delta_1 = (-1)^q\delta$ (this of course leaves the spaces of boundaries and cycles unchanged). We start analyzing the spectral sequences from the terms E_1. For the first, we have

$$E_1^{p,q} \simeq H^q(K^{p,\bullet}, d) \simeq \prod_{i_0<\cdots<i_p} H^q_{DR}(U_{i_0\dots i_p})\,.$$

Since all $U_{i_0\dots i_p}$ are contractible we have

$$\begin{aligned} E_1^{p,0} &\simeq \check{C}^p(\mathfrak{U},\mathbb{R}) \\ E_1^{p,q} &= 0 \text{ for } q \neq 0\,. \end{aligned}$$

As a consequence we have $E_2^{p,q} = 0$ for $q \neq 0$, while

$$E_2^{p,0} \simeq H^p(\check{C}^\bullet(\mathfrak{U},\mathbb{R}),\delta) = H^p(\mathfrak{U},\mathbb{R})\,.$$

This implies that $d_2 = 0$, so that the spectral sequence degenerates at the second step, and $E_\infty^{p,q} = 0$ for $q \neq 0$ and $E_\infty^{p,0} \simeq H^p(\mathfrak{U},\mathbb{R})$. The resulting filtration of $H^p(T,D)$ has only one nonzero quotient, so that $H^p(T,D) \simeq H^p(\mathfrak{U},\mathbb{R})$.

Let us now consider the second spectral sequence. We have

$${}'E_1^{p,q} \simeq H^q(K^{\bullet,p},\delta) = H^q(\check{C}^\bullet(\mathfrak{U},\Omega^p),\delta) = H^q(\mathfrak{U},\Omega^p)\,.$$

Since the sheaves Ω^p are acyclic, we have

$$\begin{aligned} E_1^{p,0} &\simeq H^0(\mathfrak{U},\Omega^p) \simeq \Omega^p(X) \\ E_1^{p,q} &= 0 \text{ for } q \neq 0\,. \end{aligned}$$

At next step we have therefore ${}'E_2^{p,q} = 0$ for $q \neq 0$, and

$${}'E_2^{p,0} \simeq H^p(\Omega^\bullet(X),d)) \simeq H^p_{DR}(X)\,.$$

Again the spectral sequence degenerates at the second step, and we have $H^p(T,D) \simeq H^p_{DR}(X)$. Comparing with what we got from the first sequence, we obtain $H^p_{DR}(X) \simeq H^p(\mathfrak{U},\mathbb{R})$. Taking a direct limit on good covers, we obtain $H^p(X,\mathbb{R}) \simeq H^p_{DR}(X)$.

[3] Here a notational conflict arises, so that we shall denote by D the differential of the total complex T.

REMARK 5.2. From this example we may get the general result that if at step r, with $r \geq 1$, we have $E_r^{p,q} = 0$ for $q \neq 0$ (or for $p \neq 0$) then the sequence degenerates at step r, and $E_r^{p,0} \simeq H^p(T,d)$ (or $E_r^{0,q} \simeq H^q(T,d)$).

5. Some applications

5.1. The spectral sequence of a resolution. In this section we extend Example 5.1 to a much general situation. Let $(\mathcal{L}^\bullet, f)$ be a complex of sheaves on a paracompact topological space X, and let $\mathfrak{U}$ be an open cover of X. We introduce the double complex $K^{p,q} = \check{C}^p(\mathfrak{U}, \mathcal{L}^q)$. We shall denote by $\mathcal{H}^q(\mathcal{L}^\bullet)$ the cohomology sheaves of the complex $\mathcal{L}^\bullet$. These are the sheaves associated with the quotient presheaves

$$\tilde{\mathcal{H}}^q(U) = \frac{\ker f \colon \mathcal{L}^q(U) \to \mathcal{L}^{q+1}(U)}{\operatorname{Im} f \colon \mathcal{L}^{q-1}(U) \to \mathcal{L}^q(U)}\,.$$

The E_1 term of the first spectral sequence is

$$E_1^{p,q} \simeq H^q(K^{p,\bullet}, \delta_2) = H^q(\check{C}^p(\mathfrak{U}, \mathcal{L}^\bullet), f)) \simeq \check{C}^p(\mathfrak{U}, \tilde{\mathcal{H}}^q(\mathcal{L}^\bullet))\,.$$

The second term of the sequence is

$$E_2^{p,q} \simeq H^p(E_1^{\bullet,q}, \delta_1) \simeq H^p(\check{C}^\bullet(\mathfrak{U}, \tilde{\mathcal{H}}^q(\mathcal{L}^\bullet)), \delta) \simeq H^p(\mathfrak{U}, \mathcal{H}^q(\mathcal{L}^\bullet))$$

where, since X is paracompact, we have replaced the presheaves $\tilde{\mathcal{H}}^\bullet$ with the corresponding sheaves $\mathcal{H}^\bullet$ (possibly replacing the cover $\mathfrak{U}$ by a suitable refinement).

For the second spectral sequence we have

$${}'E_1^{p,q} \simeq H^q(K^{\bullet,p}, \delta_1) \simeq H^q(\check{C}^\bullet(\mathfrak{U}, \mathcal{L}^p), \delta_1) \simeq H^p(\mathfrak{U}, \mathcal{L}^q)$$

$${}'E_2^{q,p} \simeq H^p({}'E_1^{q,\bullet}, \delta_2) \simeq H^p(H^q(\mathfrak{U}, \mathcal{L}^\bullet), f)\,.$$

Let assume now that $\mathcal{L}^\bullet$ is a resolution of a sheaf $\mathcal{F}$; then $\mathcal{H}^q(\mathcal{L}^\bullet) = 0$ for $q \neq 0$, and $\mathcal{H}^0(\mathcal{L}^\bullet) \simeq \mathcal{F}$. The first spectral sequence degenerates at the second step, and we have $E_2^{p,q} = 0$ for $q \neq 0$ and $E_2^{p,0} \simeq H^p(\mathfrak{U}, \mathcal{F})$. The second spectral sequence does not degenerate, but we may say that it converges to the graded group $H^\bullet(\mathfrak{U}, \mathcal{F})$ (since the same does the first sequence). By taking direct limit over the cover $\mathfrak{U}$, we have:

PROPOSITION 5.1. *Given a resolution $\mathcal{L}^\bullet$ of a sheaf $\mathcal{F}$ on a paracompact space X, there is a spectral sequence $\mathfrak{E}_\bullet$ whose second term is $\mathfrak{E}_2^{p,q} = H^q(H^p(X, \mathcal{L}^\bullet), f)$, which converges to the graded group $H^\bullet(X, \mathcal{F})$.*

The canonical filtrations of a double complex always satisfy the hypothesis of Lemma 5.2. So, considering the first spectral sequence, we obtain morphisms (again taking a direct limit)

$$H^q(\mathcal{L}^\bullet(X), f) \to H^q(X, \mathcal{F})\,.$$

In general these are not isomorphisms. The same morphisms could be obtained by breaking the exact sequence $0 \to \mathcal{F} \to \mathcal{L}^\bullet$ into short exact sequences, taking the associated long exact cohomology sequences and suitably composing the morphisms, as in the proof of the abstract de Rham theorem 3.14.

A further specialization is obtained if the resolution $\mathcal{L}^\bullet$ is *acyclic*; then the second spectral sequence degenerates at the second step as well, and we get isomorphisms $H^p(X, \mathcal{F}) \simeq H^p(\mathcal{L}^\bullet(X), f)$, i.e., we have another proof of the abstract de Rham theorem 3.14.

5.2. The spectral sequence of a fibred space. Let $\mathcal{F}$ be a sheaf on a paracompact space X and $\pi\colon X \to Y$ a continuous map, where Y is a second paracompact space. We shall use the fact that every sheaf of abelian groups on space admits flabby resolutions (cf. Sections 3.2.5 and 3.3). We shall associate a spectral sequence to these data. We consider the complex

$$0 \to \pi_*\mathcal{F} \to \pi_*\mathcal{L}_0 \xrightarrow{f} \pi_*\mathcal{L}_1 \xrightarrow{f} \dots \tag{5.12}$$

where $(\mathcal{L}^\bullet, f)$ is a flabby resolution of $\mathcal{F}$. The morphism $\pi_*\mathcal{F} \to \pi_*\mathcal{L}_0$ is injective, but otherwise the complex (5.12) is no longer exact. However, the sheaves $\pi_*\mathcal{L}^\bullet$ are flabby. We denote by $R^k\pi_*\mathcal{F}$ the cohomology sheaves $\mathcal{H}^k(\pi_*\mathcal{L}^\bullet)$.These sheaves are called the *higher direct images* of $\mathcal{F}$. Note that $R^0\pi_*\mathcal{F} \simeq \pi_*\mathcal{F}$.

PROPOSITION 5.2. *The sheaf $R^k\pi_*\mathcal{F}$ is isomorphic to the sheaf associated with the presheaf $\mathcal{P}^k$ on Y defined by $\mathcal{P}^k(U) = H^k(\pi^{-1}(U), \mathcal{F})$.*

This implies that the sheaves $R^k\pi_*\mathcal{F}$ do not depend, up to isomorphism, on the choice of the resolution.

PROOF. $R^k\pi_*\mathcal{F}$ is by definition the sheaf associated with the presheaf

$$U \rightsquigarrow \frac{\ker f\colon \mathcal{L}^k(\pi^{-1}(U)) \to \mathcal{L}^{k+1}(\pi^{-1}(U))}{\operatorname{Im} f\colon \mathcal{L}^{k-1}(\pi^{-1}(U)) \to \mathcal{L}^k(\pi^{-1}(U))} = H^k(\mathcal{L}^\bullet(\pi^{-1}(U), f)\,.$$

Since the restriction of a flabby sheaf to an open subset is flabby, by the abstract de Rham theorem we have isomorphisms

$$H^k(\mathcal{L}^\bullet(\pi^{-1}(U), f) \simeq H^k(\pi^{-1}(U), \mathcal{F})\,,$$

whence the claim follows. □

Let us consider the double complex $\check{C}^p(\mathfrak{U}, \pi_*\mathcal{L}^q)$, where $\mathfrak{U}$ is a locally finite open cover of Y. The two spectral sequences we have previously studied yield at the second term

$$E_2^{p,q} \simeq H^p(\mathfrak{U}, R^q\pi_*\mathcal{F})$$

$${}'E_2^{p,q} \simeq H^q(H^p(\mathfrak{U}, \pi_*\mathcal{L}^\bullet), f)$$

Since the sheaves $\pi_*\mathcal{L}^\bullet$ are soft (hence acyclic) the second spectral sequence degenerates, and one has ${}'E_\infty^{p,q} = 0$ for $p \neq 0$, and

$$\begin{aligned} {}'E_\infty^{0,q} &\simeq {}'E_2^{0,q} \simeq H^q(H^0(Y, \pi_*\mathcal{L}^\bullet), f) \\ &\simeq H^q(\mathcal{L}^\bullet(X), f) \simeq H^q(X, \mathcal{F}) \,. \end{aligned}$$

Again after taking a direct limit, we have:

PROPOSITION 5.3. *Given a continuous map of paracompact spaces $\pi \colon X \to Y$ and a sheaf $\mathcal{F}$ on X, there is a spectral sequence $\mathfrak{E}_\bullet$ whose second term is $\mathfrak{E}_2^{p,q} = H^p(Y, R^q\pi_*\mathcal{F})$, which converges to the graded group $H^\bullet(X, \mathcal{F})$.*

We describe without proof the relation between the stalks of the sheaf $R^k\pi_*\mathcal{F}$ at points $y \in Y$ and the cohomology groups $H^k(\pi^{-1}(y), \mathcal{F})$; here $\mathcal{F}$ is to be considered as restricted to π^{-1}, i.e., more precisely we should write $H^k(\pi^{-1}(y), i_y^{-1}\mathcal{F})$ where $i_y \colon \pi^{-1}(y) \to X$ is the inclusion. Since

$$(R^k\pi_*F)_y = \varinjlim_{y \in U} (R^k\pi_*\mathcal{F})(U) \simeq \varinjlim_{y \in U} H^k(\pi^{-1}(U), \mathcal{F}) \,,$$

while $H^k(\pi^{-1}(y), \mathcal{F})$ is the direct limit of the groups $H^k(V, \mathcal{F})$ where V ranges over *all* open neighbourhoods of $\pi^{-1}(y)$, there is a natural map

$$(R^k\pi_*\mathcal{F})_y \to H^k(\pi^{-1}(y), \mathcal{F}) \,. \tag{5.13}$$

This is an isomorphism under some conditions, e.g., if Y is locally compact and π is proper (cf. [**6**]). This happens for instance when both X and Y are compact.

As a simple Corollary to Proposition 5.3 one obtains *Leray's theorem:*

COROLLARY 5.4. *If every point $y \in Y$ has a system of neighbourhoods whose preimages are acyclic for $\mathcal{F}$, then $H^k(X, \mathcal{F}) \simeq H^k(Y, \pi_*\mathcal{F})$ for all $k \geq 0$.*

PROOF. The hypothesis of the Corollary means that every $y \in Y$ has a system of neighbourhoods $\{U\}$ such that $H^k(\pi^{-1}(U), \mathcal{F}) = 0$ for all $k > 0$. This implies that $R^k\pi_*\mathcal{F} = 0$ for $k > 0$, so that the only nonzero terms in the spectral sequence $\mathfrak{E}_2$ are $\mathfrak{E}_2^{p,0} \simeq H^p(Y, \pi_*\mathcal{F})$. The sequence degenerates and the claim follows. □

5.3. The Künneth theorem. Let X, Y be topological spaces, and G an abelian group. We shall denote by the same symbol G the corresponding constant sheaves on the spaces X, Y and $X \times Y$. The Künneth theorem computes the cohomology groups $H^\bullet(X \times Y, G)$ in terms of the groups $H^\bullet(X, \mathbb{Z})$ and $H^\bullet(Y, G)$.

We shall need the following version of the *universal coefficient theorem.*

PROPOSITION 5.5. *If X is a paracompact topological space and G a torsion-free group, then $H^k(X, G) \simeq H^k(X, \mathbb{Z}) \otimes_\mathbb{Z} G$ for all $k \geq 0$.*

PROOF. Cf. [**21**]. □

PROPOSITION 5.6. *Assume that the groups $H^\bullet(Y,G)$ have no torsion over $\mathbb{Z}$, and that X and Y are compact Hausdorff and locally Euclidean. Then,*

$$H^k(X\times Y,G)\simeq\bigoplus_{p+q=k}H^p(X,\mathbb{Z})\otimes H^q(Y,G)\,.$$

PROOF. Let $\pi\colon X\times Y\to X$ be the projection onto the first factor. If U is a contractible open set in U, then by the homotopic invariance of the cohomology with coefficients in a constant sheaf (which follows e.g. from its isomorphism with singular cohomology) we have $H^\bullet(U\times Y,G)\simeq H^\bullet(Y,G)$. If $V\subset U$, the morphism $H^\bullet(U\times Y,G)\to H^\bullet(V\times Y,G)$ corresponds to the identity of $H^\bullet(Y,G)$. Under the present hypotheses the morphism (5.13) is an isomorphism. These facts imply that $R^p\pi_*G$ is the constant sheaf on X with stalk $H^p(Y,G)$. The second term of the spectral sequence of Proposition 5.3 becomes $\mathfrak{E}_2^{p,q}\simeq H^p(X,H^q(Y,G))$. By the universal coefficient theorem, since the groups $H^q(Y,G)$ have no torsion over $\mathbb{Z}$, we have $\mathfrak{E}_2^{p,q}\simeq H^p(X,\mathbb{Z})\otimes_{\mathbb{Z}}H^q(Y,G)$. □

Part 2

Introduction to algebraic geometry

CHAPTER 6

Complex manifolds and vector bundles

In this chapter we give a sketchy introduction to complex manifolds. The reader is assumed to be acquainted with the rudiments of the theory of differentiable manifolds.

1. Basic definitions and examples

1.1. Holomorphic functions. Let $U \subset \mathbb{C}$ be an open subset. We say that a function $f\colon U \to \mathbb{C}$ is holomorphic if it is C^1 and for all $x \in U$ its differential $Df_x\colon \mathbb{C} \to C$ is not only $\mathbb{R}$-linear but also $\mathbb{C}$-linear. If elements in $\mathbb{C}$ are written $z = x + iy$, and we set $f(x, y) = \alpha(x, y) + i\beta(x, y)$, then this condition can be written as

$$\alpha_x = \beta_y, \qquad \alpha_y = -\beta_x \tag{6.1}$$

(these are the *Cauchy-Riemann conditions*). If we use z, $\bar{z}$ as variables, the Cauchy-Riemann conditions read $f_{\bar{z}} = 0$, i.e. the holomorphic functions are the C^1 function of the variable z. Moreover, one can show that holomorphic functions are analytic.

The same definition can be given for holomorphic functions of several variables.

Definition 6.1. *Two open subsets U, V of $\mathbb{C}^n$ are said to biholomorphic if there exists a bijective holomorphic map $f\colon U \to V$ whose inverse is holomorphic. The map f itself is then said to be biholomorphic.*

1.2. Complex manifolds. Complex manifolds are defined as differentiable manifolds, but requiring that the local model is $\mathbb{C}^n$, and that the transition functions are biholomorphic.

Definition 6.2. *An n-dimensional complex manifold is a second countable Hausdorff topological space X together with an open cover $\{U_i\}$ and maps $\psi_i\colon U_i \to \mathbb{C}^n$ which are homeomorphisms onto their images, and are such that all transition functions*

$$\psi_i \circ \psi_j^{-1}\colon \psi_j(U_i \cap U_j) \to \psi_i(U_i \cap U_j)$$

are biholomorphisms.

Example 6.3. (The Riemann sphere) Consider the sphere in $\mathbb{R}^3$ centered at the origin and having radius $\frac{1}{2}$, and identify the tangent planes at $(0, 0, \frac{1}{2})$ and $(0, 0, -\frac{1}{2})$ with $\mathbb{C}$. The stereographic projections give local complex coordinates z_1, z_2; the transition function $z_2 = 1/z_1$ is defined in $\mathbb{C}^\star = \mathbb{C} - \{0\}$ and is biholomorphic.

1-dimensional complex manifolds are called *Riemann surfaces.* Compact Riemann surfaces play a distinguished role in algebraic geometry; they are all algebraic (i.e. they are sets of zeroes of systems of homogeneous polynomials), as we shall see in Chapter 8.

Example 6.4. (Projective spaces) We define the n-dimensional complex projective space as the space of complex lines through the origin of $\mathbb{C}^{n+1}$, i.e.

$$\mathbb{P}_n = \frac{\mathbb{C}^{n+1} - \{0\}}{\mathbb{C}^*}\,.$$

By standard topological arguments $\mathbb{P}_n$ with the quotient topology is a Hausdorff second-countable space.

Let $\pi\colon \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}_n$ be the projection, If $w = (w^0, \dots, w^n) \in \mathbb{C}^{n+1}$ we shall denote $\pi(w) = [w^0, \dots, w^n]$. The numbers $(w^0, \dots, w^n)$ are said to be the *homogeneous coordinates* of the point $\pi(w)$. If $(u^0, \dots, u^n)$ is another set of homogeneous coordinates for $\pi(w)$, then $u^i = \lambda w^i$, with $\lambda \in \mathbb{C}^*$ $(i = 0, \dots, n)$.

Denote by $\tilde U_i \subset \mathbb{C}^{n+1}$ the open set where $w^i \neq 0$, let $U_i = \pi(\tilde U_i)$, and define a map

$$\psi_i\colon U_i \to \mathbb{C}^n, \qquad \psi([w^0, \dots, w^n]) = \left(\frac{w^0}{w^i}, \dots, \frac{w^{i-1}}{w^i}, \frac{w^{i+1}}{w^i}, \dots, \frac{w^n}{w^i}\right).$$

The sets U_i cover $\mathbb{P}_n$, the maps ψ_i are homeomorphisms, and their transition functions

$$\psi_i \circ \psi_j^{-1}\colon \quad \psi_j(U_j) \to \psi_i(U_i),$$

$$\psi_i \circ \psi_j^{-1}(z^1, \dots, z^n) = \left(\frac{z^1}{z^i}, \dots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \dots, \underset{\substack{\uparrow \\ j\text{-th argument}}}{\frac{1}{z^i}}, \dots \frac{z^n}{z^i}\right),$$

are biholomorphic, so that $\mathbb{P}_n$ is a complex manifold (we have assumed that $i < j$). The map π restricted to the unit sphere in $\mathbb{C}^{n+1}$ is surjective, so that $\mathbb{P}_n$ is compact. The previous formula for $n = 1$ shows that $\mathbb{P}_1$ is biholomorphic to the Riemann sphere.

The coordinates defined by the maps ψ_i, usually denoted $(z^1, \dots, z^n)$, are called *affine* or *Euclidean coordinates.*

Example 6.5. (The general linear complex group). Let

$$M_{k,n} = \{k \times n \text{ matrices with complex entries, } k \leq n\}$$

$$\hat M_{k,n} = \{\text{matrices in } M_{k,n} \text{ of rank } k\}, \quad \text{i.e.}$$

$$\hat M_{k,n} = \bigcup_{i=1}^{\ell} \{A \in M_{k,n} \quad \text{such that} \quad \det A_i \neq 0\}$$

where $A_i, \dots, A_\ell$ are the $k \times k$ minors of A. $M_{k,n}$ is a complex manifold of dimension kn; $\hat M_{k,n}$ is an open subset in $M_{k,n}$, as its second description shows, so it is a complex manifold of dimension kn as well. In particular, the general linear group $Gl(n, \mathbb{C}) = \hat M_{n,n}$ is a complex manifold of dimension n^2. Here are some of its relevant subgroups:

(i) $U(n) = \{A \in Gl(n, \mathbb{C}) \quad \text{such that} \quad AA^\dagger = I\}$;

(ii) $SU(n) = \{A \in U(n) \quad \text{such that} \quad \det A = 1\}$;

these two groups are real (not complex!) manifolds, and $\dim_{\mathbb{R}} U(n) = n^2$, $\dim_{\mathbb{R}} SU(n) = n^2 - 1$.

(iii) the group $Gl(k, n; \mathbb{C})$ formed by invertible complex matrices having a block form

$$M = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \tag{6.2}$$

where the matrices A, B, C are $k \times k$, $(n-k) \times k$, and $(n-k) \times (n-k)$, respectively. $Gl(k, n; \mathbb{C})$ is a complex manifold of dimension $k^2 + n^2 - nk$. Since a matrix of the form (6.2) is invertible if and only if A and C are, while B can be any matrix, $Gl(k, n; \mathbb{C})$ is biholomorphic to the product manifold $Gl(k, \mathbb{C}) \times Gl(n-k, \mathbb{C}) \times M_{k,n}$. □

1.3. Submanifolds. Given a complex manifold X, a submanifold of X is a pair (Y, ι), where Y is a complex manifold, and $\iota\colon Y \to X$ is an injective holomorphic map whose jacobian matrix has rank equal to the dimension of Y at any point of Y (of course Y can be thought of as a subset of X).

Example 6.6. $Gl(k, n; \mathbb{C})$ is a submanifold of $Gl(n, \mathbb{C})$.

Example 6.7. For any $k < n$ the inclusion of $\mathbb{C}^{k+1}$ into $\mathbb{C}^{n+1}$ obtained by setting to zero the last $n-k$ coordinates in $\mathbb{C}^{n+1}$ yields a map $\mathbb{P}_k \to \mathbb{P}_n$; the reader may check that this realizes $\mathbb{P}_k$ as a submanifold of $\mathbb{P}_n$.

Example 6.8. (Grassmann varieties) Let

$$G_{k,n} = \{\text{space of } k\text{-dimensional planes in } \mathbb{C}^n\}$$

(so $G_{1,n} \equiv \mathbb{P}_n - 1$). This is the Grassmann variety of k-planes in $\mathbb{C}^n$. Given a k-plane, the action of $Gl(n, \mathbb{C})$ on it yields another plane (possibly coinciding with the previous one). The subgroup of $Gl(n, \mathbb{C})$ which leaves the given k-plane fixed is isomorphic to $Gl(k, n; \mathbb{C})$, so that

$$G_{k,n} \simeq \frac{Gl(n, \mathbb{C})}{Gl(k, n; \mathbb{C})}\,.$$

As the reader may check, this representation gives $G_{k,n}$ the structure of a complex manifold of dimension $k(n-k)$. Since in the previous reasoning $Gl(n, \mathbb{C})$ can be replaced by $U(n)$, and since $Gl(k, n; \mathbb{C}) \cap U(n) = U(k) \times U(n-k)$, we also have the representation

$$G_{k,n} \simeq \frac{U(n)}{U(k) \times U(n-k)}$$

showing that $G_{k,n}$ is compact.

An element in $G_{k,n}$ singles out (up to a complex factor) a decomposable element in $\Lambda^k \mathbb{C}^n$,

$$\lambda = v_1 \wedge \cdots \wedge v_k$$

where the v_i are a basis of tangent vectors to the given k-plane. So $G_{k,n}$ imbeds into $\mathbb{P}(\Lambda^k\mathbb{C}^n) = \mathbb{P}_N$, where $N = \left(\binom{n}{k}\right) - 1$ (this is called the *Plücker embedding.* If a basis $\{v_1, \dots, v_n\}$ is fixed in $\mathbb{C}^n$, one has a representation

$$\lambda = \sum_{i_1,\dots,i_k=1}^{n} P_{i_1\dots i_k}\, v_{i_1} \wedge \cdots \wedge v_{i_k}\,;$$

the numbers $P_{i_1\dots i_k}$ are the *Plücker coordinates* on the Grassmann variety.

2. Some properties of complex manifolds

2.1. Orientation. All complex manifolds are oriented. Consider for simplicity the 1-dimensional case; the jacobian matrix of a transition function $z' = f(z) = \alpha(x,y) + i\beta(x,y)$ is (by the Cauchy-Riemann conditions)

$$J = \begin{pmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{pmatrix} = \begin{pmatrix} \alpha_x & \alpha_y \\ -\alpha_y & \alpha_x \end{pmatrix}$$

so that $\det J = \alpha_x^2 + \alpha_y^2 > 0$, and the manifold is oriented.

Notice that we may always conjugate the complex structure, considering (e.g. in the 1-dimensional case) the coordinate change $z \mapsto \bar z$; in this case we have $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so that the orientation gets reversed.

2.2. Forms of type (p,q). Let X be an n-dimensional complex manifold; by the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, and since a biholomorphic map is a C^∞ diffeomorphism, X has an underlying structure of $2n$-dimensional real manifold. Let TX be the smooth tangent bundle (i.e. the collection of all ordinary tangent spaces to X). If $(z^1, \dots, z^n)$ is a set of local complex coordinates around a point $x \in X$, then the complexified tangent space $T_xX \otimes_{\mathbb{R}} \mathbb{C}$ admits the basis

$$\left(\left(\frac{\partial}{\partial z^1}\right)_x, \dots, \left(\frac{\partial}{\partial z^n}\right)_x, \left(\frac{\partial}{\partial \bar z^1}\right)_x, \dots, \left(\frac{\partial}{\partial \bar z^n}\right)_x \right).$$

This yields a decomposition

$$TX \otimes \mathbb{C} = T'X \oplus T''X$$

which is intrinsic because X has a complex structure, so that the transition functions are holomorphic and do not mix the vectors $\frac{\partial}{\partial z^i}$ with the $\frac{\partial}{\partial \bar z^i}$. As a consequence one has a decomposition

$$\Lambda^i T^*X \otimes \mathbb{C} = \bigoplus_{p+q=i} \Omega^{p,q}X \qquad \text{where} \qquad \Omega^{p,q}X = \Lambda^p(T'X)^* \otimes \Lambda^q(T''X)^*\,.$$

The elements in $\Omega^{p,q}X$ are called *differential forms of type* (p,q), and can locally be written as

$$\eta = \eta_{i_1\dots i_p, j_1\dots j_q}(z,\bar z)\, dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar z^{j_1} \wedge \cdots \wedge d\bar z^{j_q}.$$

The compositions

$$\begin{array}{ccccc} & & & & \Omega^{p+1,q}X \\ & & & \overset{\partial}{\nearrow} & \\ \Omega^{p,q}X & \xrightarrow{d} & \Lambda^{p+q+1}T^*X & & \\ & & & \underset{\bar\partial}{\searrow} & \\ & & & & \Omega^{p,q+1}X \end{array}$$

define differential operators ∂, $\bar\partial$ such that

$$\partial^2 = \bar\partial^2 = \partial\bar\partial + \bar\partial\partial = 0$$

(notice that the Cauchy-Riemann condition can be written as $\bar\partial f = 0$).

3. Dolbeault cohomology

Another interesting cohomology theory one can consider is the *Dolbeault cohomology* associated with a complex manifold X. Let $\Omega^{p,q}$ denote the sheaf of forms of type (p,q) on X. The Dolbeault (or Cauchy-Riemann) operator $\bar\partial\colon \Omega^{p,q} \to \Omega^{p,q+1}$ squares to zero. Therefore, the pair $(\Omega^{p,\bullet}(X), \bar\partial)$ is for any $p \geq 0$ a cohomology complex. Its cohomology groups are denoted by $H^{p,q}_{\bar\partial}(X)$, and are called the *Dolbeault cohomology groups* of X.

We have for this theory an analogue of the Poincaré Lemma, which is sometimes called the $\bar\partial$-Poincaré Lemma (or Dolbeault or Grothendieck Lemma).

Proposition 6.1. *Let Δ be a polycylinder in $\mathbb{C}^n$ (that is, the cartesian product of disks in $\mathbb{C}$). Then $H^{p,q}_{\bar\partial}(\Delta) = 0$ for $q \geq 1$.*

Proof. Cf. [**10**]. □

Moreover, the kernel of the morphism $\bar\partial\colon \Omega^{p,0} \to \Omega^{p,1}$ is the sheaf of holomorphic p-forms Ω^p. Therefore, the Dolbeault complex of sheaves $\Omega^{p,\bullet}$ is a resolution of Ω^p, i.e. for all $p = 0, \dots, n$ (where $n = \dim_{\mathbb{C}} X$) the sheaf sequence

$$0 \to \Omega^p \to \Omega^{p,0} \xrightarrow{\bar\partial} \Omega^{p,1} \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \Omega^{p,1} \to 0$$

is exact. Moreover, the sheaves $\Omega^{p,q}$ are fine (they are $\mathcal{C}^\infty_X$-modules). Then, exactly as one proves the de Rham theorem (Theorem 3.3.15), one obtains the *Dolbeault theorem:*

Proposition 6.2. *Let X be a complex manifold. For all $p, q \geq 0$, the cohomology groups $H^{p,q}_{\bar\partial}(X)$ and $H^q(X, \Omega^p)$ are isomorphic.* □

4. Holomorphic vector bundles

4.1. Basic definitions. Holomorphic vector bundles on a complex manifold X are defined in the same way than smooth complex vector bundles, but requiring that all the maps involved are holomorphic.

Definition 6.1. *A complex manifold E is a rank n holomorphic vector bundle on X if there are*

(i) an open cover $\{U_\alpha\}$ of X

(ii) a holomorphic map $\pi\colon E \to X$

(iii) holomorphic maps $\psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$

such that

(i) $\pi = \mathrm{pr}_1 \circ \psi_\alpha$, where $= \mathrm{pr}_1$ is the projection onto the first factor of $U_\alpha \times \mathbb{C}^n$;

(ii) for all $p \in U_\alpha \cap U_\beta$, the map

$$\mathrm{pr}_2 \circ \psi_\beta \circ \psi_\alpha^{-1}(p, \bullet)\colon \mathbb{C}^n \to \mathbb{C}^n$$

is a linear isomorphism.

Vector bundles of rank 1 are called *line bundles.*

With the data that define a holomorphic vector bundle we may construct holomorphic maps

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to Gl(n, \mathbb{C})$$

given by

$$g_{\alpha\beta}(p) \cdot x = \mathrm{pr}_2 \circ \psi_\alpha \circ \psi_\beta^{-1}(\psi, x)\,.$$

These maps satisfy the cocycle condition

$$g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = \mathrm{Id} \qquad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma\,.$$

The collection $\{U_\alpha, \psi_\alpha\}$ is a *trivialization* of E.

For every $x \in X$, the subset $E_x = \pi^{-1}(x) \subset E$ is called the *fibre of E over x*. By means of a trivialization around x, E_x is given the structure of a vector space, which is actually independent of the trivialization.

A *morphism* between two vector bundles E, F over X is a holomorphic map $f\colon E \to F$ such that for every $x \in X$ one has $f(E_x) \subset F_x$, and such that the resulting map $f_x\colon E_x \to F_x$ is linear. If f is a biholomorphism, it is said to be an isomorphism of vector bundles, and E and F are said to be isomorphic.

A *holomorphic section* of E over an open subset $U \subset X$ is a holomorphic map $s\colon U \to E$ such that $\pi \circ s = \mathrm{Id}$. With reference to the notation previously introduced, the maps

$$s_{(\alpha)i}\colon U_\alpha \to E, \qquad s_{(\alpha)i}(x) = \psi_\alpha^{-1}(x, e_i), \qquad i = 1, \ldots, n$$

where $\{e_i\}$ is the canonical basis of $\mathbb{C}^n$, are sections of E over U_α. Let $E(U_\alpha)$ denote the set of sections of E over U_α; it is a free module over the ring $\mathcal{O}(U_\alpha)$ of holomorphic functions on U_α, and its subset $\{s_{(\alpha)i}\}_{i=1,\ldots,n}$ is a basis. On an intersection $U_\alpha \cap U_\beta$ one has the relation

$$s_{(\alpha)i} = \sum_{k=1}^{n} (g_{\alpha\beta})_{ik}\, s_{(\beta)k}.$$

EXERCISE 6.2. Show that two trivializations are equivalent (i.e. describe isomorphic bundles) if there exist holomorphic maps $\lambda_\alpha\colon U_\alpha \to Gl(n, \mathbb{C})$ such that

$$g'_{\alpha\beta} = \lambda_\alpha \, g_{\alpha\beta} \, \lambda_\beta^{-1} \tag{6.3}$$

□

EXERCISE 6.3. Show that the rule that to any open subset $U \subset X$ assigns the $\mathcal{O}_X^\infty(U)$-module of sections of a holomorphic vector bundle E defines a sheaf $\mathcal{E}$ (which actually is a sheaf of $\mathcal{O}_X$-modules).

If E is a holomorphic (or smooth complex) vector bundle, with transition functions $g_{\alpha\beta}$, then the maps

$$g'_{\alpha\beta} = (g_{\alpha\beta}^T)^{-1} \tag{6.4}$$

(where T denotes transposition) define another vector bundle, called the *dual vector bundle* to E, and denoted by E^*. Sections of E^* can be paired with (or act on) sections of E, yielding holomorphic (smooth complex-valued) functions on (open sets of) X.

EXAMPLE 6.4. The space $E = X \times \mathbb{C}^n$, with the projection onto the first factor, is obviously a holomorphic vector bundle, called the *trivial vector bundle* of rank n. We shall denote such a bundle by $\underline{\mathbb{C}}^n$ (in particular, $\underline{\mathbb{C}}$ denotes the trivial line bundle). A holomorphic vector bundle is said to be trivial when it is isomorphic to $\underline{\mathbb{C}}^n$.

Every holomorphic vector bundle has an obvious structure of smooth complex vector bundle. A holomorphic vector bundle may be trivial as a smooth bundle while not being trivial as a holomorphic bundle. (In the next sections we shall learn some homological techniques that can be used to handle such situations).

EXAMPLE 6.5. (The tangent and cotangent bundles) If X is a complex manifold, the "holomorphic part" $T'X$ of the complexified tangent bundle is a holomorphic vector bundle, whose rank equals the complex dimension of X. Given a holomorphic atlas for X, the locally defined holomorphic vector fields $\frac{\partial}{\partial z^1} \ldots, \frac{\partial}{\partial z^n}$ provide a holomorphic trivialization of X, such that the transition functions of $T'X$ are the jacobian matrices of the transition functions of X. The dual of $T'X$ is the holomorphic cotangent bundle of X.

EXAMPLE 6.6. (The tautological bundle) Let $(w^1, \ldots, w^{n+1})$ be homogeneous coordinates in $\mathbb{P}_n$. If to any $p \in \mathbb{P}_n$ (which is a line in $\mathbb{C}^{n+1}$) we associate that line we obtain a line bundle, the *tautological line bundle* L of $\mathbb{P}_n$. To be more concrete, let us exhibit a trivialization for L and the related transition functions. If $\{U_i\}$ is the standard cover of $\mathbb{P}_n$, and $p \in U_i$, then w^i can be used to parametrize the points in the line p. So if p has homogeneous coordinates $(w^0, \ldots, w^n)$, we may define $\psi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ as $\psi_i(u) = (p, w^i)$ if $p = \pi(u)$. The transition function is then $g_{ik} = w^i/w^k$. The dual bundle $H = L^*$ acts on L, so that its fibre at $p = \pi(u)$, $u \in \mathbb{C}^{n+1}$ can be regarded as the space of linear functionals on the line $\mathbb{C}u \equiv L_p$, i.e. as hyperplanes in $\mathbb{C}^{n+1}$. Hence H is called the *hyperplane bundle.* Often L is denoted $\mathcal{O}(-1)$, and H is denoted $\mathcal{O}(1)$ — the reason of this notation will be clear in Chapter 7.

In the same way one defines a tautological bundle on the Grassmann variety $G_{k,n}$; it has rank k.

EXERCISE 6.7. Show that that the elements of a basis of the vector space of global sections of L can be identified homogeneous coordinates, so that $\dim H^0(\mathbb{P}_n, L) = n + 1$. Show that the global sections of H can be identified with the linear polynomials in the homogeneous coordinates. Hence, the global sections of H^r are homogeneous polynomials of order r in the homogeneous coordinates. □

4.2. More constructions. Additional operations that one can perform on vector bundles are again easily described in terms of transition functions.

(1) Given two vector bundles E_1 and E_2, of rank r_1 and r_2, their direct sum $E_1 \oplus E_2$ is the vector bundle of rank $r_1 + r_2$ whose transition functions have the block matrix form

$$\begin{pmatrix} g^{(1)}_{\alpha\beta} & 0 \\ 0 & g^{(2)}_{\alpha\beta} \end{pmatrix}$$

(2) We may also define the tensor product $E_1 \otimes E_2$, which has rank $r_1 r_2$ and has transition functions $g^{(1)}_{\alpha\beta}\, g^{(2)}_{\alpha\beta}$. This means the following: assume that E_1 and E_2 trivialize over the same cover $\{U_\alpha\}$, a condition we may always meet, and that in the given trivializations, E_1 and E_2 have local bases of sections $\{s_{(\alpha)i}\}$ and $\{t_{(\alpha)k}\}$. Then $E_1 \otimes E_2$ has local bases of sections $\{s_{(\alpha)i} \otimes t_{(\alpha)k}\}$ and the corresponding transition functions are given by

$$s_{(\alpha)i} \otimes t_{(\alpha)k} = \sum_{m=1}^{r_1} \sum_{n=1}^{r_2} (g^{(1)}_{\alpha\beta})_{im} (g^{(2)}_{\alpha\beta})_{kn}\, s_{(\beta)m} \otimes t_{(\beta)n}\,.$$

In particular the tensor product of line bundles is a line bundle. If L is a line bundle, one writes L^n for $L \otimes \cdots \otimes L$ (n factors). If L is the tautological line bundle on a projective space, one often writes $L^n = \mathcal{O}(-n)$, and similarly $H^n = \mathcal{O}(n)$ (notice that $\mathcal{O}(-n)^* = \mathcal{O}(n)$).

(3) If E is a vector bundle with transition functions $g_{\alpha\beta}$, we define its determinant $\det E$ as the line bundle whose transition functions are the functions $\det g_{\alpha\beta}$. The determinant bundle of the holomorphic tangent bundle to a complex manifold is called the *canonical bundle* K.

EXERCISE 6.8. Show that the canonical bundle of the projective space $\mathbb{P}_n$ is isomorphic to $\mathcal{O}(-n-1)$.

EXAMPLE 6.9. Let $\pi\colon \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}_n$ be the usual projection, and let $(w^1, \ldots, w^{n+1})$ be homogeneous coordinates in $\mathbb{P}_n$. The tangent spaces to $\mathbb{P}_n$ are generated by the vectors $\pi_* \frac{\partial}{\partial w^i}$, and these are subject to the relation

$$\sum_{i=1}^{n+1} w^i \, \pi_* \frac{\partial}{\partial w^i} = 0 \,.$$

If ℓ is a linear functional on $\mathbb{C}^{n+1}$ the vector field

$$v(w) = \ell(w) \, \frac{\partial}{\partial w^i}$$

(i is fixed) satisfies $v(\lambda \, w) = \lambda \, v(w)$ and therefore descends to $\mathbb{P}_n$. One can then define a map

$$E \colon H^{\oplus(n+1)} \to T\mathbb{P}_n$$
$$(\sigma_1, \dots, \sigma_{n+1}) \mapsto \sum_{i=1}^{n+1} \sigma_i(w) \, \frac{\partial}{\partial w^i}$$

(recall that the sections of H can be regarded as linear functionals on the homogeneous coordinates). The map E is apparently surjective. Its kernel is generated by the section $\sigma_i(w) = w^i$, $i = 1, \dots, n+1$; notice that this is the image of the map

$$\underline{\mathbb{C}} \to H^{\oplus(n+1)}, \qquad 1 \mapsto (w^1, \dots, w^{n+1}) \,.$$

The morphism $H^{\oplus(n+1)} \to T\mathbb{P}_n$ may be regarded as a sheaf morphism $\mathcal{O}_{\mathbb{P}_n}(1)^{\oplus(n+1)} \to T\mathbb{P}_n$, the second sheaf being the tangent sheaf of $\mathbb{P}_n$, i.e., the sheaf of germs of holomorphic vector fields on $\mathbb{P}_n$, and one has an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}_n} \to \mathcal{O}_{\mathbb{P}_n}(1)^{\oplus(n+1)} \to T\mathbb{P}_n \to 0$$

called the *Euler sequence.* □

5. Chern class of line bundles

5.1. Chern classes of holomorphic line bundles. Let X a complex manifold. We define $\operatorname{Pic}(X)$ (the Picard group of X) as the set of holomorphic line bundles on X modulo isomorphism. The group structure of $\operatorname{Pic}(X)$ is induced by the tensor product of line bundles $L \otimes L'$; in particular one has $L \otimes L^* \simeq \underline{\mathbb{C}}$ (think of it in terms of transition functions — here $\underline{\mathbb{C}}$ denotes the trivial line bundle, whose class $[\underline{\mathbb{C}}]$ is the identity in $\operatorname{Pic}(X)$), so that the class $[L^*]$ is the inverse in $\operatorname{Pic}(X)$ of the class $[L]$.

Let $\mathcal{O}$ denote the sheaf of holomorphic functions on X, and $\mathcal{O}^*$ the subsheaf of nowhere vanishing holomorphic funtions. If $L \simeq L'$ then the transition functions $g_{\alpha\beta}$, $g'_{\alpha\beta}$ of the two bundles with respect to a cover $\{U_\alpha\}$ of X are 2-cocycles $\mathcal{O}^*$, and satisfy

$$g'_{\alpha\beta} = g_{\alpha\beta} \frac{\lambda_\alpha}{\lambda_\beta} \qquad \text{with} \qquad \lambda_\alpha \in \mathcal{O}^*(U_\alpha),$$

so that one has an identification $\operatorname{Pic}(X) \simeq H^1(X, \mathcal{O}^*)$. The long cohomology sequence associated with the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

(where $\exp f = e^{2\pi i f}$) contains the segment

$$H^1(X,Z) \to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}^*) \xrightarrow{\delta} H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O})$$

where δ is the connecting morphism. Given a line bundle L, the element

$$c_1(L) = \delta([L]) \in H^2(X,\mathbb{Z})$$

is the *first Chern class*[1] of L. The fact that δ is a group morphism means that

$$c_1(L \otimes L') = c_1(L) + c_1(L')\,.$$

In general, the morphism δ is neither injective nor surjective, so that

(i) the first Chern class does not classify the holomorphic line bundles on X; the group

$$\mathrm{Pic}^0(X) = \ker \delta \simeq H^1(X,\mathcal{O})/\operatorname{Im} H^1(X,\mathbb{Z})$$

classifies the line bundles having the same first Chern class.

(ii) not every element in $H^2(X,\mathbb{Z})$ is the first Chern class of a holomorphic line bundle.

The image of c_1 is a subgroup $\mathrm{NS}(X)$ of $H^2(X,\mathbb{Z})$, called the *Néron-Severi group* of X.

Exercise 6.1. Show that all line bundles on $\mathbb{C}^n$ are trivial.

Exercise 6.2. Show that there exist nontrivial holomorphic line bundles which are trivial as smooth complex line bundles. □

Notice that when X is compact the sequence

$$0 \to H^0(X,\mathbb{Z}) \to H^0(X,\mathcal{O}) \to H^0(X,\mathcal{O}^*) \to 0$$

is exact, so that $\mathrm{Pic}^0(X) = H^1(X,\mathcal{O})/H^1(X,\mathbb{Z})$. If in addition $\dim X = 1$ we have $H^2(X,\mathcal{O}) = 0$, so that every element in $H^2(X,\mathbb{Z})$ is the first Chern class of a holomorphic line bundle.[2]

From the definition of connecting morphism we can deduce an explicit formula for a Čech cocycle representing $c_1(L)$ with respect to the cover $\{U_\alpha\}$:

$$\{c_1(L)\}_{\alpha\beta\gamma} = \tfrac{1}{2\pi i}\left(\log g_{\alpha\beta} + \log g_{\beta\gamma} + \log g_{\gamma\alpha}\right).$$

From this one can easily prove that, if $f : X \to Y$ is a holomorphic map, and L is a line bundle on Y, then

$$c_1(f^*L) = f^\sharp(c_1(L))\,.$$

[1]This allows us also to define the first Chern class of a vector bundle E of any rank by letting $c_1(E) = c_1(\det E)$.

[2]Here we use the fact that if X is a complex manifold of dimension n, then $H^k(X,\mathcal{O}) = 0$ for all $k > n$.

5.2. Smooth line bundles. The first Chern class can equally well be defined for smooth complex line bundles. In this case we consider the sheaf $\mathcal{C}$ of complex-valued smooth functions on a differentiable manifold X, and the subsheaf $\mathcal{C}^*$ of nowhere vanishing functions of such type. The set of isomorphism classes of smooth complex line bundles is identified with the cohomology group $H^1(X, \mathcal{C}^*)$. However now the sheaf $\mathcal{C}$ is acyclic, so that the obstruction morphism δ establishes an isomorphism $H^1(X, \mathcal{C}^*) \simeq H^2(X, \mathbb{Z})$. The first Chern class of a line bundle L is again defined as $c_1(L) = \delta([L])$, but now $c_1(L)$ classifies the bundle (i.e. $L \simeq L'$ if and only if $c_1(L) = c_1(L')$).

EXERCISE 6.3. (A rather pedantic one, to be honest...) Show that if X is a complex manifold, and L is a holomorphic line bundle on it, the first Chern classes of L regarded as a holomorphic or smooth complex line bundle coincide. (Hint: start from the inclusion $\mathcal{O} \hookrightarrow \mathcal{C}$, write from it a diagram of exact sequences, and take it to cohomology ...) $\square$

6. Chern classes of vector bundles

In this section we define higher Chern classes for complex vector bundles of any rank. Since the Chern classes of a vector bundle will depend only on its smooth structure, we may consider a smooth complex vector bundle E on a differentiable manifold X. We are already able to define the first Chern class $c_1(L)$ of a line bundle L, and we know that $c_1(L) \in H^2(X, \mathbb{Z})$. We proceed in two steps:

(1) we first define Chern classes of vector bundles that are direct sums of line bundles;

(2) and then show that by means of an operation called *cohomology base change* we can always reduce the computation of Chern classes to the previous situation.

Step 1. Let σ_i, $i = 1 \dots k$, denote the symmetric function of order i in k arguments.[3]. Since these functions are polynomials with integer coefficients, they can be regarded as functions on the cohomology ring $H^\bullet(X, \mathbb{Z})$. In particular, if $\alpha_1, \dots, \alpha_k$ are classes in $H^2(X, \mathbb{Z})$, we have $\sigma_i(\alpha_1, \dots, \alpha_k) \in H^{2i}(X, \mathbb{Z})$.

If $E = L_1 \oplus \dots \oplus L_k$, where the L_i's are line bundles, for $i = 1...k$ we define the i-th Chern class of E as

$$c_i(E) = \sigma_i(c_1(L_1), \dots, c_1(L_k)) \in H^{2i}(X, \mathbb{Z})\,.$$

[3]The symmetric functions are defined as

$$\sigma_i(x_1, \dots, x_k) = \sum_{1 \le j_1 < \dots < j_i \le n} x_{j_1} \cdot \dots \cdot x_{j_i}.$$

Thus, for instance,

$$\begin{aligned}
\sigma_1(x_1, \dots, x_k) &= x_1 + \dots + x_k \\
\sigma_2(x_1, \dots, x_k) &= x_1x_2 + x_1x_3 + \dots + x_{k-1}x_k \\
&\dots \\
\sigma_k(x_1, \dots, x_k) &= x_1 \cdot \dots \cdot x_k.
\end{aligned}$$

As a first reference for symmetric functions see e.g. [**24**].

We also set $c_0(E) = 1$; identifying $H^0(X, \mathbb{Z})$ with $\mathbb{Z}$ (assuming that X is connected) we may think that $c_0(E) \in H^0(X, \mathbb{Z})$.

Step 2 relies on the following result (sometimes called the *splitting principle*), which we do not prove here.

PROPOSITION 6.1. *Let E be a complex vector bundle on a differentiable manifold X. There exists a differentiable map $f\colon Y \to X$, where Y is a differentiable manifold, such that*

(1) the pullback bundle f^*E is a direct sum of line bundles;

(2) the morphism $f^\sharp\colon H^\bullet(X, \mathbb{Z}) \to H^\bullet(Y, \mathbb{Z})$ is injective;

(3) the Chern classes $c_i(f^*E)$ lie in the image of the morphism $f^\sharp$.

DEFINITION 6.2. *The i-th Chern class $c_i(E)$ of E is the unique class in $H^{2i}(X, \mathbb{Z})$ such that $f^\sharp(c_i(E)) = c_i(f^*E)$.*

We also define the total Chern class of E as

$$c(E) = \sum_{i=0}^{k} c_i(E) \in H^\bullet(X, \mathbb{Z}).$$

The main property of the Chern classes are the following.

(1) If two vector bundles on X are isomorphic, their Chern classes coincide.

(2) Functoriality: if $f\colon Y \to X$ is a differentiable map of differentiable manifolds, and E is a complex vector bundle on X, then

$$f^\sharp(c_i(E)) = c_i(f^*E).$$

(3) Whitney product formula: if E, F are complex vector bundles on X, then

$$c(E \oplus F) = c(E) \cup c(F).$$

(4) Normalization: identify the cohomology group $H^2(\mathbb{P}_n, \mathbb{Z})$ with $\mathbb{Z}$ by identifying the class of the hyperplane H with $1 \in \mathbb{Z}$. Then $c_1(H) = 1$.

These properties characterize uniquely the Chern classes (cf. e.g. [**14**]). Notice that, in view of the splitting principle, it is enough to prove the properties (1), (2), (3) when E and F are line bundles. Then (1) and (2) are already known, and (3) follows from elementary properties of the symmetric functions.

The reader can easily check that all Chern classes (but for c_0, obviously) of a trivial vector bundle vanish. Thus, Chern classes in some sense measure the twisting of a bundle. It should be noted that, even in smooth case, Chern classes do not in general classify vector bundles, even as smooth bundles (i.e., generally speaking, $c(E) = c(F)$ does not imply $E \simeq F$). However, in some specific instances this may happen.

EXERCISE 6.3. Prove that for any vector bundle E one has $c_1(E) = c_1(\det E)$. □

7. Kodaira-Serre duality

In this section we introduce Kodaira-Serre duality, which will be one of the main tools in our study of algebraic curves. To start with a simple situation, let us study the analogous result in de Rham theory. Let X be a differentiable manifold. Since the exterior product of two closed forms is a closed form, one can define a bilinear map

$$H^i_{DR}(X)\otimes H^j_{DR}(X)\to H^{i+j}_{DR}(X),\qquad [\tau]\otimes[\omega]\to[\tau\wedge\omega].$$

As we already know, via the Čech-de Rham isomorphism this product can be identified with the cup product. If X is compact and oriented, by composition with the map[4]

$$\int_X\colon H^n_{DR}(X)\to\mathbb{R},\qquad \int_X[\omega]=\int_X\omega$$

where $n=\dim X$, we obtain a pairing

$$H^i_{DR}(X)\otimes H^{n-i}_{DR}(X)\to\mathbb{R},\qquad [\tau]\otimes[\omega]\to\int_X[\tau\wedge\omega]$$

which is quite easily seen to be nondegenerate. Thus one has an isomorphism

$$H^i_{DR}(X)^*\simeq H^{n-i}_{DR}(X)$$

(this is a form of Poincaré duality).

If X is an n-dimensional compact complex manifold, in the same way we obtain a nondegenerate pairing between Dolbeault cohomology groups

$$H^{p,q}_{\bar\partial}(X)\otimes H^{n-p,n-q}_{\bar\partial}(X)\to\mathbb{C},\tag{6.5}$$

and a duality

$$H^{p,q}_{\bar\partial}(X)^*\simeq H^{n-p,n-q}_{\bar\partial}(X).$$

Exercise 6.1. (1) Let E be a holomorphic vector bundle on a complex manifold X, denote by $\mathcal{E}$ the sheaf of its holomorphic sections, and by $\mathcal{E}^\infty$ the sheaf of its smooth sections. Show (using a local trivialization and proving that the result is independent of the trivialization) that one can define a $\mathbb{C}$-linear sheaf morphism

$$\bar\partial_E\colon\mathcal{E}^\infty\to\Omega^{0,1}\otimes\mathcal{E}^\infty\tag{6.6}$$

which obeys a Leibniz rule

$$\bar\partial_E(fs)=f\bar\partial_E s+\bar\partial f\otimes s$$

for $s\in\mathcal{E}^\infty(U)$, $f\in C^\infty(U)$.

(2) Show that $\bar\partial_E$ defines an exact sequence of sheaves

$$0\to\Omega^p\otimes\mathcal{E}\to\Omega^{p,0}\otimes\mathcal{E}^\infty\xrightarrow{\bar\partial_E}\Omega^{p,1}\otimes\mathcal{E}^\infty\xrightarrow{\bar\partial_E}\dots\xrightarrow{\bar\partial_E}\Omega^{p,n}\otimes\mathcal{E}^\infty\to 0.\tag{6.7}$$

[4] This map is well defined because different representatives of $[\omega]$ differ by an exact form, whose integral over X vanishes.

Here Ω^p is the sheaf of holomorphic p-forms. In particular, $\mathcal{E} = \ker(\bar\partial_E \colon \mathcal{E}^\infty \to \Omega^{0,1} \otimes \mathcal{E}^\infty)$.

(3) By taking global sections in (6.7), and taking coholomology from the resulting (in general) non-exact sequence, one defines Dolbeault cohomology groups with coefficients in E, denoted $H^{p,q}_{\bar\partial}(X, E)$. Use the same argument as in the proof of de Rham's theorem to prove an isomorphism

$$H^{p,q}_{\bar\partial}(X, E) \simeq H^q(X, \Omega^p \otimes \mathcal{E}). \tag{6.8}$$

□

By combining the pairing (6.5) with the action of the sections of E^* on the sections of E we obtain a nondegenerate pairing

$$H^{p,q}_{\bar\partial}(X, E) \otimes H^{n-p,n-q}_{\bar\partial}(X, E^*) \to \mathbb{C}$$

and therefore a duality

$$H^{p,q}_{\bar\partial}(X, E)^* \simeq H^{n-p,n-q}_{\bar\partial}(X, E^*).$$

Using the isomorphism (6.8) we can express this duality in the form

$$H^p(X, \Omega^q \otimes \mathcal{E})^* \simeq H^{n-p}(X, \Omega^{n-q} \otimes \mathcal{E}^*).$$

This is the Kodaira-Serre duality. In particular for $q = 0$ we get (denoting $K = \Omega^n = \det T^*X$, the *canonical bundle of* X)

$$H^p(X, \mathcal{E})^* \simeq H^{n-p}(X, K \otimes \mathcal{E}^*).$$

This is usually called *Serre duality.*

8. Connections

In this section we give the basic definitions and sketch the main properties of connections. The concept of connection provides the correct notion of differential operator to differentiate the sections of a vector bundle.

8.1. Basic definitions. Let E a complex, in general smooth, vector bundle on a differentiable manifold X. We shall denote by $\mathcal{E}$ the sheaf of sections of E, and by Ω^1_X the sheaf of differential 1-forms on X. A connection is a sheaf morphism

$$\nabla \colon \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$$

satisfying a Leibniz rule

$$\nabla(fs) = f\nabla(s) + df \otimes s$$

for every section s of E and every function f on X (or on an open subset). The Leibniz rule also shows that ∇ is $\mathbb{C}$-linear. The connection ∇ can be made to act on all sheaves $\Omega^k_X \otimes \mathcal{E}$, thus getting a morphism

$$\nabla \colon \Omega^k_X \otimes \mathcal{E} \to \Omega^{k+1}_X \otimes \mathcal{E}\,,$$

by letting

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \otimes \nabla(s).$$

If $\{U_\alpha\}$ is a cover of X over which E trivializes, we may choose on any U_α a set $\{s_\alpha\}$ of basis sections of $\mathcal{E}(U_\alpha)$ (notice that this is a set of r sections, with $r = \operatorname{rk} E$). Over these bases the connection ∇ is locally represented by matrix-valued differential 1-forms ω_α:

$$\nabla(s_\alpha) = \omega_\alpha \otimes s_\alpha\,.$$

Every ω_α is as an $r \times r$ matrix of 1-forms. The ω_α's are called *connection 1-forms.*

Exercise 6.1. Prove that if $g_{\alpha\beta}$ denotes the transition functions of E with respect to the chosen local basis sections (i.e., $s_\alpha = g_{\alpha\beta}\, s_\beta$), the transformation formula for the connection 1-forms is

$$\omega_\alpha = g_{\alpha\beta}\, \omega_\beta\, g_{\alpha\beta}^{-1} + dg_{\alpha\beta}\, g_{\alpha\beta}^{-1}\,. \tag{6.9}$$

The connection is not a tensorial morphism, but rather satifies a Leibniz rule; as a consequence, the transformation properties of the connection 1-forms are inhomogeneous and contain an affine term.

Exercise 6.2. Prove that if E and F are vector bundles, with connections ∇_1 and ∇_2, then the rule

$$\nabla(s \otimes t) = \nabla_1(s) \otimes t + s \otimes \nabla_2(t)$$

(minimal coupling) defines a connection on the bundle $E \otimes F$ (here s and t are sections of E and F, respectively).

Exercise 6.3. Prove that is E is a vector bundle with a connection ∇, the rule

$$< \nabla^*(\tau), s >= d < \tau, s > - < \tau, \nabla(s) >$$

defines a connection on the dual bundle E^* (here τ, s are sections of E^* and E, respectively, and $<\,,\,>$ denotes the pairing between sections of E^* and E). □

It is an easy exercise, which we leave to the reader, to check that the square of the connection

$$\nabla^2 \colon \Omega^k_X \otimes \mathcal{E} \to \Omega^{k+2}_X \otimes \mathcal{E}$$

is f-linear, i.e., it satisfies the property

$$\nabla^2(fs) = f\nabla^2(s)$$

for every function f on X. In other terms, ∇^2 is an endomorphism of the bundle E with coefficients in 2-forms, namely, a global section of the bundle $\Omega^2_X \otimes \operatorname{End}(E)$. It is called the *curvature* of the connection ∇, and we shall denote it by Θ. On local basis sections s_α it is represented by the *curvature 2-forms* Θ_α defined by

$$\Theta(s_\alpha) = \Theta_\alpha \otimes s_\alpha.$$

EXERCISE 6.4. Prove that the curvature 2-forms may be expressed in terms of the connection 1-forms by the equation (Cartan's structure equation)

$$\Theta_\alpha = d\omega_\alpha - \omega_\alpha \wedge \omega_\alpha \,. \tag{6.10}$$

EXERCISE 6.5. Prove that the transformation formula for the curvature 2-forms is

$$\Theta_\alpha = g_{\alpha\beta}\,\Theta_\beta\,g_{\alpha\beta}^{-1}\,.$$

Due to the tensorial nature of the curvature morphism, the curvature 2-forms obey a homogeneous transformation rule, without affine term. □

Since we are able to induce connections on tensor products of vector bundles (and also on direct sums, in the obvious way), and on the dual of a bundle, we can induce connections on a variety of bundles associated to given vector bundles with connections, and thus differentiate their sections. The result of such a differentiation is called the *covariant differential* of the section. In particular, given a vector bundle E with connection ∇, we may differentiate its curvature as a section of $\Omega^2_X \otimes \operatorname{End}(E)$.

PROPOSITION 6.6. *(Bianchi identity) The covariant differential of the curvature of a connection is zero,* $\nabla\Theta = 0$.

PROOF. A simple computation shows that locally $\nabla\Theta$ is represented by the matrix-valued 3-forms

$$d\Theta_\alpha + \omega_\alpha \wedge \Theta_\alpha - \Theta_\alpha \wedge \omega_\alpha \,.$$

By plugging in the structure equation (6.10) we obtain $\nabla\Theta = 0$. □

8.2. Connections and holomorphic structures. If X is a complex manifold, and E a C^∞ complex vector bundle on it with a connection ∇, we may split the latter into its (1,0) and (0,1) parts, ∇' and ∇'', according to the splitting $\Omega^1_X \otimes \mathbb{C} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$. Analogously, the curvature splits into its (2,0), (1,1) and (0,2) parts,

$$\Theta = \Theta^{2,0} + \Theta^{1,1} + \Theta^{0,2}\,.$$

Obviously we have

$$\Theta^{2,0} = (\nabla')^2, \qquad \Theta^{1,1} = \nabla' \circ \nabla'' + \nabla'' \circ \nabla', \qquad \Theta^{0,2} = (\nabla'')^2.$$

In particular ∇'' is a morphism $\Omega^{p,q}_X \otimes \mathcal{E} \to \Omega^{p,q+1}_X \otimes \mathcal{E}$. If $\Theta^{0,2} = 0$, then ∇'' is a differential for the complex $\Omega^{p,\bullet}_X \otimes \mathcal{E}$. The same condition implies that the kernel of the map

$$\nabla'' \colon \mathcal{E} \to \Omega^{0,1}_X \otimes \mathcal{E} \tag{6.11}$$

has enough sections to be the sheaf of sections of a holomorphic vector bundle.

Proposition 6.7. *If $\Theta^{0,2} = 0$, then the C^∞ vector bundle E admits a unique holomorphic structure, such that the corresponding sheaf of holomorphic sections is isomorphic to the kernel of the operator (6.11). Moreover, under this isomorphism the operator (6.11) concides with the operator $\bar{\delta}_E$ defined in Exercise 6.1.*

Proof. Cf. [**18**], p. 9. □

Conversely, if E is a holomophic vector bundle, a connection ∇ on E is said to be *compatible* with the holomorphic structure of E if $\nabla'' = \partial_E$.

8.3. Hermitian bundles. A Hermitian metric h of a complex vector bundle E is a global section of $E \otimes \overline{E^*}$ which when restricted to the fibres yields a Hermitian form on them (more informally, it is a smoothly varying assignation of Hermitian structures on the fibres of E). On a local basis of sections $\{s_\alpha\}$, of E, h is represented by matrices h_α of functions on U_α which, when evaluated at any point of U_α, are Hermitian and positive definite. The local basis is said to be *unitary* if the corresponding matrix h is the identity matrix.

A pair (E, h) formed by a holomorphic vector bundle with a hermitian metric is called a *hermitian bundle.* A connection ∇ on E is said to be *metric* with respect to h if for every pair s, t of sections of E one has

$$dh(s,t) = h(\nabla s, t) + h(s, \nabla t)\,.$$

In terms of connection forms and matrices representing h this condition reads

$$dh_\alpha = \tilde{\omega}_\alpha\, h_\alpha + h_\alpha\, \bar{\omega}_\alpha \tag{6.12}$$

where ˜ denotes transposition and ¯ denotes complex conjugation (but no transposition, i.e., it is not the hermitian conjugation). This equation implies right away that *on a unitary frame, the connection forms are skew-hermitian matrices.*

Proposition 6.8. *Given a hermitian bundle (E, h), there is a unique connection ∇ on E which is metric with respect to h and is compatible with the holomorphic structure of E.*

Proof. If we use holomophic local bases of sections, the connection forms are of type (1,0). Then equation (6.12) yields

$$\tilde{\omega}_\alpha = \partial h_\alpha\, h_\alpha^{-1} \tag{6.13}$$

and this equations shows the uniqueness. As for the existence, one can easily check that the connection forms as defined by equation (6.13) satisfy the condition (6.9) and therefore define a connection on E. This is metric w.r.t. h and compatible with the holomorphic structure of E by construction. □

EXAMPLE 6.9. (Chern classes and Maxwell theory) The Chern classes of a complex vector bundle E can be calculated in terms of a connection on E via the so-called *Chern-Weil representation theorem.* Let us discuss a simple situation. Let L be a complex line bundle on a smooth 2-dimensional manifold X, endowed with a connection, and let F be the curvature of the connection. F can be regarded as a 2-form on X. In this case the Chern-Weil theorem states that

$$c_1(L) = \frac{i}{2\pi} \int_X F \tag{6.14}$$

where we regard $c_1(L)$ as an integer number via the isomorphism $H^2(X, \mathbb{Z}) \simeq \mathbb{Z}$ given by integration over X. Notice that the Chern class of F is independent of the connection we have chosen, as it must be. Alternatively, we notice that the complex-valued form F is closed (Bianchi identity) and therefore singles out a class $[F]$ in the complexified de Rham group $H^2_{DR}(X) \otimes_{\mathbb{R}} \mathbb{C} \simeq H^2(X, \mathbb{C})$; the class $\frac{i}{2\pi}[F]$ is actually real, and one has the equality

$$c_1(L) = \frac{i}{2\pi}[F]$$

in $H^2_{DR}(X)$. If we consider a static spherically symmetric magnetic field in $\mathbb{R}^3$, by solving the Maxwell equations we find a solution which is singular at the origin. If we do not consider the dependence from the radius the vector potential defines a connection on a bundle L defined on an S^2 which is spanned by the angular spherical coordinates. The fact that the Chern class of L as given by (6.14) can take only integer values is known in physics as the *quantization of the Dirac monopole.*

CHAPTER 7

Divisors

Divisors are a powerful tool to study complex manifolds. We shall start with the one-dimensional case. The notion will be later generalized to higher dimensional manifolds.

1. Divisors on Riemann surfaces

Let S be a Riemann surface (a complex manifold of dimension 1). A divisor D on S is a locally finite formal linear combinations of points of S with integer coefficients,

$$D = \sum a_i p_i, \qquad a_i \in \mathbb{Z}, \quad p_i \in S,$$

where "locally finite" means that every point p in S has a neighbourhood which contains only a finite number of p_i's. If S is compact, this means that the number of points is finite. We say that the divisor D is *effective* if $a_i \geq 0$ for all i. We shall then write $D \geq 0$.

The set of all divisors of S forms an abelian group, denoted by $\mathrm{Div}(S)$.

Let f a holomorphic function defined in a neighbourhood of p, and let z be a local coordinate around p. There exists a unique nonnegative integer a and a holomorphic function h such that

$$f(z) = (z - z(p))^a\, h(z)$$

and $h(p) \neq 0$. We define

$$\mathrm{ord}_p\, f = a.$$

Notice that

$$\mathrm{ord}_p\, fg = \mathrm{ord}_p\, f + \mathrm{ord}_p\, g. \tag{7.1}$$

If f is a meromorphic function which in a neighbourhood of p can be written as $f = g/h$, with g and h holomorphic, we define

$$\mathrm{ord}_p\, f = \mathrm{ord}_p\, g - \mathrm{ord}_p\, h.$$

We say that f has a zero of order a at p if $\mathrm{ord}_p\, f = a > 0$ (then f is holomorphic in a neighbourhood of p), and that it has a pole of order a if $\mathrm{ord}_p\, f = -a < 0$.

With each meromorphic function f we may associate the divisor

$$(f) = \sum_{p \in S} \mathrm{ord}_p\, f \cdot p;$$

if $f = g/h$ with g and h relatively prime, then $(f) = (g) - (h)$.

1.1. Sheaf-theoretic description of divisors. The group of divisors Div can be described in sheaf-theoretic terms as follows. Let $\mathcal{M}^*$ be the sheaf of meromorphic functions that are not identically zero. We have an exact sequence

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0$$

of sheaves of abelian groups (notice that the group structure is multiplicative).

Proposition 7.1. *There is a group isomorphism* $\operatorname{Div}(S) \simeq H^0(S, \mathcal{M}^*/\mathcal{O}^*)$.

Proof. Given a cover $\mathfrak{U} = \{U_\alpha\}$ of X, one has a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & H^0(S, \mathcal{M}^*/\mathcal{O}^*) & & \\
 & & & & \downarrow & & \\
 & & C^0(\mathfrak{U}, \mathcal{M}^*) & \longrightarrow & C^0(\mathfrak{U}, \mathcal{M}^*/\mathcal{O}^*) & \longrightarrow & \prod_\alpha H^1(U_\alpha, \mathcal{O}^*) = 0 \\
 & & {\scriptstyle\delta}\downarrow & & {\scriptstyle\delta}\downarrow & & \\
C^1(\mathfrak{U}, \mathcal{O}^*) & \longrightarrow & C^1(\mathfrak{U}, \mathcal{M}^*) & \longrightarrow & C^1(\mathfrak{U}, \mathcal{M}^*/\mathcal{O}^*) & &
\end{array}$$

where $H^1(U_\alpha, \mathcal{O}^*) = 0$ because $U_\alpha \simeq \mathbb{C}$ holomorphically (here δ denotes the Čech cohomology operator). This diagram shows that a global section $s \in H^0(S, \mathcal{M}^*/\mathcal{O}^*)$ can be represented by a 0-cochain $\{f_\alpha \in \mathcal{M}^*(U_\alpha)\} \in \check{C}^0(\mathfrak{U}, \mathcal{M}^*)$ subject to the condition $f_\alpha/f_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$, so that $\operatorname{ord}_p f_\alpha$ does not depend on α, and the quantity $\operatorname{ord}_p s$ is well defined. We set $D = \sum_p \operatorname{ord}_p s \cdot p$.

Conversely, given $D = \sum a_i p_i$, we may choose an open cover $\{U_\alpha\}$ such that each U_α contains at most one p_i, and functions $g_{i\alpha} \in \mathcal{O}(U_\alpha)$ such that that $g_{i\alpha}$ has a zero of order one at p_i if $p_i \in U_\alpha$. We set

$$f_\alpha = \prod_i g_{i\alpha}^{a_i}.$$

Then $f_\alpha/f_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$, so that $\{f_\alpha\}$ determines a global section of $\mathcal{M}^*/\mathcal{O}^*$.

The two constructions are one the inverse of the other, so that they establish an isomorphism of sets. The fact that this is also a group homomorphism follows from the formula (7.1), which holds also for meromorphic functions. □

1.2. Correspondence between divisors and line bundles. Let $D \in \operatorname{Div}(S)$, and let $\{U_\alpha\}$ be an open cover of S with meromorphic functions $\{f_\alpha\}$ which define the divisor, according to Proposition 7.1. Then the functions

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$$

obviously satisfy the cocycle condition, and define a line bundle, which we denote by $[D]$. The line bundle $[D]$ in independent, up to isomorphism, of the set of functions defining D; if $\{f'_\alpha\}$ is another set, then $\operatorname{ord}_{p_i} f_\alpha = \operatorname{ord}_{p_i} f'_\alpha$, so that the functions $h_\alpha = f_\alpha/f'_\alpha$ are holomorphic and nowhere vanishing, and

$$g'_{\alpha\beta} = \frac{f'_\alpha}{f'_\beta} = \frac{f_\alpha}{f_\beta}\frac{h_\beta}{h_\alpha} = g_{\alpha\beta}\frac{h_\beta}{h_\alpha},$$

so that the transition functions $g'_{\alpha\beta}$ define an isomorphic line bundle.

If $D = D^{(1)} + D^{(2)}$ then $f_\alpha = f^{(1)}_\alpha f^{(2)}_\alpha$ by eq. (7.1), so that $[D^{(1)} + D^{(2)}] = [D^{(1)}] \otimes [D^{(2)}]$, and one has a homomorphism $\operatorname{Div}(S) \to \operatorname{Pic}(S)$.

We offer now a sheaf-theoretic description of this homomorphism. Let $f = \{f_\alpha\} \in H^0(S, \mathcal{M}^*)$; let us set $f_\alpha = g_\alpha/h_\alpha$, with g_α,$h_\alpha \in \mathcal{O}(U_\alpha)$ relatively prime. We have $(f) = (g) - (h)$, with (g) and (h) effective divisors. The line bundle $[(f)]$ has transition functions

$$g_{\alpha\beta} = \frac{g_\alpha}{g_\beta}\frac{h_\beta}{h_\alpha} = \frac{f_\alpha}{f_\beta} = 1$$

(since f is a Čech cocycle) so that $[(f)] = \underline{\mathbb{C}}$, i.e. $[(f)]$ is the trivial line bundle.

Conversely, let D be a divisor such that $[D] = \underline{\mathbb{C}}$; then the transition functions of $[D]$ have the form

$$g_{\alpha\beta} = \frac{h_\alpha}{h_\beta} \qquad \text{with} \quad h_\alpha \in \mathcal{O}^*(U_\alpha).$$

Let $\{f_\alpha\}$ be meromorphic functions which define D, so that one also has $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$, and

$$\frac{f_\alpha}{h_\alpha} = g_{\alpha\beta}\frac{f_\beta}{h_\alpha} = \frac{f_\beta}{h_\beta};$$

the quotients $\frac{f_\alpha}{h_\alpha}$ therefore determine a global nonzero meromorphic function, namely:

Proposition 7.2. *The line bundle associated with a divisor D is trivial if and only if D is the divisor of a global meromorphic function.*

In view of the identifications $\operatorname{Div}(S) \simeq H^0(S, \mathcal{M}^*/\mathcal{O}^*)$ and $\operatorname{Pic}(S) \simeq H^1(S, \mathcal{O}^*)$ this statement is equivalent to the exactness of the sequence

$$H^0(S, \mathcal{M}^*) \to H^0(S, \mathcal{M}^*/\mathcal{O}^*) \to H^1(S, \mathcal{O}^*).$$

Definition 7.3. *Two divisors D, $D' \in \operatorname{Div}(S)$ are linearly equivalent if $D' = D + (f)$ for some $f \in H^0(S, \mathcal{M})$.*

Quite evidently, D and D' are linearly equivalent if and only if $[D] \simeq [D']$, so that there is an injective group homomorphism

$$\operatorname{Div}(S)/\{\text{linear equivalence}\} \to \operatorname{Pic}(S).$$

1.3. Holomorphic and meromorphic sections of line bundles. If L is a line bundle on S, we denote by $\mathcal{O}(L)$ the sheaf of its holomorphic sections, and by $\mathcal{M}(L)$ the sheaf of its meromorphic sections, the latter being defined as $\mathcal{M}(L) = \mathcal{O}(L) \otimes_{\mathcal{O}} \mathcal{M}$. If L has transition functions $g_{\alpha\beta}$ with respect to a cover $\{U_\alpha\}$ of S, then a global holomorphic section $s \in H^0(S, \mathcal{O}(L))$ of L corresponds to a collection of functions $\{s_\alpha \in \mathcal{O}(U_\alpha)\}$ such that $s_\alpha = g_{\alpha\beta} s_\beta$ on $U_\alpha \cap U_\beta$. The same holds for meromorphic sections. A first consequence of this is that, if $s, s' \in H^0(S, \mathcal{M}(L))$, we have

$$\frac{s_\alpha}{s'_\alpha} = \frac{g_{\alpha\beta} s_\beta}{g_{\alpha\beta} s'_\beta} = \frac{s_\beta}{s'_\beta} \qquad \text{on} \quad U_\alpha \cap U_\beta,$$

so that the quotient of s and s' is a well-defined global meromorphic function on S.

Let $s \in H^0(S, \mathcal{M}(L))$; we have

$$\frac{s_\alpha}{s_\beta} = g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$$

so that

$$\operatorname{ord}_p s_\alpha = \operatorname{ord}_p s_\beta \qquad \text{for all} \quad p \in U_\alpha \cap U_\beta;$$

the quantity $\operatorname{ord}_p s$ is well defined, and we may associate with s the divisor

$$(s) = \sum_{p \in S} \operatorname{ord}_p s \cdot p.$$

By construction we have $[(s)] \simeq L$. Obviously, s is holomorphic if and only if (s) is effective.

So we have

Proposition 7.4. *A line bundle L is associated with a divisor D (i.e. $L = [D]$ for some $D \in \operatorname{Div}(S)$) if and only if it has a global nontrivial meromorphic section. L is the line bundle associated with an effective divisor if and only if it has a global nontrivial holomorphic section.*

Proof. The "if" part has already been proven. For the "only if" part, let $L = [D]$ with D a divisor with local equations $f_\alpha = 0$. Then $f_\alpha = g_{\alpha\beta} f_\beta$, where the functions $g_{\alpha\beta}$ are transition functions for L; the functions f_α glue to yield a global meromorphic section s of L. If D is effective the functions f_α are holomorphic so that s is holomorphic as well. □

Corollary 7.5. *The line bundle $[p]$ trivializes over the cover $\{U_1, U_2\}$, where $U_1 = S - \{p\}$ and U_2 is a neighbourhood of p, biholomorphic to a disc in $\mathbb{C}$.*

Proof. Since $[p]$ is effective it has a global holomorphic section which vanishes only at p, so that $[p]$ is trivial on U_1. Of course it is trivial on U_2 as well. □

So the same happens for the line bundles $[kp]$, $k \in \mathbb{Z}$.

For the remainder of this section we assume that S is compact. Let us define the *degree* of a divisor $D = \sum a_i p_i$ as the integer

$$\deg D = \sum a_i \,.$$

For simplicity we shall write $\mathcal{O}(D)$ for $\mathcal{O}([D])$.

COROLLARY 7.6. *If* $\deg D < 0$*, then* $H^0(S, \mathcal{O}(D)) = 0$. □

If L is a line bundle we denote by $\int_S c_1(L)$ the number obtained by integrating over S a differential 2-form which via de Rham isomorphism represents[1] the Čech cohomology class $c_1(L)$ regarded as an element in $H^2(S, \mathbb{R})$.

PROPOSITION 7.7. *For any* $D \in \operatorname{Div}(S)$ *one has*

$$\int_S c_1(D) = \deg D.$$

Before proving this result we need some preliminaries. We define a *hermitian metric* on a line bundle L as an assignment of a hermitian scalar product in each L_p which is C^∞ in p; thus a hermitian metric is a C^∞ section h of the line bundle $L^* \otimes L^*$ such that each $h(p)$ is a hermitian scalar product in L_p. In terms of a local trivialization over an open cover $\{U_\alpha\}$ a hermitian metric is represented by nonvanishing real functions h_α on U_α. On $U_\alpha \cap U_\beta$ one has $h_\alpha = |g_{\alpha\beta}|^2 h_\beta$, so that the 2-form $\frac{i}{2\pi}\bar\partial\partial \log h_\alpha$ does not depend on α, and defines a global closed 2-form on S, which we denote by Θ.

LEMMA 7.8. *The class of* Θ *is the image in* $H^2_{DR}(S)$ *of* $c_1(L)$.

PROOF. We need the explicit form of the de Rham correspondence. One has exact sequences

$$0 \to \mathbb{R} \to \mathcal{C}^\infty \to \mathcal{Z}^1 \to 0, \qquad 0 \to \mathcal{Z}^1 \to \Omega^1 \to \mathcal{Z}^2 \to 0\,. \tag{7.2}$$

(Here Ω^1 is the sheaf of smooth real-valued 1-forms.) From the long exact cohomology sequences of the second sequence we get

$$H^0(S, \Omega^1) \to H^0(S, \mathcal{Z}^2) \to H^1(S, \mathcal{Z}^1) \to 0$$

so that the connecting morphism $H^0(S, \mathcal{Z}^2) \to H^1(S, \mathcal{Z}^1)$ induces an isomorphism $H^2_{DR}(S) \to H^1(S, \mathcal{Z}^1)$. Since we may write $\Theta = \frac{i}{2\pi} d\partial \log h_\alpha$ a cocycle representing the image of $[\Theta]$ in $H^1(S, \mathcal{Z}^1)$ is $\{\theta_\alpha - \theta_\beta\}$, with

$$\theta_\alpha = \tfrac{i}{2\pi}\partial \log h_\alpha.$$

Notice that

$$\theta_\alpha - \theta_\beta = \tfrac{i}{2\pi}\partial\left(\log h_\alpha - \log h_\beta\right) = \tfrac{i}{2\pi}\, d \log g_{\alpha\beta}$$

so that $d(\theta_\alpha - \theta_\beta) = 0$.

[1] The reader should check that the integral does not depend on the choice of the representative.

If we consider now the first of the sequences (7.2) we obtain from its long cohomology exact sequence a segment

$$0 \to H^1(S, \mathcal{Z}^1) \to H^2(S, \mathbb{R}) \to 0$$

so that the connecting morphism is now an isomorphism. If we apply it to the 1-cocycle $\{\theta_\alpha - \theta_\beta\}$ we get the 2-cocycle of $\mathbb{R}$

$$\tfrac{1}{2\pi i} \log g_{\alpha\beta} + \tfrac{1}{2\pi i} \log g_{\beta\gamma} + \tfrac{1}{2\pi i} \log g_{\gamma\alpha} = (c_1(L))_{\alpha\beta\gamma}.$$

□

Proof of Proposition 7.7: Since c_1 and deg are both group homomorphisms, it is enough to consider the case $D = [p]$. Consider the open cover $\{U_1, U_2\}$, where $U_1 = S - \{p\}$, and U_2 is a small patch around p. Then

$$\int_S c_1(D) = \int_S \Theta = \tfrac{i}{2\pi} \lim_{\epsilon \to 0} \int_{S - B(\epsilon)} d\partial \log h_1$$

where $B(\epsilon)$ is the disc $|z| < \epsilon$, with z a local coordinate around p, and $z(p) = 0$. Since $\bar\partial\partial = \frac{1}{2} d(\partial - \bar\partial)$, and assuming that $h_{1|U_2 - B(\epsilon)} = |z|^2$, which can always be arranged, we have

$$\int_S c_1(D) = \tfrac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{\partial B(\epsilon)} \partial \log z\bar z = \tfrac{1}{2\pi i} \int_{\partial B(\epsilon)} \frac{dz}{z} = 1$$

having used Stokes' theorem and the residue theorem (note a change of sign due to a reversal of the orientation of $\partial B(\epsilon)$). □

This result suggests to set

$$\deg L = \int_S c_1(L)$$

for all line bundles on S.

Corollary 7.9. *If* $\deg L < 0$, *then* $H^0(S, \mathcal{O}(L)) = 0$.

Proof. If there is a nonzero $s \in H^0(S, \mathcal{O}(L))$, then $L = [D]$ with $D = (s)$. Since $\deg D < 0$ by the previous Proposition, this contradicts Corollary 7.6. □

Corollary 7.10. *A global meromorphic function on a compact Riemann surface has the same number of zeroes and poles (both counted with their multiplicities).*

Proof. If f global meromorphic function, we must show that $\deg(f) = 0$. But f is a global meromorphic section of the trivial line bundle $\underline{\mathbb{C}}$, whence

$$\deg(f) = \int_S c_1(\underline{\mathbb{C}}) = 0\,.$$

□

1.4. The fundamental exact sequence of an effective divisor. Let us first define for all $p \in S$ the sheaf k_p as the *1-dimensional skyscraper sheaf concentrated at* p, namely, the sheaf

$$k_p(U) = \mathbb{C} \quad \text{if} \quad p \in U, \qquad k_p(U) = 0 \quad \text{if} \quad p \notin U.$$

k_p has stalk $\mathbb{C}$ at p and stalk 0 elsewhere.

Let $D = \sum a_i p_i$ be an effective divisor. Then the line bundle $L = [D]$ has at least one section s; this allows one to define a morphism $\mathcal{O} \to \mathcal{O}(D)$ by letting $f \mapsto f\, s_{|U}$ for every $f \in \mathcal{O}(U)$. We also define the skyscraper sheaf $k_D = \sum_i (k_{p_i})^{a_i}$ concentrated on D.

PROPOSITION 7.11. *The sequence*

$$0 \to \mathcal{O} \to \mathcal{O}(D) \to k_D \to 0 \tag{7.3}$$

is exact.

PROOF. We shall actually prove the exactness of the sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to k_D \to 0 \tag{7.4}$$

from which the previous sequence is obtained by tensoring by $\mathcal{O}(D)$.[2] Notice also that $k_D \otimes_{\mathcal{O}} \mathcal{O}(D) \simeq k_D$ because in a neighbourhood of every point p_i the sheaf $\mathcal{O}(D)$ is isomorphic to $\mathcal{O}$.

The exactness of the sequence (7.4) follows from the fact the any local holomorphic function can be represented around p_i in the form (Taylor polynomial)

$$f(z) = f(z_0) + \sum_{k=1}^{a_i - 1} \frac{1}{k!} f^{(k)}(z_0)\,(z - z_0)^k + (z - z_0)^{a_i}\, g(z)$$

where $z_0 = z(p)$, and g is a holomorphic function. The term $(z - z_0)^{a_i}\, g(z)$ is a section of $\mathcal{O}(-D)$, while the first two terms on the right single out a section of k_D. □

The sheaf $\mathcal{O}(-D)$ can be regarded as the sheaf of holomorphic functions which at p_i have a zero of order at least a_i. Since $\mathcal{O}(D) \simeq \mathcal{O}(-D)^*$, the $\mathcal{O}(D)$ may be identified with the sheaf of meromorphic functions which at p_i have a pole of order at most a_i.

In particular one may write

$$0 \to \mathcal{O}(-2p) \to \mathcal{O} \to k_p \oplus T_p^* S \to 0$$

where $T_p^* S$ is considered as a skyscraper sheaf concentrated at p (indeed the quantity $f'(z_0)$ determines an element in $T_p^* S$).

If E is a holomorphic vector bundle on S, let us denote $E(D) = E \otimes [D]$. Then by tensoring the exact sequence (7.4) by $\mathcal{O}(E)$ we get

[2] Here we use the fact that tensoring all elements of an exact sequence by the sheaf of sections of a vector bundle preserves exactness. This is quite obvious because by the local triviality of E the stalk of $\mathcal{O}(E)$ at p is $\mathcal{O}_p^k$, with k the rank of E.

$$0 \to \mathcal{O}(E(-D)) \to \mathcal{O}(E) \to E_D \to 0$$

where $E_D = \oplus_i E_{p_i}^{\oplus a_i}$ is a skyscraper sheaf concentrated on D.

2. Divisors on higher-dimensional manifolds

We start with some preparatory material.

Definition 7.1. *An analytic subvariety V of a complex manifold X is a subset of X which is locally defined as the zero set of a finite collection of holomorphic functions.*

An analytic subvariety V is said to be reducible if $V = V_1 \cup V_2$ with V_1 and V_2 properly contained in V. V is said to be irreducible if it is not reducible.

A point $p \in V$ is a smooth point of V if around p the subvariety V is a submanifold, namely, it can be written as $f_1(z^1, \dots, z^n) = \dots f_k(z^1, \dots, z^n) = 0$ with $\operatorname{rank} J = k$, where $\{z^1, \dots, z^n\}$ is a local coordinate system for X around p, and J is the jacobian matrix of the functions $f_1, \dots f_k$. The set of smooth points of V is denoted by V^*; the set $V_s = V - V^*$ is the singular locus of V. The dimension of V is by definition the dimension of V^*.

If $\dim V = \dim X - 1$, V will be called an analytic hypersurface.

Proposition 7.2. *Any analytic subvariety V can be expressed around a point $p \in V$ as the union of a finite number of analytic subvarieties V_i which are irreducible around p, and are such that $V_i \not\subset V_j$.*

Proof. This follows from the fact that the stalk $\mathcal{O}_p$ is a unique factorization domain ([**10**] page 12).[3] Let us sketch the proof for hypersurfaces. In a neighbourhood of p the hypersurface V is given by $f = 0$. Denoting by the same letter the germ of f in p, since $\mathcal{O}_p$ (where $\mathcal{O}$ is the sheaf of holomorphic functions on X) is a unique factorization domain we have

$$f = f_1 \cdot \dots \cdot f_m,$$

where the f_i's are irreducible in $\mathcal{O}_p$, and are defined up to multiplication by invertible elements in $\mathcal{O}_p$; if V_i is the locus of zeroes of f_i, then $V = \cup_i V_i$. Since f_i irreducible, V_i is irreducible as well; since it is not true that $f_j = g f_i$ for some $g \in \mathcal{O}_p$ which vanishes at p, we also have $V_i \not\subset V_j$. □

We may now give the general definition of divisor:

[3] Let us recall this notion: one says that a ring R is an integral domain if $uv = 0$ implies that either $u = 0$ or $v = 0$. An element $u \in R$ in an integral domain is said to be irreducible if $u = vw$ implies that v or w is a unit; R is a unique factorization domain if any element u can be written as a product $u = u_1 \cdot \dots \cdot u_m$, where the u_i are irreducible and unique up to multiplication by units.

DEFINITION 7.3. *A divisor D on a complex manifold X is a locally finite formal linear combination with integer coefficients $D = \sum a_i V_i$, where the V_i's are irreducible analytic hypersurfaces in X.*

If $V \subset X$ is an analytic irreducible hypersurface, and $p \in V$, we may choose around p a coordinate system $\{w, z^2, \ldots, z^n\}$ such that V is given around p by $w = 0$. Given a function f defined in a neighbourhood of p, let a be the greatest integer such that

$$f(w, z^2, \ldots, z^n) = w^a \, h(w, z^2, \ldots, z^n)$$

with $h(p) \neq 0$. The function f has the same representation in all nearby points of V, so that a is constant on the connected components of V, namely, it is constant on V, so that we may define

$$\operatorname{ord}_V f = a.$$

With this proviso all the theory previously developed applies to this situation; the only definition which no longer applies is that of degree of a line bundle, in that $c_1(L)$ is still represented by a 2-form, while the quantities that can be integrated on X are the $2n$-forms if $\dim_{\mathbb{C}} X = n$. Proposition 7.7 must now be reformulated as follows. Let $D = \sum a_i V_i$ be a divisor, and let V_i^* be the smooth locus of V_i. We then have:

PROPOSITION 7.4. *For any divisor $D \in \operatorname{Div}(X)$ and any $(2n-2)$-form ϕ on X,*

$$\int_X c_1(D) \wedge \phi = \sum_i a_i \int_{V_i^*} \phi.$$

PROOF. The proof is basically the same as in Proposition 7.7 (cf. [**10**] page 141). □

3. Linear systems

In this section we consider a compact complex manifold X of arbitrary dimension. Let $D = \sum a_i V_i \in \operatorname{Div}(X)$, and define $|D|$ as the set of all effective divisors linearly equivalent to D. We start by showing that there is an isomorphism

$$\lambda \colon \mathbb{P}H^0(X, \mathcal{O}(D)) \to |D| \,.$$

We fix a global meromorphic section s_0 of $[D]$, and set

$$s \in H^0(X, \mathcal{O}(D)) \mapsto \left(\frac{s}{s_0}\right) + D \in |D| \,; \qquad (7.5)$$

one should notice that $\operatorname{ord}_{p_i}\left(\frac{s}{s_0}\right) \geq -a_i$ if $p_i \in V_i$ so that $\left(\frac{s}{s_0}\right) + D$ is indeed effective. If $s' = \alpha\, s$ with $\alpha \in \mathbb{C}^*$ then $\left(\frac{s}{s_0}\right) = \left(\frac{s'}{s_0}\right)$ so that equation (7.5) does define a map $\mathbb{P}H^0(X, \mathcal{O}(D)) \to |D|$. This map is

(i) injective because if $\lambda(s_1) = \lambda(s_2)$ then s_1/s_2 is a global nonvanishing holomorphic function, i.e. $s_1 = \alpha\, s_2$ with $\alpha \in \mathbb{C}^*$.

(ii) Surjective because if $D_1 \in |D|$ then $D_1 = D + (f)$ for a global meromorphic function f with $\mathrm{ord}_{p_i}(f) \geq -a_i$ if $p_i \in V_i$. So fs_0 is a global holomorphic section of $[D]$.

Definition 7.1. *A linear system is the set of divisors corresponding to a linear subspace of* $\mathbb{P}H^0(X, \mathcal{O}(D))$. *A linear system is said to be complete if it corresponds to the whole of* $\mathbb{P}H^0(X, \mathcal{O}(D))$.

So a linear system is of the form $E = \{D_\lambda\}_{\lambda \in \mathbb{P}_m}$ for some m. The number m is called the dimension of E. A one-dimensional linear system is called a *pencil,* a two-dimensional one a *net*, and a three-dimensional one a *web.* Since all divisors in a linear system have the same degree, one can associate a degree to a linear system.

Remark 7.2. If the elements $\lambda_0, \ldots, \lambda_m$ are independent in $\mathbb{P}_m$ (which means that they are images of linearly independent elements in $\mathbb{C}^{m+1}$), and $E = \{D_\lambda\}_{\lambda \in \mathbb{P}_m}$ is a linear system, then

$$D_{\lambda_0} \cap \cdots \cap D_{\lambda_m} = \bigcap_{\lambda \in \mathbb{P}_m} D_\lambda.$$

For instance, if $m = 1$, and D_{λ_0} and D_{λ_1} have local equations $f = 0$ and $g = 0$, then D_λ has local equation $c_0 f + c_1 g = 0$ if $\lambda = c_0\lambda_0 + c_1\lambda_1$. So $D_{\lambda_0} \cap D_{\lambda_1} \subset \cap_{\lambda \in \mathbb{P}_1} D_\lambda$, which implies $D_{\lambda_0} \cap D_{\lambda_1} = \cap_{\lambda \in \mathbb{P}_1} D_\lambda$.

Definition 7.3. *If* $E = \{D_\lambda\}_{\lambda \in \mathbb{P}_m}$ *is a linear system, we define its base locus as* $B(E) = \cap_{\lambda \in \mathbb{P}_m} D_\lambda$.

Example 7.4. If $E = \{D_\lambda\}_{\lambda \in \mathbb{P}_1}$ is a pencil, every $p \in X - B(E)$ lies on a unique D_λ, so that there is a well-defined map $X - B(E) \to \mathbb{P}_1$. This map is holomorphic. We may indeed write a local equation for D_λ in the form

$$f(z^1, \ldots, z^n) + \lambda g(z^1, \ldots, z^n) = 0 \tag{7.6}$$

where f and f are local defining functions for D_0 and D_∞ (holomorphic because the divisors in E are effective). f and g do not vanish simultaneously on $X - B(E)$, so that they do not vanish separately either. Then the above map is given by $\lambda = -f(z^1, \ldots, z^n)/g(z^1, \ldots, z^n)$. □

Example 7.5. Since $H^1(\mathbb{P}_n, \mathcal{O}) = H^2(\mathbb{P}_n, \mathcal{O}) = 0$, the line bundles on $\mathbb{P}_n$ are classified by $H^2(\mathbb{P}_n, \mathbb{Z}) \simeq \mathbb{Z}$. Moreover, since $c_1(H) = 1$ under this identification (i.e. $\deg H = 1$), all divisors are linearly equivalent to multiples of H; in other terms, on $\mathbb{P}_n$ the only complete linear system of degree d is $|dH|$.

Notice that $|H|$ is base-point free, i.e. $B(|H|) = \emptyset$. □

A fundamental result in the theory of linear systems is the following.

Proposition 7.6. (Bertini's theorem) *The generic element of a linear system is smooth away from the base locus.*

By this we mean that the set of divisors in a linear system E which have singular points outside the base locus form a subvariety of E of dimension strictly smaller than that of E.

Proof. If E is linear system, and $D \in E$ has singularities outside $B(E)$, Bertini's theorem would be violated by all pencils containing D. It is therefore sufficient to prove the theorem for pencils; in this case genericity means that the divisors having singularities out of the base locus are finite in number.

So let $E = \{D_\lambda\}_{\lambda\in\mathbb{P}_1}$ be a pencil, locally described by eq. (7.6), where the coordinates $\{z^1, \ldots, z^n\}$ can be defined on an open subset $\Delta \subset X$ whose image in $\mathbb{C}^n$ is a polydisc. Let p_λ be a singular point of D_λ which is not contained in the base locus. We have the conditions

$$f(p_\lambda) + \lambda g(p_\lambda) = 0 \tag{7.7}$$

$$\frac{\partial f}{\partial z^i}(p_\lambda) + \lambda \frac{\partial f}{\partial z^i}(p_\lambda) = 0, \quad i = 1, \ldots, n \tag{7.8}$$

$$f(p_\lambda), g(p_\lambda) \neq 0.$$

We then have $\lambda = -f(p_\lambda)/g(p_\lambda)$, so that

$$\frac{\partial f}{\partial z^i} - \frac{f}{g}\frac{\partial g}{\partial z^i} = 0 \qquad \text{in} \quad p_\lambda,$$

and

$$\frac{\partial}{\partial z^i}\left(\frac{f}{g}\right) = 0 \qquad \text{in} \quad p_\lambda. \tag{7.9}$$

Let Y be the locus in $\Delta \times \mathbb{P}_1$ cut out by the conditions (7.7) and (7.8); Y is an analytic variety, so the same holds true for its image V in Δ. Actually V is nothing but the locus of all singular points of the divisors D_λ. Equation (7.9) shows that f/g is constant on the connected components of $V - B$, that is, every connected component of $V - B$ meets only one divisor of the pencil. Since the connected components of $V - B$ are finitely many by Proposition 7.2, the divisors which have singularities outside $B(E)$ are finite in number. □

4. The adjunction formula

If V is a smooth analytic hypersurface in a complex manifold X, we may relate the canonical bundles K_V and K_X. We shall denote by $\iota_V \colon V \to X$ the inclusion; one has an injective morphism $TV \to \iota_V^* TX$ of bundles on V. If we choose around $p \in V$ a coordinate system $(z^1, \ldots, z^n)$ for X such that $z^1 = 0$ locally describes V, then the vector field $\dfrac{\partial}{\partial z^1}$ restricted to V locally generates the quotient sheaf $N_V = \iota_V^* TX/TV$, so that N_V is the sheaf of sections of a line bundle, which is called the *normal bundle* to V.

The dual N_V^*, the *conormal bundle* to V, is the subbundle of $\iota_V^* T^*X$ whose sections are holomorphic 1-forms which are zero on vectors tangent to V.

We first prove the isomorphism

$$N_V^* \simeq \iota_V^*[-V]. \tag{7.10}$$

We consider the exact sequence of vector bundles on V

$$0 \to N_V^* \to \iota_V^* T^*X \to T^*V \to 0$$

whence we get[4]

$$\iota_V^* K_X \simeq K_V \otimes N_V^*. \tag{7.11}$$

If, relative to an open cover $\{U_\alpha\}$ of X, the divisor V is locally given by functions $f_\alpha \in \mathcal{O}(U_\alpha)$, the line bundle $[V]$ has transition functions $g_{\alpha\beta} = f_\alpha/f_\beta$. The 1-form $df_{\alpha|V\cap U_\alpha}$ is a section of $N^*_{V|V\cap U_\alpha}$, which never vanishes because V is smooth. On $U_\alpha \cap U_\beta$ we have

$$df_\alpha = d(g_{\alpha\beta} f_\beta) = dg_{\alpha\beta}\, f_\beta + g_{\alpha\beta}\, df_\beta = g_{\alpha\beta}\, df_\beta$$

the last equality holding on $V \cap U_\alpha \cap U_\beta$. This equation shows that the 1-forms df_α do not glue to a global section of N_V^*, but rather to a global section of the line bundle $N_V^* \otimes \iota_V^*[V]$, so that this bundle is trivial, and the isomorphism (7.10) holds.

By combining the formula (7.10) with the isomorphism (7.11) we obtain the *adjunction formula:*

$$K_V \simeq \iota_V^*(K_X \otimes [V]). \tag{7.12}$$

Sometimes an additive notation is used, and then the adjunction formula reads

$$K_V = K_{X|V} + [V]_{|V}\,.$$

Example 7.1. Let V be the divisor cut out from $\mathbb{P}_3$ by the quartic equation

$$w_0^4 + w_1^4 + w_2^4 + w_3^4 = 0 \tag{7.13}$$

where the w_i's are homogeneous coordinates in $\mathbb{P}_3$. It is easily shown the V is smooth, and it is of course compact: so it is a 2-dimensional compact complex manifold, called the *Fermat surface*. By a nontrivial result, known as Lefschetz hyperplane theorem ([**10**] p. 156) one has $H^1(V, \mathbb{R}) = 0$, so that $H^1(V, \mathcal{O}_V) = 0$. Then the group $\mathrm{Pic}^0(V)$, which classifies the line bundles whose first Chern classes vanishes, is trivial: if a line bundle L on V is such that $c_1(L) = 0$, then it is trivial, and every line bundle is fully classified by its first Chern class. (The same happens on $\mathbb{P}_3$, since $H^1(\mathbb{P}_3, \mathcal{O}_{\mathbb{P}_3}) = 0$).

[4] We use the fact that whenever

$$0 \to E \to F \to G \to 0$$

is an exact sequence of vector bundles, then $\det F \simeq \det E \otimes \det G$, as one can prove by using transition functions.

We also know that $K_{\mathbb{P}_3} = \mathcal{O}_{\mathbb{P}_3}(-4H)$, where H is any hyperplane in $\mathbb{P}_3$. Therefore $\iota_V^* K_X \simeq \mathcal{O}_V(-4H_V)$, where $H_V = H \cap V$ is a divisor in V.

Let us compute $c_1([V]_{|V}) = \iota_V^* c_1([V])$. We use the following fact: if D_1, D_2, D_3 are irreducible divisors in $\mathbb{P}_3$, then we can move the divisors inside their linear equivalence classes in such a way that they intersects at a finite number of points. This number is computed by the integral

$$\int_{\mathbb{P}_3} c_1([D_1]) \wedge c_1([D_2]) \wedge c_1([D_3])$$

where one considers the Chern classes $c_1([D_i])$ as de Rham cohomology classes. If we take $D_1 = V$, $D_2 = D_3 = H$ the number of intersection points is 4, because such is the degree of the algebraic system formed by the equation (7.13) and by the equations of two (different) hyperplanes. Since the class h, where $h = c_1([H])$, generates $H^2(\mathbb{P}_3, \mathbb{Z})$, we have $c_1([V]) = 4h$, that is, $V \sim 4H$. Then $[V]_{|V} \simeq \mathcal{O}_V(4H_V)$.

From the adjunction formula we get $K_V \simeq \underline{\mathbb{C}}$: the canonical bundle of V is trivial. Since we also have $H^1_{DR}(V) = 0$, V is an example of a *K3 surface*.

CHAPTER 8

Algebraic curves I

The main purpose of this chapter is to show that compact Riemann surfaces can be imbedded into projective space (i.e. they are *algebraic curves*), and to study some of their basic properties.

1. The Kodaira embedding

We start by showing that any compact Riemann surface can be embedded as a smooth subvariety in a projective space $\mathbb{P}_N$; this is special instance of the so-called Kodaira's embedding theorem. Together with Chow's Lemma this implies that every compact Riemann surface is algebraic.

We recall that, given two complex manifolds X and Y, we say that (Y,ι) is a submanifold of X is ι is an injective holomorphic map $Y \to X$ whose differential $\iota_{*p}: T_pY \to T_{\iota(p)}X$ is of maximal rank (given by the dimension of Y) at all $p \in Y$. In other terms, ι maps isomorphically Y onto a smooth subvariety of X.

PROPOSITION 8.1. *Any compact Riemann surface can be realized as a submanifold of $\mathbb{P}_N$ for some N.*

PROOF. Pick up a line bundle L on S such that $\deg L > \deg K + 2$ (choose an effective divisor D with enough points, and let $L = [D]$). By Serre duality we have

$$H^1(S, \mathcal{O}(L-2p)) \simeq H^0(S, \mathcal{O}(L-2p)^{-1} \otimes K)^* = 0 \tag{8.1}$$

for any $p \in S$, since $\deg(K - L + 2p) < 0$ (here $L - 2p = L \otimes [-2p]$). Consider now the exact sequence

$$0 \to \mathcal{O}(L-2p) \to \mathcal{O}(L) \xrightarrow{d_p \oplus \mathrm{ev}_p} T^*_pS \oplus L_p \to 0$$

(the morphism d_p is Cartan's differential followed by evaluation at p, while ev_p is the evaluation of sections at p). Due to (8.1) we get

$$0 \to H^0(S, \mathcal{O}(L-2p)) \to H^0(S, \mathcal{O}(L)) \xrightarrow{d_p \oplus \mathrm{ev}_p} T^*_pS \oplus L_p \to 0$$

so that $\dim|D| \geq 1$. Let $N = \dim|D|$, and let $\{s_0, \ldots, s_N\}$ be a basis of $|D|$. If U is an open neighbourhood of p, and $\phi\colon L_{|U} \to U \times \mathbb{C}$ is a local trivialization of L, the quantity

$$[\phi \circ s_0, \ldots, \phi \circ s_N] \in \mathbb{P}_N \tag{8.2}$$

does not depend on the trivialization ϕ; we have therefore established a (holomorphic) map $\iota_L \colon S \to \mathbb{P}_N$.[1] We must prove that (1) ι_L is injective, and (2) the differential $(\iota_L)_*$ never vanishes. (1) It is enough to prove that, given any two points $p, q \in S$, there is a section $s \in H^0(S, \mathcal{O}(L))$ such that $s(p) \neq \lambda s(q)$ for all $\lambda \in \mathbb{C}^*$; this in turn implied by the surjectivity of the map

$$H^0(S, \mathcal{O}(L)) \xrightarrow{r_{p,q}} L_p \oplus L_q, \qquad s \mapsto s(p) + s(q).$$

To show this we start from the exact sequence

$$0 \to \mathcal{O}(L - p - q) \to \mathcal{O}(L) \xrightarrow{r_{p,q}} L_p \oplus L_q \to 0$$

and note that in coholomology we have

$$H^0(S, \mathcal{O}(L - p - q)) \xrightarrow{r_{p,q}} L_p \oplus L_q \to H^1(S, \mathcal{O}(L - p - q)) = 0$$

since

$$H^1(S, \mathcal{O}(L - p - q)) \simeq H^0(S, \mathcal{O}(L - p - q)^{-1} \otimes K)^* = 0$$

because $\deg(L - p - q)^{-1} \otimes K = \deg K - \deg L + 2 < 0$.

(2) We shall actually show that the adjoint map $(\iota_L)^* \colon T^*_{\iota_L(p)}\mathbb{P}_N \to T^*_p S$ is surjective. The cotangent space $T^*_p S$ can be realized as the space of equivalence classes of holomorphic functions which have the same value at p (e.g; the zero value) and have a first-order contact (i.e. they have the same differential at p). Let ϕ be a trivializing map for L around p; we must find a section $s \in H^0(S, \mathcal{O}(L))$ such that $\phi \circ s(p) = 0$ (i.e. $s(p) = 0$) and $(\phi \circ s)^*$ is surjective at p. This is equivalent to showing that the map $H^0(S, \mathcal{O}(L - p)) \xrightarrow{d_p} T^*_p S$ is surjective, since $\mathcal{O}(L - p)$ is the sheaf of holomorphic sections of L vanishing at p. We consider the exact sheaf sequence

$$0 \to \mathcal{O}(L - 2p) \to \mathcal{O}(L - p) \xrightarrow{d_p} T^*_p S \to 0;$$

by Serre duality,

$$H^1(S, \mathcal{O}(L - 2p))^* \simeq H^0(S, O(-L + 2p + K)) = 0$$

so that $H^0(S, \mathcal{O}(L - p)) \xrightarrow{d_p} T^*_p S$ is surjective. □

Given any complex manifold X, one says that a line bundle L on X is *very ample* if the construction (8.2) defines an imbedding of X into $\mathbb{P}H^0(X, \mathcal{O}(L))$. A line bundle L is said to be *ample* if L^n is very ample for some natural n. A sufficient condition for a line bundle to be ample may be stated as follows (cf. [**10**]).

DEFINITION 8.2. *A (1,1) form ω on a complex manifold is said to be positive if it can be locally written in the form*

$$\omega = i\, \omega_{ij}\, dz^i \wedge d\bar{z}^j$$

[1]This map actually depends on the choice of a basis of $|D|$; however, different choices correspond to an action of the group $\mathbb{P}Gl(N+1, \mathbb{C})$ on $\mathbb{P}_N$ and therefore produce isomorphic subvarieties of $\mathbb{P}_N$.

with ω_{ij} a positive definite hermitian matrix.

PROPOSITION 8.3. *If the first Chern class of a line bundle L on a complex manifold can be represented by a positive 2-form, then L is ample.* □

While we have seen that any compact Riemann surface carries plenty of very ample line bundles, this in general is not the case: there are indeed complex manifolds which cannot be imbedded into any projective space.

A first consequence of the imbedding theorem expressed by Proposition 8.1 is that any line bundle on a compact Riemann surface comes from a divisor, i.e. $\mathrm{Div}(S)$/linear equivalence $\simeq \mathrm{Pic}(S)$.

PROPOSITION 8.4. *If M is a smooth 1-dimensional[2] analytic submanifold of projective space $\mathbb{P}_n$ (i.e. M is the imbedding of a compact Riemann surface into $\mathbb{P}_n$), and L is a line bundle on M, there is a divisor D on M such that $L = [D]$.*

PROOF. We must find a global meromorphic section of L. Let H_M be the restriction to M of the hyperplane bundle H of $\mathbb{P}_n$, and let V be the intersection of M with a hyperplane in $\mathbb{P}_n$ (so $[V] \simeq H_M$, and since V is effective, H_M has global holomorphic sections). We shall show that for a big enough integer m the line bundle $L + mH_M$ ($= L \otimes H_M^m$) has a global holomorphic section s; if t is a holomorphic section of H_M, the required meromorphic section of L is s/t^m.

We have an exact sequence

$$0 \to \mathcal{O}_M(-H_M) \xrightarrow{s} \mathcal{O}_M \to k_V \to 0$$

so that after tensoring by $L + mH_M$,

$$0 \to \mathcal{O}_M(L + (m-1)H_M) \xrightarrow{s} \mathcal{O}_M(L + mH_M) \to k_V \to 0. \tag{8.3}$$

(Here $\xrightarrow{s}$ denotes the morphism given by multiplication by s). The associated long cohomology exact sequence contains the segment

$$H^0(M, \mathcal{O}_M(L + mH_M)) \xrightarrow{r} \mathbb{C}^N \to H^1(M, \mathcal{O}_M(L + (m-1)H_M))$$

where $N = \deg V$. But

$$H^1(M, \mathcal{O}_M(L + (m-1)H_M)) \simeq H^0(M, K_M \otimes \mathcal{O}(-L - (m-1)H_M))^* = 0$$

by Serre duality and the vanishing theorem (if m is big enough, $\deg K_M \otimes \mathcal{O}(-L - (m-1)H_M) < 0$). Therefore the morphism r in (8.3) is surjective, and $H^0(M, \mathcal{O}_M(L + mH_M)) \neq 0$. □

We shall now proceed to identify compact Riemann surfaces with (smooth) algebraic curves. Given a homogeneous polynomial F on $\mathbb{C}^{n+1}$ the zero locus of F in $\mathbb{P}_n$ is by definition the projection to $\mathbb{P}_n$ of the zero locus of F in $\mathbb{C}^{n+1}$.

[2]This result is actually true whatever is the dimension of M, cf. [**10**].

DEFINITION 8.5. *A (projective) algebraic variety is a subvariety of $\mathbb{P}_n$ which is the zero locus of a finite collection of homogeneous polynomials. We shall say that an algebraic variety is smooth if it is so as an analytic subvariety of $\mathbb{P}_n$.*

The dimension of an algebraic variety is its dimension as an analytic subvariety of $\mathbb{P}_n$. A one-dimensional algebraic variety is called an *algebraic curve.*

The following fundamental result, called *Chow's lemma*, it is not hard to prove; we shall anyway omit its proof for the sake of brevity (cf. [**10**] page 167).

PROPOSITION 8.6. (Chow's lemma) *Any analytic subvariety of $\mathbb{P}_n$ is algebraic.*

EXERCISE 8.7. Use Chow's lemma to show that $H^0(\mathbb{P}_n, H^d)$ — where H is the hyperplane line bundle — can be identified with the space of homogeneous polynomials of degree d on $\mathbb{C}^{n+1}$. □

Using Chow's lemma together with the imbedding theorem (Proposition 8.1) we obtain

COROLLARY 8.8. *Any compact Riemann surface is a smooth algebraic curve.*

We switch from the terminology "compact Riemann surface" to "algebraic curve", understanding that we shall only consider smooth algebraic curves.[3]

We shall usually denote an algebraic curve by the letter C.

2. Riemann-Roch theorem

A fundamental result in the study of algebraic curves in the Riemann-Roch theorem. Let C be an algebraic curve, and denote by K its canonical bundle.[4] We denote $g = h^0(K)$, and call it the *arithmetic genus* of C (this number will be shortly identified with the topological genus of C).

PROPOSITION 8.1. (Riemann-Roch theorem) *For any line bundle L on C one has*

$$h^0(L) - h^1(L) = \deg L - g + 1.$$

PROOF. If $L = \underline{\mathbb{C}}$ is the trivial line bundle, the result holds obviously (notice that $H^1(C, \mathcal{O})^* \simeq H^0(C, K)$ by Serre duality). Exploiting the fact that $L = [D]$ for some divisor D, it is enough to prove that if the results hold for $L = [D]$, then it also holds for $L' = [D + p]$ and $L'' = [D - p]$.

In the first case we start from the exact sequence

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D + p) \to k_p \to 0$$

[3]Strictly speaking an algebraic curve consists of more data than a compact Riemann surface S, since the former requires an imbedding of S into a projective space, i.e. the choice of an ample line bundle.

[4]We introduce the following notation: if $\mathcal{E}$ is a sheaf of $\mathcal{O}_C$-modules, then $h^i(\mathcal{E}) = \dim H^i(C, \mathcal{E})$.

which gives (since $H^1(C, k_p) = 0$)

$$0 \to H^0(S, \mathcal{O}(D)) \to H^0(S, \mathcal{O}(D+p)) \to \mathbb{C} \to H^1(S, \mathcal{O}(D)) \to H^1(S, \mathcal{O}(D+p)) \to 0$$

whence

$$h^0(L') - h^1(L') = h^0(L) - h^1(L) + 1 = \deg L - g + 2 = \deg L' - g + 1.$$

Analogously for L''. $\square$

By using the Riemann-Roch theorem and Serre duality we may compute the degree of K, obtaining

$$\deg K = 2g - 2.$$

This is called the *Riemann-Hurwitz formula.* It allows us to identify g with the topological genus g_{top} of C regarded as a compact oriented 2-dimensional real manifold S. To this end we may use the *Gauss-Bonnet theorem*, which states that the integral of the Euler class of the real tangent bundle to S is the Euler characteristic of S, $\chi(S) = 2 - 2g_{\text{top}}$. On the other hand the complex structure of C makes the real tangent bundle into a complex holomorphic line bundle, isomorphic to the holomorphic tangent bundle TC, and under this identification the Euler class corresponds to the first Chern class of TC. Therefore we get $\deg K = 2g_{\text{top}} - 2$, namely,[5]

$$g = g_{\text{top}}.$$

3. Some general results about algebraic curves

Let us fix some notations and give some definitions.

3.1. The degree of a map. Let C be an algebraic curve, and ω a smooth 2-form on C, such that $\int_C \omega = 1$; the de Rham cohomology class $[\omega]$ may be regarded as an element in $H^2(C, \mathbb{Z})$, and actually provides a basis of that space, allowing an identification $H^2(C, \mathbb{Z}) \simeq \mathbb{Z}$. If $f \colon C' \to C$ is a nonconstant holomorphic map between two algebraic curves, then $f^\sharp[\omega]$ is a nonzero element in $H^2(C', \mathbb{Z})$, and there is a well defined integer n such that

$$f^\sharp[\omega] = n[\omega'],$$

where ω' is a smooth 2-form on C' such that $\int_{C'} \omega' = 1$. If $p \in C$ we have

$$\deg f^*(p) = \int_{C'} c_1(f^*[p]) = \int_{C'} f^\sharp c_1([p]) = n \int_C c_1([p]) = n,$$

so that the map f takes the value p exactly n times, including multiplicities in the sense of divisors; we may say that f covers C n times.[6] The integer n is called the *degree* if f.

[5] This need not be true if the algebraic curve C is singular. However the Riemann-Roch theorem is still true (provided we know what a line bundle on a singular curve is!) with g the arithmetic genus.

[6] Since two holomorphic functions of one variable which agree on a nondiscrete set are identical, and since C' is compact, the number of points in $f^{-1}(p)$ is always finite.

3.2. Branch points. Given again a nonconstant holomorphic map $f\colon C' \to C$, we may find a coordinate z around any $q \in C'$ and a coordinate w around $f(q)$ such that locally f is described as

$$w = z^r. \tag{8.4}$$

The number $r-1$ is called the *ramification index* of f at q (or at $p = f(q)$), and $p = f(q)$ is said to be a *branch point* if $r(p) > 1$. The *branch locus* of f is the divisor in C'

$$B' = \sum_{q \in C'} (r(q) - 1) \cdot q$$

or its image in C

$$B = \sum_{q \in C'} (r(q) - 1) \cdot f(q).$$

For any $p \in C$ we have

$$f^*(p) = \sum_{q \in f^{-1}(p)} r(q) \cdot q$$

$$\deg f^*(p) = \sum_{q \in f^{-1}(p)} r(q) = n.$$

From these formulae we may draw the following picture. If $p \in C'$ does not lie in the branch locus, then exactly n distinct points of C' are mapped to $f(p)$, which means that $f\colon C' - B' \to C - B$ is a covering map.[7] It $p \in C'$ is a branch point of ramification index $r-1$, at p exactly r sheets of the covering join together.

There is a relation between the canonical divisors of C' and C and the branch locus. Let η be a meromorphic 1-form on C, which can locally be written as

$$\eta = \frac{g(w)}{h(w)} dw.$$

From (8.4) we get

$$f^*\eta = \frac{g(z^r)}{h(z^r)} dz^r = r z^{r-1} \frac{g(z^r)}{h(z^r)} dz$$

so that

$$\operatorname{ord}_p f^*\eta = \operatorname{ord}_{f(p)} \eta + r - 1.$$

This implies the relation between divisors

$$(f^*\eta) = f^*(\eta) + \sum_{p \in C'} (r(p) - 1) \cdot p.$$

On the other hand the divisor (η) is just the canonical divisor of C, so that

$$K_{C'} = f^* K_C + B'.$$

[7] A (holomorphic) covering map $f\colon X \to Y$, with X connected, is a map such that each $p \in Y$ has a connected neighbourhood U such that $f^{-1}(U) = \cup_\alpha U_\alpha$ is the disjoint union of open subsets of X which are biholomorphic to U via f.

From this formula we may draw an interesting result. By taking degree we get

$$\deg K_{C'} = n \deg K_C + \sum_{p \in C'} (r(p) - 1);$$

by using the Riemann-Hurwitz formula we obtain

$$g(C') = n(g(C) - 1) + 1 + \tfrac{1}{2} \sum_{p \in C'} (r(p) - 1)\,. \tag{8.5}$$

EXERCISE 8.1. Prove that if $f\colon C' \to C$ is nonconstant, then $f^\sharp\colon H^0(C, K_C) \to H^0(C', K_{C'})$ is injective. (Hint: a nonzero element $\omega \in H^0(C, K_C)$ is a global holomorphic 1-form on C which is different from zero at all points in an open dense subset of C. Write an explicit formula for $f^*\omega$....) □

Both equation (8.5) and the previous Exercise imply

$$g(C') \geq g(C).$$

3.3. The genus formula for plane curves. An algebraic curve C is said to be plane if it can be imbedded into $\mathbb{P}_2$. Its image in $\mathbb{P}_2$ is the zero locus of a homogeneous polynomial; the degree d of this polynomial is by definition the degree of C. As a divisor, C is linearly equivalent to dH (indeed, since $\operatorname{Pic}(\mathbb{P}_2) \simeq \mathbb{Z}$, any divisor D on $\mathbb{P}_2$ is linearly equivalent to mH for some m; if D is effective, m is the number of intersection points between D and a generic hyperplane in $\mathbb{P}_2$, and this is given by the degree of the polynomial cutting D). [8]

We want to show that for smooth plane curves the following relation between genus and degree holds:

$$g(C) = \tfrac{1}{2}(d-1)(d-2). \tag{8.6}$$

(For singular plane curves this formula must be modified.) We may prove this equation by using the adjunction formula: C is imbedded into $\mathbb{P}_2$ as a smooth analytic hypersurface, so that

$$K_C = \iota^*(K_{\mathbb{P}_2} + C),$$

where $\iota\colon C \to \mathbb{P}_2$. Recalling that $K_{\mathbb{P}_2} = -3H$ we then have $K_C = (d-3)\iota^* H$.

[8] We are actually using here a piece of intersection theory. The fact is that any k-dimensional analytic subvariety V of an n-dimensional complex manifold X determines a homology class $[V]$ in the homology group $H_{2k}(X, \mathbb{Z})$. Assume that X is compact, and let W be an $(n-k)$-dimensional analytic subvariety of X; the homology cap product $H_{2k}(X, \mathbb{Z}) \cap H_{2n-2k}(X, \mathbb{Z}) \to \mathbb{Z}$, which is dual to the cup product in cohomology, associates the integer number $[V] \cap [W]$ with the two subvarieties. One may pick up different representatives V' and W' of $[V]$ and $[W]$ such that V' and W' meet transversally, i.e. they meet at a finite number of points; then the the number $[V] \cap [W]$ counts the intersection points (cf. [**10**] page 49).

In our case the number of intersection points is given by the number of solutions to an algebraic system, given by the equation of C in $\mathbb{P}_2$ (which has degree d) and the linear equation of a hyperplane. For a generic choice of the hyperplane, the number of solutions is d.

To carry on the computation, we notice that, as a divisor on C, $\iota^* H = C \cap H$, so that

$$\deg \iota^* H = d,$$

and

$$\deg K_C = d(d-3) = 2g-2$$

whence the formula (8.6).

EXAMPLE 8.2. Consider the affine curve in $\mathbb{C}^2$ having equation

$$y^2 = x^6 - 1\,.$$

By writing this equation in homogeneous coordinates one obtain a curve in $\mathbb{P}_2$ which is a double covering of $\mathbb{P}_1$ branched at 6 points. By the Riemann-Hurwitz formula we may compute the genus, obtaining $g = 2$. Thus the formula (8.6), which would yield $g = 10$, fails in this case. This happens because the curve is singular at the point at infinity. □

3.4. The residue formula. A meromorphic 1-form on an algebraic curve C is a meromorphic section of the canonical bundle K. Given a point $p \in C$, and a local holomorphic coordinate z such that $z(p) = 0$, a meromorphic 1-form φ is locally written around p in the form $\varphi = f\,dz$, where f is a meromorphic function. Let a be coefficient of the z^{-1} term in the Laurent expansion of f around p, and let B a small disc around p; by the Cauchy formula we have

$$a = \int_{\partial B} \varphi$$

so that the number a does not depend on the representation of φ. We call it the *residue* of φ at p, and denote it by $\mathrm{Res}_p(\varphi)$.

Given a meromorphic 1-form φ its polar divisor is $D = \sum_i p_i$, where the p_i's are the points where the local representatives of φ have poles of order 1.

PROPOSITION 8.3. *Let $D = \sum_i p_i$ be the polar divisor of a meromorphic 1-form φ. Then $\sum_i \mathrm{Res}_{p_i}(\varphi) = 0$.*

PROOF. Choose a small disc B_i around each point p_i. Then

$$\sum_i \mathrm{Res}_{p_i}(\varphi) = \int_{\partial \cup_i B_i} \varphi = -\int_{C - \cup_i B_i} d\varphi = 0\,.$$

□

3.5. The $g = 0$ case. We shall now show that all *algebraic curves of genus zero are isomorphic to the Riemann sphere* $\mathbb{P}_1$. Pick a point $p \in C$; the line bundle $[p]$ is trivial on $C - \{p\}$, and has a holomorphic section s_0 which is nonzero on $C - \{p\}$ and has a simple zero at p (this means of course that $(s_0) = p$). On the other hand, since by Serre duality $h^1(\mathcal{O}) = h^0(K) = 0$, by taking the cohomology exact sequence associated with the sequence

$$0 \to \mathcal{O} \to \mathcal{O}(p) \to k_p \to 0$$

we obtain the existence of a global section s of $[p]$ which does not vanish at p. Of course s vanishes at some other point s_0. Then the quotient $f = s/s_0$ is a global meromorphic function, with a simple pole at p and a zero at p_0.[9] By considering ∞ as the value of f at p, we may think of f as a holomorphic nonconstant map $f : C \to \mathbb{P}_1$; this map takes the value ∞ only once. Suppose that f takes the same value α at two distinct points of C; then then function $f - \alpha$ has two zeroes and only one simple pole, which is not possible. Thus f is injective. The following Lemma implies that f is surjective as well, so that it is an isomorphism.

LEMMA 8.4. *Any holomorphic map between compact complex manifolds of the same dimension whose Jacobian determinant is not everywhere zero is surjective.*

PROOF. Let $f : X \to Y$ be such a map, and let $n = \dim X = \dim Y$. Let ω be a volume form on Y; since the Jacobian determinant of f is not everywhere zero, and where it is not zero is positive, we have $\int_X f^*\omega > 0$. Assume $q \neq \operatorname{Im} f$. Since $H^{2n}(Y - \{q\}, \mathbb{R}) = 0$ (prove it by using a Mayer-Vietoris argument), we have $\omega = d\eta$ on $Y - \{q\}$. But then

$$\int_X f^*\omega = \int_{\partial X} f^*\eta = 0,$$

a contradiction. □

[9]Otherwise one can directly identify the sections of L with meromorphic functions having (only) a single pole at p, since such functions can be developed around p in the form

$$f(z) = \frac{a}{z} + g(z)\,,$$

where g is a holomorphic function. $a \in \mathbb{C}$ should be indentified with the projection of f onto k_p. (Here z is a local complex coordinate such that $z(p) = 0$.)

CHAPTER 9

Algebraic curves II

In this chapter we further study the geometry of algebraic curves. Topics covered include the Jacobian variety of an algebraic curve, some theory of elliptic curves, and the desingularization of nodal plane singular curves (this will involve the introduction of the notion of *blowup* of a complex surface at a point).

1. The Jacobian variety

A fundamental tool for the study of an algebraic curve C is its Jacobian variety $J(C)$, which we proceed now to define. Let V be an m-dimensional complex vector space, and think of it as an abelian group. A *lattice* Λ in V is a subgroup of V of the form

$$\Lambda = \left\{ \sum_{i=1}^{2m} n_i\, v_i, \quad n_i \in \mathbb{Z} \right\} \tag{9.1}$$

where $\{v_i\}_{i=1,\dots,2m}$ is a basis of V as a real vector space. The quotient space $T = V/\Lambda$ has a natural structure of complex manifold, and one of abelian group, and the two structures are compatible, i.e. T is a compact abelian complex Lie group. We shall call T a *complex torus.* Notice that by varying the lattice Λ one gets another complex torus which may not be isomorphic to the previous one (the complex structure may be different), even though the two tori are obviously diffeomorphic as real manifolds.

Example 9.1. If C is an algebraic curve of genus g, the group $\operatorname{Pic}^0(C)$, classifying the line bundles on C with vanishing first Chern class, has a structure of complex torus of dimension g, since it can be represented as $H^1(C,\mathcal{O})/H^1(C,\mathbb{Z})$, and $H^1(C,\mathbb{Z})$ is a lattice in $H^1(C,\mathcal{O})$. This is the *Jacobian variety* of C. In what follows we shall construct this variety in a more explicit way. □

Consider now a smooth algebraic curve C of genus $g \geq 1$. We shall call *abelian differentials* the global sections of K (i.e. the global holomorphic 1-forms). If ω in abelian differential, we have $d\omega = 0$ and $\omega \wedge \omega = 0$; this means that ω singles out a cohomology class $[\omega]$ in $H^1(C,\mathbb{C})$, and that

$$\int_C \omega \wedge \omega = 0. \tag{9.2}$$

Moreover, since locally $\omega = f(z)\,dz$, we have

$$i\int_C \omega\wedge\bar\omega > 0 \qquad \text{if} \qquad \omega\neq 0. \tag{9.3}$$

If γ is a smooth loop in C, and $\omega\in H^0(C,K)$, the number $\int_\gamma\omega$ depends only on the homology class of γ and the cohomology class of ω, and expresses the pairing $<,>$ between the Poincaré dual spaces $H_1(C,\mathbb{C}) = H_1(C,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}$ and $H^1(C,\mathbb{C})$.

Pick up a basis $\{[\gamma_1],\dots,[\gamma_{2g}]\}$ of the $2g$-dimensional free $\mathbb{Z}$-module $H_1(C,\mathbb{Z})$, where the γ_i's are smooth loops in C, and a basis $\{\omega_1,\dots,\omega_g\}$ of $H^0(C,K)$. We associate with these data the $g\times 2g$ matrix Ω whose entries are the numbers

$$\Omega_{ij} = \int_{\gamma_j}\omega_i .$$

This is called the *period matrix*. Its columns Ω_j are linearly independent over $\mathbb{R}$: if for all $i = 1,\dots g$

$$0 = \sum_{j=1}^{2g}\lambda_j\Omega_{ij} = \sum_{j=1}^{2g}\lambda_j\int_{\gamma_j}\omega_i$$

then also $\sum_{j=1}^{2g}\lambda_j\int_{\gamma_j}\bar\omega_i = 0$. Since $\{\omega_i,\bar\omega_i\}$ is a basis for $H^1(C,\mathbb{C})$, this implies $\sum_{j=1}^{2g}\lambda_j[\gamma_j] = 0$, that is, $\lambda_j = 0$. So the columns of the period matrix generate a lattice Λ in $\mathbb{C}^g$. The quotient complex torus $J(C) = \mathbb{C}^g/\Lambda$ is the *Jacobian variety* of C.

Define now the *intersection matrix* Q by letting $Q^{-1}_{ij} = [\gamma_j]\cap[\gamma_i]$ (this is the $\mathbb{Z}$-valued "cap" or "intersection" product in homology). Notice that Q is antisymmetric. Intrinsically, Q is an element in $\mathrm{Hom}_{\mathbb{Z}}(H^1(C,\mathbb{Z}),H_1(C,\mathbb{Z}))$. Since the cup product in cohomology is Poincaré dual to the cap product in homology, for any abelian differentials ω, τ we have

$$[\omega]\cup[\tau] = < Q[\omega],[\tau] > .$$

The relations (9.2), (9.3) can then be written in the form

$$\Omega\,Q\,\tilde\Omega = 0, \qquad i\,\Omega\,Q\,\Omega^\dagger > 0$$

(here $\tilde{}$ denotes transposition, and † hermitian conjugation). In this form they are called *Riemann bilinear relations.*

A way to check that the construction of the Jacobi variety does not depend on the choices we have made is to restate it invariantly. Integration over cycles defines a map

$$i\colon H_1(C,\mathbb{Z})\to H^0(C,K)^*, \qquad i([\gamma])(\omega) = \int_\gamma\omega .$$

This map is injective: if $i([\gamma])(\omega) = 0$ for a given γ and all ω then γ is homologous to the constant loop. Then we have the representation $J(C) = H^0(C,K)^*/H_1(C,\mathbb{Z})$.

Exercise 9.2. By regarding $J(C)$ as $H^0(C,K)^*/H_1(C,\mathbb{Z})$, show that Serre and Poincaré dualities establish an isomorphism $J(C)\simeq \mathrm{Pic}^0(C)$. □

1.1. The Abel map. After fixing a point p_0 in C and a basis $\{\omega_1, \ldots, \omega_g\}$ in $H^0(C, K)$ we define a map

$$\mu\colon C \to J(C) \tag{9.4}$$

by letting

$$\mu(p) = \left(\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_g \right).$$

Actually the value of $\mu(p)$ in $\mathbb{C}^g$ will depend on the choice of the path from p_0 to p; however, if δ_1 and δ_2 are two paths, the oriented sum $\delta_1 - \delta_2$ will define a cycle in homology, the two values will differ by an element in the lattice, and $\mu(p)$ is a well-defined point in $J(C)$.

From (9.4) we may get a group homomorphism

$$\mu\colon \operatorname{Div}(C) \to J(C) \tag{9.5}$$

by letting

$$\mu(D) = \sum_i \mu(p_i) - \sum_j \mu(q_j) \qquad \text{if} \qquad D = \sum_i p_i - \sum_j q_j.$$

All of this depends on the choice of the base point p_0, note however that if $\deg D = 0$ then the choice of p_0 is immaterial.

Proposition 9.3. (Abel's theorem) *Two divisors $D, D' \in \operatorname{Div}(C)$ are linearly equivalent if and only if $\mu(D) = \mu(D')$.*

Proof. For a proof see [**10**] page 232. □

Corollary 9.4. *The Abel map $\mu\colon C \to J(C)$ is injective.*

Proof. If $\mu(p) = \mu(q)$ by the previous Proposition $p \sim q$ as divisors, but since $g(C) \geq 1$ this implies $p = q$ (this follows from considerations analogous to those in subsection 8.3.5). □

Abel's theorem may be stated in a fancier language as follows. Let $\operatorname{Div}_d(C)$ be the subset of $\operatorname{Div}(C)$ formed by the divisors of degree d, and let $\operatorname{Pic}^d(C)$ be the set of line bundles of degree d.[1] One has a surjective map $\ell\colon = \operatorname{Div}_d(C) \to \operatorname{Pic}^d(C)$ whose kernel is isomorphic to $H^0(C, \mathcal{M}^*)/H^0(C, \mathcal{O}^*)$. Then μ filters through a morphism $\nu\colon \operatorname{Pic}^d(C) \to J(C)$, and one has a commutative diagram

$$\begin{array}{ccc} \operatorname{Div}_d(C) & \xrightarrow{\ \ell\ } & \operatorname{Pic}^d(C) \\ & \searrow^{\mu} & \downarrow{\scriptstyle\nu} \\ & & J(C) \end{array} \quad ;$$

[1]Notice that $\operatorname{Pic}^d(C) \simeq \operatorname{Pic}^{d'}(C)$ as sets for all values of d and d'.

moreover, the morphism ν is injective (if $\nu(L) = 0$, set $L = \ell(D)$ (i.e. $L = [D]$); then $\mu(L) = 0$, that is, L is trivial).

We can actually say more about the morphism ν, namely, that it is a bijection. It is enough to prove that ν is surjective for a fixed value of d (cf. previous footnote).

Let C^d be the d-fold cartesian product of C with itself. The symmetric group S_d of order d acts on C^d; we call the quotient $\mathrm{Sym}^d(C) = C^d/S_d$ the *d-fold symmetric product* of C. $\mathrm{Sym}^d(C)$ can be identified with the set of effective divisors of C of degree d. The map μ defines a map $\mu_d \colon \mathrm{Sym}^d(C) \to J(C)$.

Any local coordinate z on C yields a local coordinate system $\{z^1, \ldots, z^d\}$ on C^d,

$$z^i(p_1, \ldots, p_d) = z(p_i),$$

and the elementary symmetric functions of the coordinates z^i yield a local coordinate system for $\mathrm{Sym}^d(C)$. Therefore the latter is a d-dimensional complex manifold. Moreover, the holomorphic map

$$C^d \to J(C), \qquad (p_1, \ldots, p_d) \mapsto \mu(p_1) + \cdots + \mu(p_d)$$

is S_d-invariant, hence it descends to a map $\mathrm{Sym}^d(C) \to J(C)$, which coincides with μ_d. So the latter is holomorphic.

Exercise 9.5. Prove that $\mathrm{Sym}^d(\mathbb{P}_1) \simeq \mathbb{P}_d$. (Hint: write explicitly a morphism in homogeneous coordinates.) □

The surjectivity of ν follows from the following fact, usually called *Jacobi inversion theorem.*

Proposition 9.6. *The map* $\mu_g \colon \mathrm{Sym}^g(C) \to J(C)$ *is surjective.*

Proof. Let $D = \sum p_i \in \mathrm{Sym}^g(C)$, with all the p_i's distinct, and let z^i be a local coordinate centred in p_i; then $\{z^1, \ldots, z^g\}$ is a local coordinate system around D. If D' is near D we have

$$\frac{\partial}{\partial z^i}(\mu_g(D'))^j = \frac{\partial}{\partial z^i}\int_{p_0}^{p_i'} \omega_j = h_{ji} \tag{9.6}$$

where h_{ji} is the component of ω_j on dz^i.

Consider now the matrix

$$\begin{pmatrix} \omega_1(p_1) & \ldots & \omega_1(p_g) \\ \ldots & \ldots & \ldots \\ \omega_g(p_1) & \ldots & \omega_g(p_g) \end{pmatrix} \tag{9.7}$$

We may choose p_1 so that $\omega_1(p_1) \neq 0$, and then subtracting a suitable multiple of ω_1 from $\omega_2, \ldots, \omega_g$ we may arrange that $\omega_2(p_1) = \cdots = \omega_g(p_1) = 0$. We next choose p_2 so that $\omega_2(p_2) \neq 0$, and arrange that $\omega_3(p_2) = \cdots = \omega_g(p_3) = 0$, and so on. In this way the matrix (9.7) is upper triangular. With these choices of the abelian differentials ω_i and of the points p_i the Jacobian matrix $\{h_{ji}\}$ is upper triangular as well, and since $\omega_i(p_i) \neq 0$, its diagonal elements h_{ii} are nonzero at D, so that at the point D corresponding to our choices the Jacobian determinant is nonzero. This means that the determinant is not everywhere zero, and by Lemma 8.4 one concludes. □

PROPOSITION 9.7. *The map μ_g is generically one-to-one.*

PROOF. Let $u \in J(C)$, and choose a divisor $D \in \mu_g^{-1}(u)$. By Abel's theorem the fibre $\mu_g^{-1}(u)$ is formed by all effective divisors linearly equivalent to D, hence it is a projective space. But since $\dim J(C) = \dim \operatorname{Sym}^d(C)$ the fibre of μ_g is generically 0-dimensional, so that generically it is a point. □

This means that μ_g establishes a biholomorphic correspondence between a dense subset of $\operatorname{Sym}^d(C)$ and a dense subset of $J(C)$; such maps are called *birational.*

COROLLARY 9.8. *Every divisor of degree $\geq g$ on an algebraic curve of genus g is linearly equivalent to an effective divisor.*

PROOF. Let $D \in \operatorname{Div}_d(C)$ with $d \geq g$. We may write $D = D' + D''$ with $\deg D' = g$ and $D'' \geq 0$. By mapping D' to $J(C)$ by Abel's map and taking a counterimage in $\operatorname{Sym}^g(C)$ we obtain an effective divisor E linearly equivalent to D'. Then $E + D''$ is effective and linearly equivalent to D. □

COROLLARY 9.9. *Every elliptic smooth algebraic curve (i.e. every smooth algebraic curve of genus 1) is of the form $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$.*

PROOF. We have $J(C) = \mathbb{C}/\Lambda$, and the map μ_1 concides with μ,

$$\mu(p) = \int_{p_0}^{p} \omega.$$

By Abel's theorem, $\mu(p) = \mu(q)$ if and only if there is on C a meromorphic function f such that $(f) = p - q$; but on C there are no meromorphic functions with a single pole, so that μ is injective. μ is also surjective by Lemma 8.4 (this is a particular case of Jacobi inversion theorem), hence it is bijective. □

COROLLARY 9.10. *The canonical bundle of any elliptic curve is trivial.*

PROOF. We represent an elliptic curve C as a quotient $\mathbb{C}/\Lambda$. The (trivial) tangent bundle to $\mathbb{C}$ is invariant under the action of Λ, therefore the tangent bundle to C is trivial as well. □

Another consequence is that if C is an elliptic algebraic curve and one chooses a point $p \in C$, the curve has a structure of abelian group, with p playing the role of the identity element.

1.2. Jacobian varieties are algebraic. According to our previous discussion, any 1-dimensional complex torus is algebraic. This is no longer true for higher dimensional tori. However, the Jacobian variety of an algebraic curve is always algebraic.

Let Λ be a lattice in $\mathbb{C}^n$. Any point in the lattice singles out univoquely a cell in the lattice, and two opposite sides of the cell determine after identification a closed smooth loop in the quotient torus $T = \mathbb{C}^n/\Lambda$. This provides an identification $\Lambda \simeq H_1(T, \mathbb{Z})$.

Let now ξ be a skew-symmetric $\mathbb{Z}$-bilinear form on $H_1(T, \mathbb{Z})$. Since $\mathrm{Hom}_{\mathbb{Z}}(\Lambda^2 H_1(T, \mathbb{Z}), \mathbb{Z}) \simeq H^2(T, \mathbb{Z})$ canonically (check this isomorphism as an exercise), ξ may be regarded as a smooth complex-valued differential 2-form on T.

PROPOSITION 9.11. *The 2-form ξ which on the basis $\{e_j\}$ is represented by the intersection matrix Q^{-1} is a positive (1,1) form.*

PROOF. If $\{e_j,\, j = 1 \dots 2n\}$ are the real basis vectors in $\mathbb{C}^n$ generating the lattice, they can be regarded as basis in $H_1(T, \mathbb{Z})$. They also generate $2n$ real vector fields on T (after identifying $\mathbb{C}^n$ with its tangent space at 0 the e_j yield tangent vectors to T at the point corresponding to 0; by transporting them in all points of T by left transport one gets $2n$ vector fields, which we still denote by e_j). Let $\{z^1, \dots, z^n\}$ be the natural local complex coordinates in T; the period matrix may be described as

$$\Omega_{ij} = \int_{e_j} dz^i.$$

After writing ξ on the basis $\{dz^i, d\bar{z}^j\}$ one can check that the stated properties of ξ are equivalent to the Riemann bilinear relations.[2] $\square$

There exists on $J(C)$ a (in principle smooth) line bundle L whose first Chern class is the cohomology class of ξ. This line bundle has a connection whose curvature is (cohomologous to) $\frac{2\pi}{i}\xi$; since this form is of type (1,1), L may be given a holomorphic structure. With this structure, it is ample by Proposition 8.3.[3] This defines a projective imbedding of $J(C)$, so that the latter is algebraic.

2. Elliptic curves

Consider the curve C' in $\mathbb{C}^2$ given by an equation

$$y^2 = P(x), \tag{9.8}$$

[2]So we are not only proving that the Jacobian variety of an algebraic curve is algebraic, but, more generally, that any complex torus satisfying the Riemann bilinear relations is algebraic.

[3]We are using the fact that if a smooth complex vector bundle E on a complex manifold X has a connection whose curvature has no (0,2) part, then the complex structure of X can be "lifted" to E. Cf. [**19**]. Otherwise, we may use the fact that the image of the map c_1 in $H^2(J(C), \mathbb{Z})$ (the Néron-Severi group of $J(C)$, cf. subsection 6.5.1) may be represented as $H^2(J(C), \mathbb{Z}) \cap H^{1,1}(J(C), \mathbb{Z})$, i.e., as the group of integral 2-classes that are of Hodge type (1,1). The class of ξ is clearly of this type.

where x, y are the standard coordinates in $\mathbb{C}^2$, and $P(x)$ is a polynomial of degree 3. By writing the equation (9.8) in homogeneous coordinates, C' may be completed to an algebraic curve C imbedded in $\mathbb{P}_2$ — a cubic curve in $\mathbb{P}_2$. Let us assume that C is smooth. By the genus formula we see that C is an elliptic curve.

EXERCISE 9.1. Show that $\omega = dx/y$ is a nowhere vanishing abelian differential on C. After proving that all elliptic curves may be written in the form (9.8), this provides another proof of the triviality of the canonical bundle of an elliptic curve. (Hint: around each branch point, $z = \sqrt{P(x)}$ is a good local coordinate...)

The equation (9.8) moreover exhibits C as a cover of $\mathbb{P}_1$, which is branched of order 2 at the points where $y = 0$ and at the point at infinity. One also checks that the point at infinity is a smooth point. We want to show that every smooth elliptic curve can be realized in this way.

So let C be a smooth elliptic curve. If we fix a point p in C and consider the exact sequence of sheaves on C

$$0 \to \mathcal{O}(p) \to \mathcal{O}(2p) \to k_p \to 0\,,$$

proceeding as usual (Serre duality and vanishing theorem) one shows that $H^0(C, \mathcal{O}(2p))$ is nonzero. A nontrivial section f can be regarded as a global meromorphic function holomorphic in $C - \{p\}$, having a double pole at p. Moreover we fix a nowhere vanishing holomorphic 1-form ω (which exists because K is trivial). We have

$$\mathrm{Res}_p(f\omega) = 0\,.$$

We realize C as $\mathbb{C}/\Lambda$; these singles out a complex coordinate z on the open subset of C corresponding to the fundamental cell of the lattice Λ. Then we may choose $\omega = dz$, and f may be chosen in such a way that

$$f(z) = \frac{1}{z^2} + O(z)\,.$$

On the other hand, the meromorphic function df/ω is holomorphic outside p, and has a triple pole at p. We may choose constants a, b, c such that

$$\tilde{f} = a\frac{df}{\omega} + bf + c = \frac{1}{z^3} + O(z)\,.$$

The line bundle $\mathcal{O}(3p)$ is very ample, i.e., its complete linear system realizes the Kodaira imbedding of C into projective space. By Riemann-Roch and the vanishing theorem we have $h^0(3p) = 3$, so that C is imbedded into $\mathbb{P}_2$. To realize explicitly the imbedding we may choose three global sections corresponding to the meromorphic functions 1, f, $\tilde{f}$. We shall see that these are related by a polynomial identity, which then expresses the equation cutting out C in $\mathbb{P}_2$.

We indeed have, for suitable constants α, β, γ,

$$\tilde{f}^2 = \frac{1}{z^6} + \frac{\alpha}{z^2} + O(\frac{1}{z}), \qquad f^3 = \frac{1}{z^6} + \frac{\beta}{z^3} + \frac{\gamma}{z^2} + O(\frac{1}{z})\,,$$

so that, setting $\delta = \alpha - \beta$,

$$\tilde{f}^2 + \beta \tilde{f} - f^3 + \delta f = O(\frac{1}{z}) .$$

So the meromorphic function in the left-hand side is holomorphic away from p, and has at p a simple pole. Such a function must be constant, otherwise it would provide an isomorphism of C with the Riemann sphere.

Thus C may be described as a locus in $\mathbb{P}_2$ whose equation in affine coordinates is

$$y^2 + \beta y = x^3 - \delta x + \epsilon \tag{9.9}$$

for a suitable constant ϵ. By a linear transformation on y we may set $\beta = 0$, and then by a linear transformation of x we may set the two roots of the polynomial in the right-hand side of (9.9) to 0 and 1. So we express the elliptic curve C in the standard form (Weierstraß representation)[4]

$$y^2 = x(x-1)(x-\lambda) . \tag{9.10}$$

Exercise 9.2. Determine for what values of the parameter λ the curve (9.10) is smooth.

We want to elaborate on this construction. Having fixed the complex coordinate z, the function f is basically fixed as well. We call it the *Weierstraß $\mathcal{P}$-function.* Its derivative is $\mathcal{P}' = -2\tilde{f}$. Notice that $\mathcal{P}$ cannot contain terms of odd degree in its Laurent expansion, otherwise $\mathcal{P}(z) - \mathcal{P}(-z)$ would be a nonconstant holomorphic function on C. So

$$\begin{aligned}
\mathcal{P}(z) &= \frac{1}{z^2} + az^2 + bz^4 + O(z^6) \\
\mathcal{P}'(z) &= -\frac{2}{z^3} + 2az + 4bz^3 + O(z^5) \\
(\mathcal{P}(z))^3 &= \frac{1}{z^6} + \frac{3a}{z^2} + 3b + O(z^2) \\
(\mathcal{P}'(z))^2 &= \frac{4}{z^6} - \frac{8a}{z^2} - 16b + O(z)
\end{aligned}$$

for suitable constants a, b. From this we see that $\mathcal{P}$ satisfies the condition

$$(\mathcal{P}')^2 - 4\mathcal{P}^3 + 20\, a = \text{constant}'$$

one usually writes g_2 for $20\, a$ and g_3 for the constant in the right-hand side.

In terms of this representation we may introduce a map $j \colon \mathcal{M}_1 \to \mathbb{C}$, where $\mathcal{M}_1$ is the set of isomorphism classes of smooth elliptic curves (the moduli space of genus one

[4]Even though the Weierstraß representation only provides the equation of the affine part of an elliptic curve, the latter is nevertheless completely characterized. It is indeed true that any affine plane curve can be uniquely extended to a compact curve by adding points at infinity, as one can check by elementary considerations.

curves) [5]

$$j(C) = \frac{1728\, g_2^3}{g_2^3 - 27\, g_3^2}\,.$$

One shows that this map is bijective; in particular $\mathcal{M}_1$ gets a structure of complex manifold. The number $j(C)$ is called the *j-invariant* of the curve C. We may therefore say that the moduli space $\mathcal{M}_1$ is isomorphic to $\mathbb{C}$. [6]

EXERCISE 9.3. Write the j-invariant as a function of the parameter λ in equation (9.10). Do you think that λ is a good coordinate on the moduli space $\mathcal{M}_1$?

The holomorphic map

$$\psi\colon C \to \mathbb{P}_2, \qquad z \mapsto [1, \mathcal{P}(z), \mathcal{P}'(z)]$$

imbeds C into $\mathbb{P}_2$ as the cubic curve cut out by the polynomial

$$F = y^2 - 4x^3 + g_2 x + g_3$$

(we use the same affine coordinates as in the previous representation). Since $\tilde{f} = df/\omega$ we have

$$\omega = \frac{dx}{y}$$

and the inverse of ψ is the Abel map,[7]

$$\psi^{-1}(p) = \int_{p_0}^{p} \frac{dx}{y} \quad \bmod \Lambda$$

having chosen p_0 at the point at infinity, $p_0 = \psi(0) = [0, 0, 1]$.

In terms of this construction we may give an elementary geometric visualization of the group law in an elliptic curve. Let us choose p_0 as the identity element in C. We shall denote by $\bar{p}$ the element $p \in C$ regarded as a group element (so $\bar{p}_0 = 0$). By Abel's theorem, Proposition 9.3, we have that

$$\bar{p}_1 + \bar{p}_2 + \bar{p}_3 = 0 \quad \text{if and and only if} \quad p_1 + p_2 + p_3 \sim 3\, p_0$$

(indeed one may think that $\bar{p} = \mu(p)$, and one has $\mu(p_1 + p_2 + p_3 - 3\, p_0) = 0$).

Let $M(x, y) = mx + ny + q$ be the equation of the line in $\mathbb{P}_2$ through the points p_1, p_2, and let p_4 be the further intersection of this line with $C \subset \mathbb{P}_2$. The function $M(z) = M(\mathcal{P}(z), \mathcal{P}'(z))$ on C vanishes (of order one) only at the points p_1, p_2, p_4, and has a pole at p_0. This pole must be of order three, so that the divisor of $M(z)$ is $p_1 + p_2 + p_4 - 3\, p_0$, i.e; $p_1 + p_2 + p_4 - 3\, p_0 \sim 0$.

[5] The fancy coefficient 1728 comes from arithmetic geometry, where the theory is tailored to work also for fields of characteristic 2 and 3.

[6] By uniformization theory one can also realize this moduli space as a quotient $\mathbb{H}/Sl(2, \mathbb{Z})$, where $\mathbb{H}$ is the upper half complex plane. This is not contradictory in that the quotient $\mathbb{H}/Sl(2, \mathbb{Z})$ is biholomorphic to $\mathbb{C}$! (Notice that on the contrary, $\mathbb{H}$ and $\mathbb{C}$ are not biholomorphic). Cf. [**11**].

[7] One should bear in mind that we have identified C with a quotient $\mathbb{C}/\Lambda$.

If $p_1 + p_2 + p_3 \sim 3\,p_0$, then $p_3 \sim p_4$, so that $p_3 = p_4$, and p_1, p_2, p_3 are collinear. *Vice versa,* if p_1, p_2, p_3 are collinear, $p_1 + p_2 + p_3 - 3\,p_0$ is the divisor of the meromorphic function M, so that $p_1 + p_2 + p_3 - 3\,p_0 \sim 0$. We have therefore shown that $\bar{p}_1 + \bar{p}_2 + \bar{p}_3 = 0$ *if and only if* p_1, p_2, p_3 *are collinear points in* $\mathbb{P}_2$.

EXAMPLE 9.4. Let C be an elliptic curve having a Weierstraß representation $y^2 = x^3 - 1$. C is a double cover of $\mathbb{P}_1$, branched at the three points

$$p_1 = (1, 0), \qquad p_2 = (\alpha, 0), \qquad p_3 = (\alpha^2, 0)$$

(where $\alpha = e^{2\pi i/3}$) and at the point at infinity p_0. The points p_1, p_2, p_3 are collinear, so that $\bar{p}_1 + \bar{p}_2 + \bar{p}_3 = 0$.

The two points $q_1 = (0, i)$, $q_2 = (0, -i)$ lie on C. The line through q_1, q_2 intersects C at the point at infinity, as one may check in homogeneous coordinates. So in this case the elements $\bar{q}_1$, $\bar{q}_2$ are one the inverse of the other, and $q_1 + q_2 \sim 2\,p_0$. More generally, if $q \in C$ is such that $\bar{q} = -\bar{p}$, then $p + q \sim 2\,p_0$, and q is the further intersection of C with the line going through p, p_0; if $p = (a, b)$, then $q = (a, -b)$. So the branch points p_i are 2-torsion elements in the group, $2\,\bar{p}_i = 0$. □

3. Nodal curves

In this section we show how (plane) curve singularities may be resolved by a procedure called *blowup.*

3.1. Blowup. Blowing up a point in a variety[8] means replacing the point with all possible directions along which one can approach it while moving in the variety. We shall at first consider the blowup of $\mathbb{C}^2$ at the origin; since this space is 2-dimensional, the set of all possible directions is a copy of $\mathbb{P}_1$. Let x, y be the standard coordinates in $\mathbb{C}^2$, and w_0, w_1 homogeneous coordinates in $\mathbb{P}_1$. The blowup of $\mathbb{C}^2$ at the origin is the subvariety Γ of $\mathbb{C}^2 \times \mathbb{P}_1$ defined by the equation

$$x\,w_1 - y\,w_0 = 0\,.$$

To show that Γ is a complex manifold we cover $\mathbb{C}^2 \times \mathbb{P}_1$ with two coordinate charts, $V_0 = \mathbb{C}^2 \times U_0$ and $V_1 = \mathbb{C}^2 \times U_1$, where U_0, U_1 are the standard affine charts in $\mathbb{P}_1$, with coordinates $(x, y, t^0 = w_1/w_0)$ and $(x, y, t^1 = w_0/w_1)$. Γ is a smooth hypersurface in $\mathbb{C}^2 \times \mathbb{P}_1$, hence it is a complex surface. On the other hand if we put homogeneous coordinates (v_0, v_1, v_2) in $\mathbb{C}^2$, then Γ can be regarded as a open subset of the quadric in $\mathbb{P}_2 \times \mathbb{P}_1$ having equation $v_1\,w_1 - v_2\,w_0 = 0$, so that Γ is actually *algebraic.*

[8]Our treatment of the blowup of an algebraic variety is basically taken from [**1**].

Since Γ is a subset of $\mathbb{C}^2 \times \mathbb{P}_1$ there are two projections

$$\begin{array}{ccc} \Gamma & \xrightarrow{\pi} & \mathbb{P}_1 \\ {\scriptstyle\sigma}\downarrow & & \\ \mathbb{C}^2 & & \end{array} \tag{9.11}$$

which are holomorphic. If $p \in \mathbb{C}^2 - \{0\}$ then $\sigma^{-1}(p)$ is a point (which means that there is a unique line through p and 0), so that

$$\sigma \colon \Gamma - \sigma^{-1}(0) \to \mathbb{C}^2 - \{0\}$$

is a biholomorphism.[9] On the contrary $\sigma^{-1}(0) \simeq \mathbb{P}_1$ is the set of lines through the origin in $\mathbb{C}^2$.

The fibre of π over a point $(w_0, w_1) \in \mathbb{P}_1$ is the line $x\,w_1 - y\,w_0 = 0$, so that π makes Γ into the total space of a line bundle over $\mathbb{P}_1$. This bundle trivializes over the cover $\{U_0, U_1\}$, and the transition function $g \colon U_0 \cap U_1 \to \mathbb{C}^*$ is $g(w_0, w_1) = w_0/w_1$, so that the line bundle is actually the tautological bundle $\mathcal{O}_{\mathbb{P}_1}(-1)$.

This construction is local in nature and therefore can be applied to any complex surface X (two-dimensional complex manifold) at any point p. Let U be a chart around p, with complex coordinates (x, y). By repeating the same construction we get a complex manifold U' with projections

$$\begin{array}{ccc} U' & \xrightarrow{\pi} & \mathbb{P}_1 \\ {\scriptstyle\sigma}\downarrow & & \\ U & & \end{array}$$

and

$$\sigma \colon U' - \sigma^{-1}(p) \to U - \{p\}$$

is a biholomorphism, so that one can replace U by U' inside X, and get a complex manifold X' with a projection $\sigma \colon X' \to X$ which is a biholomorphism outside $\sigma^{-1}(p)$. The manifold X' is the *blowup* of X at p. The inverse image $E = \sigma^{-1}(p)$ is a divisor in X', called the *exceptional divisor,* and is isomorphic to $\mathbb{P}_1$. The construction of the blowup Γ shows that X' is algebraic if X is.

EXAMPLE 9.1. The blowup of $\mathbb{P}_2$ at a point is an algebraic surface X_1 (an example of a *Del Pezzo surface*); the manifold Γ, obtained by blowing up $\mathbb{C}^2$ at the origin, is biholomorphic to X_1 minus a projective line (so X_1 is *a compactification* of Γ). □

3.2. Transforms of a curve. Let C be a curve in $\mathbb{C}^2$ containing the origin. We denote as before Γ the blowup of $\mathbb{C}^2$ at the origin and make reference to the diagram (9.11). Notice that the inverse image $\sigma^{-1}(C) \subset \Gamma$ contains the exceptional divisor E, and that $\sigma^{-1}(C) \setminus E$ is isomorphic to $C - \{0\}$.

[9] So, according to a terminology we have introduce in a previous chapter, the map σ is a *birational morphism.*

DEFINITION 9.2. *The curve $\sigma^{-1}(C) \subset \Gamma$ is the total transform of C. The curve obtained by taking the topological closure of $\sigma^{-1}(C) \setminus E$ in Γ is the strict transform of C.*

We want to check what points are added to $\sigma^{-1}(C) \setminus E$ when taking the topological closure. To this end we must understand what are the sequences in $\mathbb{C}^2$ which converge to 0 that are lifted by σ to convergent sequences. Let $\{p_k = (x_k, y_k)\}_{k\in\mathbb{N}}$ be a sequence of points in $\mathbb{C}^2$ converging to 0; then $\sigma^{-1}(x_k, y_k)$ is the point (x_k, y_k, w_0, w_1) with $x_k\, w_1 - y_k\, w_0 = 0$. Assume that for k big enough one has $w_0 \neq 0$ (otherwise we would assume $w_1 \neq 0$ and would make a similar argument). Then $w_1/w_0 = y_k/x_k$, and $\{\sigma^{-1}(p_k)\}$ converges if and only if $\{y_k/x_k\}$ has a limit, say h; in that case $\{\sigma^{-1}(p_k)\}$ converges to the point $(0, 0, 1, h)$ of E. This means that the lines r_k joining 0 to p_k approach the limit line r having equation $y = hk$. So a sequence $\{p_k = (x_k, y_k)\}$ convergent to 0 lifts to a convergent sequence in Γ if and only if the lines r_k admit a limit line r; in that case, the lifted sequence converges to the point of E representing the line r.

The strict transform C' of C meets the exceptional divisor in as many points as are the directions along which one can approach 0 on C, namely, as are the tangents at C at 0. So, if C is smooth at 0, its strict transform meets E at one point. Every intersection point must be counted with its multiplicity: if at the point 0 the curve C has m coinciding tangents, then the strict transform meets the exceptional divisor at a point of multiplicity m.

DEFINITION 9.3. *Let the (affine plane) curve C be given by the equation $f(x, y) = 0$. We say that C has multiplicity m at 0 if the Taylor expansion of f at 0 starts at degree m.*

This means that the curve has m tangents at the point 0 (but some of them might coincide). By choosing suitable coordinates one can apply this notion to any point of a plane curve.

EXAMPLE 9.4. A curve is smooth at 0 if and only if its multiplicity at 0 is 1. The curves $xy = 0$, $y^2 = x^2$ and $y^2 = x^3$ have multiplicity 2 at 0. The first two have two distinct tangents at 0, the third has a double tangent. □

If the curve C has multiplicity m at 0 than it has m tangents at 0, and its strict transform meets the exceptional divisor of Γ at m points (notice however that these points are all distinct only if the m tangents are distincts).

DEFINITION 9.5. *A singular point of a plane curve C is said to be* nodal *if at that point C has multiplicity 2, and the two tangents to the curve at that point are distinct.*

EXERCISE 9.6. With reference to equation (9.10), determine for what values of λ the curve has a nodal singularity.

EXERCISE 9.7. Show that around a nodal singularity a curve is isomorphic to an open neighbourhood of the origin of the curve $xy = 0$ in $\mathbb{C}^2$.

EXAMPLE 9.8. (Blowing up a nodal singularity.) We consider the curve $C \subset \mathbb{C}^2$ having equation $x^3 + x^2 - y^2 = 0$. This curve has multiplicity 2 at the origin, and its two tangents at the origin have equations $y = \pm x$. So C has a nodal singularity at the origin. We recall that Γ is described as the locus

$$\{(u, v, w_0, w_1) \in \mathbb{C}^2 \times \mathbb{P}_1 \mid u\, w_0 = v\, w_1\}\,.$$

The projection σ is described as

$$\begin{cases} x = u \\ y = u\, w_0/w_1 \end{cases} \qquad \begin{cases} x = v\, w_1/w_0 \\ y = v \end{cases} \tag{9.12}$$

in $\Gamma \cap V_1$ and $\Gamma \cap V_0$, respectively. By substituting the first of the representations (9.12) into the equation of C we obtain the equation of the restriction of the total transform to $\Gamma \cap U_1$:

$$u^2(u + 1 - t^2) = 0$$

where $t = w_0/w_1$. $u^2 = 0$ is the equation of the exceptional divisor, so that the equation of the strict transform is $u + 1 - t^2 = 0$. By letting $u = 0$ we obtain the points $(0, 0, 1, 1)$ and $(0, 0, 1, -1)$ as intersection points of the strict transform with the exceptional divisor. By substituting the second representation in eq. (9.12) we obtain the equation of the total transform in $\Gamma \cap U_0$; the strict transform now has equation $t^3\, v + t^2 - 1$, yielding the same intersection points.

The total transform is a reducible curve, with two irreducible components which meet at two points.

EXERCISE 9.9. Repeat the previous calculations for the nodal curve $xy = 0$. In particular show that the total transform is a reducible curve, consisting of the exceptional divisor and two more genus zero components, each of which meets the exceptional divisor at a point.

EXAMPLE 9.10. (The cusp) Let C be curve with equation $y^2 = x^3$. This curve has multiplicity 2 at the origin where it has a double tangent.[10] Proceeding as in the previous example we get the equation $v\, t^3 = 1$ for C' in $\Gamma \cap V_0$, so that C' does not meet E in this chart. In the other chart the equation of C' is $t^2 = u$, so that C' meets E at the point $(0, 0, 0, 1)$; we have one intersection point because the two tangents to C at the origin coincide.

The strict transform is an irreducible curve, and the total transform is a reducible curve with two components meeting at a (double) point. □

[10] Indeed this curve can be regarded as the limit for $\alpha \to 0$ of the family of nodal curves $x^3 + \alpha^2\, x^2 - y^2 = 0$, which at the origin are tangent to the two lines $y = \pm\alpha\, x$.

3.3. Normalization of a nodal plane curve. It is clear from the previous examples that the strict transform of a plane nodal curve C (i.e., a plane curve with only nodal singularities) is again a nodal curve, with one less singular point. Therefore after a finite number of blowups we obtain a smooth curve N, together with a birational morphism $\pi\colon N \to C$. N is called the *normalization* of C.

EXAMPLE 9.11. Let us consider the smooth curve C_0 in $\mathbb{C}^2$ having equation $y^2 = x^4 - 1$. Projection onto the x-axis makes C_0 into a double cover of $\mathbb{C}$, branched at the points $(\pm 1, 0)$ and $(\pm i, 0)$. The curve C_0 can be completed to a projective curve simply by writing its equation in homogeneous coordinates (w_0, w_1, w_2) and considering it as a curve C in $\mathbb{P}_2$; we are thus compactifying C_0 by adding a point at infinity, which in this case is not a branch point. The equation of C is

$$w_0^2\, w_2^2 - w_1^4 + w_0^4 = 0\,.$$

This curve has genus 1 and is singular at infinity (as one could have alredy guessed since the genus formula for smooth plane curves does not work); indeed, after introducing affine coordinates $\xi = w_0/w_2$, $\eta = w_1/w_2$ (in this coordinates the point at infinity on the x-axis is $\eta = \xi = 0$) we have the equation

$$\xi^2 = \eta^4 - \xi^4$$

showing that C is indeed singular at infinity. One can redefine the coordinates ξ, η so that C has equation

$$(\xi - \eta^2)(\xi + \eta^2) = 0$$

showing that C is a nodal curve. Then it can be desingularized as in Example 9.8. □

A genus formula. We give here, without proof, a formula which can be used to compute the genus of the normalization N of a nodal curve C. Assume that N has t irreducible components $N_1, \dots, N_t$, and that C has δ singular points. Then:

$$g(C) = \sum_1^t g(N_i) + 1 - t + \delta.$$

For instance, by applying this formula to Example 9.8, we obtain that the normalization is a projective line.

Bibliography

[1] F. Acquistapace, F. Broglia & F. Lazzeri, *Topologia delle superficie algebriche in* $\mathbb{P}_3(\mathbb{C})$, Quaderni dell'Unione Matematica Italiana **13**, Pitagora, Bologna 1979.

[2] C. Bartocci, U. Bruzzo & D. Hernández Ruipérez, *The Geometry of Supermanifolds,* Kluwer Acad. Publishers, Dordrecht 1991.

[3] R. Bott & L. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, New York 1982.

[4] G.E. Bredon, *Sheaf theory,* McGraw-Hill, New York 1967.

[5] J. Dieudonné, *Une généralisation des espaces compacts,* J. Math. Pures Appl. **23** (1944) 65–76.

[6] R. Godement, *Théorie des Faisceaux,* Hermann, Paris 1964.

[7] B. Gray, *Homotopy theory,* Academic Press, New York 1975.

[8] M.J. Greenberg, *Lectures on Algebraic Topology,* Benjamin, New York 1967.

[9] A. Grothendieck, *Sur quelques points d'algèbre homologique,* Tôhoku Math. J. **9** (1957) 119-221.

[10] P. A. Griffiths & J. Harris, *Principles of Algebraic Geometry*, Wiley, New York 1994.

[11] R. Hain, *Moduli of Riemann surfaces, transcendental aspects.* ICTP Lectures Notes, School on Algebraic Geometry, L. Göttsche ed., ICTP, Trieste 2000 (available from `http://www.ictp.trieste.it/ pub_off/lectures/`).

[12] R. Hartshorne, *Algebraic geometry,* Springer-Verlag, New York 1977.

[13] P. J. Hilton & U. Stammbach, *A Course in Homological Algebra,* Springer-Verlag, New York 1971.

[14] F. Hirzebruch, *Topological Methods in Algebraic Geometry,* Springer-Verlag, Berlin 1966.

[15] S.-T. Hu, *Homotopy theory,* Academic Press, New York 1959.

[16] D. Husemoller, *Fibre Bundles,* McGraw-Hill, New York 1966.

[17] J. Kelley, *General Topology,* Springer-Verlag, Berlin Heidelberg New York xxxx.

[18] S. Kobayashi, *Differential Geometry of Complex Vector Bundles,* Princeton University Press, Princeton 1987.

[19] S. Kobayashi & K. Nomizu, *Foundations of Differential Geometry, Vol. I,* Interscience Publ., New York 1963.

[20] W.S. Massey, *Exact couples in algebraic topology, I, II,* Ann. Math. **56** (1952), 3363-396; **57** (1953), 248-286.

[21] E.H. Spanier,*Algebraic topology,* Corrected reprint, Springer-Verlag, New York-Berlin 1981.

[22] S. Sternberg, *Lectures on differential geometry,* Chelsea, New York 1983.

[23] B.R. Tennison, *Sheaf theory,* London Math. Soc. Lect. Notes Ser. **20**, Cambridge Univ. Press, Cambridge 1975.

[24] B.L. Van der Waerden, *Algebra,* Frederick Ungar Publ. Co., New York 1970.

[25] R.O. Wells Jr., *Differential analysis on complex manifolds*, Springer-Verlag, New York-Berlin 1980.