Categorical Heisenberg Actions on Hilbert Schemes of points

(with T Licata)

Background. S = smooth surface.

$S^{[n]}$ = Hilbert scheme of n pts.

$S \rightsquigarrow \mathcal{H}_S$ = heisenberg algebra.

eg $S = \mathbb{C}^2 \rightsquigarrow \mathcal{H}_{\mathbb{C}^2}$ = gens. $a(m)$ $m \in \mathbb{Z}^{\times}$

rels $[a(m), a(n)] = \delta_{m-n,0}\, n$

Fact (Nakajima, Grojnowski) $\exists$ action of $\mathfrak{h}_S$ on $\bigoplus_{n\geq 0} H^*(S^{[n]}, \mathbb{C})$.

Idea: $W = \{(I_1, I_2) : I_2 \subset I_1 : \underset{\substack{\| \\ k\cdot \text{ fixed pt}}}{\mathrm{supp}(I_1/I_2)}\}$

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ S^{[n]} & & S^{[n+k]} \end{tikzcd}$$

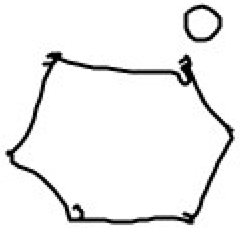
<u>**Question:**</u> Lift to K-theory and $DCoh(S^{[n]})$.

today: lift action in case

$S_\Gamma = \widehat{\mathbb{C}^2/\Gamma}$ $\longleftarrow$ min resolution

$\Gamma \overset{\text{finite}}{\subset} SL_2\mathbb{C}$.

<u>Recall</u>. $\Gamma \subset SL_2\mathbb{C} \overset{\text{McKay}}{\longleftrightarrow}$ affine Dynkin diagram I_Γ

ex. $\Gamma = \mathbb{Z}/n\mathbb{Z}$. $\longleftrightarrow$  $\Big\}$ n vertices.

$$\begin{tikzcd}
S_\Gamma = \widehat{\mathbb{C}^2/\Gamma} \arrow[d, "\pi"] & \pi^{-1}(0) \arrow[l] \arrow[d] \\
\mathbb{C}^2/\Gamma & 0 \arrow[l]
\end{tikzcd}$$

$\pi^{-1}(0) =$ chain of $\mathbb{P}^1$'s indexed by $i \in I_\Gamma \setminus 0$.

eg $\Gamma = \mathbb{Z}/4 \rightsquigarrow \pi^{-1}(0) = \begin{cases} \mathbb{P}^1 = E_1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 = E_3 \end{cases}$

denote $E_i = \mathbb{P}^1$ indexed by $i \in I_\Gamma \setminus 0$.

$$W_i^n = \{ (I_1, I_2) : I_2 \subset I_1 : \operatorname{supp}(I_1/I_2) \in E_i \} \xrightarrow{p} E_i$$

$$\begin{array}{ccc} & W_i^n & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ S_\Gamma^{[n]} & & S_\Gamma^{[n+1]} \end{array}$$

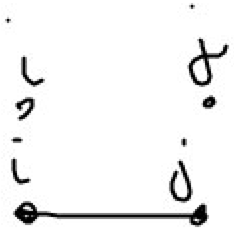
Define: $D^{\mathbb{C}^\times}Coh(S_\Gamma^{[n]}) \underset{Q_i}{\overset{P_i}{\rightleftarrows}} D^{\mathbb{C}^\times}Coh(S_\Gamma^{[n+1]})$

$$P_i(\cdot) = R\pi_{2*}\left(L\pi_1^*(\cdot) \otimes p^*\mathcal{O}_{E_i}(-1)\right)$$
$$Q_i(\cdot) = R\pi_{1*}\left(L\pi_2^*(\cdot) \otimes p^*\mathcal{O}_{E_i}(-1)\right)$$

__Lemma__ $\quad P_i P_j \simeq P_j P_i \qquad i, j \in I_\Gamma$

$Q_i Q_j \simeq Q_j Q_i$

$$Q_j P_i \cong P_i Q_j \oplus \begin{cases} 0 \\ \mathrm{Id} \\ \mathrm{Id}[1]\langle -1\rangle \oplus \mathrm{Id}[-1]\langle 1\rangle = \bigoplus_{[2]} \mathrm{Id} \end{cases}$$



$i = j$

P_i and Q_i are biadjoint (up to shift)

Higher Structure

What are natural transformations?

$\mathcal{F}$ $\mathrm{End}(P_i^n) = $?

<u>Lemma:</u> $\mathbb{C}[S_n] \hookrightarrow \mathrm{End}(P_i^n)$.

symm. group

Cor for any partition $\lambda \vdash n$.

get $P_i^{(\lambda)} = \mathrm{Im}(e_\lambda)$ ← min. idemp. in $\mathbb{C}[S_n]$ corresp. to λ.

Lemma 3 $Q_i^{(n)} P_i^{(m)} = \bigoplus_{k \geq 0} \bigoplus_{[k]!} P_i^{(m-k)} Q_i^{(n-k)}$

eg.

eg $$Q_i^{(2)} P_i^{(2)} \simeq P_i^{(2)} Q_i^{(2)} \oplus \begin{array}{c} P_i Q_i[1]\langle -1\rangle \\ P_i Q_i[-1]\langle 1\rangle \end{array} \oplus \underbrace{\begin{array}{c} \mathrm{Id}[2]\langle -2\rangle \\ \mathrm{Id} \\ \mathrm{Id}[-2]\langle 2\rangle \end{array}}_{\bigoplus_{[3]} \mathrm{Id}}$$

<u>Summary</u>. This gives cat action of $\mathcal{H}_\Gamma$ on $\bigoplus_{[n]} D^{\mathbb{C}^\times}\mathrm{Coh}(S_\Gamma^{[n]})$

where $\mathcal{H}_\Gamma$: gens $P_i^{(m)}$ $Q_i^{(m)}$ $i \in I_\Gamma$ $m \geq 0$

quantum heisenberg (has t).

rels last lemma.

$t \longleftrightarrow [1]\langle -1 \rangle.$

(half) Vertex ops

$\mathcal{H}_q$ not in usual presentation.

↳ "usual" presentation:

gens: $a_i(m)$ $\quad m \in \mathbb{Z}^\times \quad i \in I_\Gamma$

rels $\quad [a_i(m), a_j(n)] = \delta_{m+n,0}\,[n\langle i,j\rangle]\,\frac{[n]}{n}$

where $\langle i,j\rangle = \begin{cases} 0 & \text{if } \bullet \quad \bullet \\ -1 & \text{if } \bullet\!-\!\bullet \\ 2 & \text{if } i=j \end{cases}$

then half vertex op

$$\begin{cases} \displaystyle\sum_{n\geq 0} P_i^{(n)} z^n = \exp\left(\sum_{m\geq 1} \frac{a_i(-m)}{[m]} z^m\right) \\ \displaystyle\sum_{n\geq 0} Q_i^{(n)} z^n = \exp\left(-\sum \frac{a_i(m)}{[m]} z^m\right) \end{cases}$$

Frenkel-Kac-Segal

define $E_i = \exp\left(+\sum \frac{a_i(-m)}{[m]} z^m\right) \exp\left(-\sum \frac{a_i(m)}{[m]} z^m\right)$

F_i = similar

E_i and F_i given action of

$\mathcal{U}_q(\mathfrak{g}_\Gamma)$ (eg $\Gamma = \mathbb{Z}/n\mathbb{Z}$ then $\mathfrak{g}_\Gamma = \widehat{\mathfrak{sl}}_n$)

quantum affine algebras.

In our cat. setting you have

finite cpx

$$E_i = \left[\cdots \to P_i^{(2)} Q_i^{(1^{n+2})} \to P_i Q_i^{(1^{n+1})} \to Q_i^{(1^n)} \right]$$

F_i = similar

<u>Cor</u> get cat. action of $U_t(\hat{\mathfrak{g}})$.
on $\bigoplus D^{\mathbb{C}^\times}\mathrm{Coh}(S_\Gamma^{[n]})$

<u>Summary</u>: Categorification $\longleftrightarrow$ study of natural transf.