Parabolic Refined Invariants and Macdonald polynomials

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Parabolic
Higgs
bundles

Motivic
(refined)
DT invariants

String theory
framework

Hausel Letellier
Rodriguez-Villegas (HLRV)

nested H-scheme

Md polynomials

HLRV Formula

$\vec{\mu} = (\mu_1, \ldots, \mu_k)$

C: smooth proj curve / $\mathbb{C}$

$D = p_1 + \ldots + p_k$

$p_i \rightsquigarrow \gamma_i \in \pi_1(C \setminus D)$

$C_i \subset GL(r)$ semisimple conj cls

$\mu_i \vdash r$

$$\left\{ \varphi : \pi_1(C \setminus D) \to GL(r) \;\middle|\; \varphi(\gamma_i) \in C_i \right\} / \text{conj}$$

$$= \mathcal{C}_{\vec{\mu}}$$

HLRV ; C_i suff generic $\Rightarrow$ $\mathcal{C}_{\vec{\mu}} \neq \emptyset$

is a ζ-proj smooth

var$/\mathbb{C}$

$W. H^{*}_{cpt}(\mathcal{C}_{\vec{\mu}})$ weight filtr

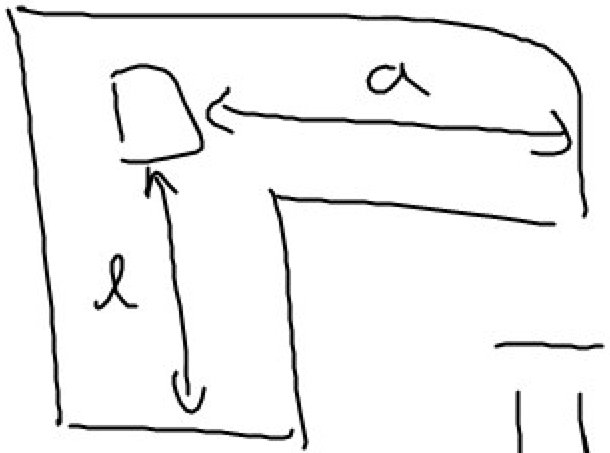
Mixed Poincaré polynomial

$$P(\mathcal{C}_{\vec{\mu}}, u^{1/2}, t) = \sum_{i,m} \dim\left(Gr^{W}_{i} H^{m}_{c}\right) u^{i/2} (-t)^{m}$$

HLRV

$$Z_{HLRV} = \exp\left(F_{HLRV}\right)$$

$$Z_{HLRV} = \sum_{\lambda} \Omega^{g}_{\lambda}(z,w) \prod_{i=1}^{k} \tilde{H}_{\lambda}(z^2,w^2;X_i)$$



$$\Omega^{g}_{\lambda}(z,w) = \prod_{\square\in\lambda} \frac{\left(z^{2a+1}-w^{2\ell+1}\right)^{2g}}{\left(z^{2a+2}-w^{2\ell}\right)\left(z^{2a}-w^{2\ell+2}\right)}$$

$$F_{HLRV} = \sum_{k \geq 1} \frac{1}{k} \sum_{\mu} \frac{w^{-k d_{\mu} - 1} \cdot P_c\left(z^{-2k}, (zw)^k\right)}{(1 - z^{2k})(w^{2k} - 1)} \prod_{i=1}^{k} m_{\mu_i}(x_i^k)$$

$$Z_{HLRV} = \exp(F_{HLRV})$$

$\lambda \vdash r$

$\widetilde{H}_\lambda(q_1, q_2; x)$

$$\begin{tikzcd} S_r \circlearrowright \widetilde{H}^r(\mathbb{C}^2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\supset"] & (\mathbb{C}^2)^r \circlearrowleft S_r \arrow[d] \\ H^r(\mathbb{C}^2) \arrow[r] & S^r(\mathbb{C}^2) \end{tikzcd}$$

Thm (Haiman)

$$\begin{array}{c} \widetilde{H}^r(\mathbb{C}^2)_{red} \\ \big\downarrow \\ [I_\lambda] \in H^r(\mathbb{C}^2) \circlearrowleft T = \mathbb{C}^\times \times \mathbb{C}^\times \end{array}$$

irred, normal, CM, Gorenstein 2-proj var

$$S_r \curvearrowright \mathcal{P} = \pi_* \mathcal{O}_{\widetilde{H}^r(\mathbb{C}^2)_{red}}$$ Procesi bundle

$$\mathcal{P}\big|_{[I_\lambda]} = \bigoplus_{\rho \vdash r} V_{\rho,\lambda} \otimes R_\rho$$

$\mathbb{C}^\times \times \mathbb{C}^\times$

irr of S_r

Thm (Haiman)

$$\sum_{\rho \vdash r} ch_T(V_{\rho,\lambda}) S_\rho(x) = \widetilde{H}_\lambda(q_1, q_2, x)$$

Plan

Plan · construct a CY_3 orbifold Y

Conj: $Z_{HLRV} \underset{\text{change var.}}{=} Z^{ref}_{PT}(Y)$

motivic DT of Kontsevich, Soibelman, Bussi, Joyce, Meinhardt

2)

$$Z_{HLRV} = \exp(F_{HLRV})$$

change variables

BPS states

$$Z^{ref}_{P_I}(Y) = \exp\left(F^{ref}_{GV}(Y)\right)$$

Char. var $\longleftrightarrow$ parabolic Higgs bundles

$p_i \in C$

$p_i \neq \infty \in C$

$$\left\{ \varphi : \pi_1(C \setminus D \setminus \infty) \longrightarrow GL(r) \;\middle|\; \varphi(\gamma_i) \in C_i,\ \varphi(\gamma_\infty) = \exp\left(\frac{2\pi\sqrt{-1}\,\xi}{r}\right) \right\}$$

conj

$$C_i = \mathrm{diag}\left(e^{2\pi\sqrt{-1}\,\alpha_i^a} \right)_{1\le a\le r}$$

$\alpha_i^a \in (0,1)$ $\qquad \alpha_i^1 < \alpha_i^2 < \dots$

$\mathcal{C}_{\vec{\mu}}$ diffeo to mod of stable par bundles

- E v.b on C $\quad E_i^k \subset E_{p_i}$ of type μ_i
- α_i^a par weight
- $\Phi : E \to E \otimes K_C(D)$, $\mathrm{res}_{p_i}(\Phi)$ nilpotent

· stab condition = slope cond.

$$\operatorname{pardeg}(E, E_i, \alpha_i^a) =$$

$$(e=) \deg E + \sum_{i,a} \mu_i^a \alpha_i^a \quad (=\varepsilon)$$

$\Rightarrow$ generic α_i^a, primitive inv $\Rightarrow$ smooth & proj

$$\mathcal{M}_{\vec{\mu}}^{e} \xrightarrow{h} B \text{ affine space}$$

de Cataldo, Hausel, Migliorini

$\Rightarrow$ perverse filtr on $P^{\cdot} H_{cp}(\mathcal{H}^{e}_{\vec{\mu}})$

<u>Conj</u> $\qquad \mathcal{C}^{\varepsilon}_{\vec{\mu}} \cong_{\text{diffeo}} \mathcal{H}^{e}_{\vec{\mu}}$

$W_{\cdot} = P^{\cdot}$

$$
\begin{tikzcd}
\mathcal{H}^e_{\vec\mu} \arrow[d, "h"'] \\
B
\end{tikzcd}
\qquad
\begin{tikzcd}
P^\bullet H^*_{cp}(\mathcal{H}^e_{\vec\mu}) \arrow[loop below, "\mathfrak{sl}(2)\ \text{Lefschetz}"]
\end{tikzcd}
$$

$$
P_c(\mathcal{H}^e_{\mu}; z^{-2}, -wz)\Big|_{(q,y)=(z^{-1}w,\, z^{-1}w^{-1})}
$$

$$
= q^{\#} y^{\#} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}_{\geq 0}} (-1)^{j+1} N^e_{\vec\mu}(j,m)\, \mathrm{Tr}_{S_j}\!\left(q^{H_{\mathfrak{sl}(2)}}\right) y^m
$$

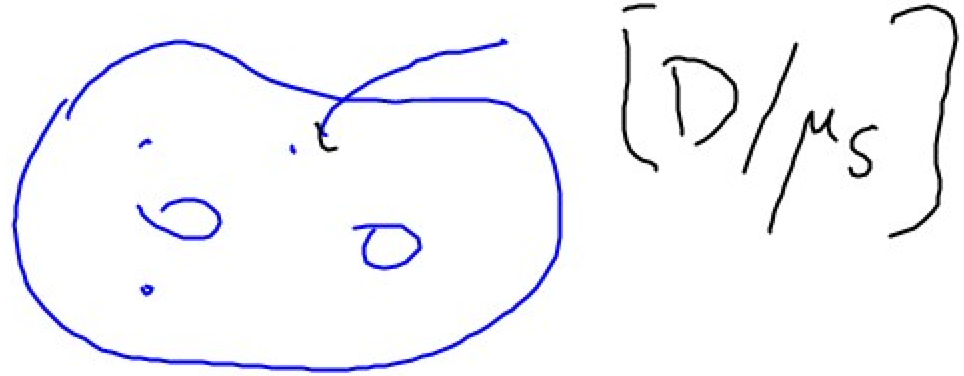
$$N^{e}_{\vec{\mu}} = \dim \operatorname{Prim} Gr^{P}_{j} H^{m}_{cp.}$$

BPS invariants

$$\begin{array}{ccc} \_\; Z_{HLRV} & = & \exp(F_{ARW}) \\ \downarrow & & \downarrow \\ --- & & --- \end{array}$$

- $\Rightarrow$ recursion formula for Poincaré polynomials of $\mathcal{C}^e_{\vec{\mu}}$ by wallcrossing (KS)

Constr of Y



Biswas, Borne, Mazatyr-Steer, M Groechenig

Parabolic Higgs bundles on CP_1 $\longleftrightarrow$ Higgs bundle on a root stack $\tilde{C}$

$Y = \text{Tot space}\left(K_{\tilde{C}} \oplus \mathcal{O}_{\tilde{C}}\right)$

Spectral covers · par Higgs bd $\leftrightarrow$ pure dim 1 sheaves on Y

loc

· cheack explicit comp $C = \mathbb{P}^1$
(Nekrasov, Okounkov) $p \in T$

· geom engineering virtual PT
ref PT of Y $\longleftrightarrow$ equiv K-theoretic inv
of $\tilde{C} \times \mathbb{C}^2$

PT on $\tilde{C} \times \mathbb{C}^2$

nested Hilb. sch on $\mathbb{C}^2$

$$Z_{\text{nested}} = Z_{\text{itLRV}}$$

Haiman's thm