$\mathbb{C}^n \circlearrowleft S_n \rightsquigarrow T^*\mathbb{C}^n \circlearrowleft S_n$, preserving symplectic structure.

DEFN: The Rational Cherednik Algebra (of type A_{n-1}) is the universal graded deformation of $\mathcal{O}(T^*\mathbb{C}^n)\# S_n$.

$\rightsquigarrow$ this produces deformations of $\mathcal{O}(T^*\mathbb{C}^n)^{S_n} = \mathcal{O}(\mathrm{Sym}^n\,\mathbb{C}^2)$

There exists a down-to-earth definition too:

$\mathbb{C}$-algebra generated by S_n, $\mathcal{O}(\mathbb{C}^n) = \mathbb{C}[X_1 \dots X_n]$, $\mathcal{O}((\mathbb{C}^n)^*) = \mathbb{C}[Y_1 \dots Y_n]$ subject to

$$\sigma X_i = X_{\sigma(i)}\sigma, \quad \sigma Y_i = Y_{\sigma(i)}\sigma, \quad [Y_i, X_j] = c\,(i\,j) \quad i \neq j \quad \& \quad [Y_i, X_i] = 1 - c\sum_{k\neq i}(i\,k)$$

Denoted H_c or $H_c(S_n)$. (Etingof-Ginzburg)

As a vector space

$$H_c \underset{v.s}{\cong} \mathbb{C}[T^*\mathbb{C}^n] \# S_n$$

Set $e = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \in \mathbb{C}S_n \subset H_c$

$U_c := e H_c e$ (the spherical R.C.A.)

As a vector space

$$U_c \underset{v.s.}{\cong} e(\mathbb{C}[T^*\mathbb{C}^n] \# S_n) e \cong \mathbb{C}(T^*\mathbb{C}^n)^{S_n}.$$

Remark : $c = 0 \longrightarrow H_0 = \mathcal{D}^{poly}(\mathbb{C}^n) \# S_n, \quad U_0 = \mathcal{D}^{poly}(\mathbb{C}^n)^{S_n}$

$c = \frac{1}{n}$: $H_0 \curvearrowright \mathbb{C}$ with $X_i, Y_i \mapsto 0, \quad \sigma \mapsto 1.$

$$Q: \begin{tikzcd} \bullet \arrow[loop above] \\ \bullet \arrow[u] \end{tikzcd} \quad \rightsquigarrow \quad \mathrm{Rep}(Q, n\delta+\varepsilon_\infty): \begin{tikzcd} \mathbb{C}^n \arrow[loop above, "X"] \\ \mathbb{C} \arrow[u, "v"'] \end{tikzcd}$$

$$\mathrm{Rep}(Q, n\delta+\varepsilon_\infty) = R \circlearrowleft G = GL(n) = GL(n\delta)$$

Differentiating: $\mu^\#: \mathfrak{g} \ni x \longmapsto x_R \in \mathrm{Vect}(R) \subset \mathcal{O}(T^*R)$

& $\mu: T^*R \longrightarrow \mathfrak{g}^*$ is G-equivariant

Nakajima: • $\mu^{-1}(0)/\!\!/G \simeq \mathrm{Sym}^n(\mathbb{C}^2) =: X$

• G acts freely on $T^*R^{\theta-ss}$ ($\theta = \det$) and $\mu^{-1}(0)^{\theta-ss}/G \simeq \mathrm{Hilb}^n(\mathbb{C}^2) =: Y$

Recall for vector space U the deformation quantization:

$$W(T^*U) = \mathbb{C}[[h]]\langle x_1,\dots,x_d, y_1,\dots, y_d\rangle \Big/ (y_i x_j - x_j y_i - h^2\delta_{ij})$$

which becomes a sheaf of algebras, $\mathcal{W}$, over T^*U: $y_i = h\frac{\partial}{\partial x_i}$; $x_i = hX_i$

$$\mathcal{W}/h\mathcal{W} \cong \mathcal{O}_{T^*U}$$

So it's easy to apply this to Hilb

$$\tau : \mathfrak{g} \ni x \longmapsto h x_R \in h\,\mathrm{Vect}(X) \subset \mathcal{W}(T^*R)$$

Define:

$$\mathcal{W}_{\mathrm{Hilb}^n} = \left(\mathcal{W}|_{T^*R^{\theta\text{-}ss}} \Big/ \tau\mathcal{W}|_{T^*R^{\theta\text{-}ss}} \right)^G$$

deformation quantization of $\mathrm{Hilb}^n\,\mathbb{C}^2$

Generally, for rep^n theory we like a $\mathbb{C}^*$-action on

$$\pi : Y \longrightarrow X$$

contracting to a single fixed point on X and scaling the symplectic form positively $\lambda \cdot \omega_Y = \lambda^m \omega_Y$, $m > 0$.

Bezrukavnikov-Kaledin: $\mathbb{C}^*$-equivariant deformation quantizations labelled by $H^2_{dR}(Y, \mathbb{C})$.

i.e. $H^2(\mathrm{Hilb}^n \mathbb{C}^2, \mathbb{C}) = \mathbb{C}$, so 1 parameter family:

$$\mathcal{W}^c_{\mathrm{Hilb}^n} := \left(\mathcal{W}|_{T^*V^{\theta\text{-}ss}} \Big/ (\tau - c\,\mathrm{tr})\mathcal{W}|_{T^*V^{\theta\text{-}ss}} \right)^G$$

$\mathbb{C}^*$-action allows us to "kill" the variable h:

W^c_{Hilb}-mod : $\mathbb{C}^*$-equivariant, $W^c_{Hilb^n}[h^{-1}]$-modules admitting f.g. $W^c_{Hilb^n}$-lattice.

This category admits a functor of invariant global sections

$$\Gamma^{\mathbb{C}^*} : W^c_{Hilb}\text{-mod} \longrightarrow U_c\text{-mod} \quad M \longmapsto \Gamma(Hilb, M)^{\mathbb{C}^*}$$

where $U_c := \left(\left(W(T^*R) \big/ (\mathcal{T}-c\,tr) W(T^*R) \right)[h^{-1}]\right)^{\mathbb{C}^*}$, a finitely generated algebra

[e.g. $W(T^*R)^{\mathbb{C}^*} = \mathbb{C}\langle h^{-1}x_i, h^{-1}y_i \rangle = D^{poly}(R)$.]

THM (Etingof–Ginzburg, Bezrukavnikov–Finkelberg–Ginzburg)

U_c is the spherical rational Cherednik algebra.

U_c-modules with filtration $\xrightarrow{①} W^c_{Hilb}$-mod $\xrightarrow{②}$ Coh $Hilb^n(\mathbb{C}^2)$

$$M \longmapsto W^c_{Hilb}[h^{-1}] \otimes_{U_c} M =: \mathcal{M} \longmapsto \mathcal{M}/h\mathcal{M}$$

So Cherednik algebras are machines for producing coherent sheaves on $Hilb^n \mathbb{C}^2$, which are even $\mathbb{C}^*$-equivariant.

① Localization (cf Beilinson-Bernstein in Lie theory)

② Information contained e.g. in filtrations.

THM (McGerty–Nevins) (At a great level of generality)

$$R\Gamma^{\mathbb{C}^*} : D(W^c_{\mathrm{Hilb}^n}\text{-mod}) \longrightarrow D(U_c\text{-mod})$$

is an equivalence iff U_c has finite global dimension. ∎

e.g. $U_0 = \left(D(R) / \tau D(R) \right)^G \cong D(\mathbb{C}^n)^{S_n} \underset{\mathrm{Mor}}{\sim} D(\mathbb{C}^n) \# S_n$

$N^{S_n} \longleftarrow N$

finite global dimension

Generally: U_c fin. gl. dim. iff $c \notin \{ -\frac{a}{b} : 1 \leq a < b \leq n \}$

Bridgeland stability cond^{ns} on $D_{\pi^{-1}(0)}(\mathrm{Hilb}^n \mathbb{C}^2)$ attached to each connected component of $\mathbb{R} \setminus \{ \frac{a}{b} : 1 < b \leq n \,\&\, (a,b) = 1 \}$. [Bezrukavnikov]

In this specific example, the derived equivalence is not surprising:

$$\mathcal{D}(\mathrm{Coh}\,\mathrm{Hilb}^n\mathbb{C}^2) \xrightarrow{\sim} \mathcal{D}_{S_n}(T^*\mathbb{C}^n) = \mathcal{D}(\mathbb{C}[T^*\mathbb{C}^n]\#S_n\text{-mod})$$

via $\mathcal{P}$, Procesi bundle, $\mathcal{M} \longmapsto R\mathrm{Hom}(\mathcal{P},\mathcal{M})$

$\mathrm{End}(\mathcal{P}) = \mathbb{C}[T^*\mathbb{C}^n]\#S_n$ $\qquad$ $\mathrm{Ext}^i(\mathcal{P},\mathcal{P}) = 0 \quad i>0.$

$\Downarrow$

deforms! over the quantization $\mathcal{W}^c_{\mathrm{Hilb}^n}$ & stays tilting

$\Downarrow$

produces deformation $\mathbb{C}[T^*\mathbb{C}^n]\#S_n$ i.e. Rational Cherednik algebra.

i.e. $\mathcal{D}(\mathcal{W}^c_{\mathrm{Hilb}^n}\text{-mod}) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{U}_c\text{-mod}) = \mathcal{D}^b(H_c\text{-mod})$

is quantization of classical derived equivalence.

THM (G-Stafford, Gan-Ginzburg, Kashiwara-Rouquier, McGerty-Nevins)

For $c \notin \left\{ -\frac{a}{b} : 1 < b \leq n, (a,b) = 1, a > 0 \right\}$

$$\Gamma^{\mathbb{C}^*} : W^c_{\mathrm{Hilb}^n}\text{-mod} \longrightarrow U_c\text{-mod}$$

is an equivalence of categories. $\blacksquare$

Intuitively:

$$\begin{tikzcd}
W^c_{\mathrm{Hilb}}\text{-mod} \arrow[r, "\Gamma^{\mathbb{C}^*}"] \arrow[dr, phantom, "?"] & U_c\text{-mod} \\
\left(W|_{T^*R^{\theta\text{-ss}}}, G\right)_c\text{-mod} \arrow[u, "? \text{ free action}"'] & (W, G)_c\text{-mod} \arrow[l, "\text{restriction}"'] \arrow[u, "\Gamma^{G \times \mathbb{C}^*}"']
\end{tikzcd}$$

a) Show restriction is surjective (Kirwan surjectivity)

b) Show anything in kernel has no G-invariants (local coh)

$\Longrightarrow \Gamma^{\mathbb{C}^*}$ is exact

Then use derived theorem.

So R.C.A. reps = sheaves on quantization of Hilbert scheme

and filtrations allow degenerations to sheaves on the honest Hilbert scheme.

Extended example:

$\exists$ hamiltonian $\mathbb{C}^*$-action on $T^*\mathfrak{h}$, $\mathrm{Hilb}^n\mathbb{C}^2$, etc. Call this T

$$Z = \{ p \in \mathrm{Hilb}^n\mathbb{C}^2 : \lim_{t\to 0} t.p \ \exists \} \subset \mathrm{Hilb}^n\mathbb{C}^2$$

$$\overset{\|}{\pi^{-1}\left((\mathbb{C}^n \times \{0\})/S_n \right)}$$

DEF^N: Category $\mathcal{G}_c$ is the full subcategory of $W^c_{\mathrm{Hilb}^n}$-mod whose objects M admit T-equivariant lattice such that $\mathrm{supp}(M/hM) \subseteq Z$.

Rk: Under equivalence to H_c-mod, these are the H_c-modules M for which $y_1 \dots y_n$ act nilpotently on any element $m \in M$.

$Z \supset Z^{reg} := \pi^{-1}(\mathbb{C}^n_{reg} \times \{0\}/S_n) \cong \mathbb{C}^n_{reg}/S_n$ and T^*Z^{reg} is open in $\mathrm{Hilb}^n \mathbb{C}^2$

$$W^c|_{T^*Z^{reg}}\text{-mod} \cong D^{poly}(\mathbb{C}^n_{reg}/S_n)\text{-mod}$$

which produces, when restricted to $\mathcal{O}_c$, a functor

$$\mathcal{O}_c \xrightarrow[T^*Z^{reg}]{\text{res}^n \text{ to}} \mathrm{Conn}^{r.s.}(\mathbb{C}^n_{reg}/S_n)$$

Thanks to Ginzburg-Guay-Opdam-Rouquier & McCarty, these produce the Knizhnik-Zamolodchikov connection on S_n-rep^{ns}

$$KZ_c : \mathcal{O}_c \longrightarrow \mathbb{C}[B_n = \pi_1(\mathbb{C}^n_{reg}/S_n)] \Big/ \langle (T_s - 1)(T_s + \exp(2\pi\sqrt{-1}c)) \rangle \text{-mod} =$$

$$\| \quad \mathcal{H}_q(S_n)\text{-mod} \quad (q = \exp(2\pi\sqrt{-1}c))$$

This functor is fully faithful on projectives ($\Rightarrow$ controls $\mathcal{O}_c$)

Rep^n theory of $\mathcal{H}_q(S_n)$ is "combinatorial", depending critically on the order of $q \in \mathbb{C}^*$ alone.

$\mathbb{C} \in H_c$-mod for $c = \frac{1}{n}$ i.e. q primitive n^{th} root of 1

$\Rightarrow \exists$ f.d. rep^n $L_c \in H_c$-mod for $c = \frac{a}{n}$ with $(a,n) = 1$.

— it turns out to be of dim^n a^{n-1}.

Moreover, $\exists$ natural filtration on L_c (Gorsky-Oblomkov-Rasmussen-Shende, Berest-Etingof-Ginzburg, G)

$\rightsquigarrow$ $(\mathbb{C}^*)^2$-equivariant coherent sheaves on $\mathrm{Hilb}^n \mathbb{C}^2$

· if $a = nk+1$ these are: $\mathcal{P} \otimes \mathcal{O}(k-1)\big|_{\pi^{-1}(0)}$

· conjectural description of Gorsky-Negut related to (m,n)-torus knot invariants arising from refined Chern-Simons theory.