Refined curve counting; Hilbert schemes, tropical Geometry and Fock space

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(in order of appearance)

S proj. surface over $\mathbb{C}$; L very ample line bundle on S
$|L|$ complete linear system
L suff. ample $\Longrightarrow$ δ-nodal curves occur in codimension δ in $|L|$
Let $\mathbb{P}^\delta \subset |L|$ general δ-dimensional linear subspace

Severi degree:

$$n_{(S,L),\delta} := \#\{\delta\text{-nodal curves in } \mathbb{P}^\delta\}$$

Conjecture

There exists a universal polyn. $n_\delta^{(S,L)}$ in $L^2, LK_S, K_S^2, c_2(S)$ computing $n_{(S,L),\delta}$ for L-sufficiently ample (δ-very ample).

Proven by Tzeng, Kool-Shende-Thomas
Kool-Shende-Thomas use Euler numbers of relative Hilbert schemes of points to define the polynomials $n_\delta^{(S,L)}$

Give refinement motivated by the K-S-T proof replacing Euler number by χ_{-y}-genus

$S^{[n]}$ =Hilbert scheme of points on S

$\mathcal{C} = \{(p,[C]) | p \in C\} \subset S \times \mathbb{P}^\delta$ universal curve

$\mathcal{C}^{[n]} = \{([Z],[C]) | Z \subset C\} \subset S^{[n]} \times \mathbb{P}^\delta$ relative Hilbert scheme

χ_{-y}-genus $\chi_{-y}(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X) y^q$

Conjecture

There exist $N_0^{\mathcal{C}}(y), \ldots, N_\delta^{\mathcal{C}}(y) \in \mathbb{Z}[y]$ s.th.

$$\sum_{n \geq 0} \chi_{-y}(\mathcal{C}^{[n]}) t^n = \sum_{l=0}^{\delta} N_l^{\mathcal{C}}(y) t^l ((1-t)(1-yt))^{g(L)-l-1}$$

$g(L)$ =genus of smooth curve in $|L|$

Conjecture

There exist $N_0^{\mathcal{C}}(y), \ldots, N_\delta^{\mathcal{C}}(y) \in \mathbb{Z}[y]$ *s.th.*

$$\sum_{n\geq 0} \chi_{-y}(\mathcal{C}^{[n]})t^n = \sum_{l=0}^{\delta} N_l^{\mathcal{C}}(y) t^l((1-t)(1-yt))^{g(L)-l-1}$$

Theorem

The conjecture is true

1. *If* δ *is replaced by* $g(L)$
2. *if* K_S *is numerically trivial*
3. *modulo* t^{11} *for all surfaces*

Conjecture

$$\sum_{n\geq 0}\chi_{-y}(\mathcal{C}^{[n]})t^n=\sum_{l=0}^{\delta}N_l^{\mathcal{C}}(y)t^l((1-t)(1-yt))^{g(L)-l-1}$$

Definition

The *refined invariant* is $N_\delta^{(S,L)}(y):=N_\delta^{\mathcal{C}}(y)/y^\delta$ (symmetric Laurent polynomial)

If L suff. ample $N_\delta^{(S,L)}(1)=n_{(S,L),\delta}$ Severi degree

What is counted at other values of y

What is the meaning of the polynomial $N_\delta^{(S,L)}(y)$?

Conjectural generating function for refined invariants $N^{(S,L)}_{\delta}(y)$
Proven for $K3$- and Abelian surfaces

$$\Delta(y,q) := q\prod_{n\geq 1}(1-q^n)^{20}(1-q^ny)^2(1-q^n/y)^2$$

$$\widetilde{DG}_2(y,q) = \sum_{n\geq 1} q^n \sum_{d|n}\frac{(y^{d/2}-y^{-d/2})^2}{(y^{1/2}-y^{-1/2})^2}$$

Theorem

1. *(S_g, L_g) K3 surface with ample irred. l.b. of genus g*

$$\sum_{g\geq k} N^{(S_g,L_g)}_{g-k}(y)q^{g-1} = \frac{\widetilde{DG}_2(y,q)^k}{\Delta(y,q)}$$

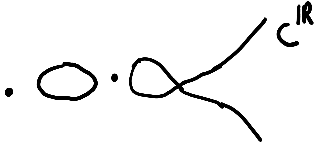
2. *(A_g, L_g) Abelian surface with ample irred. l.b. of genus g*

$$\sum_{g\geq k+2} N^{(A_g,L_g)}_{g-k-2}(y)q^{g-1} = \widetilde{DG}_2(y,q)^k q\frac{\partial}{\partial q}\widetilde{DG}_2(y,q)$$

Show: $N_\delta^{(S,L)}(y)$ related to real algebraic and tropical geometry

Let S real algebraic surface; complex conj. τ maps S to S
real algebraic curve = curve C such that $\tau(C) = C$
Real locus of C: $C^{\mathbb{R}} = C^\tau$
P configuration of dim $|L| - \delta$ real points of S
Welschinger invariants: $W_{(S,L),\delta}(P) = \sum_C (-1)^{s(C)}$

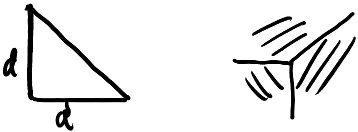
sum is over all real δ-nodal curves C in $|L|$ though P
$s(C) = \#\{\text{isolated nodes of } C\}$



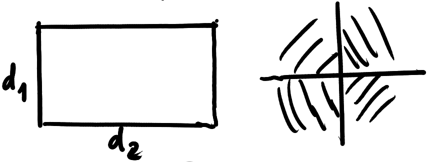
$s(C) = 2$

From now on S toric surface:
$\mathbb{C}^* \times C^* \subset S$ open dense, action extends to S
Automatically real surface
Correspondence:
$\{$Convex lattice polygons $\Delta \subset \mathbb{R}^2\} \longleftrightarrow$
$\{$pairs $(S(\Delta), L(\Delta))$ toric surface, ample toric line bundle$\}$
$S(\Delta)$ toric surface defined by the normal fan to Δ

1. $\mathbb{P}^2$, $L = \mathcal{O}(d)$



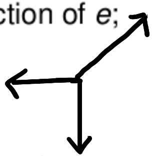
2. $\mathbb{P}^1 \times \mathbb{P}^1$, $L = \mathcal{O}(d_1, d_2)$



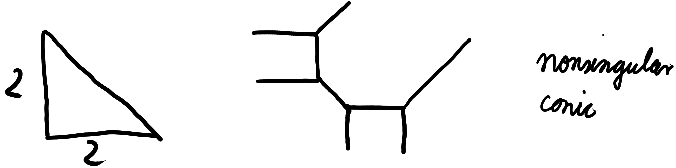
$\#(\Delta \cap \mathbb{Z}^2) = h^0(S(\Delta), L(\Delta))$, $\quad \#(\mathrm{int}(\Delta) \cap \mathbb{Z}^2) = g(L(\Delta))$

plane tropical curve of degree Δ (Δ conv. lattice polyg.):
piecewise linear graph Γ immersed in $\mathbb{R}^2$ s.t.

1. the edges e of Γ have rational slope
2. they have weight $w(e) \in \mathbb{Z}_{>0}$
3. **balancing condition:**
   let $p(e)$ primitive integer vector in direction of e;
   for all vertices v of Γ:
   $$\sum_{e \text{ at } v} p(e)w(e) = 0.$$



4. for every edge of Δ (of lattice length n) Γ has n unbouded edges in corresponding outer normal direction



There is notion of number of nodes of tropical curve Γ
A **simple** tropical curve is trivalent
Known: through $\#(\Delta \cap \mathbb{Z}^2) - 1 - \delta$ general points in $\mathbb{R}^2$, there are finitely many δ-nodal degree Δ tropical curves, all simple
Tropical Severi degree: Let Γ simple tropical curve, v vertex, e_1, e_2, e_3 edges at v

$$m(v) := w(e_1)w(e_2)|\det(p(e_1), p(e_2))|, \qquad m(\Gamma) = \prod_{v \text{ vertex}} m(v)$$

Tropical Severi degree: $n^{trop}_{\Delta,\delta} := \sum_{\Gamma} m(\Gamma)$

sum over all δ-nodal, degree Δ tropical curves through $\#(\Delta \cap \mathbb{Z}^2) - 1 - \delta$ general points in R^2.

Let Γ simple tropical curve, v vertex

$$\omega(v) := \begin{cases} (-1)^{(m(v)-1)/2} & m(v) \text{ odd} \\ 0 & m(v) \text{ even} \end{cases}$$

$$\omega(\Gamma) = \prod_{v \text{ vertex}} \omega(v)$$

Tropical Severi degree: $W^{trop}_{\Delta,\delta} := \sum_{\Gamma} \omega(\Gamma)$

sum over all δ-nodal, degree Δ tropical curves through $\#(\Delta \cap \mathbb{Z}^2) - 1 - \delta$ general points in R^2.

Mikhalkin: The Severi degree is equal to the tropical Severi degree and the Welschinger invariants are equal to the tropical Welschinger invariants.

$$n_{S(\Delta),L(\Delta),\delta} = n^{trop}_{\Delta,\delta}$$

$$W_{S(\Delta),L(\Delta),\delta} = W^{trop}_{\Delta,\delta}$$

We know, for Δ sufficiently ample $N^{(S,L)}_{\delta}(1) = n_{S(\Delta),L(\Delta),\delta}$,

Conjecture

For Δ sufficiently ample $N^{(S,L)}_{\delta}(-1) = W_{S(\Delta),L(\Delta),\delta}$.

quantum number: $[n]_y := \frac{y^{n/2}-y^{-n/2}}{y^{1/2}-y^{-1/2}}$

By definition $[n]_1 = n$, $[n]_{-1} = \begin{cases}(-1)^{(n-1)/2} & n \text{ odd}\\ 0 & n \text{ even}\end{cases}$

Let Γ simple tropical curve, v vertex

$$M(v) := [m(v)]_y, \qquad M(\Gamma) = \prod_{v \text{ vertex}} M(v)$$

Tropical Severi degree:

$$N^{trop}_{\Delta,\delta}(y) := \sum_{\Gamma} M(\Gamma)$$

sum as above

By definition $N^{trop}_{\Delta,\delta}(1) = n^{trop}_{\Delta,\delta} = n_{(S(\Delta),L(\Delta)),\delta}$, $N^{trop}_{\Delta,\delta}(-1) = W^{trop}_{\Delta,\delta}$

Conjecture

For Δ sufficiently ample $N^{(S,L)}_{\delta}(y) = N^{trop}_{\Delta,\delta}(y)$

H deformed Heisenberg algebra for hyperbolic lattice, i.e.
gen. by $a_n, b_n, \quad n \in \mathbb{Z}$
$a_{-n},\ b_{-n}$ with $n > 0$ are called **creation operators**
$a_n,\ b_n$ with $n > 0$ are called **annihilation operators**
commutation relations

$$[a_n, a_m] = 0 = [b_n, b_m], \qquad [a_n, b_m] = [n]_y \delta_{n,-m}$$

Fock space: F generated by **creation operators** a_{-n}, b_{-n}
acting on vacuum vector $v_\emptyset$
H-module by $a_n v_\emptyset := 0$, $b_n v_\emptyset := 0$ for $n \geq 0$
(concatenate and apply commutation relations)

F has $\mathbb{Q}[y^{\pm 1/2}]$ basis paramtr. by pairs of partitions
$\mu = (1^{\mu_1}, 2^{\mu_2}, \ldots)$, $\nu = (1^{\nu_1}, 2^{\nu_2}, \ldots)$
$a_\mu := \prod_i \frac{a_i^{\mu_i}}{\mu_i!}$, $a_{-\mu} := \prod_i \frac{a_{-i}^{\mu_i}}{\mu_i!}$, similarly for b_ν, $b_{-\nu}$
$v_{\mu,\nu} := a_{-\mu} b_{-\nu} v_\emptyset$ basis for F
inner product $\langle v_\emptyset | v_\emptyset \rangle = 1$; a_n, b_n adjoint to a_{-n}, b_{-n}.

Cooper-Pandharipande:
For $S = \mathbb{P}^1 \times \mathbb{P}^1$ formula for $n_{(S,L),\delta}$ in terms of Fock space
Generalize this to refined Severi degrees and to large class of toric surfaces (h-transversal lattice polygons)
I will state only for $\mathbb{P}^2$ and rational ruled surfaces.

Case of $\mathbb{P}^2$

$$H(t) := \sum_{k>0} b_k b_{-k} + t \sum_{\|\mu\|=\|\nu\|-1} a_\nu a_{-\mu}$$

$\|\mu\| := \sum_i i\mu_i;$ sum includes $\mu = \emptyset$

Theorem

1. $$N^{trop}_{d,\delta}(y) = \langle v_\emptyset | \mathrm{Coeff}_{t^d} H(t)^{d(d+3)/2-\delta} v_{(1^d),\emptyset}\rangle$$

2. $$\sum_{d,g} N_d^{g,trop}(y) \frac{t^d u^{3d-1+g}}{(3d-1+g)!} = \langle v_\emptyset | \exp(uH(t)) \exp(a_{-1}) v_\emptyset\rangle$$

$g = d(d-1)/2 - \delta$

Hirzebruch surface Σ_e

F fibre; let E section with self intersection $-e$; $H = E + eF$

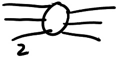
$$H_e(t) = \sum_{k>0} b_k b_{-k} + t \sum_{\|\mu\|=\|\nu\|-e} a_\nu a_{-\mu}$$

Theorem

$$N^{g,trop}_{(\Sigma_e, dH+mF)}(y) = \langle v_{(1^m,\emptyset)} | \mathrm{Coeff}_{t^d}[H_e(t)^{g+2(d+m)+ed-1}] v_{(1^{m+ed},\emptyset)} \rangle$$

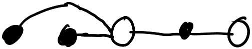
Feynman diagrams: To each monomial M in the $b_k b_{-k}$, $a_\nu a_{-\mu}$ associate diagrams:

- for $b_k b_{-k}$ write 

e.g. for $a_{(1^2,2)} a_{-(1^3)}$ 

- write vertices in order they are in the monomial
- connect all vertices so that edges connect only vertices of different colour, and the weights match

$(b_1 b_{-1})^2 a_{(1^2)} a_{-1} b_1 b_{-1} a_1$

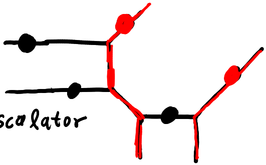


count the diagrams with multiplicity $m(\Gamma) := \prod_{e \text{ edges}} [w(e)]_y$.

Proposition (Wicks Theorem)

$$\langle v_\emptyset | M v_\emptyset \rangle = \sum_{\Gamma \text{ Graphs for } M} m(\Gamma)$$

To Γ tropical curve through horizontally stretched conf. of points associate marked floor diagram.



escalators: horizontal segments of Γ
floors: conn. comp. of complem. of escalators. One marked point on each floor and escalator
Floor diagram: black vertex for escalator
white vertex for floor
connect if escalator connects to floor, keep weight

Put $m(\Lambda) := \prod_{e \text{ edges}} [w(e)]_y$

Proposition

$$N^{trop}_{\Delta,\delta}(y) = \sum_{\Lambda \text{ floor diagrams}} m(\Lambda)$$

Claim: floor diagrams = Feynman diagrams