Three Stratifications for representation spaces of quivers

I Geometric Inv thy

$$\underset{\text{cx red gp}}{G} \overset{\text{lin}}{\curvearrowright} \underset{\text{cx aff. sp}}{V}$$

Use $\rho : G \to \mathbb{C}^\times$ to linearise

ie. $G \curvearrowright L = V \times \mathbb{C}$

$g \cdot (v, c) = (gv, \rho(g)c)$

Mumford's GIT $V^{\rho\text{-ss}} \longrightarrow V/\!/_\rho G$

Hilbert-Mumford (King) v is ρ-ss $\iff (\rho, \lambda) \geq 0$ $\forall$ 1-PS $\lambda : \mathbb{C}^* \to G$

s.t $\lim_{t \to 0} \lambda(t) \cdot v$ exists

$$\rho \circ \lambda(t) = t^{(\rho, \lambda)}$$

Ex: $Q = (V, A, h, t)$ $\qquad \underline{d}$

$$V = \mathrm{Rep}(Q, \underline{d}) = \bigoplus_{a \in A} \mathrm{Hom}\left(\mathbb{C}^{d_{t(a)}}, \mathbb{C}^{d_{h(a)}}\right)$$

$$G = \prod_{v \in V} GL(\mathbb{C}^{d_v})$$

$$\rho : G \longrightarrow \mathbb{C}^\times \qquad \rho_v \in \mathbb{Z} \qquad \rho(g_v) = \prod_v (\det g_v)^{\rho_v}$$

assume: $\sum \rho_v d_v = 0$

Thm (King) A rep W of dim $\underline{d}$ is ρ-ss $\Longleftrightarrow$ $\forall\, W' \subseteq W$

$$\rho(W') = \sum \rho_v \dim W'_v \geq 0$$

__Instability__ Choose G-inv norm $\|-\|$ on $X_*(G)_{\mathbb{R}}$

Let $M^\rho(v) = \inf\limits_{\substack{\text{1-PS } \lambda \\ \text{s.t } \lim_{t\to 0} \lambda(t)\cdot v \\ \text{exists}}} \dfrac{(\rho, \lambda)}{\|\lambda\|}$ HM: v is ρ-ss $\iff M^\rho(v) \geq 0$

__Defn__: A 1-PS λ is __adapted__ to v (a ρ-unstable pt)

If $M^\rho(v) = \dfrac{(\rho, \lambda)}{\|\lambda\|}$ & $\lim\limits_{t\to 0} \lambda(t)\cdot v$ exists

$\Lambda^\rho(v) = \{\text{non-div adapted 1-PS to } v\} \underset{\text{Kempf}}{\neq} \emptyset$

$[\lambda]$ = conj class of λ

Hesselink's Stratification $V - V^{\rho-ss} = \coprod_{[\lambda]} S^H_{[\lambda]}$

where $S^H_{[\lambda]} = \{ v : \Lambda^\rho(v) \cap [\lambda] \neq \phi \}$ $\quad S^H_{[0]} = V^{\rho-ss}$

Structure of strata?

$$\begin{array}{ccc} V^\lambda_+ = \{ v : \lim_{t \to 0} \lambda(t) \cdot v \text{ exists} \} & \supseteq S_\lambda = \{ v : \lambda \in \Lambda^\rho(v) \} & \\ \downarrow p_\lambda & \downarrow & \\ V^\lambda & Z_\lambda = \{ v : \lambda \in \Lambda^\rho(v) \cap G_v \} & \end{array}$$

Prop 1) Z_λ = GIT ss for smaller red gp $\curvearrowright V^\lambda$

2) $S^H_{[\lambda]} = G S_\lambda \qquad S_\lambda = p_\lambda^{-1}(Z_\lambda)$

II Moment maps

$\underset{\text{compact}}{K} \overset{\text{symp?}}{\curvearrowright} (V, \omega)$

cx v. sp sympl

For $\rho: K \to S^1$, m. map $\mu^\rho : V \to \mathfrak{k}^*$

given by $\mu(v)\cdot A = \frac{1}{2}\omega(Av, v) - \frac{1}{2\pi i} d\rho \cdot A$

Fix ρ & $\mu = \mu^\rho$.

Choose K-inv norm on $\mathfrak{k}$

<u>Idea</u>: $\|\mu\|^2 : V \to \mathbb{R} \rightsquigarrow$ Morse strat of V

As $\|\mu\|^2$ is K-inv, so is $\mathrm{crit}\,\|\mu\|^2 = \bigsqcup_{K\cdot\beta} C_{K\cdot\beta}$

Morse strat: $V = \bigsqcup_{K\cdot\beta} S^M_{K\cdot\beta}$

pts flowing to $C_{K\cdot\beta}$

+ Structure results.

III Comparison

Aff case: $G \overset{\text{lin}}{\curvearrowright} V$ $\quad \omega = \operatorname{Im} H$

G cx red $\qquad$ V cx v.space $\qquad$ assume: $K \subseteq G$ maxl compact acts sympl

K-inv norm $\|-\|$ on $\mathfrak{k} \rightsquigarrow \|-\|$ on $X_*(G)_{\mathbb{R}}$

$\lambda : \mathbb{C}^\times \to G \qquad \lambda' = g\lambda g^{-1} \quad K \to S^1$

$$\|\lambda\| = \|d\lambda'(2\pi i)\|$$

Thm (H). In this affine set up, for any ρ & $\|-\|$,

Hesselink's strat = Morse strat.

Idea of pf:

- Affine Kempf–Ness wrt ρ: lowest strata agree $V^{\rho\text{-}ss} = S_0^M \leftarrow \mu^{-1}(0)$
- Structure results of strata

$$S^M_{K\cdot\beta} = G S_\beta \qquad S_\beta = p_\beta^{-1}(Z_\beta)$$ lowest morse strata

$$Z_\beta = Z_\lambda$$ where λ = 1-PS assoc to a +ve multiple of $\beta \in R^*$

IV Quiver reps

$G(Q,\underline{d}) \curvearrowright V = \mathrm{Rep}(Q,\underline{d})$

$\rho \longleftrightarrow \quad \rho_v \in \mathbb{Z}$ s.t $\sum \rho_v d_v = 0$

$\|-\|_\alpha \longleftrightarrow \quad \alpha_v \in \mathbb{N}$

Every rep W has a ! HN filtr wrt (ρ & α)

<u>Thm</u>(H): The stratification of V by HN types agrees with the Morse = Hess strat.