Curves on surfaces

§1. S sm. pr. sf. / $\mathbb{C}$, $\beta \in H_2(S)$

$H_\beta := \mathrm{Hilb}_\beta(S) = \{C \text{ eff. div. s.t. } [C] = \beta\}$

$\mathrm{Pic}^\beta(S) = \{[L] \in \mathrm{Pic}(S) \mid c_1(L) = \beta\}$

$$\begin{array}{ccccc} AJ : & H_\beta & \longrightarrow & \mathrm{Pic}^\beta, & C \mapsto [\mathcal{O}_S(C)] \\ & \cup\!\mid & & \cup\!\mid & \\ \mathbb{P}^{h^0(L)-1} \cong & |L| & \longrightarrow & [L] & \end{array}$$

$C \subseteq S$, def $H^0(N_{C/S})$, $N_{C/S} \cong \mathcal{O}_C(C)$

ob $H^1(N_{C/S})$

$\mathcal{C} \to M_\beta$ univ.

$\pi : M_\beta \times S \to M_\beta$ $\rightsquigarrow$ $R\pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{C})$

Dürr–Kab.–Okounk: gives perf ob thy

$F^\bullet = R\pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{C})^\vee \longrightarrow \mathbb{L}_{M_\beta}$.

Def. (Behrend-Fantechi).

M scheme

$E^\bullet = \{E^{-1} \to E^{0}\}$ loc. free

$\phi : E^\bullet \longrightarrow \underbrace{\mathbb{L}_M}_{\text{cot. cx. of } M}$ mph. in $D^b(X)$, s.t. $h^0(\phi)$ iso, $h^{-1}(\phi)$ surj.

perf ob thy on M

$\rightsquigarrow$ - $p \in M$, def. $h^0(E^\vee|_p)$

ob $h^1(E^\vee|_p)$

- $[M]^{vir} \in A_{vd}(M)$ $vd = rk\, E =$ dim defs $-$ dim ob

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$C \subseteq S$, $0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$

$\sigma: H^1(N) \to H^2(\mathcal{O})$, semi-reg map

$ob \in \ker \sigma$

$A \subseteq S$ div.

$\gamma := \beta + [A]$, $\forall L \in \mathrm{Pic}^\gamma$: $h^1(L) = h^2(L) = 0$

M_γ sm, $M_\beta \overset{\iota}{\hookrightarrow} M_\gamma$, $C \mapsto A + C$

$D \in M_\gamma$: $D \in \iota(M_\beta) \iff s_D|_A = 0 \in H^0(\mathcal{O}_A(D))$

$\mathcal{D} \to M_\gamma$

$\pi: M_\gamma \times A \to M_\gamma$

$\pi_* \mathcal{O}(\mathcal{D})|_{M_\gamma \times A}$ sheaf

$\downarrow \uparrow s$

$M_\gamma \supseteq s^{-1}(0) = M_\beta$

(ASL) : $\forall L \in \mathrm{Pic}^{\beta}$ eff. $h^2(L) = 0$

$\rightsquigarrow$ $\pi_* \mathcal{G}_A(\mathcal{L})$ v.b. in ngh. $\iota(M_\beta)$

$\rightsquigarrow$ pot on M_β $\quad \sigma$: ker σ

$\iota_*[M]^{vir} = c_{top}(E)$

$E^\bullet$ p.o.t.

E v.b.

$\downarrow \circlearrowleft s$

$A \underset{sm}{\supseteq} s^{-1}(0) = M$

$$\begin{tikzcd}
E^*|_M \arrow[r] \arrow[d, "s^*"'] & \Omega_A|_M \arrow[d, equal] \\
I_M/I_M^2|_M \arrow[r, "d"] & \Omega_A|_M
\end{tikzcd} \Big\} \quad \begin{tikzcd} E^\bullet \arrow[d, "\phi"] \\ \mathbb{L}_M \end{tikzcd}$$

$F \rightsquigarrow [M_\beta]^{vir}$ $\qquad$ $\dfrac{\beta(\beta-k_S)}{2} = vd$

$F^{red} \rightsquigarrow [M_\beta]^{red}$ $\qquad$ ——"—— $+ h^{0,2}(S) = vd$

—— (DKO) $\qquad$ $k_S := c_1(\mathcal{O}(K_S))$

§2. DKO: $[M_\beta]^{vir} \rightsquigarrow P^{\pm}_S(\beta) \in \Lambda^* H^2(S,\mathbb{Z})$

$[M_{k_S-\beta}]^{vir}$

Poincaré inv.

Conj (DKO) If $h^{0,2}(S)>0$: $P^+_S(\beta) = P^-_S(\beta) = SW_S(\beta)$

If $h^{0,2}(S)=0$: $P^{\pm}_S(\beta) = SW^{\pm}_S(\beta)$

SW = Seiberg-Witten

Thm. (Chang-Kiem). True

$$\underset{\text{0 dim.}}{Z} \subseteq C \subseteq S \quad , \quad \underset{s^{-1}(0)}{/\!/} \; \mathrm{Hilb}^n(\mathcal{C}/M_\beta) \subseteq S^{[n]} \times M_\beta \circlearrowright E_{\mathcal{M}}$$

rel. Hilb n pts on fibres $\mathcal{C} \to M_\beta$

Hilb n pts on S on sm.

$\rightsquigarrow$ rel. p.o.t. on $\mathrm{Hilb}^n(\mathcal{C}/M_\beta)$

<u>Thm 1.</u> (K-Thomas). This rel. p.o.t. + $[M_\beta]^{vir}$, $[M_\beta]^{red}$ gives $[\mathrm{Hilb}^n(\mathcal{C}/M_\beta)]^{vir/red}$.

PT thy.

$\{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X)$

X CY3, (F, s) : – F pure dim 1 sheaf.

– $s \in H^0(F)$ w/ 0-dim coher.

$\mathrm{supp}\, F = \beta$

$\chi(F) = \chi$ $\rightsquigarrow$ $P_\chi(X, \beta)$ comp. mod. sp. of cx. in $D^b(X)$

Le Potier

Thm (PT) $\exists$ pot. on $P_\chi(X, \beta)$ of vd$=0$.

$PT_{\beta\chi}(X) = \int_{[P_\chi]^{vir}} 1$ PT inv.

$$GW_{g,\beta}(X) := \int_{[\overline{M}_{g,0}(X,\beta)]^{vir}} 1$$ GW inv. (disconn.)

$$PT_\beta(X) = \sum_\chi PT_{\beta,\chi}(X)\, q^\chi, \quad GW_\beta(X) := \sum_g GW_{g,\beta}(X)\, u^{2g-2}$$

<u>Conj</u> (PT/GW corr.). For $-q = e^{iu}$: $PT_\beta(X) = GW_\beta(X)$

$$\mathbb{C}^* \circlearrowright X = \mathrm{Tot}(K_S), \quad P_\chi(X,\beta) \supseteq P_\chi(X,\beta)^{\mathbb{C}^*}$$

non-cpt. CY3   non-cpt.   cpt.

$$P_\chi(X,\beta)^{\mathbb{C}^*} = \underbrace{P_\chi(S,\beta)}_{\cong \mathrm{Hilb}^n(\mathcal{C}/M_\beta)} \sqcup \cdots\cdots$$

$\chi = 1-h+n$

h arith. gen of cv. in M_β

$\sigma_1, \ldots, \sigma_m \in H^*(X) \rightsquigarrow PT^{red}_{\beta,\chi}(X, \sigma_1 \cdots \sigma_m)$ — def by Graber–Pand. loc. formula via int on $[P_\chi(X,\beta)^{\mathbb{C}^*}]^{vir}_{red}$

$P_\chi(X,\beta) \longmapsto E_{PT,X}$ ordinary PT thy.

$\hookrightarrow E^{\text{red.}}_{PT,X}$ red. PT thy.

(AS2) : $\cup\beta : H^1(T_S) \twoheadrightarrow H^2(\mathcal{O}_S)$

(AS2) $\Rightarrow$ (AS1)

Thm. 2 (K-Thomas)

$E_{PT,X}\big|^{fix}_{P_\chi(S,\beta)}$ gives $[\mathrm{Hilb}^n(\mathcal{C}/H_\beta)]^{vir}$

of Thm. 1. (Same for red.)

§3. Thm. 3 (K)

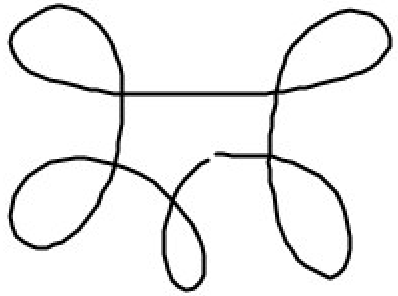
$\forall S, \beta$ s.t. ① β irred., or
② $-K_S$ nef and β is suff ample w.r.t h.

$\left(X = \mathrm{Tot}(K_S), \; h \text{ arith. gen. } \beta, \; m := \frac{\beta(\beta - k)}{2} \right)$

true: Tanbes

Then: the lowest order terms $GW_\beta(X, [pt]^m)$, $PT_\beta(X, [pt]^m)$, agree $\Longleftrightarrow$ $SW_\beta(S) = \int_{[\overline{M}_{h,m}(X,\beta)]^{vir}} \prod_{i=1}^{m} ev_i^* [pt]$.

Göttsche:



$S, \beta, |L|, c_1(L) = \beta$

$\delta > 0$, assume L suff ample wrt. δ

then $\underset{\text{general}}{\mathbb{P}^\delta \subseteq |L|}$ has finite $\#$ δ-nodals

$a_\delta^L(S)$ Severi degrees

$$\overline{M}_{h-\delta,\bullet}(S,\beta) \ni h : C \to S$$

want: $h(C) \in \mathbb{P}^\delta \subseteq |L| \subseteq M_\beta$

insertion: $\prod_{i=1}^{b_1} ev_i^* \gamma_i \cdot \prod_{j=b_1}^{b_1+\chi(L)-1-\delta} ev_j^*(pt) =: (\cdots)$

$\bullet = b_1 + \chi(L) - 1 - \delta$

γ_i basis $H_1(S)$

$\deg(\cdots) = [\overline{M}_{h-\delta,\bullet}(S,\beta)]^{red}$

Fact: $a^L_\delta(S) = \int_{[\overline{M}_{h-\delta,\cdot}(X,\beta)]^{red}} (\cdots)$

<u>Thm 4.</u> (K-Thomas).

The lowest order terms of $GW^{red}_\beta(X,(\cdots))$ and $PT^{red}_\beta(X,(\cdots))$ agree $(-q=e^{iu})$ $\Longleftrightarrow$

$a^L_\delta(S) =$ <u>specific</u> lin. of $e(\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta))$.
$i=0,\ldots,\delta$

$q_\delta^L(y) = \cdots$

true: K-Shende-Thomas