Homology of Hilbert Schemes of a Loc. Planar Curve

C alg curve$/\mathbb{C}$, proper, red. irr., loc pl. sings

$\dim_{\mathbb{C}} \mathfrak{m}_P/\mathfrak{m}_P^2 \leq 2 \quad \forall P \in C.$

$C^{[n]} = \mathrm{Hilb}^n C.$

C sm. $\quad C^{[n]} = \mathrm{Sym}^n C = C^n/S_n$

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$$H_*(C^{(n)}, \mathbb{Q}) = H_*(C^n)^{S_n} = \bigoplus_{k=0}^{n} \mathrm{Sym}^k(H_{even}(C)) \otimes \Lambda^{n-k}(H_{odd}(C)).$$

C sing. Topology of $C^{[n]}$ dep. on sings.

$$H_*(C^{[n]}) = ?$$

Cptfied Jac: J = Sch. of torsion fr. rk 1 and deg 0 coh. sh. on C.

C loc. pl. $\Rightarrow$ · $\mathrm{Pic}^0 C \hookrightarrow \bar{J}$ open dense

· $\bar{J}$ proper, red. irr. lci

(Altman-Iarrobino-Kleiman '77)

· $C^{(n)}$ ———— $\cdot\,\cdot$ ————

- $\exists$ Abel-Jacobi map $AJ : C^{[n]} \longrightarrow J$

$$Z \longmapsto I_Z \otimes \mathcal{O}(nP)$$

$F \in J$

$AJ^{-1}(F) = \mathbb{P}(\mathrm{Hom}(F, \mathcal{O}(nP)))$ $\quad$ P fixed n.s. pt $\in C$.

$\rightsquigarrow$ $n \geq 2g(C) - 1$, AJ is $\mathbb{P}^N$-bdl.

· $\exists$ deformation fam. of curves
$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\ 0 \arrow[r, phantom, "\in"] & B \end{tikzcd}$$

s.t. $\begin{tikzcd} \mathcal{C}^{[n]} \arrow[d] \\ B \end{tikzcd}$ & rel. cpt. Jac $\begin{tikzcd} \mathcal{Y} \arrow[d] \\ B \end{tikzcd}$

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are nonsing.

$$\mathcal{C}^{[n]} \longrightarrow S^{[n]}$$

Heisenberg action on $H_*(C^{[n]})$

(cf. Nakajima, Grojnowski for $H_*(S^{[n]})$)

Define 4 operators

$$\mu_\pm[pt] : H_*(C^{[n]}) \longrightarrow H_{*-1\pm1}(C^{[n\pm1]})$$

$$\mu_\pm[C] : H_*(C^{[n]}) \longrightarrow H_{*+1\pm1}(C^{[n\pm1]})$$

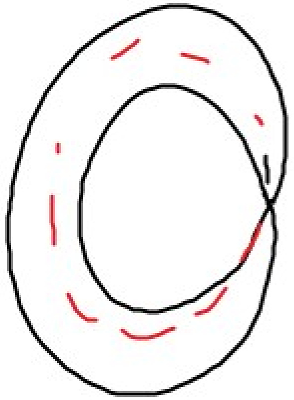
$\mu_\pm[pt]$: Fix ns $P \in C \rightsquigarrow i : C^{[n]} \hookrightarrow C^{[n+1]}$

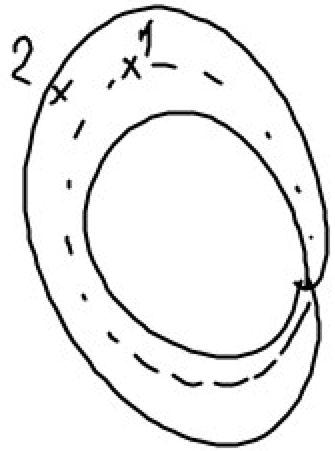
$$Z \longmapsto Z \cup P$$

$\underline{\text{Def}}$ $\mu_+[pt] = i_*$ $\quad \mu_-[pt] = i^!$ $\quad$ ($C^{[n]} \subset C^{[n+1]}$ Cartier div.)

$\mathcal{M}_{\pm}[C]$:

$\mathcal{M}_{+}[C]$





$$\alpha \in H_1(C^{[1]}) \longrightarrow H_3(C^{[2]})$$

$$\begin{tikzcd} & C^{[n,n+1]} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ C^{[n]} & & C^{[n+1]} \end{tikzcd} \quad := \{ (Z, Z') \in C^{[n]} \times C^{[n+1]} , Z \subset Z' \}$$

$$\mu_+[C] = q_* p^! \qquad \mu_-[C] = p_* q^!$$

<u>Def</u> $p^!, q^!$

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\ 0 \arrow[r, phantom, "\in"] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} C^{[n,n+1]} \arrow[r, hook] \arrow[d, "p"] & \mathcal{C}^{[n,n+1]} \arrow[d] \\ C^{[n]} \arrow[r, hook] & \mathcal{C}^{[n]} \end{tikzcd}$$

ns.

$$p^!(\alpha) = \alpha \cdot [\mathcal{C}^{[n,n+1]}] \quad \text{ref. int. prod.}$$

$V(C) := \bigoplus_{n\geq 0} H_*(C^{[n]})$, ops. act on $V(C)$

<u>**Thm (R)**</u> i) The ops satisfy comm. rels.

$$[\mu_\mp(pt), \mu_\pm[C]] = \pm\mathrm{id}$$

all other pairs commute.

ii) $W := \ker \mu_-(pt) \cap \ker \mu_-[C]$. The map

$$W \otimes \mathbb{Q}[\mu_+(pt), \mu_+[C]] \longrightarrow V(C)$$

is iso.

iii) The map $AJ : C^{[n]} \longrightarrow J \rightsquigarrow AJ_* : V(C) \longrightarrow H_*(J)$

induces $W \xrightarrow{\sim} H_*(J)$.

$V(C)$ is gr. by $V(C)_n = H_*(C^{[n]}) \rightsquigarrow W$ is n-gr.

$\rightsquigarrow H_*(J)$ is n-gr. $H_*(J) = \bigoplus H_*(J)_n$

<u>Cor</u> (Maulik-Yun, Migliorini-Shende '11)

$$H_*(C^{[n]}) = \bigoplus_k H_*(J)_k \otimes \mathrm{Sym}^{n-k}(\mathbb{Q} \oplus \mathbb{Q}[-2])$$

$$\begin{array}{ccc} C \longrightarrow \mathcal{C} & & J \hookrightarrow \mathcal{J} \\ \downarrow & \rightsquigarrow & \downarrow \quad \downarrow f \\ B & & 0 \underset{i}{\hookrightarrow} B \end{array}$$

$Rf_* \mathbb{Q}_{\mathcal{J}} \in D^b_c(B)$ has perverse coh. filtr.

$i^* Rf_* \mathbb{Q}_{\mathcal{J}} = H^*(J)$ —— $\shortparallel$ ——

Pf. of comm rels.

Ops. are correspondences

$$\begin{tikzcd} & C^{[n-1,n]} \arrow[dl] \arrow[dr] & \\ C^{[n]} & C^{[n-1]} \arrow[l, hook'] & C^{[n]} \end{tikzcd}$$

$\rightsquigarrow$ Compute composition as corr.

E.g. $[\mu_-[pt], \mu_+[c]] = id$

$\mu_-[pt]\,\mu_+[c]$

$$\begin{tikzcd} & C^{[n,n+1]} \arrow[dl] \arrow[d] & X \arrow[l, "="'] \arrow[d] \arrow[dl, phantom, "\lrcorner"] \\ C^{[n]} & C^{[n+1]} & C^{[n]} \arrow[l, hook'] \end{tikzcd}$$

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ C^{[n]} & & C^{[n]} \end{tikzcd}$$

$\mu_+[C]\,\mu_-[pt]$

$$\begin{tikzcd} & C^{[n-1,n]} \arrow[dl] \arrow[dr] & \\ C^{[n]} & & C^{[n]} \end{tikzcd}$$

Im. of X = Im. of $C^{[n-1,n]} \cup \Delta$

$[X] = [C^{[n-1,n]}] + [\Delta]$

iii) Follows from $C^{[n]} \xrightarrow{AJ} J$ is $\mathbb{P}^{r}$-bdl.

$\rightsquigarrow \dim W = \dim H_*(J)$.

$AJ_* : W \to H_*(J)$ inj. <u>easy</u>.

$H^*(J) = \bigoplus_n H^*(J)_n$

$H^*(J)_{\leq n}$ is per. filt.

$H^*(J)_{\geq n}$?

Conj (M-Y) $F_n = H^*(J)_{\geq n}$

F is finest filtr. s.t. hard Lefschetz holds for $gr_F H^*(J)$ wrt. action of $\Theta \in H^2(J)$.

True if $AJ_*([C^{[n]}]) = \frac{\Theta^n}{n!} \in H_*(J)$.