2010 Workshop on Algebraic Geometry and Physics

DEFORMATION, QUANTIZATION AND ALGEBRAIC INDEX THEOREMS

Saint Jean de Monts, France, June 7 to 11, 2010

Abstracts of talks

Anton Alekseev: Deformation quantization and flat connections. I shall describe a flat connection on the configuration space of points in the plane which naturally arises from Kontsevich's graphical calculus. This connection is a close relative of the Knizhnik-Zamolodchikov connection, and defines a new explicit solution of Drinfeld's associator axioms. This talk is based on a joint work with Charles Torossian, and on recent results by Pavol Severa and Thomas Willwacher.

Mihai Anastasiei: Deformation quantization of Hamilton-Lagrange spaces and Finsler geometry. We provide a method for converting Lagrange and Finsler spaces and their Legendre transforms to Hamilton and Cartan spaces into almost Kähler structures on tangent and cotangent bundles. In particular cases, the Hamilton spaces contain nonholonomic lifts of (pseudo) Riemannian/Einstein metrics on effective phase spaces. This allows us to define the corresponding Fedosov operators and develop deformation quantization schemes for nonlinear mechanical and gravity models on Lagrange and Hamilton-Fedosov manifolds

Paul Bressler: Formality for algebroids. The celebrated formality theorem of Kontsevich establishes a bijection beween the collection of equivalence classes of (non-commutative) deformations of the algebra of functions on a manifold on the one hand, and the collection of equivalence classes of deformations of the trivial Poisson bracket. I will describe a generalization of this result to algebroids which are twisted forms of the sheaf of functions on the manifold (a.k.a. gerbes).

Damien Calaque (minicourse): Duflo's isomorphism and Căldăraru's conjecture. These lectures will be about joint work with C. Rossi and M. Van den Bergh. We will start with a reminder on the Duflo isomorphism, its (co)homological variants, and an analogous result in algebraic geometry known as Căldăraru's conjecture. We will present a uniform approach to these statements using sheaves of DG commutative algebras. We will then review the natural algebraic structures one can put on Hochschild chains and cochains, and finally sketch the proof of a very general Duflo-type isomorphism for sheaves of DG commutative algebras. It will make use of Kontsevich's weights and formal geometry, two ingredients of particular importance in deformation quantization.
Main references (if needed):

- Hochschild cohomology and Atiyah classes, arXiv:0708.2725
- Compatibility with cap-products in Tsygan's formality and homological Duflo isomorphism, arXiv:0805.3444
- Caldararu's conjecture and Tsygan's formality, arXiv:0904.4890

• Lectures on Duflo isomorphisms in Lie algebras and complex geometry.

Andrea D'Agnolo (minicourse): D-modules and (deformation-)quantization. We shall describe the (deformation-)quantization algebroid of a complex symplectic manifold X using the algebras of microdifferential operators on local contactifications. For X compact, regular holonomic quantization modules provide a Calabi-Yau category of the same dimension as X. On the other hand, functorial operations on deformation-quantization modules extend to the Poisson case and provide a powerful framework for dealing with index theorems.

Essential bibliography:
M. Kashiwara, "Quantization of contact manifolds", Publ. Res. Inst. Math. Sci. 32 (1996), 1–7
M. Kashiwara, "D-modules and microlocal calculus", Translations of Mathematical Monographs, 217, 2003.
M. Kashiwara and P. Schapira, "Deformation quantization module", `arXiv:1003.3304`
P. Polesello and P. Schapira, "Stacks of quantization-deformation modules on complex symplectic manifold", Int. Math. Res. Not. 49 (2004), 2637–2664.

Giuseppe Dito: Wave maps and deformation quantization. Abstract: An approach for constructing deformations on the space of Cauchy data of covariant wave equations will be presented. It is based on existence results of wave maps for a large family of classical wave equations relevant to physics. We will discuss a candidate for the "Wightman distributions" in this context and present some aspects of their perturbative expansion.

Benjamin Enriquez: Elliptic associators. We develop an elliptic analogue of associator theory. We introduce the notion of an elliptic structure over a braided monoidal category (such structures give rise to representations of braid groups in genus 1). The corresponding automorphism group is an elliptic analogue GT_{ell} of the Grothendieck-Teichmueller group GT and can be expressed as a semidirect product of GT with a radical. We introduce a scheme $Ell(-)$ of elliptic associators, which is a torsor under the action of a prounipotent version $GT_{ell(-)}$ of GT_{ell}. This enables to prove that the Lie algebra of GT_ell is isomorphic to a graded Lie algebra grt_{ell}. We exhibit elements in $Ker(grt_{ell} \to grt)$ and conjecture that they generate this Lie algebra. We relate our construction to the universal KZB connection introduced in our earlier joint work with Calaque and Etingof, by constructing a map from the Poincaré half-plane to $Ell(CC)$. This enables us to express a morphism $GL_2(ZZ) \to GRT_{ell}(CC)$, analogous to the morphism $\{1, -1\} \to GRT(CC)$ underlying the construction of Drinfeld generators for grt_1, in terms of iterated Mellin transforms of Eisenstein series. This gives rise to relations between these numbers and MZV's.

Boris Fedosov: On a spectral theorem for deformation quantization. We give a construction of an eigenstate for a non-critical level of Morse Hamiltonian function and define the contributions of critical points to the spectral decomposition giving the dicrete spectrum. For these discrete eigenstates we develop a perturbation theory resembling that of quantum field theory. We also dicuss some links to the index formula for deformation quantization.

Yaël Frégier: Formal geometry approach to homotopy representations and layer cake homotopy algebras. Representations up to homotopy of Lie algebras have recently attracted much attention. On the other hand J. Baez has introduced a way to build a homotopy Lie algebra out of a Lie algebra and an n-cocycle. We shall show in this talk a common framework enabling to generalize both notions and extend them to other types of algebras. This is a joint work with J. Baez.

Dmitri Gurevich: Braiding geometry arising from a quantization. By braided geometry I mean a sort of geometry dealing with a braiding (a solution of the Quantum Yang-Baxter Equation) instead of the usual flip. In my talk I plan to consider braided analogs of certain classical objects and operators (traces, Lie algebras, orbits of coadjoint action of $GL(n)$) arising from a quantization. In particular, I shall discuss the problem of regularity of "braided orbits".

Giovanni Landi: Dimensional reduction over the quantum sphere and non-abelian q-vortices. We extend equivariant dimensional reduction techniques to the case of quantum spaces which are the product of a Kaehler manifold M with the quantum two-sphere. We work out the reduction of bundles which are equivariant under the natural action of the quantum SU(2) group, and also of invariant gauge connections on these bundles. The reduction of Yang–Mills gauge theory on the product space leads to q-deformation of the usual quiver gauge theories on M. In particular, we formulate generalized instanton equations on the quantum space and show that they correspond to q-deformations of the usual holomorphic quiver chain vortex equations on M. We study some topological stability conditions for the existence of solutions to these equations, and prove that the corresponding vacuum moduli spaces are generally better behaved than their undeformed counterparts, but much more constrained by the q-deformation. We work out several explicit examples, including new examples of non-abelian vortices on Riemann surfaces, and q-deformations of instantons whose moduli spaces admit the standard hyper-Kaehler quotient construction.

Takashi Kimura: On stringy algebraic structures in equivariant K-theory. We will introduce and explain some new algebraic structures associated to a smooth variety with a proper action of an algebraic group that are invariants of the associated quotient stack. Such algebraic structures arise from their orbifold K-theory ring, a K-theoretic variant of Chen-Ruan orbifold cohomology (which can be interpreted as a kind of equivariant topological field theory), and it suggests some interesting questions and directions.

Pierre Martinetti: Noncommutative geometry distance in the Moyal plane. We investigate the metric aspect of the Moyal plane, from the point of view of the distance formula introduced by Connes in noncommutative geometry (which is a generalization to the noncommutative setting of the Monge-Kantorovich distance in optimal transport theory). We show that the eigenstates of the quantum harmonic oscillator form a one dimensional lattice inside the set P of pure states of the Moyal algebra. The diameter of P is infinite, and the topology induced by the distance is not the weak* topology. Thus the Moyal plane is not a quantum metric space in the sense of Rieffel. However we show how to truncate the Moyal plane so to obtain compact quantum metric spaces.

Christoph Nölle: Deformation and geometric quantization. There are two main approaches to the quantization of a general symplectic manifold. The formal deformation quantization gives rise to a non-commutative deformation of the algebra of smooth functions, whereas the emphasis of geometric quantization is on the construction of a Hilbert space, the "space of states". In the talk I will sketch an idea how to relate the two constructions, focusing on the example of cotangent bundles, which admits explicit calculations.

Michael Pezvner: Spectral approach to composition formulas. We shall discuss an alternative approach to the composition formulas (*-products) of quantized operators based on the representation theory of the underlying Lie groups. Two examples leading to interactions with number theory will be presented.

Hessel Posthuma: The higher index theorem for orbifolds. I will discuss the higher algebraic index theorem on orbifolds using the language of groupoids. After that, I will explain the definition of higher indices of elliptic operators on orbifolds and how these can be computed using the algebraic index theorem.

Vladimir Salnikov: (Twisted) Poisson sigma model and equivariant cohomology. Inspired by the relation of gauging the Wess-Zumino term in the G/G WZW model with equivariant cohomology we try to implement a similar idea for the twisted Poisson sigma model. By reformulating and generalizing the usual formalism into a supergeometrical setting, we show that the latter sigma model can be likewise understood as an (appropriately defined) equivariantly closed extension of the 3-form used as a Wess-Zumino term. An important role in this context is played by a Lie algebra extension of the infinite dimensional Lie algebra corresponding to the Dirac structure of the twisted Poisson manifold by 1-forms and 2-tensors.

Pierre Schapira: Quantization of Hamiltonian isotopies and application to non-displaceability results in symplectic topology. The classical Arnold conjecture on the non-displaceability of the zero-section of the cotangent bundle to a compact manifold has been proved for long, but recently Tamarkin presented a new approach based on the microlocal theory of sheaves. For that purpose he had to adapt this theory which relies on the homogeneous symplectic (contact) structure to the non homogeneous (symplectic) case. Here we remain in the homogeneous setting which makes the proofs much easier. The main tool is a quantization of Hamiltonian isotopies in the framework of sheaves. This is a joint work in progress with Stéphane Guillermou and Masaki Kashiwara.

Martin Schlichenmaier: Berezin-Toeplitz quantization of moduli spaces. As was shown by Bordemann, Meinrenken and Schlichenmaier, the Berezin-Toeplitz (BT) operator quantization and its associated star product give a unique natural quantization for a quantizable compact Kaehler manifold. In the talk an overview of BT quantization is given. The procedure is applied to the moduli space of gauge equivalence classes of SU(N) connections on a fixed Riemann surface. In the language of algebraic geometry this moduli space is the moduli space of semi-stable vector bundles over a smooth projective curve. In this context the Verlinde spaces and the Verlinde bundle over Teichmueller space show up. Recent results of J. Andersen on the asymptotic faithfulness of the representation of the mapping class group on the space of covariantly constant sections of the Verlinde bundle are presented.

Artur Sergyeyev: Generalized Stäckel transform: an integrability-preserving transformation for finite-dimensional dynamical systems. We introduce multiparameter generalized Stäckel transform – a noncanonical transformation of a special kind relating the sets of (not necessarily commuting) integrals of motion and leaving the phase space coordinates intact. It is shown that under certain conditions the transformation in question preserves Liouville integrability, noncommutative integrability and superintegrability. For instance, when applied to an n-tuple of Poisson commuting Hamiltonians, this transformation yields a (new) n-tuple of Poisson commuting Hamiltonians. In this way one can construct new integrable systems from the old ones and also find new links among known integrable systems using the transformation under study. The associated transformation for the equations of motion proves to be a reciprocal transformation of a special form. This is joint work with Maciej Blaszak.

Junwu Tu: Matrix factorizations via Koszul duality. Recently, two approaches in the study of the category of matrix factorizations (also called Landau-Ginzburg models in physics) have appeared. It is expected that these two approaches should be related by Koszul duality. We will confirm this assertion in this talk. Along the way, we also recover several known results in a conceptual way and explain a subtlety when computing the Hochschild (co)homology of matrix factorizations.

Sergiu Vacaru: Commutative Ricci flows of Fedosov-Finsler Spaces and quantum gravity. Using nonholonomic distributions, the Einstein, Finsler and Lagrange spaces, and generalizations, can be equivalently represented as almost Kähler manifolds. The goals of this communication are: 1) to review some models of Fedosov-type deformation quantization and 2) Ricci flows theory of Einstein and Fedosov-Lagrange-Hamilton spaces; 3) to provide an unified scheme both for noncommutative geometry and geometric evolution theories. Such constructions can be related to standard theories in physics working on nonholonomic (semi) Riemannian manifolds and their quantum deformations (we use some important methods from the geometry of nonlinear connections originally developed in Finsler-Riemannian geometry). Alternatively, various types of theories with violation of local Lorentz symmetry can be considered if the main geometric arena is derived from (non) commutative tangent/vector bundle spaces (such studies also present interest in modern geometry and physics). Finally, we shall outline a general geometric metric (of noholonomic deformations) of constructing exact solutions in gravity and Ricci flow theories with nonholonomic and/or quantum variables.

Dmitri Vassilievich: Covariant star-products. I will introduce star products on tensor fields over a symplectic manifold, demonstrate that under certain conditions such product is practically unique, and discuss applications to noncommutative gravities.

Boris Zhilinskii: Qualitative theory of quantum energy level patterns and their rearrangements. A short review of the qualitative approach to the description of generic patterns of molecular energy levels will be given. The relation between Hamiltonian monodromy and the rearrangements of energy bands under variation of a control parameter will be the central point of the discussion. Purely classical, semi-classical, semi-quantum and totally quantum approaches are used together in order to understand the universality of the observed physical phenomenon of redistrubution and to generalise it and to find appropriate mathematical tools for its modeling. On this way such new notions as fractional monodromy and bidromy are introduced. The description of rearrangement of bands within semi-quantum model leads naturally to the mathematical question of quantization of vector bundles and to generic modification of Chern classes for bands within parametric families of Hamiltonians. Illustration of mathematical models will be given by using spectra of relatively simple molecular and atomic systems (hydrogen atom in fields, CO2, or CH4 molecules).

Friedrich Wagemann: On Lie group cohomology. This is a report on joint work with Christoph Wockel (Hamburg). There are several competing definitions of Lie group cohomology. Let us consider two of them. Let G be a (possibly infinite dimensional) Lie group and A a smooth G-module. On the one hand, locally smooth cohomology takes into account group cochains $c : G^p \to A$ which are smooth only in a neighborhood of (1,...,1) in G^p. On the other hand, Brylinski shows how to construct a simplicial sheaf out of A on the simplicial manifold BG associated to G. Differentiable cohomology (in the sense of Brylinski) is then by definition the hypercohomology of this sheaf. We show how to compare the two cohomologies and some cases in which they give the same result.