A Geometric Introduction to F–Theory

Lectures Notes
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Introduction

In these notes we give a general introduction to F–theory including some of the more recent developments.

The focus of the lectures is on aspects of F–theory which are potentially relevant for the *real world phenomenology.* In spite of this, we find convenient to adopt complex geometry as the basic language and tool. Indeed, geometry is the easiest and more illuminating language to do physics ($\equiv$ to compute basic observables) in this context. Each phenomenological requirement may be stated as a geometrical property, and the subtle relations between the different physical mechanisms become much more transparent when reinterpreted goemetrically.

Phenomenology. From the phenomenological standpoint adopted here, the aim of F–theory is to reproduce — starting from a fully consistent UV complete quantum theory containing gravity — the Minimal Supersymmetric Standard Model (MSSM), which is a theory already much studied in the phenomenological literature, and which is generally considered a viable and promising possibility for the physics beyond the standard model.

Viewed as a field theory, the MSSM has many free parameters. Starting from the more fundamental F–theory, one would hope to be able to predict most of these parameters and to compare them with the 'experimental' values.

For the gauge couplings associated with the three factor groups of the SM, $SU(3)$, $SU(2)$, and $U(1)_Y$, the fundamental theory has to explain the remarkable fact that they seem to unify at a scale which is significantly lower than the Planck scale, and this without introducing unwanted aspects (such as: colored Higgs fields, proton decay,...).

For the Yukawa couplings it has to explain the well–known pattern of the fermionic masses (the jerarchy $m_\tau : m_\mu : m_e$) and of CKM matrices.

Of course, to make contact with the real world, we also need a viable supersymmetry breaking mechanism, which should produce the right soft SUSY breaking terms, with coefficients of the correct size.

Phenomenologically, the basic assumption in the game is that supersymmetry is broken to a scale low enough that an intermediate *supersymmetric* effective field theory — like the MSSM — is physically relevant. If this is not true in the real world, F–theory still remains a good candidate for the fundamental theory, but our ability to extract explicit phenomenological conclusions out of it (namely, to compute something measurable) is greatly diminished.

Typically, string theory is not very predictive phenomenologically: There are so many interesting vacua, each with its own low–energy physics, which virtually any outcome of an experiment may be consistent with the theory, in one region or the other of its huge vacuum landscape. In this respects, F–theory (as recently applied to real world physics by Cumrun Vafa and coworkers) looks quite the opposite: the relevant solutions are very few, and the expertimental predictions are expected to be rather sharp.

Prerequisites and reading conventions

Generally speaking, the present notes are a follow–up and an application of my SISSA course *Geometric Structures in Supersymmetric (Q)FTs*, whose lecture notes (available on–line) will be referred to as [GSSFT].

Students familiar with the kind of material covered in [GSSFT] (in its more recent versions) should have no problem in attending the present course.

Alternatively, it will suffice for the student to have a rather vague knowledge of the basic material covered (say) in Chapters 0 and 1 of P. Griffiths and J. Harris, *Principles of Algebraic Geometry.*

Reading conventions.

- An asterisk ($*$) in the title of a section/subsection means that it is additional material that is *wise not to read.*
- ADDENDUM in the title of a section/subsection means that it is spurios material that *only a crazy guy will read.*
- The symbol ($\mathbb{J}$) in the title of a section/subsection means that it is very well known stuff that everybody would prefer *to jump over.*
- Remarks/footnotes labelled *'for the pedantic reader'* are really meant for this category of very attentive readers.

CHAPTER 1

From Type IIB to F–theory

In this introductory chapter we explain how F–theory arises as a *non–perturbative completion* of the usual Type IIB superstring. The first two sections are very quick reviews of well–known facts, stated in a language convenient for our (geometric) purposes. Starting from section 3 we try to be more detailed and precise.

By a *non–perturbative completion* of Type IIB supertring we mean any theoretical scheme with a *fully consistent* interpretation as a *physical theory* which agrees with the Type IIB superstring in some asymptotic limit. Of course, we do not have a proper non–perturbative definition of F–theory. F–theory, like its female[1] couterpart M–theory, is a "misterious" object. But this is not a limitation to our ability to make experimental predictions out of it. One starts by determining some *necessary conditions* that any consistent completion of Type IIB should satisfy, the most important being: *i)* $(2,0)$ $D = 10$ *local* supersymmetry, and *ii)* Cumrun Vafa's finite volume condition. Local supersymmetry requires the presence of a massless gravitino and hence of a massless graviton. The infrared couplings of a massless spin–two particle is governed by universal theorems; applying $(2,0)$ SUSY to them, we get theorems governing the soft–physics of light states of any spin. The precise form of the soft theorems is governed by the Vafa finite volume property.

True, we get just infrared theorems. But 'infrared' here means all the physics up to the Planck scale.

The original paper about F–theory is reference [**1**].

1. Type IIB superstring

1.1. The massless spectrum. Type IIB superstring is a theory of supersymmetric closed oriented strings in $D = 10$ space–time having, from the space–time viewpoint, $(2,0)$ supersymmetry. That is, we have *two* Majorana–Weyl supercharges of the *same* ten–dimensional chirality (say $\Gamma_{11}Q = +Q$). The number of real supercharges is then 32. The $(2,0)$ SUSY algebra has a $U(1)_R \simeq SO(2)$ automorphism group rotating the two supercharges.

[1] According to a tradition, M–theory stands for *the Mother of all theories,* while F–theory is *the Father of all theories.*

The $D = 10$ $(2,0)$ superalgebra has a unique linear representation (supermultiplet) with spins ≤ 2. This supermultiplet is automatically massless and contains a graviton. In terms of representations of the little bosonic group $SO(8) \times U(1)_R$, the $D = 10$ $(2,0)$ massless graviton supermultiplet decomposes as (cfr. the $D = 10$ $(2,0)$ entry in Table 1 of ref.[**2**])

$$\begin{aligned} 2^8 = \mathbf{1}_{-4} \oplus (\mathbf{28}_v)_{-2} \oplus (\mathbf{35}_v)_0 \oplus (\mathbf{35}_-)_0 \oplus (\mathbf{28}_v)_2 \oplus \mathbf{1}_4 \oplus \\ \oplus (\mathbf{8}_+)_{-3} \oplus (\mathbf{56}_+)_{-1} \oplus (\mathbf{56}_+)_1 \oplus (\mathbf{8}_+)_3, \end{aligned} \tag{1.1}$$

where the first line corresponds to bosonic states and the second to the fermionic ones.

By supersymmetry, eqn.(1.1) also corresponds to the massless spectrum of Type IIB superstring (in flat $D = 10$ space).

1.2. Light fields. The massless fields arising from the superstring Neveu–Schwarz–Neveu–Schwarz (NS–NS) sector are easily read from their covariant vertices. They are the metric $g_{\mu\nu}$, a two–form $B_{\mu\nu}$ with the Abelian gauge symmetry $B \to B + d\Lambda$, and the dilaton ϕ.

The Ramond–Ramond (R–R) massless spectrum can be read directly from the covariant $2d$ superconformal vertices of the associated field–strengths[2]

$$V(p)_{\mu_1\mu_2\cdots\mu_k} := S_\alpha \, (C\Gamma_{\mu_1\mu_2\cdots\mu_k})^{\alpha\beta} \, \widetilde{S}_\beta \, e^{-(\varphi+\widetilde{\varphi})/2} \, e^{ip\cdot X} \tag{1.2}$$

where S_α (resp. $\widetilde{S}_\alpha$) is the left–moving (resp. right–moving) spin–field, $\varphi, \widetilde{\varphi}$ are the $2d$ chiral scalars bosonizing the superconformal ghosts, and α is a Weyl spinor index taking 16 values.

From the Dirac matrix algebra it follows that $(C\Gamma_{\mu_1\mu_2\cdots\mu_k})^{\alpha\beta}$ is not zero if and only if k is odd. Moreover,

$$(C\Gamma_{\mu_1\mu_2\cdots\mu_k})^{\alpha\beta} = -\frac{(-1)^{k(k-1)/2}}{(10-k)!} \, \epsilon_{\mu_1\cdots\mu_k\mu_{k+1}\cdots\mu_{10}} (C\Gamma^{\mu_{k+1}\mu_{k+2}\cdots\mu_{10}})^{\alpha\beta}, \tag{1.3}$$

Thus: *In type IIB, the field–strenghts of the R–R massless bosons are forms of* odd *degree* $k = 1, 3, 5, 7, 9$. *The field–strenghts of degree* k *and* $10 - k$ *are dual. In particular, the field strength of degree* 5 *is (anti)self–dual.* To avoid any missunderstanding, we stress that this result holds *at the linearized level* around the trivial (flat) background.

The independent R–R potentials are a zero–form C_0 (the *axion*), a second two–form C_2, and a 4–form C_4 whose field–strength satisfies a (non–linear version of the) self–duality constraint.

Finally, the fermions are two Majorana–Weyl gravitini of chirality[3] -1 and two Majorana–Weyl fermions of chirality $+1$ (called *dilatini*).

[2] That is: the vertex of the $(k-1)$–form field, $C_{k-1}^{\mu_1\cdots\mu_{k-1}}$, is $W_{\mu_1\cdots\mu_{k-1}} := V_{\mu_1\mu_2\cdots\mu_k} p^{\mu_k}$ which is automatically transverse, $p^{\mu_1} W_{\mu_1\cdots\mu_{k-1}} = 0$, as required for the vertex of a gauge field (to decouple the longitudinal component).

[3] In the standard conventions. See [**3**].

1.3. Anomalies. The theory is chiral, so we may wonder about anomalies. However the field content is such that the gravitational anomalies cancel. The contribution from a single chiral 4–form[4] precisely cancel those from the chiral gravitini and dilatini [**4**].

More precisely, according to the general *Russian formula* [**5**], we can encode the contribution of a chiral field χ to the gravitational anomalies of a $D = 10$ theory in a 12–form $I_\chi(R)$. For a chirality $+1$ Majorana–Weyl spinor λ,

$$I_\lambda(R_2) = \frac{\mathrm{tr}(R^6)}{725760} - \frac{\mathrm{tr}(R^4)\,\mathrm{tr}(R^2)}{552960} + \frac{\big[\mathrm{tr}(R^2)\big]^3}{1327104} \tag{1.4}$$

while for a chirality $+1$ Majorana-Weyl gravitino

$$I_\psi(R_2) = -495\,\frac{\mathrm{tr}(R^6)}{725760} - 225\,\frac{\mathrm{tr}(R^4)\,\mathrm{tr}(R^2)}{552960} - 63\,\frac{\big[\mathrm{tr}(R^2)\big]^3}{1327104} \tag{1.5}$$

and for a self–dual 4–form

$$I_C(R_2) = 992\,\frac{\mathrm{tr}(R^6)}{725760} + 448\,\frac{\mathrm{tr}(R^4)\,\mathrm{tr}(R^2)}{552960} + 128\,\frac{\big[\mathrm{tr}(R^2)\big]^3}{1327104} \tag{1.6}$$

In our convention, the chiral fields of the theory are: two Majorana–Weyl spinors of chirality $+1$, two gravitini of chirality -1, and an *anti*selfdual 4–form. Hence the total gravitational anomaly is

$$2\,I_\lambda(R) - 2\,I_\psi(R) - I_C(R) \equiv 0, \tag{1.7}$$

and the theory is anomaly–free.

1.4. R–R charges. The fact that in the $2d$ superconformal theory we have directly the vertices $V(p)_{\mu_1\mu_2\cdots\mu_k}$ for the R–R *field–strenghts,* rather than those for the *form potentials,* means that no perturbative state carries charges (either electric or magnetic) with respect to these gauge fields C_k. However, the string theory does contain objects — with masses of order $O(1/g)$, and hence non–perturbative in the string coupling[5] g — which are electrically and magnetically charged with respect to the R–R gauge fields, the most well–known such objects being the D–branes [**6**].

The fact that both electric and magnetic charges are present, implies a Dirac–like quantization condition. Hence the R–R charges must take values in a suitable integral lattice. This fact will be crucial below.

[4] By a *chiral* k–form in $D = 2k + 2$ we mean a k–form A whose field strenght dA is (anti)self–dual.

[5] Although they may be perturbative from other points of view.

2. The low–energy effective theory

The massless sector of the theory contains, in particular, the graviton and two gravitini. Consistency then requires the low–energy effective theory to be a *supergravity.* Indeed, the field contend we deduced in §. 1.1 is precisely that of the Type IIB supergravity. This is a tricky field theory, even at the classical level. The fact that it contains a *chiral* four–form C_4, means that it has no *standard* covariant Lagrangian formulation. For our purposes, the (covariant) equations of motion will suffice, and we will not attempt subtler constructions.

2.1. The global $SL(2,\mathbb{R})$ symmetry. The formulation of the effective supergravity theory is simplified once we understand the large symmetry it should enjoy.

2.1.1. *The scalars' manifold $\mathcal{M}$.* As already mentioned above, the $(2,0)$ superPoincaré algebra in $D = 10$ — much as the $\mathcal{N} = 1$ SUSY algebra in $d = 4$ — has a $U(1)_R$ automorphism group. Just as in $d = 4$, this fact implies that the scalars' manifold $\mathcal{M}$ should be a Kähler space. This is already evident from eqn.(1.1): The two scalars have charge ± 4 under $U(1)_R$; on general grounds, we know that the R–symmetry group $U(1)_R$ should act as (part of) the holonomy group of $\mathcal{M}$ (see [GSSFT] for full details).

Besides, $\mathcal{M}$ should be *locally* isometric to a symmetric manifold[6], and negatively curved[7]. The theory has two physical scalars, ϕ and C_0, so $\dim_{\mathbb{R}} \mathcal{M} = 2$. By the Riemann uniformization theorem, there is only one *simply–connected* such manifold, namely the *upper half–plane*

$$\mathcal{H} = SL(2,\mathbb{R})/U(1) = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}, \tag{2.1}$$

equipped with the Poincaré $SL(2,\mathbb{R})$–invariant metric

$$ds^2 = \frac{dz\, d\bar{z}}{\big(\operatorname{Im} z\big)^2}. \tag{2.2}$$

The group $SL(2,\mathbb{R})/\{\pm 1\}$ acts on $\mathcal{H}$ by Möbius transformations,

$$z \mapsto \frac{a\,z+b}{c\,z+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}), \tag{2.3}$$

which are easily seen to be isometries of the Poincaré metric (2.2).

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[6] *Why?* Reduce the theory to $d = 3$ on a flat seven–torus. We get a $\mathcal{N} = 16$ supergravity whose scalars' manifold M_3 should have holomomy Spin(16). By Berger's theorem, M_3 locally is a symmetric space. Since $\mathcal{M}$ is a totally geodesic submanifold of M_3, it is also locally symmetric. We refer to the Lectures Notes [GSSFT]. for details on the statements contained in the present footnote.

[7] *Why?* In the present case, the holonomy group of $\mathcal{M}$ is just the SUSY automorphism group $U(1)_R$. In SUGRA, the curvature of the connection gauging the automorphism group of any supersymmetry is always negative, as we see from its universal tt^* form (see again the Lectures Notes [GSSFT] for details).

Thus the most general scalars' manifold $\mathcal{M}$ compatible with $(2,0)$ $D=10$ supersymmetry is the double–coset

$$\boxed{\mathcal{M}=\Gamma\Big\backslash SL(2,\mathbb{R})\Big/U(1)} \tag{2.4}$$

where Γ is a discrete subgroup of $SL(2,\mathbb{R})$ (*i.e.* a Fucsian group).

Determining Γ is a far–reaching dynamical problem. Two configurations differing by the action of an element of Γ are *physically* identified, that is, *the action of* Γ *commutes with all physical observables.*

Then Γ should be an invariance of the *full* non–perturbative theory, not just of its massless sector ($\equiv$ IIB SUGRA). If we restrict to the strictly massless sector, the full group $SL(2,\mathbb{R})$ is a symmetry, but this is obviously not true in the complete theory[8].

Thus we learn that the scalars' kinetic terms should take the form (setting $z=x+iy$, with $y>0$)

$$y^{-2}\Big(\partial_\mu x\,\partial^\mu x+\partial_\mu y\,\partial^\mu y\Big). \tag{2.5}$$

Forgetting for the moment subtleties related to the global identifications under the action of the Fucsian group Γ, we see that

$$x \leftrightarrow -x \tag{2.6}$$

$$x \rightarrow x+\text{const.} \tag{2.7}$$

are symmetries of the scalars' sector (with all the other fields set to zero). They should be matched with the analogous properties of the perturbative amplitudes of the Type IIB superstring. Since, *at the perturbative level,* the axion C_0 may appear in the effective Lagrangian only trough its 1–form field–strenght $\partial_\mu C_0$, we are led to the identification

$$\mathrm{Re}\,z=\ \text{the axion field } C_0. \tag{2.8}$$

Then y should be a function of the dilaton field ϕ. Which function can be seen by requiring that the equations of motion, after setting the R–R fields to zero, are invariant under $\phi\mapsto\phi+$const. (since this would add to the $2d$ σ–model action just a topological term proportional to $\chi(\Sigma)$ which cannot affect the β–functions). Moreover, the SUGRA's σ–model fields, x and y, become free as $y\rightarrow\infty$. These two conditions uniquely fix $y^{-1}=e^{\phi}$ (up to an irrelevant additive constant in ϕ), and finally

$$\boxed{z=\ C_0+i\,e^{-\phi}} \tag{2.9}$$

[8] By a $SL(2,\mathbb{R})$ transformation, we may set the dilaton, hence the string coupling to any prescribed value, while from string perturbation theory we know that the tree–level physical amplitudes do depend on the string coupling.

2.1.2. *The global and local symmetries.* Quite generally (see [GSSFT] for details) one shows that, whenever in supergravity the scalars' manifold is a symmetric space G/H, the theory has a natural formulation with symmetry

$$G_{\mathrm{global}} \times H_{\mathrm{local}} \tag{2.10}$$

where we represent the scalars' fields as a map from spacetime to the Lie group G

$$x \mapsto \mathcal{E}(x) \in G. \tag{2.11}$$

$\mathcal{E}(x)$ is called the *vielbein.* The symmetry (2.10) acts on the scalars as

$$\mathcal{E}(x) \mapsto g\,\mathcal{E}(x)\,h(x), \qquad g \in G_{\mathrm{global}},\ h(x) \in H_{\mathrm{local}}. \tag{2.12}$$

H_{local} acts on the fermions as a *local* (*i.e.* gauged) R–symmetry and leaves invariant the non–scalar bosonic fields. Consistency requires the bosonic fields to organize in definite representations of G_{global}, while the fermions are inert under this global symmetry.

Specializing to Type IIB SUGRA, we have a symmetry

$$SL(2,\mathbb{R})_{\mathrm{global}} \times U(1)_{\mathrm{local}}. \tag{2.13}$$

The non–scalar bosonic fields should organize themselves into definite representations of $SL(2,\mathbb{R})_{\mathrm{global}}$. Since the action of $SL(2,\mathbb{R})_{\mathrm{global}}$ commutes with the Lorentz group, the action must be linear. Then the metric $g_{\mu\nu}$ and the self–dual 5–form field–strenght F_5 are automatically singlets. However, we have *two* three–form field–strenths, $H_3 = dB$ and $F_3 = dC_2$, and these may well transform in the 2–dimensional representation of $SL(2,\mathbb{R})_{\mathrm{global}}$.

The simplest way to see that this should be the case is the so–called *'target space equivalence principle'* advocated in [GSSFT]: Linearize the theory around a point in G/H which, up to symmetry, we may take in the equivalence class of the identity 1. This 'vacuum' is invariant under the diagonal subgroup $H \subset G_{\mathrm{global}} \times H_{\mathrm{local}}$. Since H_{local} is the automorphism group of the SUSY algebra, this diagonal action is precisely the one induced on the linearized spectrum by the SUSY automorphism group. The contend of the various supermultiplets in terms of H–representations can be read from the tables of algebraic (linear) representations of SUSY [**2**]. Notice that, since G/H is symmetric, H contains a *maximal torus* of G, thus from the H–representations we may read directly the G–weights, and hence reconstruct the G–representation contend of the corresponding field realization.

In the IIB case, the linear SUSY representation is given by eqn. (1.1). The two–forms, B and C_2, correspond to the $\mathbf{28}_v$ of $SO(8)$ (*i.e.* the antisymmetric representation $\wedge^2\mathbf{8}_v$). From eqn. (1.1) we see that they have $U(1)_R$ charges ± 2, which is half the charges of the scalars (± 4). From eqn. (2.11) it is obvious that the scalars correspond to the adjoint of $SL(2,\mathbb{R})_{\text{local}}$. Then ± 2 are the weights of the fundamental (doublet) representation.

In conclusion: *The two 2–form fields B and C_2 make a doublet under $SL(2,\mathbb{R})_{\text{global}}$.*

We write B_a ($a = 1,2$) for the two–forms corresponding to the standard basis of $SL(2,\mathbb{R})$. The $SL(2,\mathbb{R}) \simeq Sp(1,\mathbb{R})$ indices will be raised/lowered with the invariant symplectic tensor ϵ_{ab}.

We can use the vielbein $\mathcal{E}$ to convert the global $SL(2,\mathbb{R})$ indices into local $U(1)_R$ indices and *vice versa*[9]. In particular, we define the $U(1)_R$ covariant field–strenghts

$$G_3^{\pm} := \epsilon^{ab}\,(\mathcal{E})_a{}^{\pm}\,H_b, \qquad (H_a := dB_a) \tag{2.14}$$

which are inert under G_{global} and transform as

$$G_3^{\pm} \to e^{\pm 2i\alpha(x)}\,G_3^{\pm} \tag{2.15}$$

under $U(1)_{\text{local}}$.

Finally, we may read the $U(1)_{\text{local}}$ transformations of the fermions directly in eqn. (1.1). In a complex basis,

$$\psi_\mu \to e^{i\alpha(x)}\,\psi_\mu \tag{2.16}$$

$$\lambda \to e^{3i\alpha(x)}\,\lambda. \tag{2.17}$$

2.1.3. *Explicit formulae.* As in [GSSFT] we obtain the scalars' couplings by decomposing the Maurier–Cartan form[10]

$$\mathcal{E}^{-1}\partial_\mu\mathcal{E} = \sigma_{2a-1}\,P^a_\mu + i\,\sigma_2\,Q_\mu. \tag{2.18}$$

Q_μ is the $U(1)_R$ connection entering in the covariant derivatives acting on the fermions

$$D_\mu\psi_\nu = \left(\nabla_\mu - \tfrac{1}{2}Q_\mu\right)\psi_\nu \tag{2.19}$$

$$D_\mu\lambda = \left(\nabla_\mu - \tfrac{3}{2}Q_\mu\right)\lambda, \tag{2.20}$$

while P^a_μ is the pull–back to spacetime of the metric vielbeins on $\mathcal{M}$. Then the scalars' kinetic terms read just

$$\tfrac{1}{2}P^a_\mu P^{a\,\mu}. \tag{2.21}$$

To reduce this general expression to the Poincaré form, eqn. (2.2), we must fix a specific $U(1)_R$ gauge for the scalars[11]. This is the Iwasawa gauge [GSSFT]. Let us state the Iwasawa decomposition for $SL(2,\mathbb{R})$ (in the basis used in automorphic representation theory):

[9] This is why it is called a *vielbein*.

[10] σ_1, σ_2 and σ_3 are, of course, the standard Pauli matrices!

[11] The symmetries of the theory are much more manifest in the gauge independent formulation based on $\mathcal{E}$, than in its more frequently used gauged fixed version.

PROPOSITION 2.1 (Iwasawa decompostion for $SL(2,\mathbb{R})$, see §. 1.2 of ref. [**27**]). *Any $SL(2,\mathbb{R})$ matrix may be* <u>*uniquely*</u> *decomposed in the form*

$$\begin{pmatrix} y^{1/2} & x\,y^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad x, y, \theta \in \mathbb{R},\ y > 0. \tag{2.22}$$

Thus, as a choice of $U(1)_R \simeq SO(2)$ gauge, we may take

$$\mathcal{E} = \begin{pmatrix} y^{1/2} & x\,y^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}. \tag{2.23}$$

In this gauge the scalars fields are precisely x and y. Then, under $SL(2,\mathbb{R})_{\text{global}}$ acting on the left, $z = x + iy$ transforms as in eqn. (2.3) and the invariant metric takes the usual Poincaré form.

In this gauge, the relation between the H_a's and the $G^{\pm}$'s is

$$G^{\pm}(1 \mp \sigma_2) = (H_1, H_2)(i\sigma_2) \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} (1 \mp \sigma_2) \tag{2.24}$$

or

$$G^+ = -iy^{-1/2}(H_1 + z\,H_2) \tag{2.25}$$

$$G^- = \ iy^{-1/2}(H_1 + \bar{z}\,H_2). \tag{2.26}$$

The two field strenghts H_1 and H_2 appearing in these SUGRA expressions correspond to the field–strenghts $H_{RR} = dC_2$ and $H_{NSNS} = dB$ of the superstring. *Which is which?* In the low energy effective Lagrangian of the string the NSNS and RR field–strenghts appear with different powers of the string coupling e^{ϕ}. In the string frame, the NSNS terms have an overall factor $e^{-2\phi}$, while no such factor is present for the RR ones. In our present formalism, at $C_0 = 0$, the *Einstein frame* Lagrangian is proportional to

$$|G^+|^2 + \cdots = e^{\phi}\left(H_1^2 + e^{-2\phi}\,H_2\right) + \cdots \tag{2.27}$$

and hence in the string frame to

$$\left(H_1^2 + e^{-2\phi}\,H_2\right) + \cdots \tag{2.28}$$

from which we see that H_2 is the NSNS field H_{NSNS}, while H_1 is the RR field H_{RR}.

2.1.4. *Caley transform.* Another way of writing $SL(2,\mathbb{R})/SO(2) \simeq SU(1,1)/U(1)$ is as the unit disk with the Poincaré metric

$$\mathcal{D} := \{\, w \in \mathbb{C} \,|\, |w| < 1 \,\} \tag{2.29}$$

$$ds^2 = \frac{dw\,d\bar{w}}{(1 - |w|^2)^2}. \tag{2.30}$$

The two representations are related by the Caley transformation [GSSFT]

$$w = \frac{z - i}{z + i}. \tag{2.31}$$

In the new basis, an element of $SL(2,\mathbb{R}) \simeq SU(1,1)$ is written as

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \qquad \text{with } |a|^2 - |b|^2 = 1, \tag{2.32}$$

while $U(1)_{\text{local}}$ acts diagonally as $\exp\big(i\alpha(x)\,\sigma_3\big)$ on the right.

This Cayley rotated formulation is more frequent in the SUGRA literature [**3**], but it is less convenient for the present applications.

2.2. Higher gauge symmetries. From the free massless spectrum, we know that, *at the linearized level,* the field forms B_2^a ($a = 1, 2$) and C_4 should have a gauge–symmetry of the form

$$\delta B_2^a = d\Lambda_1^a + \cdots \tag{2.33}$$

$$\delta C_4 = d\Lambda_3 + \cdots\,, \tag{2.34}$$

with local parameters Λ_1^a (a 1–form which is a doublet of $SL(2,\mathbb{R})$) and Λ_3 (a singlet 3–form).

However, the gauge transformations may look quite different at the full non–linear level. The gauge transformation of B_2^a cannot have non–linear corrections when $C_4 = 0$, since in this case we can write a $2d$ σ–model (rotating B_2^a in the NS–NS direction), and $\delta B_2 = d\Lambda_1$ is an exact invariance of the $2d$ QFT. Moreover, the possible transformation laws are restricted by the conditions that the two gauge transformations commute. In conclusion, one infers that

$$\delta B_2^a = d\Lambda_1^a \tag{2.35}$$

is exact.

The second equation, (2.34), however, has a natural non–linear modification

$$\delta C_4 = d\Lambda_3 + a\,\epsilon_{ab}\, H_3^a \wedge \Lambda_1^b, \qquad \text{where } H_3^a \equiv dB_2^a, \tag{2.36}$$

where a is some numerical coefficient to be fixed. To compute it, and to verify that $a \neq 0$, one could procede in various ways: One can enforce the hidden E_8 symmetry [GSSFT], or ask for the closure of the gauge superalgebra as in the original paper [**3**]. These methods, although deep, are computationally very messy. So we look for a short–cut.

We start with the following observation: in $D = 9$ there is *only one* supergravity 'with 32 supercharges'. Hence the toroidal compactification of our $D = 10$ Type IIB SUGRA should agree with the toroidal compactification of the (unique) $D = 11$ SUGRA. As it is well–known [**7**], the $D = 11$ theory has a unique form–field, a 3–form C_3, which enters in the Lagrangian trough the usual kinetic term plus a cubic Chern–Simons coupling

$$\mathcal{L}_{D=11}\Big|_{\text{terms containing } C} = \tfrac{1}{2} F_4 \wedge *F_4 + \frac{1}{3}\lambda\, C_3 \wedge F_4 \wedge F_4, \tag{2.37}$$

where $F_4 \equiv dC_3$ and λ is a certain constant.

The field–forms of the $D = 9$ theory are as in the table

degree	from $D = 11$	from $D = 10$ Type IIB
1	$(C_3)_{\mu\,10\,11},\ g_{\mu\,10},\ g_{\mu\,11}$	$g_{\mu\,10},\ (B_2^1)_{\mu\,10},\ (B_2^2)_{\mu\,10}$
2	$(C_3)_{\mu\,\nu\,10},\ (C_3)_{\mu\,\nu\,11}$	$(B_2^1)_{\mu\,\nu},\ (B_2^2)_{\mu\,\nu}$
3	$(C_3)_{\mu\,\nu\,\rho}$	$(C_4)_{\mu\,\nu\,\rho\,10}$
4	–	$(C_4)_{\mu\nu\rho\sigma}$

The two columns agree. It may seem that the degree 4 line is a mismatch, but recall that, in Type IIB, C_4 is *not* an ordinary 4–form field propagating on–shell 70 degrees of freedom, but rather a *chiral* 4–form which propagates only 35 degrees of freedom. Now, a 4–form field in $D = 9$ propagates precisely 35 degrees of freedom, so the $D = 9$ field $(C_4)_{\mu\nu\rho\sigma}$ in the last line of the table propagates 35 degrees of freedom which (by duality) are physically identified with those associated to the 3–form field $(C_4)_{\mu\nu\rho\,10}$ (again 35 d.o.f.). Thus, in the second column we have a double–counting. Correcting this aspect, we have exact agreement.

In conclusion: we have *two* possible formulations of the $D = 9$ SUGRA, one with a 3–form field and one with a 4–form. The two formulations are related by a *duality transformation.* Let us choose the formulation with the 3–form which is more directly related to the $D = 11$ theory.

The equations of motion of the $D = 9$ 3–form field C_3 can be directly read from the $D = 11$ Lagrangian (2.37). Setting the 1–form fields to zero, we get

$$d\big(\ast_{11}\, dC_3\big) + \lambda\,\epsilon_{ab}\, dC_2^a \wedge dC_2^b = 0, \tag{2.38}$$

where C_k denote the k–form fields of the $D = 9$ sugra as obtained from the $D = 11$ perspective.

Now consider the *dual* formulation in terms of a 4–form field C_4. Under duality

$$\text{(equations of motion)} \;\longleftrightarrow\; \text{(Bianchi identities)}, \tag{2.39}$$

so eqn.(2.38) should be interpreted as the Bianchi identity for the *gauge invariant* field–strenght F_5 of C_4

$$F_5 = \ast_{11} dC_3 + \cdots \tag{2.40}$$

(the ellipsis being terms containing fields that we set to zero). The dual Bianchi identity then reads

$$dF_5 + \lambda\,\epsilon_{ab}\, H_3^a \wedge H_3^b = 0. \tag{2.41}$$

This Bianchi identity can be solved in terms of a 4–form C_4 as

$$F_5 = dC_4 - \lambda\,\epsilon_{ab}\,H_3^a \wedge B_2^b. \tag{2.42}$$

The gauge invariance of F_5 implies the following gauge transformation of C_4

$$\delta C_4 = d\Lambda_3 + \lambda\,\epsilon_{ab}\,H_3^a \wedge \Lambda_2^a, \tag{2.43}$$

which can then be lifted to $D = 10$, giving eqn.(2.36) with

$$a = \lambda. \tag{2.44}$$

As a normalization condition on the fields, we take $\lambda = 1/4$.

2.3. The complete equations of motion and susy transformation. Now we have all the ingredients to formulate the complete non–linear Type IIB SUGRA.

2.3.1. *Equations of motion of the bosonic fields.* Forgetting about the subtlety with the chiral 4–form, it is easy to write down a Lagrangian which has the above symmetries and leads to the correct equations of motion

$$\begin{aligned}\int \sqrt{-g}\Big(\tfrac{1}{2}R - \tfrac{1}{4}P_\mu\,\overline{P}^\mu - \tfrac{1}{4}|G^+|^2 - \tfrac{1}{4}|F_5|^2\Big) - \\ - \int \frac{\epsilon_{ab}}{4}\,C_4 \wedge H_3^a \wedge H_3^b + \text{fermions}\end{aligned} \tag{2.45}$$

Notice that the equations of motion of C_4 reads

$$d * F_5 - \frac{\epsilon_{ab}}{4}\,H_3^a \wedge H_3^b = 0 \tag{2.46}$$

Comparing with the Bianchi identity, we see that it is consistent to take F_5 to be anti–self–dual. Notice that the anti–self–duality condition should be imposed *after* varying the action. (More satisfatory formulations exist, but we shall not need them for our purposes).

For future reference, we write the scalars' and Einstein equations

$$\ldots\ldots\ldots\ldots\ldots \tag{2.47}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \tag{2.48}$$

2.4. Susy transformations. From eqn.(1.1), we see that in the normalization in which the SUSY parameter ϵ has $U(1)_R$ charge $+1$, the complex (Weyl) dilatino λ (with $\gamma_{11}\lambda = +\lambda$) has charge $+3$. From the same equation (and the 'target space equivalence principle') we see that the $U(1)_R$–covariant quantities P_μ and $G_{\mu\nu\rho} \equiv G^+_{\mu\nu\rho}$ have, respectively, $U(1)_R$ charge $+4$ and $+2$. Then the only locally covariant expression for the SUSY transformation of the dilatino is

$$\delta\lambda = i\gamma^\mu\epsilon^*\hat{P}_\mu - ia\,\gamma^{\mu\nu\rho}\epsilon\,\hat{G}_{\mu\nu\rho} \tag{2.49}$$

where a is a numerical constant and the *hat* stands for the covariantization of the derivatives with respect supersymmetry (see [GSSFT]). Explicitly,

$$\hat{P}_\mu = P_\mu - \bar{\psi}^*_\mu \lambda \tag{2.50}$$

$$\hat{G}_{\mu\nu\rho} = G_{\mu\nu\rho} - 3\bar{\psi}_{[\mu}\gamma_{\nu\rho]}\lambda - 6i\bar{\psi}^*_{[\mu}\gamma_\nu\psi_{\rho]}. \tag{2.51}$$

a can be fixed from the linearized theory to $1/24$ [**3**].

As always, the SUSY transformation of the gravitino is more involved. One gets [**3**]

$$\delta\psi_\mu = D_\mu\epsilon + \frac{i}{480}\gamma^{\rho_1\cdots\rho_5}\epsilon\, F_{\rho_1\cdots\rho_5} + \frac{1}{96}\Big(\gamma_\mu{}^{\nu\rho\lambda}\, G_{\nu\rho\lambda} - 9\gamma^{\rho\lambda}\, G_{\mu\rho\lambda}\Big)\epsilon^* + \cdots \tag{2.52}$$

where the $\cdots$ stand for terms trilinear in fermions. The numerics of the coefficients can be 'easily' obtained from the linear theory and the γ–logy.

The covariant derivative in eqn.(2.52) is again

$$D_\mu\epsilon = \big(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{rs}\gamma_{rs} - \frac{i}{2}Q_\mu\big)\epsilon. \tag{2.53}$$

3. The modular symmetry Γ

The most important physical datum of the theory is the discrete subgroup $\Gamma \subset SL(2,\mathbb{R})$ which specifies which configurations should be considered physically equivalent.

3.1. Perturbative consistency: $\mathbb{Z}_2 \times \mathbb{Z} \subset \Gamma$. The massless sector of the theory is invariant under the translation $x \to x + \text{const.}$. We ask whether a discrete subgroup of translations map a field configuration into a physically equivalent one. In other words, we ask whether $x \equiv C_0$ is a *periodic scalar* taking values in a circle S^1 of some radius R.

As we argued in §. 1.4, in string theory there are objects (like the D–branes) which carry electric and magnetic charges under all the R–R gauge fields. The usual Dirac argument then shows that (in suitable units) the gauge–invariant field–strenghts F_k whose Bianchi identities take the simple form $dF_k = 0$ should be *integral*

$$F_k \in H^k(\text{spacetime}, \mathbb{Z}), \tag{3.1}$$

so, for any closed k–cycle γ,

$$\int_\gamma F_k = \text{integer}. \tag{3.2}$$

The Dirac argument applies to $F_1 = dC_0 = dx$. Let γ be a closed path $[0,1] \to$ (spacetime), with $\gamma(1) = \gamma(0) = x_0$. One has

$$\Delta C_0(x_0) = C_0\big(\gamma(1)\big) - C_0\big(\gamma(0)\big) = \int_\gamma dC_0 = \text{integer}, \tag{3.3}$$

so the value of C_0 at one point is well–defined modulo an integer, that is our Type IIB scalar x takes value in a circle S^1 of lenght 1, and

$$z \sim z + 1. \tag{3.4}$$

For instance, we have $\Delta C_0 = 1$ along a path γ which encircles a $D7$–brane once (since a $D7$ brane is *magnetically* charged with respect to $F_1 = dC_0$ with unit charge). Then the periodicity $x \sim x + 1$ reflects the physical consistency of the $D7$–brane, an object which may be construct by 'perturbative' techniques.

Thus we have found a parabolic subgroup B_∞ of Γ

$$B_\infty \equiv \{T^n,\ n \in \mathbb{Z}\} \subset \Gamma \tag{3.5}$$

$$T \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{3.6}$$

The Dirac argument also holds for the H_a's which should also represent *integral* cohomology classes. Since, under an $SL(2,\mathbb{R})$,

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \tag{3.7}$$

the integrality of H_1, H_2 requires a, b, c, and d to be integers. Thus

$$B_\infty \subset \Gamma \subset SL(2,\mathbb{Z}). \tag{3.8}$$

The modular group $SL(2,\mathbb{Z})$ is generated by T and the transformation

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.9}$$

So, to show that Γ is the full modular group, it is enough to show that $S \in \Gamma$.

Notice that, at $C_0 = 0$, S would act by sending the string coupling $g = e^\phi$ to $1/g$. So, if S is a symmetry at all, it must be a highly non–perturbative *weak/strong coupling duality*. In particular, such a would–be symmetry cannot be deduced from perturbative Type IIB superstring. However, if the S duality is present, S^2 will also be a duality invariance. S^2 leaves τ, and hence g, fixed and must be visible already in string perturbation theory. It acts as

$$S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad i.e.\ H_a \to -H_a. \tag{3.10}$$

That S^2 is indeed a duality is implied by the perturbative consistency of the orbifold 7–planes O7 (going around such an object H returns to minus itself), much as the consistency of the D7–brane implies the invariance under T.

Then 'perturbative' consistency already shows $\mathbb{Z}_2 \times \mathbb{Z} \subset \Gamma$, where $\mathbb{Z}_2 \times \mathbb{Z}$ is the group generated by $T, S^2 \in SL(2,\mathbb{R})$.

3.2. Brane spectrum: $\Gamma \subseteq SL(2,\mathbb{Z})$. To extablish S as a true duality, one has to go non–perturbative. Between the quantities that may be reliably computed at strong coupling there is the spectrum of BPS (extended) objects. In fact their masses/tensions are protected against quantum corrections by the extended supersymmetry. Therefore, a *necessary* condition for $SL(2,\mathbb{Z})$ to be a symmetry is that the *spectrum of BPS objects is $SL(2,\mathbb{Z})$–invariant.*

The 3–form field–strengths H_3^a transform as a doublet of $SL(2,\mathbb{Z})$. Hence, in particular, the electric/magnetic 2–form charges of the BPS objects should make full orbits under the fundamental action of $SL(2,\mathbb{Z})$. The fundamental superstring is electrically charged under the NS–NS two–form B, and not charged under the R–R 2–form. Hence its 2–form charges have the form $\binom{0}{1}$. Under a $SL(2,\mathbb{Z})$ transformation

$$\begin{pmatrix}0\\1\end{pmatrix} \longmapsto \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}. \tag{3.11}$$

Since, given two integers (b,d), we can find two integers a, c such that

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL(2,\mathbb{Z}) \tag{3.12}$$

<u>if and only if</u> $\gcd(b,d)=1$, we deduce that: *If $SL(2,\mathbb{Z})$ is a symmetry, we must have a BPS string with 2–form electric charges (p,q) for each pair of* coprime *integers p and q.* Dually, the same thing should be true for the *magnetic* 2–form charges and hence for the BPS 5–branes.

The above statements are true in Type IIB string theory where the (p,q) string (resp. 5–brane) is a bound state of p fundamental strings (resp. NS 5–brane) and q D1–branes (resp. D5–branes). The restriction to *coprime* integers is easy to understand from the BPS mass formula for a (p,q) object (which follows from the SUSY representation theory)

$$M^2_{(p,q)} = p^2 M^2_{(1,0)} + q^2 M^2_{(0,1)} \leq \left(\sum_{\substack{\sum p_i = p\\ \sum q_i = q}} M_{(p_i,q_i)}\right)^2 \tag{3.13}$$

with equality precisely iff $(p,q)=(rp',rq')$ and $p_i=p'$, $q_i=q'$ for all i. Thus, a charge (p,q) BPS object can decay into objects of smaller charge/mass precisely iff the integers p,q are not coprime. If they *are* coprime, a BPS objects with charges (p,q) is necessarily stable.

Finally, there is also a notion of BPS (p,q) *seven*–branes. Seven branes will be between the heros of our novel.

3.3. The (p,q)–seven branes. In eqns.(3.2)(3.3) we have seen that, going around a D7–brane (in the orthogonal plane), the complex scalar z changes as $z \to z+1$. The D7–brane configuration is obviously invariant under the parabolic subgroup B_∞. Thus the different seven–brane *species* which form the $SL(2,\mathbb{Z})$–orbit of the basic D7–brane are in one–to–one correspondence with the points of the the coset

$$SL(2,\mathbb{Z})\big/B_\infty. \tag{3.14}$$

Taking the matrix inverse, we map this coset in the more canonical one $B_\infty\big\backslash SL(2,\mathbb{Z})$. It is a well–known fact that the points of this coset are in one–to–one correspondence with the pairs of *coprime* integers (p,q). The simplest way to see this, is to rewrite the Möbius action of $\left(\begin{smallmatrix} a & b\\ p & q\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, with $p>0$, in the form

$$z \xrightarrow{\left(\begin{smallmatrix} a & b\\ p & q\end{smallmatrix}\right)} \frac{a}{p} - \frac{1}{p(pz+q)}, \qquad a\,q \equiv 1 \mod p, \tag{3.15}$$

so, mod 1 we can replace a with the *unique* inverse of q in $(\mathbb{Z}/p\,\mathbb{Z})^\times$, $\overline{q}$,

$$z \mapsto \frac{\overline{q}}{p} - \frac{1}{p(pz+q)} \mod 1, \tag{3.16}$$

so a modular transformation is defined, up to an element of B_∞ precisely by two coprime integers. Correspondingly, the BPS seven–branes are also classified by *a pair of coprime integers* (p,q).

Let $S_{p,q}$ be any matrix in the coset associated to the pair (p,q). Going around a (p,q) seven brane, the field z will transform according to the modular transformation

$$S_{p,q}\,T\,S_{p,q}^{-1} \;\in\; SL(2,\mathbb{Z}). \tag{3.17}$$

Notice that this elements is well–defined, independently of the choice of the coset representative.

3.4. Normal subgroups of $SL(2,\mathbb{Z})$. However, if $SL(2,\mathbb{Z})$ *is* a symmetry of the non–perturbative theory (as suggested in §. 3.2) it cannot be not just a symmetry in the *ordinary* sense. Indeed,

FACT 3.1. *If $SL(2,\mathbb{Z})$ is a symmetry, it is a superselection group (namely, it commutes with all physical observables).*

Indeed, the group of symmetries which *act trivially on the physical observables* should be a *normal* subgroup of the group of all symmetries. The typical example is the ordinary spin group $Spin(3)$, whose superselection subgroup is ± 1 (*i.e.* the rotations by a multiple of 2π, which leave invariant all observables) which is indeed a normal subgroup.

Let $\mathfrak{S} \subset SL(2,\mathbb{Z})$ be the superselection group. In §. 3.1 we saw that we must identify two field configurations which differ by the action of T or $-\mathbf{1}$ as *physically equivalent.* Thus T and $-\mathbf{1}$ are elements of $\mathfrak{S}$.

Then FACT 3.1 follows from the

LEMMA 3.1. *Let N be a normal subgroup of $SL(2,\mathbb{Z})$ containing T and $-\mathbf{1}$. Then $N \equiv SL(2,\mathbb{Z})$.*

PROOF. Since N is normal, it contains $-\mathbf{1}$, T, and $S^{-1}TS$. Then it contains also $-\mathbf{1}\,T(S^{-1}TS)T \equiv S$. But S and T generate $SL(2,\mathbb{Z})$. $\square$

4. The finite volume property

Cumrun Vafa has proposed the following very general and profound conjecture [**8**] (see discussion in [GSSFT]):

CONJECTURE 4.1 (C. Vafa). *Let $\mathcal{M}_{\text{eff}}$ be the target space of the (massless) scalars in the low–energy effective theory emerging from any superstring/M–/F–theory vacuum configuration. Equip $\mathcal{M}_{\text{eff}}$ with the metric g_{eff} appearing in the (quadratic) kinetic terms of the low–energy effective Lagrangian $\mathcal{L}_{\text{eff}}$. Then the Riemannian manifold $(\mathcal{M}_{\text{eff}}, g_{\text{eff}})$ is* non–compact, complete, *and has* finite volume.

In particular, $\mathcal{M}_{\text{eff}}$ has infinitely long cuspidal spikes. $\mathcal{M}_{\text{eff}}$ can be compactified by 'closing' the cusps. Non–compact and finite volume means, in general, that the L^2 spectrum of the scalar Laplacian Δ consists of both a continuous spectrum and a discrete one (in particular, the constants are normalizable zero–modes).

This conjecture should, in particular, hold for the effective Lagrangian of Type IIB. This requires

$$\text{Vol}\Big(\Gamma\backslash SL(2,\mathbb{R})/U(1)\Big) < \infty \quad \Rightarrow \quad \Gamma \text{ has finite index in } SL(2,\mathbb{Z}).$$

Notice that the non–compactness already follows from $\Gamma \subseteq SL(2,\mathbb{Z})$.

In view of (say) THEOREM I.6.4 of ref. [**9**], this is equivalent to saying that the fundamental domain of Γ has a finite number of sides no one being on the boundary of $\mathcal{H}$. A subgroup $\Gamma \subset SL(2,\mathbb{Z})$ is characterized by the following numbers:

- the *index* $\mu = [SL(2,\mathbb{Z}) : \Gamma] = \frac{\text{Vol}(\text{Vol}(\Gamma\backslash\mathcal{H})}{\text{Vol}(SL(2,\mathbb{Z})\backslash\mathcal{H})}$;
- the *genus* $p =$ genus of $\Gamma\backslash\mathcal{H}$;
- ν_2, the number of elliptic fixed points of order 2;
- ν_3, the number of elliptic fixed points of order 3;
- ν_∞, the number of parabolic fixed points (cusps).

These numbers are related by the identity

$$p = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}. \tag{4.1}$$

EXERCISE 4.1. Prove eqn.(4.1). HINT: use the Hurwitz formula.

REMARK. The conjecture is more powerful in the case we have a low–energy theory with a smaller number of non–compact dimensions and a large number of unbroken supersymmetries. This is the case, for instance, for $d = 4$ and $\mathcal{N} > 2$. In this case the scalars' manifold has the form $\Gamma\backslash G/H$ with G the U–duality group (having rank larger than 1) and H its maximal compact subgroup. Then, by the Margulis theorem [**10**], Γ must be an *arithmetic* subgroup and, typically, in these cases arithmetic subgroups are also congruence subgroup.

In particular, in these cases, Vafa's finite volume condition implies Dirac's quantization of charge.

4.1. Two 'proofs' of the conjecture. We give two arguments in favor of the conjecture for the special case of a *general* $D = 10$ theory having $(2, 0)$ supersymmetry, a massless graviton, and quantized magnetic sources for the 1–form field–strenght.

The two arguments are logically equivalent, but they are expressed in two different languages that will be both useful in the sequel, so the present discussion is meant as a baby example illustrating future constructions in their simplest possible context. The astute reader may recognize that, in fact, the two arguments for finite volume corresponds to two different pictures of an (elliptic) $K3$ surface, namely 1) as an elliptic complex surface with base $\mathbb{P}^1$ and numerical invariants $\chi = 24$ and $\tau = -16$, and 2) as a compact simply–connected hyperKähler surface, that is a complex *symplectic* manifold, *i.e.* the phase space of a (holomorphic) classical mechanical system which happens to be *integrable* (since the surface is elliptic over a Lagrangian base). The notation p, q alludes to the mechanical viewpoint.

Both arguments aim to show that the theory is physically sick unless the finite volume condition is satisfied. Let us breafly discuss what 'sick' means.

4.1.1. *Physical singularity of a spacetime.* By *physically sick* we mean that, if the volume of the scalars' manifold $\mathcal{M}$ is *not* finite, almost all the classical solutions are singular in a drammatic way. The definition of singularity of the space–time M is the same as in the usual *singularity theorems* of General Relativity[12]. That is, a spacetime M is singular if it is *non–complete* with respect the time–like and/or the null geodesics. In fact such a manifold has (say) time–like geodesics which cannot extended for all values of the proper time τ. This means that an observer free–falling along such a geodesic will disappear from the space–time in a finite (proper) time. This is obviously inconsistent with unitarity, and such a solution to the Einstein equation is called *singular*. At first sight, one may think that this kind of singularity is a minor problem, since the usual *singularity theorems* of General Relativity [**11**] state that similar singularities do appear for generic initial conditions, so Type IIB theory with infinite–volume scalars' manifold looks not worst than any other gravitational theory. *It is not so!* The generic singularity of General Relativity is the consequence of gravitational collapse and black–hole formation, which do happen for generic initial distribution of matter because of the attractive nature of gravity.

[12] See §. 8.1 of ref. [**11**].

Instead, the singularity in our case will already be there for (almost all) *Poincaré invariant* configurations. These singularities have no consistent physical interpretation. *Worst than that:* the black–hole singularity is expected to be smoothed out by quantum and stringy corrections In our case, the half–BPS configurations are singular, and they are supposed to be protected against the 'smoothing out' corrections!

We make the following preliminary remark: A Poincaré invariant configuration of the form $M = \mathbb{R}^{d-1,1} \times X$, is incomplete with respect to the time–like geodesics if and only if the Riemannian manifold X is incomplete in the standard Hopf–Rinow sense. Indeed, let $\gamma(s)$ be an incomplete geodesic of X with the affine parameter s equal to the arc–length. Then

$$(0, \cdots, 0, t) \times \gamma(vt) \in \mathbb{R}^{d-1,1} \times X, \qquad v < 1, \tag{4.2}$$

is an incomplete time–like geodesic on M.

To show our claim, we take the effective action

$$S = \int d^n x \sqrt{g} \left[-\frac{1}{2} R - \frac{1}{4} \frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{(\operatorname{Im} \tau)^2} + \cdots \right] \tag{4.3}$$

and consider the solutions to the equations of motion in which only the fields τ and g are non–trivial and depend only on two coordinates x_9, x_{10}. We set $z = x_9 + i x_{10}$ and take the spacetime to be a manifold of the form $X \times \mathbb{R}^{1,7}$ with X *orientable.* This corresponds to the following ansatz for the metric

$$ds^2 = dx^M \, dx_M + e^{\phi(z,\bar{z})} \, dz \, d\bar{z}. \tag{4.4}$$

In the metric (4.4), the equations for the scalar $\tau \in \mathcal{H}$ then become

$$\partial \overline{\partial} \tau + \frac{2 \, \partial \tau \, \overline{\partial} \tau}{\bar{\tau} - \tau} = 0. \tag{4.5}$$

Any holomorphic function $\tau = \tau(z)$ is a solution (in fact, the solutions of this form preserve 16 supercharges). Taking τ to be holomorphic, the Einstein equations reduce to

$$\partial \overline{\partial} \phi = \frac{\partial \tau \, \overline{\partial} \bar{\tau}}{(\tau - \bar{\tau})^2} = \partial \overline{\partial} \log \operatorname{Im} \tau. \tag{4.6}$$

4.1.2. *First argument.* The metric on X is obviously Kähler. The Einstein equations (4.6) can be written more intrisically as

$$\operatorname{Ric}_X = \frac{1}{2} \tau^* \Omega, \tag{4.7}$$

where Ric_X is the Ricci form of the Kähler manifold X and Ω is the Kähler form on $\mathcal{M} \equiv \Gamma \backslash \mathcal{H}$ corresponding to the Poincaré metric, $\Omega = y^{-2} \, dx \wedge dy$. Since the Poincaré metric is positive, $\operatorname{Ric}_X \geq 0$, with equality if and only if $\tau = \text{const}$. From the Cheeger–Gromoll like comparison theorems[13] we have

PROPOSITION 4.1. *Assume X to be complete. Then X is one of the following:*

1. *a compact space, hence S^2;*
2. *a flat space;*
3. *a space diffeomorphic to $\mathbb{R}^2$. Hence, by the Riemann mapping theorem, X, as a complex space, is either $\mathbb{C}$ or the unit disk Δ. Since there is no complete metric on Δ having positive Ricci–curvature[14], we remain with only the first possibility, $\mathbb{C}$.*

[REMARK. The last statement is a special instance[15] of the Yau's uniformization conjecture [**15**]. This conjecture is stated in all complex dimensions, so it may be used to generalize the present argument in more general contexts].

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[13] See references [**12**][**13**][**14**]. We refer, in particular, to the THEOREM on page 413 of ref. [**14**].

[14] *An argument:* Take Δ to be the unit disk with a complete Kähler metric. By averaging with respect to the compact automorphism group $U(1)$, we may assume that the *global* Kähler potential K (which exists since Δ is contractible) is *rotational invariant* $K = K(|z|^2)$. Let $V(r) = \int_{|z|\leq r} \omega$ and $R(r) = \int_{|z|\leq r} \text{Ric}$ be, respectively, the volume of the disk of radius r centered in the origin and the analogous quantity with the Kähler form ω replaced by the Ricci form. Then a simple computation gives

$$R(r) = -\pi \frac{d}{d\log r} \log\left(\frac{d}{dr^2} V(r)\right)$$

or,

$$\frac{d}{dr^2} V(r) = C \exp\left(-\frac{1}{\pi} \int_0^r \frac{R(\rho)}{\rho}\, d\rho\right),$$

where C is a *positive* integration constant. By the Rinow–Hopf theorem, the completeness of the metric implies $V(r) \to \infty$ as $r \to 1$. Then $\frac{d}{dr^2} V(r)$ should diverge as $r \to 1$. But, if $\text{Ric} > 0$, $R(r) > 0$ and $0 \leq \frac{d}{dr^2} V(r) \leq C$. Then absurt.

A BETTER *argument:* We assume the existence of a complete Kähler metric on the unit disk Δ with positive Ricci form, and get a contradiction. The condition that the Ricci tensor is positive reads, explicitly (eqn.(5.3.36) of ref. [**16**])

$$-\partial_z \partial_{\bar{z}} \phi \geq 0,$$

that is *$-\phi$ is a sub–harmonic function.* A harmonic function, for which the above inequality is saturated, satisfies the *mean value theorem:* The value of the function at a point p is equal to the mean value of the function on any circle centered at p. This, in particular, implies that the functions has its minimum and maximum values on the boundary. A sub–harmonic function is everywhere $\leq$ of the harmonic function with the same boundary values. Hence, *a fortiori*, a sub–harmonic function has its *maximum* on the boundary. But $\phi \to +\infty$ on the boundary of the disk (otherwise some points on the boundary will be at finite distance, which is not allow by completeness). Hence the maximum of $-\phi$ on Δ is $-\infty$, which is absurt.

[15] The pedantic reader who thinks that the argument in the text is not good enough, may prefer to invoke the more precise THEOREM 4.3 of ref. [**17**].

Case (3) is sick. The positivity of the scalars' kinetic terms requires $\mathrm{Im}\,\tau > 0$, that is $|\exp(\pi i\,\tau(z))|^2 < 1$ for all $z \in \mathbb{C}$, and this is not possible *unless* the holomorphic function $\tau(z)$ is actually a constant. In this last case X is flat, and we get a special instance of case (2).

It remains case (1). From the Gauss–Bonnet theorem, we know that the de Rham class of the Ricci–form of S^2 is non–trivial for *all* Kähler metrics. Viewing the Einstein equation (4.7) in cohomology, and assuming τ not to be a costant, we learn that Ω should be a non–trivial class on $\mathcal{M}$, that is, that there is *no* global Kähler potential for Ω. If the scalars' manifold $\mathcal{M} := \Gamma\backslash\mathcal{H}$ has infinite volume, a global Kähler potential Φ always exists[16]. Thus

FACT 4.1. *If X is complete, either*

1. *X is flat and $\tau = \mathrm{const.}$;*
2. *$X = \mathbb{P}^1$ and $\mathrm{Vol}(\Gamma\backslash\mathcal{H}) < \infty$.*

In the second case, we may be more precise[17]

$$\begin{aligned} 2 = \chi(S^2) &= \int_X c_1 = \frac{1}{2\pi}\int_X \mathrm{Ric}_X = \\ &= \frac{1}{4\pi}\int_X \tau^*\Omega = \frac{\deg\tau}{4\pi}\int_{\mathcal{M}}\Omega = \frac{\deg\tau}{4\pi}\,[SL(2,\mathbb{Z}):\Gamma]\,\frac{\pi}{3}, \end{aligned} \tag{4.8}$$

that is

$$\deg\tau\cdot[SL(2,\mathbb{Z}):\Gamma] = 24. \tag{4.9}$$

Thus the index μ of Γ in $SL(2,\mathbb{Z})$ should divide 24.

On the other hand, X non flat implies that the genus p of Γ is zero[18]. In view of the Hurwitz formula, eqn.(4.1), we may refine the above FACT 4.1

FACT 4.2. *If X is complete, either*

1. *X is flat and $\tau = \mathrm{const.}$;*
2. *$X = \mathbb{P}^1$ and $\mathcal{M} = \Gamma\backslash\mathcal{H}$ where Γ is a subgroup of $SL(2,\mathbb{Z})$ of (finite) index $\mu|24$ such that*

$$\mu = 3\nu_2 + 4\nu_3 + 6\nu_\infty - 12, \tag{4.10}$$

---

[16] *Justification for the pedantic:* If the volume is infinite, $\mathcal{M}$ cannot be compactified while preserving $[\omega]$. A non compact complex space of dimension 1 is automatically Stein (ref. [**18**] page 134). A Stein Kähler space has a global Kähler potential (obvious). *If the pedantic is still not satisfied:* please see eqn.(4.8).

[17] In the last equality we use the well–known fact that $\mathrm{Vol}(SL(2,\mathbb{Z})\backslash\mathcal{H}) = \pi/3$.

[18] *Justification:* If $\Gamma\backslash\mathcal{H}$ has genus $p > 0$, it has p linearly independent holomorphic 1–forms ξ_a. Its volume form is cohomologous to $i((\mathrm{Im}\,\Omega)^{-1})^{ab}\bar{\xi}_a\wedge\xi_b$. Let $\tau\colon S^2 \to \mathcal{M}$ be a holomorphic map. The volume of the image is

$$i((\mathrm{Im}\,\Omega)^{-1})^{ab}\int \tau^*(\bar{\xi}_a\wedge\xi_b) = 0$$

since $\tau^*\xi_a$ is necessarily exact, S^2 being simply connected. Then τ is the constant map by the open map theorem.

group	$\Gamma_0(2)$	$\Gamma_0(3)$	$\Gamma_0(4)$	$\Gamma_0(5)$	$\Gamma_0(6)$	$\Gamma_0(7)$
μ	3	4	6	6	12	8
group	$\Gamma_0(8)$	$\Gamma_0(9)$	$\Gamma_0(12)$	$\Gamma_1(5)$	$\Gamma_1(7)$	$\Gamma_1(8)$
μ	12	12	24	12	24	24

TABLE 1.1. Hecke congruence subgroups satisfying the conditions.

and containing T.

Since physical consistency (*i.e.* unitarity) requires metric completeness, we have just two possibilities: either (1) we adhere to the strict Type IIB perturbative paradigm and consider just vacua with τ constant, or (2) we are *forced* to identify field configurations differing by the action of some subgroup Γ having the prescribed properties.

F–theory is the non–perturbative completion of Type IIB superstring theory corresponding to the second, much more physically sound, alternative.

Just for the fun of it, we list in the table the Hecke congruence subgroups[19] which satisfy the conditions.

Of course, by far the most natural solution to the above conditions is that the group Γ is the full modular group $SL(2,\mathbb{Z})$. This is strongly suggested by the arguments of the previous section. If we assume $\Gamma = SL(2,\mathbb{Z})$ (as we shall do from now on), the equation (4.9) gives that the degree of τ is fixed to be 24 ($\equiv \chi(K3)$ not a coincidence!).

Notice, however, that the physics will not change too much if $\Gamma \neq SL(2,\mathbb{Z})$. The small modifications will be described in §. 9 below.

In conclusion: A target space of infinite volume (or even of finite volume if not equal to $\pi\,\mu/3$ with μ a divisor of 24 !) would not allow for BPS configuration in which τ varies, as it is the case for a flat D7 brane. It would make sense only in a strictly perturbative superstring theory where we decouple the branes by sending their tension to infinity. If we do not want to decouple the branes, we are forced to have a finite volume target.

The conjecture is argued.

4.1.3. *Second argument.* Locally, the solution to the Einstein equations (4.5) is

$$e^{\phi} = \operatorname{Im}\tau(z)\,|f(z)|^2, \qquad f(z) \text{ holomorphic.} \tag{4.11}$$

In a coordinate patch $U \subset X$ (taken to be a small disk) we can define *holomorphic* functions q, p by the equations

$$dq = f(z)\,dz, \qquad dp = \tau(z)\,f(z)\,dz, \tag{4.12}$$

[19] Of course, most subgroups of the modular group are NOT Hecke subgroups! (Unfortunately, I am not familiar with those more general subgroups).

by the Poincaré lemma. The Kähler form in U takes the form

$$\omega = \frac{i}{2}(dp \wedge d\bar{q} + d\bar{p} \wedge dq) \equiv i\,\partial\overline{\partial}\,\frac{1}{2}(p\bar{q} - \bar{p}q). \tag{4.13}$$

Suppose now that the field $y = \operatorname{Im}\tau$ is globally defined (while the field $x = \operatorname{Re}\tau$ may be periodic, say $x \sim x + 1$). Then

$$-\frac{i}{\operatorname{Im} z}\,\omega = |f(z)|^2 dz \wedge d\bar{z} = dq \wedge d\bar{q} \tag{4.14}$$

is also globally defined, which implies that in the overlap $U_i \cap U_j$ between two coordinates patches $dq_i = e^{i\,c_{ij}}\,dq_j$ for some real constant c_{ij}. The 1–cocyle $\{e^{ic_{ij}}\}$ is necessarily trivial[20], and hence the dq_i's can be glued into a global closed holomorphic form dq which is necessarily exact (since we assume spacetime to be simply connected). Thus q is a global holomorphic function. Then, from eqns.(4.12)

$$\text{in } U_i \cap U_j: \qquad dp_i - dp_j = n_{ij}\,dq \quad n_{ij} \in \mathbb{Z}. \tag{4.15}$$

Again, $\{n_{ij}\} \in H^1(X, \mathbb{Z}) \equiv 0$, so we may glue the dp_i's into a global holomorphic form dp which is automatically exact. We have shown the following

LEMMA 4.1. *If $y \equiv \operatorname{Im}\tau$ is globally defined on X (simply connected) there exist* global *holomorphic functions, p and q, such that the function*

$$K = -\frac{i}{2}(p\bar{q} - \bar{p}q) \tag{4.16}$$

is a global *Kähler potential.*

In fact, we may invert the logic: If $y \equiv \operatorname{Im}\tau$ is globally defined, take any two holomorphic functions p and q then the Kähler metric (4.16) and $\tau = dp/dq$ give a solution to the classical equations of motion. Physical consistency requires that the physical space, that is the region $\operatorname{Im}\tau > 0$, is geodesically complete. In the case of a Kähler space with a global Kähler potential K this requires K to diverge on the boundary of the physical region $\operatorname{Im}\tau = 0$. This does not happen, so the classical solution are inconsistent if $y \equiv \operatorname{Im}\tau$ is globally defined.

[20] *Proof for the pedantic reader.* One has $\{e^{i\,c_{ij}}\} \in H^1(X, U(1))$ where X is the complex surface parameterized by z (*i.e.* our spacetime is $X \times \mathbb{R}^{1,7}$). It is enough to show that $H^1(X, U(1)) = 0$. Consider the exact sequence of constant sheaves

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{R} \xrightarrow{e(\cdot)} U(1) \to 1,$$

from which we get the exact sequence in cohomology

$$0 \equiv H^1(X, \mathbb{R}) \xrightarrow{e(\cdot)} H^1(X, U(1)) \to H^2(X, \mathbb{Z}) \xrightarrow{i} H^2(X, \mathbb{R}) \to$$

where we used the assumption that X is simply connected. Thus

$$H^1(X, U(1)) \simeq \ker i \equiv H^2(X, \mathbb{Z})|_{\text{torsion}}.$$

From the *Universal Coefficients Theorem* (COROLLARY 15.14.1 of ref. [**19**]), $H^2(X, \mathbb{Z})|_{\text{torsion}} \simeq H_1(X, \mathbb{Z})|_{\text{torsion}}$, while (by THEOREM 17.20 of the same reference) $H_1(X, \mathbb{Z}) \equiv$ Abelianization of $\pi_1(X) \equiv 0$, since X is simply connected.

4.2. Relation to $\mathcal{N} = 2$ gauge theory (Seiberg–Witten). The Kähler geometry we found as a solution to the Einstein equations above is very special: indeed it is known as the *special Kähler geometry* [GSSFT], that is the geometry of the scalars belonging to the vector supermultiplets of a $\mathcal{N} = 2$ $D = 4$ gauge theory. The geometric aspects which are relevant here are the same which are crucial for the solution of the $\mathcal{N} = 2$ theory [**20**] (for very earlier work, see [**21**]).

To get the relation, identify our complex coordinate q with the complex field in a vector multiplet, while the holomorphic function F defined (locally) by

$$dF = p\, dq \tag{4.17}$$

(that is, the Hamilton–Jacobi function in the classical mechanical language) is identified with the prepotential function in the sense of the $\mathcal{N} = 2$ superspace,

$$L = \int d^4\theta\, F(q). \tag{4.18}$$

The reader is invited to check all the geometric relations.

In particular, the monodromies we use in the present context do correspond to the Seiberg–Witten monodromies which lead to the solution of the gauge theory.

Phrased differently,

FACT 4.3. *The solution of the Coulomb branch of any $\mathcal{N} = 2$ theory with gauge group of rank 1 gives an explicit compactification of F–theory to 8 dimensions.*

In fact, the reason why the physics is sick if the target volume is infinite, is related to the reason why the $4D$ gauge theory would be non–perturbatively inconsistent if its particle spectrum would be the perturbative one for all values of the Coulomb branch parameters.

5. The manifold $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/U(1)$

The above discussion of the modular invariance (in field space!) of our theory implies that a scalars' field configuration may be seen as a smooth map

$$\tau\colon \text{(space–time)} \;\to\; \mathfrak{F} := SL(2,\mathbb{Z})\Big\backslash SL(2,\mathbb{R})\Big/U(1). \tag{5.1}$$

Then it is relevant to discuss the geometry of the manifold $\mathfrak{F} := SL(2,\mathbb{Z})\Big\backslash SL(2,\mathbb{R})\Big/U(1)$.

$\mathfrak{F}$ is one of the most important and ubiquous spaces in mathematics and physics. For one thing, it is the moduli space of complex 1 dimensional tori, *a.k.a. elliptic curves.* Let us recall that story.

5.1. Elliptic curves. For our purposes, it is better to start from the elliptic curve side (even if it is less elementary, *sorry !).*

DEFINITION 5.1. An *elliptic curve* over the field K is defined as a nonsingular projective curve E of genus 1 together with a 'rational' point $O \in E(K)$ (that is: a point whose homogeneous coordinates take values in the ground field[21] K).

Note the crucial fact that the definition of an elliptic curve *includes the specification of a point* O.

In our applications, the ground field K will be either $\mathbb{C}$ or a function field $\mathbb{C}(B_n)$ where B_n is a compact complex manifold (typically algebraic) of dimension n. Such an elliptic curve will describe a supersymmetric compactification of F–theory down to $d = 2(5 - n)$ spacetime dimensions. However, most of the techniques now used to extract phenomenological information out of F–theory (say the *Tate algorithm*, which describes the gauge group and representations of the low–energy effective theory) were originally developed in order to study elliptic curves over fancier fields K of interest in Number Theory/Diophantine Geometry, so the abstract language is the most convenient one (in the sense that it is more directly related to the physical predictions !).

Below we shall show that an equivalent definition is

DEFINITION 5.2. An *elliptic curve* over the field K is a nonsingular projective *plane* curve $E \subset \mathbb{P}K^2$ of degree 3 together with a 'rational' point $O \in E(K)$.

5.1.1. *The Weiestrass equation.* Let E be an elliptic curve and O its preferred point. According to our definition, it has genus 1.

We consider the vector spaces $H^0(E, [k\,O])$ $(k = 0, 1, 2, 3, \dots)$, which are canonically identified with the spaces of rational[22] functions on E having, at most, a pole of order k in O and no pole elsewhere. Note that, by definition, $H^0(E, [k\,O]) \subset H^0(E, [k'\,O])$ if $k \leq k'$.

[21] *As a piece of notation:* $E(K)$ means the set of points of E whose coordinates are in K. We can consider $E(L)$, the set of points of E with coordinates in L, where L is either a subfield of K or an algebraic extension (of finite or infinite degree) of K. In mundane terms: E is defined by a set of homogeneous polynomials with coefficients in K. We can consider special solutions to these equations which belong to a subfield, or look for solutions which are algebraic over K (*e.g.* $K = \mathbb{R}$ and E is defined by equations with real coefficients; then $E(\mathbb{C})$ is the corresponding complex space whose points are *complex* solutions of the defining equations).

[22] In the complex case, *rational* is equivalently to *meromorphic*. If you feel more at ease, substitute everywhere the word *rational* with the word *meromorphic*.

By the Riemann–Roch theorem and Serre duality[23],

$$\dim H^0(E,[k\,O]) = \dim H^0(E,-[k\,O]) + k = \begin{cases} 1 & \text{for } k=0 \\ k & \text{for } k \geq 1. \end{cases} \tag{5.2}$$

The $\mathbb{C}$–space $H^0(E,[0\cdot O]) \equiv \Gamma(E,\mathcal{O})$ is spanned by the constant 1. Then 1 should also span the one–dimensional space $H^0(E,[1\,O])$.

The space $H^0(E,[2\,O])$ has dimension 2, so it is spanned 1 and a second rational function which we call X. X has a double pole at O. Then the space $H^0(E,[3\,O])$, having dimension 3, is spanned by the three functions 1, X and Y. Continuing in this way, $H^0(E,[4\,O])$ is spanned by 1, X, Y, X^2, while $H^0(E,[5\,O])$ by 1, X, Y, X^2, and XY. Finally we arrive at $H^0(E,[6\,O])$, which has dimension 6. We already have *seven* rational functions which belong to this space, namely

$$1,\ X,\ Y,\ X^2,\ XY,\ X^3,\ Y^2, \tag{5.3}$$

so there must be a linear relation between them of the form

$$a_0Y^2 + a_1XY + a_3Y = a_0'X^3 + a_2X^2 + a_4X + a_6 \tag{5.4}$$

Moreover, a_0 and a_0' should be not zero (otherwise we get that a function with a pole of order 6 at O is a linear combination of functions with poles of order ≤ 5). We are free to normalize the functions X and Y in such a way that $a_0 = a_0' = 1$.

The map $p \mapsto (X(p), Y(p)) \in K^2$ then sends E into the affine curve of equation (5.4). Its projective completion is the plane projective curve

$$\boxed{\begin{aligned} Y^2Z + a_1XYZ + a_3YZ^2 &= \\ &= a_0'X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3 \end{aligned}} \tag{5.5}$$

DEFINITION 5.3. An equation of the form (5.5) is called a *Weierstrass equation* for the elliptic curve E.

The Weierstrass equation can be further simplified (however, for many purposes, the general form, eqns.(5.4)(5.5), is more convenient and should be always kept in mind). Indeed, if K has characteristic $\neq 2, 3$ (and our ground fields will always have characteristic zero), the change of variables

$$X' = X + \frac{a_2}{3}, \qquad Y' = Y + \frac{a_1}{2}X + \frac{a_3}{2}, \qquad Z' = Z, \tag{5.6}$$

[23] Over $\mathbb{C}$, the RIEMANN–ROCH THEOREM is just the $\overline{\partial}$ index theorem that can be obtained from the usual Adler–Bardeen axial anomaly (or, in $2d$ QFT, *via* the bosonization formulae). In this context, the Serre duality expresses the CPT invariance of the $2d$ QFT.

EXERCISE 5.1. Prove Riemann–Roch using Feyman diagrams.

will eliminate the terms XYZ, X^2, and Y, leaving us with the *Weierstrass equation*

$$\boxed{Y^2Z = X^3 + AXZ^2 + BZ^2} \tag{5.7}$$

It remains to describe in this Weierstrass settings the preferred point O. In the projective set–up, the homogeneous coordinates we constructed, (X, Y, Z), are the three holomorphic[24] sections of the line bundle associated to the divisor $3[O]$, while the corresponding meromorphic Cartesian coordinates (that we also denoted by X and Y) are given by the global meromorphic functions X/Z and, respectively, Y/Z. The section Z, being associated to the function 1, is the local defining function of the divisor $3[O]$, and hence (by definition) has a zero of order 3 in O. Since X/Z has a double pole at O, the section X should have a single zero there. Therefore, in the homogeous coordinates (X, Y, Z), O is the point *at infinity,* namely

$$(0 : 1 : 0). \tag{5.8}$$

REMARK. Above we stated that we can equivalently define an elliptic curve as a nonsingular plane projective curve of degree 3 (together with a point $O \in E$). Indeed: any nonsingular plane cubic has genus 1 (by the genus formula), while any elliptic curve has a Weierstrass equation which realizes it as a plane cubic. Notice that, from the point of view of the plane curve, the point $O = (0 : 1 : 0)$ is pointed out by the fact that it is the unique *inflection point* of the cubic (namely the point in which the tangent has a contact of order 3).

Let us summarize the situation in a theorem (see *e.g.* refs. [**22**][**23**])

THEOREM 5.1. *Let K be a field of characteristic $\neq 2, 3$.*

(1) *Every elliptic curve (E, O) is isomorphic to a curve of the form*

$$E(A, B): \quad Y^2Z = X^3 + AXZ^2 + BZ^3, \quad A, B \in K, \tag{5.9}$$

pointed by $(0 : 1 : 0)$.

(2) *(Conversely) the curve $E(A, B)$ is nonsingular (and so, together with $(0 : 1 : 0)$ is an elliptic curve) if and only if*

$$\boxed{\Delta \equiv 4A^3 + 27B^2 \neq 0} \tag{5.10}$$

Δ *is called the* discriminant *of the elliptic curve.*

[24] *For the pendantic:* Yes you are right. Here I use the words HOLOMORPHIC and MEROMORPHIC in a loose sense. Working on a general ground field I must use instead the words REGULAR and, respectively, RATIONAL. However, for the sake of simplicity, I use the same terms that would be appropriate in the best known (to physicists) case $K = \mathbb{C}$.

(3) *Let $\varphi\colon E(A,B)\to E(A',B')$ be an isomorphism sending $O=(0:1:0)$ to $O'=(0:1:0)$. Then there exists $c\in K^\times$ such that*

$$A'=c^4A,\qquad B'=c^6B, \tag{5.11}$$

$$\text{and } \varphi\colon (X,Y,Z)\mapsto (c^2X,c^3Y,Z). \tag{5.12}$$

Conversely, if $A'=c^4A$, $B'=c^6B$ for some $c\in K^\times$, then $(X,Y,Z)\mapsto(c^2X,c^3Y,Z)$ is an isomorphism $E(A,B)\to E(A',B')$ with $O\mapsto O'$.

(4) *If the elliptic curve (E,O) is isomorphic to the Weierstrass curve $\big(E(A,B),(0:1:0)\big)$, we let*

$$\boxed{j(E)=\frac{1728\,(4A^3)}{4A^3+27B^2}} \tag{5.13}$$

*$j(E)\in K$ is an invariant which depends only on (E,O). It is called the j–*invariant *of the elliptic curve E. Two elliptic curves E and E' are isomorphic over $K^{\rm al}$) (!!) if and only if*

$$j(E)=j(E'). \tag{5.14}$$

Thus over $\mathbb{C}$, which is algebraically closed, the j–invariant completely characterizes the elliptic curve E, up to isomorphism. On the other hand $\mathbb{C}(B)$ is not algebraically closed, and so the j–invariant will not fully characterize an F–theory configuration. An F–theory configuration with trivial (that is constant) j–invariant which is not a trivial fibration will correspond, physically, to a Type IIB configuration in presence of *orientifold planes* (this will be discussed in chapter ....).

Proof. We already proved item (1).

(2). Let $W=ZY^2-X^3-AXZ^2-BZ^6$. At the point $(0:1:0)$ we have $\partial W/\partial Z=2\neq 0$, so the point O is always non–singular. Then the elliptic curve is nonsingular iff the corresponding affine curve

$$C\colon\quad Y^2=X^3+AX+B \tag{5.15}$$

is nonsingular. A point $(X,Y)\in C$ is singular iff

$$2Y=0,\quad 3X^2+A=0,\qquad Y^2=X^3+AX+B, \tag{5.16}$$

so X is a common zero of X^3+AX+B and its derivative, *i.e.* a double root of the cubic polynomial. Therefore

$$\begin{aligned} E \text{ nonsingular } &\iff X^3+AX+B \text{ has no multiple roots } \iff\\ &\iff \text{the discriminant } -\Delta=4A^3+27B^2\neq 0. \end{aligned} \tag{5.17}$$

(3). Consider the rational (meromorphic in the complex case) functions φ^*X', φ^*Y'. They have, respectively, a pole of order 2 and 3 at

O. Hence they have the form, respectively, $\alpha X + \beta$ and $\gamma Y + \delta X + \epsilon$. Then

$$(\gamma Y + \delta X + \epsilon)^2 - (\alpha X + \beta)^3 - A'(\alpha X + \beta) - B'. \tag{5.18}$$

subtract from this $\gamma^2(Y^2 - X^3 - AX - B) \equiv 0$. We get a linear relation between the six functions 1, X, Y, X^2, XY and X^3 which are linearly independent. Then all coefficients must vanish. In particular, $\beta = \delta = \epsilon = 0$. Set $c = \gamma/\alpha$. We get $\alpha = c^2$, $\beta = c^3$ and $A' = c^4 A$, $B' = c^6 B$.

(4). If E is isomorphic to both $E(A, B)$ and $E(A', B')$ there exists $c \in K^\times$ such that $A' = c^4 A$ and $B = c^6 B$, and $j(E)$ is equal in the two cases (since both the numerator and the denominator scale as $c^1 2$).

Conversely, suppose $j(E) = j(E')$. First notice

$$A = 0 \;\Leftrightarrow\; j(E) = 0 \;\Leftrightarrow j(E') = 0 \;\Leftrightarrow A' = 0 \tag{5.19}$$

and any two ellitpic curves of the form $ZY^2 = X^3 + BZ^3$ are isomorphic over K^{al}. Let $A, A' \neq 0$. We replace (A, B) with $(c^4 A, c^6 B)$ with $c = (A'/A)^{1/4}$, so that now $A = A'$. Then $j(E) = j(E')$ implies $B' = \pm B$. The minus sign may be removed by taking $c = \sqrt{-1}$. □

Remark. For every $j \in K$ there exists at least one elliptic curve (up to isomorphism) that has $j(E) = j$ (and precisely one if K is algebraically closed). *E.g.*

$$
\begin{aligned}
&Y^2 Z = X^3 + Z^3 && j = 0\\
&Y^2 Z = X^3 + XZ^2 && j = 1728\\
&Y^2 Z = X^3 - \frac{27}{4}\frac{j}{j - 1728} XZ^2 - \frac{27}{4}\frac{j}{j-1728} Z^3 && j \neq 0, 1728.
\end{aligned}
$$

5.1.2. *The group structure of an elliptic curve.* An elliptic curve (E, O) is an Abelian group with the point O playing the role of the zero element. There are many equivalent descriptions of the group law.

Looking at (E, O) as a plane cubic with a point O singled out, we define the *opposite* of the point $P \in E$, written $-P$, to be the third point of intersection of the cubic with the line trough P and O. (Thus $-O = O$, since the tangent in O has a triple point of contact). Then we write

$$P + Q + R = O \tag{5.20}$$

if the three points P, Q, and R are the points of intersection of a line with the cubic. That is, the point $P + Q$ is obtained by the following procedure: we draw the line trough P and Q. The third point of intersection is $-(P+Q)$. To get $P+Q$ we have to draw the line trough $-(P + Q)$ and O. The third intersection along this second line is the sum $P + Q$. If $P = Q$ we take the tangent line at the point.

In this description it is not obvious that the operation is associative (while commutativity is manifest). There is a simple geometrical proof of this fact, that we omit (see *e.g.* refs. [**22**][**23**]). This *tangent–secant formulation*, however, has some advantages: it expresses the group operations as explicit rational maps in the X, Y variables (which are rational functions on E). In §. 5.2.2 below we shall interprete these formulae as summation theorems for the Weierstrass elliptic functions. We confine these (and other) formulae in an Appendix.

There is another point of view in which the associativity is obvious. As we saw above, an elliptic curve has genus 1.

Let $aX + bY + cZ = 0$ be the (projective) line trough the points P and Q. Let $dX + eY + fZ = 0$ be the line trough the third point of intersection, $-(P + Q)$, and the point at ∞, O. Then

$$\left.\frac{aX + bY + cZ}{dX + eY + fZ}\right|_E \tag{5.21}$$

is a well defined rational (meromorphic, in the special case $K = \mathbb{C}$) function on E. It has a zero at P and Q and a pole at O and $P{+}Q$, while the third zero of the numerator $-(P{+}Q)$ cancel against a corresponding zero of the denominator. Thus, in terms of divisors on the curve

$$[P] + [Q] \sim [P + Q] + [O] \tag{5.22}$$

where $\sim$ stands for linear equivalence. Therefore, the additive group of an elliptic curve is just the additive group of divisors modulo linear equivalence, that is the group $\mathrm{Pic}_0(E) \simeq Jac(E)$. We have recovered the well–known fact that a curve of genus 1, with a point O singled out, is canonically equivalent to its Jacobian $Jac(E)$.

5.1.3. $(*)$ *Singular cubics and their group laws.* The above discussion is appropriate for an elliptic curve, that is *a nonsingular* Weierstrass cubic. Let us now discuss the singular case (which is the most relevant for F–theory), that is the case $\Delta = 0$.

Let S be a singular point in the cubic $C\colon Y^2 = X^3 + AX + B$. Essentially by definition, a line trough S would have there an intersection number with C of order ≥ 2. Taking any point $P \neq S$ on C and considering the line l_{PQ} trough S and P. l_{PQ} has a total intersection number 3 with C. Then we see that S is necessarily a double point, and P is necessarily regular, that is:

On an (irreducible) plane cubic C there is at most one singular point which is a double point.

We have already shown around eqn.(5.16) that the X–coordinate of a singular point, $X(S)$, is a common zero of the polynomial $X^3{+}AX{+}B$ and its derivative. Then we have two possibilities: Either two roots are equal and the third is different, or the three roots are all equal. Up to isomorphism, we may assume the double root to be at 0. Then

$$ZY^2 = X^3 + M\,XZ^2, \qquad M \neq 0 \qquad \text{cubic with a node} \tag{5.23}$$

$$Y^2 = X^3 \qquad \text{cubic with a cusp.} \tag{5.24}$$

As we observed in the proof of THEOREM 5.1, the point

$$O = (0:1:0)$$

is never singular. Let $P \neq S$ be a nonsingular point, and consider the line trough P and O. The third intersection, $-P$, is again nonsigular. So, if P, Q are two nonsingular points, so is the third intersection point $-(P+Q)$. Thus:

The set of nonsingular points $C^{\mathrm{ns}} = C \setminus S$ of a plane cubic curve C is an Abelian group.

Which group? The cusp case is easy. We see that the singular point $S = (0:0:1)$ is the only point on C with $Y = 0$. Thus C^{ns} is precisely the *affine* curve $C \cap \{\, Y = 0\}$, *i.e.*

$$Z = X^3. \tag{5.25}$$

Let $Z = \alpha X + \beta$ be the line trough P and Q. The coordinates of the three intersection points satisfy the equation

$$X^3 - \alpha X - \beta = 0. \tag{5.26}$$

Since the coefficient of X^2 vanishes, the sum of the X's of the three intersection points vanishes, that is

$$P + Q + R = 0 \iff X(P) + X(Q) + X(R) = 0, \tag{5.27}$$

The map $P \mapsto X(P)$ gives an isomorphism of Abelian groups

$$C^{\mathrm{ns}} \simeq K. \tag{5.28}$$

The nodal case is slightly tricker. There are two cases, either $M = \gamma^2$ is a square in K or not. If not, we work over the field $K[\gamma]$. Set

$$R + S\gamma = \frac{Y + \gamma X}{Y - \gamma Y} \tag{5.29}$$

which satisfy Pell's equation [**24**]

$$R^2 - M\,S^2 = 1. \tag{5.30}$$

The intersection of the curve (5.23) with a generic line not passing trough $S = (0:0:1)$, having equation $Z = \alpha(Y - \gamma X) + \beta(Y + \gamma X)$, is

$$\begin{aligned} 8\gamma^3(Y+\gamma X)(Y-\gamma X)\Big(\alpha(Y-\gamma X) + \beta(Y+\gamma X)\Big) = \\ = \Big((Y+\gamma X) - (Y-\gamma X)\Big)^3 \end{aligned} \tag{5.31}$$

and dividing by $(Y-\gamma X)^3$ we get

$$(R+\gamma S)^3+\cdots-1=0, \tag{5.32}$$

that is

$$\begin{aligned}&P+Q+R=0 \iff \\ &\iff \big(R(P)+\gamma S(P)\big)\big(R(Q)+\gamma S(Q)\big)\big(R(R)+\gamma S(R)\big)=1, \end{aligned} \tag{5.33}$$

namely C^{ns} is isomorphic, as an Abelian group, to the *multiplicative group* of elements (R,S) with $R^2-MS^2=1$.

If M is a square in K, the map $P\mapsto R+\gamma S\in K^\times$ gives an isomorphism of groups

$$C^{\mathrm{ns}}\simeq K^\times. \tag{5.34}$$

If M is *not* a square we get a *twisted* (*a.k.a.* non–split) multiplicative law[25].

5.2. Complex tori and elliptic curves over $\mathbb{C}$. An elliptic curve over $\mathbb{C}$ is, from the analytic point of view, just a one–dimensional torus.

5.2.1. *Lattices and complex tori.* A *lattice in* $\mathbb{C}$ is a set

$$\Lambda=\omega_1\,\mathbb{Z}\oplus\omega_2\,\mathbb{Z}$$

where $\{\omega_1,\omega_2\}$ is a basis of $\mathbb{C}$ over $\mathbb{R}$. Changing the sign to ω_2, if necessary, we may assume $\omega_1/\omega_2\in\mathcal{H}$.

Two lattices $\Lambda=\omega_1\,\mathbb{Z}\oplus\omega_2\,\mathbb{Z}$ and $\Lambda'=\omega_1'\,\mathbb{Z}\oplus\omega_2'\,\mathbb{Z}$, with $\omega_1/\omega_2\in\mathcal{H}$ and $\omega_1'/\omega_2'\in\mathcal{H}$, coincide, $\Lambda'=\Lambda$, if and only if

$$\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix},\qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z}). \tag{5.35}$$

Definition 5.4. A *complex torus* is a quotient of the complex plane by a lattice, $\mathbb{C}/\Lambda$.

In particular, a torus is an *Abelian group* [under the obvious addition $(z+\Lambda)+(z'+\Lambda)=z+z'+\Lambda$].

A nonzero holomorphic homomorphism between complex tori is called an *isogeny.*

---

[25] An example is worth one thousand explanations:

Example. Take $K=\mathbb{R}$. Then up to isomorphism, we have two possibilities $Y^2=X^3+X$ and $Y^2=X^3-X$. In the first case, $+1$ is a square, and we have $(R+S)(R-S)=1$, so $(R+S)\in\mathbb{R}^\times$ with $(R-S)=(R+S)^{-1}$, and we get the group $\mathbb{R}^\times$ isomorphic to the hyperbola $XY=1$. In the second case we get $R^2+S^2=1$, and the twisted multiplicative group is just the unit circle in the complex plane (isomorphic to the circle).

EXERCISE 5.2. Let $\varphi\colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ a holomorphic map between complex tori. Show that there exist complex numbers m and b, with $m\Lambda \subset \Lambda'$, so that $\varphi(z) = m\,z + b + \Lambda'$. Show that the map is invertible iff $m\Lambda = \Lambda'$.

In particular, the tori $\mathbb{C}/\Lambda$ and $\mathbb{C}/(m\Lambda)$ are isomorphic. Taking $m = \omega_1^{-1}$, up to isomorphism we may assume the lattice generators to be $(1, \tau)$ with $\operatorname{Im}\tau > 0$. By the exercise and eqn.(5.35), τ and τ' correspond to isomorphic tori iff

$$\tau' = \frac{a\tau + b}{c\tau + d}. \tag{5.36}$$

Since τ takes values in the upper half–plane $\mathcal{H} = SL(2,\mathbb{R})/SO(2)$, we get

THEOREM 5.2. *The isomorphism classes of complex tori of dimension* 1 *are labelled by points in the modular curve*

$$SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \tag{5.37}$$

which is precisely the massless scalars's manifold in F–theory.

5.2.2. *Complex tori as elliptic curves.* A complex torus of (complex) dimension 1, C, is obviously a Riemann surface (that is a compact complex manifold of dimension 1). Since, topologically, the genus is half the first Betti number b_1, C has genus 1. As we notice above, C is also an Abelian group, with the point $O = 0 + \Lambda$ as the zero element. Hence it should describe the same objects (*i.e* elliptic curves) we discussed in §. 5.1 with $K = \mathbb{C}$.

The meromorphic functions on the torus $f\colon \mathbb{C}/\Lambda \to \mathbb{P}^1$ are naturally identified with the meromorphic functions on the plane $\mathbb{C}$ which are Λ–periodic

$$f(z) = f(z + m\omega_1 + n\omega_2), \qquad \forall\, m, n \in \mathbb{Z}. \tag{5.38}$$

Again, take the point $O \in C$ and consider the spaces $H^0(C, k[O])$ ($k = 0, 1, 2, 3, \dots$) of the meromorphic functions on C having, at most, a pole of order k at the origin O and no other pole. As before

$$\dim H^0(C, k[O]) = \begin{cases} 1 & \text{for } k = 0 \\ k & \text{for } k \geq 1. \end{cases} \tag{5.39}$$

The $\mathbb{C}$–space $H^0(C, 0[O])$ is spanned by the constant function 1. 1 spans also the space $H^0(C, 1[O])$. The space $H^0(C, 2[O])$ has dimension 2, so it is spanned by two functions, 1 and a second meromorphic Λ–periodic function which we call $\wp(z)$. By definition, $\wp(z)$ has a double pole at the lattice points $z \in \Lambda$, and we normalize it by setting $wp(z) = 1/z^2$+less singular. Since $z \mapsto -z$ is a symmetry (a complex automorphism) of any torus, we must have $\wp(-z) = \wp(z)$.

Then the space $H^0(C, 3[O])$ is spanned by the three functions. As the third function we can take $\wp'(z)$, since this meromorphic function is Λ–periodic, has a pole or order 3 at the origin, and is regular everywhere else. From the properties of $\wp(z)$ we see that $\wp'(z)$ has the form $-21/z^3$+less singular, and has the symmetry $\wp'(-z) = -\wp'(z)$.

Going on, we arrive at $H^0(C, 6[O])$, which has dimension 6, so there must exist a linear relation between the *seven* meromorphic functions 1, $\wp(z)$, $\wp'(z)$, $\wp(z)^2$, $\wp(z)\,\wp'(z)$, $\wp(z)^3$, and $\big(\wp'(z)\big)^2$, of the form

$$\left(\frac{d\wp}{dz}\right)^2 = 4\wp^3 + c_1\,\wp\,\wp' + c_2\,\wp(z)^2 + c_3\,\wp' - g_2\,\wp - g_3, \tag{5.40}$$

where we matched the coefficients of $1/z^6$ in the two sides. The symmetry $z \leftrightarrow -z$ implies $c_1 = c_3 = 0$, while we are free to redefine what we call $\wp(z)$ by adding a constant. We choose this constant to set $c_2 = 0$. We end up with the *Weierstrass differential equation in canonical form*

$$\boxed{\left(\frac{d\wp}{dz}\right)^2 = 4\,\wp^3 - g_2\,\wp - g_3} \tag{5.41}$$

If we set $Y = \frac{1}{2}\wp'(z)$, $X = \wp(z)$, $A = -g_2/4$, $B = -g_3/4$, we get the previous Weierstrass equation. Thus the map $\mathbb{C}/\Lambda \to \mathbb{P}^2$ given by

$$z \mapsto \left(\wp(z), \frac{1}{2}\,\wp'(z), 1\right) \tag{5.42}$$

identifies the torus $\mathbb{C}/\Lambda$ with the elliptic curve

$$Y^2 = X^3 - \frac{g_1}{4}X - \frac{g_6}{4}. \tag{5.43}$$

Thus: *a complex torus is an elliptic curve (over* $\mathbb{C}$*).*

EXERCISE 5.3. Deduce the sum–formulae for the Weierstrass function $\wp$ from the group law of the corresponding elliptic curve.

The converse is also true. Before going to that, we pause a while to discuss the relation with the modular functions.

5.2.3. *$\wp$ and the modular forms.* One obvious way to construct Λ–periodic functions, is to take any function f on $\mathbb{C}$, which vanishes rapidly enough at infinity, and take the Poincaré sum

$$F(z) = \sum_{\omega\,\in\Lambda} f(z+\omega).$$

Let us apply this to the function $-2/z^3$. The corresponding Poincaré series is absolutely convergent so it defines a Λ–periodic meromorphic function $F(z)$ having a pole of order 3 at the lattice points, *i.e.* $F(z) \sim -2/(z-\omega)^3$ for $z \sim \omega \in \Lambda$. Moreover, $F(-z) = -F(z)$ by the $\Lambda \leftrightarrow -\Lambda$ symmetry of the lattice. From §. 5.2.2 we know that there is precisely one *odd* Λ–periodic function which is holomorphic for $z \notin \Lambda$ and has the form $-2/z^3$+less singular for $z \sim 0$. Therefore

$$\wp'(z) = -2 \sum_{\omega\,\in\Lambda} \frac{1}{(z+\omega)^3}. \tag{5.44}$$

Let us expand this function in Laurent series

$$\begin{aligned}\wp'(z) &= -\frac{2}{z^3} + \sum_{k=0}^{\infty} (-1)^k (k+2)(k+1)\, z^k \sum_{\omega\in\Lambda,\,\omega\neq 0} \frac{1}{\omega^{k+3}} \\ &= -\frac{2}{z^3} + \sum_{l=1}^{\infty} (2l+1)(2l)\, z^{2l-1} \sum_{\omega\in\Lambda,\,\omega\neq 0} \frac{1}{\omega^{2l+2}}\end{aligned} \tag{5.45}$$

where we used that $\sum_\Lambda^* \omega^{-(2k+1)} = 0$ by the symmetry of the lattice. The lattice sums in the RHS are known as *the Eisenstein series (of weight* $2l+2$*) of the lattice* Λ

$$G_{2k}(\Lambda) = \sum_{\omega\in\Lambda,\,\omega\neq 0} \frac{1}{\omega^{2k}}. \tag{5.46}$$

Notice the homogeneity condition $G_{2k}(m\Lambda) = m^{-2k} G_{2k}(\Lambda)$. Then

$$G_{2k}(\omega_1\,\mathbb{Z}\oplus\omega_2\,\mathbb{Z}) = (\omega_2)^{-2k}\, G_{2k}(\tau\,\mathbb{Z}\oplus\mathbb{Z}) \equiv (\omega_2)^{-2k}\, G_{2k}(\tau), \tag{5.47}$$

so we reduce to a function of τ, $G_{2k}(\tau)$ (also called *the Eisenstein series of weight* $2k$*)*, which depends only on the periods' ratio τ. Chainging basis by a $SL(2,\mathbb{Z})$ transformation,

$$G_{2k}\left(\frac{a\tau+b}{c\tau+b}\right) = (c\tau+d)^{2k}\, G_{2k}(\tau), \tag{5.48}$$

so $G_{2k}(\tau)$ is *a modular form of weight* $2k$ [**25**][**26**][**27**].

Finally, integrating (5.45), we get

$$\boxed{\wp(z) = \frac{1}{z^2} + \sum_{l=1}^{\infty} (2l+1)\, G_{2l+2}(\Lambda)\, z^{2l}} \tag{5.49}$$

LEMMA 5.1. *One has*

$$\big(\wp'(z)\big)^2 - 4\,\wp(z)^3 + 60\, G_4(\Lambda)\,\wp(z) + 140\, G_6(\Lambda) \equiv 0. \tag{5.50}$$

PROOF. It is enough to show that the Laurent series of the RHS of (5.50) contains only positive powers of z, since a holomorphic function on the torus vanishing at the origin should vanish everywhere.

Take the derivative of (5.50). It factorizes as $2\wp'(\wp''-6\wp^2+30G_4)$. From eqn.(5.49)

$$\begin{aligned}\wp'' &= \frac{6}{z^4}+6\,G_4+60\,G_6z^2+\cdots\\ \wp^2 &= \frac{1}{z^4}+6\,G_4+10\,G_6z^2+\cdots\\ &\Rightarrow\ \wp''-6\wp^2+30\,G_4=O(z^4)\quad\text{hence identically zero!}\end{aligned}$$

Thus the RHS of (5.50) is a constant. The coefficients of z^0 in the various Laurent expansions are easy to compute

$$(\wp')^2\Big|_{z^0}=-80\,G_6,\qquad \wp^3\Big|_{z^0}=15\,G_6,\qquad \wp\Big|_{z^0}=0,\tag{5.51}$$

so the constant also vanishes. □

This lemma motivates the following

Definition–Proposition 5.1. *Let Λ be a lattice in $\mathbb{C}$. Set $g_2(\Lambda)=60\,G_4(\Lambda)$ and $g_3(\Lambda)=140\,G_6(\Lambda)$. The* elliptic curve $E(\Lambda)$ *is the projective curve*

$$E(\Lambda):\quad Y^2Z=4\,X^3-g_2(\Lambda)\,XZ^2-g_3(\Lambda)\,Z^3.\tag{5.52}$$

Two lattices differing only by the overall scale, $\Lambda'=m\Lambda$, define isomorphic *elliptic curves, indeed*

$$g_2(\Lambda')=m^{-4}\,g_2(\Lambda),\qquad g_3(\Lambda')=m^{-6}\,g_3(\Lambda).\tag{5.53}$$

The discriminant $\Delta(\Lambda)$ and the j–invariant are given by

$$\Delta(\Lambda)=g_2(\Lambda)^3-27\,g_3(\Lambda)^2,\qquad j(\Lambda)=\frac{1728\,g_2(\Lambda)^3}{\Delta(\Lambda)}.\tag{5.54}$$

Notice that $j(m\Lambda)\equiv j(\Lambda)$ so the j–invariant depends only on the isomorphism class of $E(\Lambda)$.

Proof. The only thing that remains to prove is that $E(\Lambda)$ is an elliptic curve, that is that the projective curve (5.52) is non singular. Equivalently, we have to show that $\Delta(\Lambda)\neq 0$ for all lattices in $\mathbb{C}$. By construction $\Delta(\Lambda)$ is a modular function of weight 12. Then there is two ways to show that $\Delta(\Lambda)\neq 0$. From the side of the theory of the modular cusp forms or from the viewpoint of the function theory on the torus $\mathbb{C}/\Lambda$. We choose the second strategy. Let

$$e_1=\tfrac{1}{2}\omega_1,\qquad e_2=\tfrac{1}{2}\omega_2,\qquad e_3=\tfrac{1}{2}(\omega_1+\omega_2),\tag{5.55}$$

be the three points in the torus corresponding to the half–lattice (*i.e.* 2–torsion) points. One has

$$\wp'(e_i)=\wp'(e_i-2e_i)=\wp'(-e_i)=-\wp'(e_i)=0.\tag{5.56}$$

Thus the Weierstrass equation may be rewritten as

$$\big(\wp'(z)\big)^2=4\big(\wp(z)-\wp(e_1)\big)\big(\wp(z)-\wp(e_2)\big)\big(\wp(z)-\wp(e_3)\big),\tag{5.57}$$

and the condition $\Delta(\Lambda) \neq 0$ is equivalent to $\wp(e_i) \neq \wp(e_j)$ for $i \neq j$. Consider, say, the function $\wp(z) - \wp(e_1)$. It has a double pole at the origin and no other pole, so it must have two zeros. But it has two zeros at $z = e_1$, since the derivative there also vanishes. Hence it has no other zero and (say) $\wp(e_2) - \wp(e_1) \neq 0$. $\square$

REMARK. From eqn.(5.53) and the homogeneity of the Weierstrass equation (or from eqn.(5.48)) we get the following transformation formula for the Weiertrass function $\wp(z, \tau)$ on the torus with normalized periods τ and 1

$$\wp\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2\, \wp(z, \tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \quad (5.58)$$

We arrive at the result already advertised,

THEOREM 5.3. *Every elliptic curve E over $\mathbb{C}$ is isomorphic to $E(\Lambda)$ for some Λ.*

PROOF. We already know that over $\mathbb{C}$ (which is algebraically closed!) two elliptic curves are isomorphic if and only if they have the same j–invariant. So the only thing we have to show is that for all $j \in \mathbb{C}$ there exists a lattice Λ such that $j(\Lambda) = j$. Without loss of generality, we may take a normalized lattice of periods $(\tau, 1)$. Thus we have to show that there is a τ, unique up to modular equivalence, such that $j(\tau) = j$. The function $j(\tau)$ gives a holomorphic map

$$j \colon \overline{SL(2, \mathbb{Z})\backslash\mathcal{H}} \to \mathbb{P}^1, \quad (5.59)$$

where the overbar means compactification adding the cusp point at ∞. Both the source and target spaces are Riemann surfaces of genus zero. Any such map has a definite degree, d, equal to the number of poles and the number of zeros. Any point in the target $\mathbb{P}^1$ has d preimages. But d should be 1, since otherwise we may find two distinct points $\tau, \tau' \in \overline{SL(2, \mathbb{Z})\backslash\mathcal{H}}$ with $j(\tau) = j(\tau')$ impling that the two elliptic curves are isomorphic, while we know from their complex torus realization that they are not. $d = 1$ means, in particular, that the map (5.59) is surjective. $\square$

Notice that $j(\tau)$ must have just one pole (at ∞). More details on the j–function in the next subsection.

5.2.4. *The modular form Δ and the modular invariant j.* The modular functions satisfy a bunch of useful and deep identities that we shall use from time to time. We list some of them in APPENDIX A. A nice place where to look is [**28**].

5.3. The differential dX/Y. A curve of genus one has, by definition, a holomorphic differential ω without zeros. Over $\mathbb{C}$, writing the curve as a the torus $\mathbb{C}/\Lambda$, ω is just dz.

Now,

$$dz = \frac{d\wp(z)}{\wp'(z)} = 2\,\frac{dX}{Y}. \tag{5.60}$$

The expression dX/Y makes sense over any K (say of characteristic zero). Thus the holomorphic differential over the elliptic curve $Y^2 = X^3 + AX + B$ can be always written as dX/Y (up to overall normalization).

Alternatively, the formula (5.60) may be deduced from the Griffiths residue theorem (see *e.g.* ref. [**29**], vol. II, chapter 6) applied to the Weierstrass hypersurface in $\mathbb{P}^2$.

The Griffiths residue theorem also implies that $H^1(E,\mathbb{C})$ is spanned by the holomorphic form dX/Y and the *meromorphic* form $XdX/Y = d\zeta(z)$ (here $\zeta(z)$ is the Weierstrass ζ–function). There is a bilinear pairing between dX/Y and $X\,dX/Y$ corresponding to the wedge product in $H^*(E,\mathbb{C})$. This reproduces the Legendre relation of elliptic function theory.

6. F–theory: elliptic formulation

6.1. The scalar's sector. In F–theory the complex massless scalar

$$\tau = C_0 + i\,e^{-\phi} \tag{6.1}$$

is well defined only up to an $SL(2,\mathbb{Z})$ transformation. Working with a field which may jump in a complicated way is not very convenient and hides the physical content of the theory. Then one has to look for alternative representations of the scalars' configurations which are more regular and physically *intrinsic.*

At fist sight, one could think that it is enough to make a change of coordinates in target space,

$$\tau \mapsto j(\tau),$$

since the j–function is a modular invariant and hence globally defined. However such a parametrization would make us to loose some physically important information (see discussion below).

The best idea is to see the scalars' configuration

$$\tau\colon\ \text{SPACETIME} \equiv \mathcal{X} \ \to\ SL(2,\mathbb{Z})\big\backslash SL(2,\mathbb{R})\big/SO(2)$$

as a (smooth) map which associates to each point x in spacetime an elliptic curve E_x, well–defined up to isomorphism. Hence we shall write a scalars' configuration as a canonical Weierstrass curve

$$Y^2 = X^3 + A(x)\,X + B(x), \tag{6.2}$$

where $A(x)$, $B(x)$ are smooth functions well–defined up to

$$A(x) \to c(x)^4\, A(x), \qquad B(x) \to c(x)^6\, B(x), \tag{6.3}$$

where $c(x)$ is a nowhere vanishing smooth function.

Notice that the functions $A(x)$, $B(x)$ need *not* to be globally defined. If $\cup_i U_i$ is a (sufficiently fine) open cover of the $10D$ spacetime $\mathcal{X}$, and $A_i(X), B_i(x)$ are the coefficients of the Weierstrass curve over the patch U_i, we only need that there exist *never–vanishing* complex functions $c_{ij}(x)$ such that

$$A_i(x) = c_{ij}(x)^4\, A_j(x), \quad B_i(x) = c_{ij}(x)^6\, B_j(x) \quad \text{in } U_i \cap U_j. \tag{6.4}$$

The $c_{ij}(x)$'s manifestly satisfy the 1–cocycle condition. Hence they are the transition functions defining a smooth complex line bundle $\mathcal{L} \to \mathcal{X}$. Thus we learn the

General lesson 6.1. *In F–theory the massless scalars' configuration is encoded in a (smooth) line bundle*

$$\mathcal{L} \to \mathcal{X},$$

and two sections $A(x) \in C^\infty(\mathcal{X}, \mathcal{L}^4)$ and $B(x) \in C^\infty(\mathcal{X}, \mathcal{L}^6)$ with

$$0 \not\equiv 4\, A(x)^3 + 27\, B(x)^2 \ \in\ C^\infty(\mathcal{X}, \mathcal{L}^{12}), \tag{6.5}$$

trough the elliptic curve

$$Y^2 Z = X^3 + A(x)\, XZ + B(x)\, Z^3. \tag{6.6}$$

Y and X transform, respectively, as sections of $\mathcal{L}^3$ and $\mathcal{L}^2$. Hence the differentials dX/Y and $X\,dX/Y$ transform, respectively, as sections[26] *of $\mathcal{L}^{-1}$ and $\mathcal{L}$.*

What have we achieved by this re–formulation?

The list of the advantages is long:

- first of all, it is a manifestly $SL(2,\mathbb{Z})$–invariant formulation which does not loose any subtle physical information;
- it is very geometric: eqn.(6.6) describes a *twelve*–dimensional manifold $\mathcal{Y}_{12}$ with a natural projection to the physical $10D$ spacetime $\mathcal{X}$,

$$\pi \colon \mathcal{Y}_{12} \to \mathcal{X}. \tag{6.7}$$

  The fibration (6.7) has a preferred section $\sigma \colon \mathcal{X} \to \mathcal{Y}_{12}$ obtained by sending the point $x \in \mathcal{X}$ to the preferred point O_x of the elliptic curve E_x over x (that is, to the neutral element of the corresponding Abelian group).

[26] Note that dX/Y and $X\,dX/Y$ transform in opposite ways, so that their product is invariant. This is a manifestation of the invariant pairing alluded at the end of §. 5.3 (namely the intersection form in $H^1(E_x, \mathbb{C})$).

We identity the physical spacetime $\mathcal{X}$ with its immage under σ: then the spacetime is seen as a submanifold of $\mathcal{Y}_{12}$.

Gravity, described by the geometry of the spacetime manifold $\mathcal{X}$, and the scalars' dynamics are unified in the geometry of the twelve dimensional manifold $\mathcal{Y}_{12}$. This is an higher form of unification of the fundamental interactions, stronger than the one implied by the mere $(2,0)$ supersymmetry. Moreover, having geometrized the scalars' dynamics is also very convenient from a technical viewpoint: we have many tools to study the geometry of $\mathcal{Y}_{12}$.

In its more flamboyant interpretation, F–theory is a *twelve dimensional theory,* and $\mathcal{Y}_{12}$ is the fundamental spacetime. Notice, however, that two dimensions of $\mathcal{Y}_{12}$ have a rather different status with respect to the other ten. We have a projection π, a section σ, and while *a priori* spacetime is just a smooth manifold, the fibers of π come with a well–defined complex structure (but not a natural metric!);

- many physical quantities are elegantly described (and computed) in this *elliptic* framework. For instance, we have the line bundle $\mathcal{L}$ (which, *a priori,* is just a smooth one). Smooth line bundles are classified up isomorphism by their Chern class $c_1(\mathcal{L})$. What is the meaning of this topological invariant of the line bundle?

FACT 6.1. *The class $12\, c(\mathcal{L}) \in H^2(\mathcal{X},\mathbb{Z})$ is the Poincaré dual of the total* seven–brane *homology class (that is, it is the class dual to the class $\sum_i [\gamma_i]$, where $\gamma_i \subset \mathcal{X}$ stands for the world–volume of the i–th seven–brane with* sign *according orientation).*

In formulae

$$12 \int_{\mathcal{X}} c(\mathcal{L}) \wedge \alpha = \sum_i \int_{\gamma_i} \alpha, \tag{6.8}$$

for all *closed* 8–forms α.

In these lectures we will give (most of the times implicitly) many proofs of this crucial fact. Le us limit ourselves to a rough argument here[27].

[27] The argument has technical loopholes. From the mathematical side, we must assume that the discriminant $4A^3 + 27B^2$ is a *transverse* section of $\mathcal{L}^{12}$. This, in particular, means that the singularities of the elliptic fibration are in real codimension 2 (and hence *seven* branes).

Physically, the point is that for a general (that is non–BPS) configuration we have brane and anti–branes which may annihilate each other leaving behind branes of lesser dimension. So care is neeeded in defining what we mean by the brane world–volume. Having done everything properly, the result should be true.

It is enough to work in the vicinity of a seven brane. Let z be the complex coordinates in a plane locally orthogonal to the brane, localized at $z = 0$. Then

$$\partial_\mu \tau = \frac{1}{2\pi\, i}\,\partial_\mu \log z + \partial_\mu f, \qquad (f \text{ a globally defined smooth function}), \tag{6.9}$$

so $\operatorname{Im}\tau \to \infty$ as we approach the brane core.

In terms of $q(x) \equiv \exp(2\pi i\, \tau(x))$ one has

$$\begin{aligned} 4\,A(x)^3 + 27\,B(x)^2 = \\ = -\frac{(2\pi)^{12}}{16}\,q(x)\prod_{n=1}^{\infty}\Big(1-q(x)^n\Big)^{24} = -\frac{(2\pi)^{12}}{16}\,q(x) + O(q^2), \end{aligned} \tag{6.10}$$

and the LHS vanishes *precisely* on the brane locus. Therefore

$$\frac{1}{2\pi\, i}\oint_\gamma d\log\Big(4\,A^3 + 27\,B^2\Big) = \sharp\,(\text{seven branes encircled by } \gamma). \tag{6.11}$$

Since the discriminant $4\,A^3 + 27\,B^2$ is a section of $\mathcal{L}^{12}$, we get the claim. [Recall that the Chern class of a line bundle is Poincaré dual to the zero locus of a *transverse* section; see *e.g.* ref. [**19**], PROPOSITION II.12.8 and PROPERTY IV.(20.10.6).]

REMARK. The above formalism is a little funny in the sense that we have a smooth twelve dimensional manifold $\mathcal{Y}_{12}$ which has a complex structure *fiberwise*. Such a geometrical category can be defined, of course. But, geometrically, it would be much nicer to work with manifolds which are fully complex. This will be typically the case if we limit ourselves to configurations which have some residual unbroken supersymmetry (as we do most of the time: vacua, BPS objects, *etc.).* In these lecture we shall work mostly in the complex (and even *algebraic !*) category. In these cases the above geometrical facts take a much powerful and precise form. Already in the category C^ω of the real analytic manifolds we may make some stronger statement.

7. The G–flux

7.1. Completing the reformulation of F–theory. In section 6 we presented the 'elliptic' formulation of F–theory but limited ourselves to the scalars'/gravity sector. We must complete the reformulation to include the form–fields (the *fluxes).* Before doing that, we de–mythize the formalism. Indeed, in the traditional language of supergravity (see ref. [**30**] or [GSSFT]) the above 'elliptic' formalism would be expressed in less fancy terms.

As we have already reviewed, the SUGRA language is based on the symmetry $G_{\text{global}} \times H_{\text{local}}$. We can choose to work with quantities which have 'G_{global} indices' or have 'H_{local} indices', the vielbein $\mathcal{E}$ being used to convert one kind of objects into the other.

The objects of the 'elliptic' formalism are of the second kind, that is they transform in representations of $U(1)_{\text{local}}$, while are invariant under $SL(2,\mathbb{R})_{\text{global}} \supset SL(2,\mathbb{Z})_{\text{superselection}}$.

Indeed, consider the line bundle $\mathcal{L} \to \mathcal{X}$. It has structure group $\mathbb{C}^{\times}$; in general, the structure group of a vector bundle can be restricted to the corresponding unitary subgroup, in this case $U(1)$. I claim that this $U(1)$ is the same as the $U(1)_{\text{local}}$ of traditional SUGRA. In the old days one would have said that the two fields A and B have R–charge, respectively, -8 and -12 (in the normalization of eqn.(1.1)), rather that there was an elliptic fibration.

To check the correcteness of the claim, just compare the connections of the two local $U(1)$'s.

The main difference is that in traditional SUGRA, or more generally in Lagrangian Field Theory one uses *unitary* basis (*i.e.* trivializations) of vector bundle, while in geometry it is more common to use 'holomorphic' trivializations (here holomorphic in the fiberwise sense). Of course the two are perfectly equivalent.

Let us return to the standard SUGRA equations (2.25)(2.26). The metric along the fiber of $\mathcal{L}^{-1}$ is equal to $y^{-1} \equiv (\operatorname{Im}\tau)^{-1}$. Indeed, $\mathcal{L}^{-1}$ is the Hodge bundle, and the standard flat metric of constant volume[28] on the torus E_x is $|dz|^2/(\operatorname{Im}\tau)$. Then the factors $y^{-1/2}$ in front of the RHS of eqns.(2.25)(2.26) is precisely the vielbein converting from the holomorphic to the unitary trivializations[29] (in particular, the fiber metric, $(\operatorname{Im}\tau)^{-1}$, transforms as a section of $\mathcal{L} \otimes \overline{\mathcal{L}}$).

So, in the 'elliptic' language, the basic 3–form field strenghts are

$$G = -\,i(H_1 + \tau\, H_2) \qquad \text{is a 3–form with coefficients in } \mathcal{L}^{-1} \tag{7.1}$$

$$\overline{G} = \;i(H_1 + \bar{\tau}\, H_2) \qquad \text{is a 3–form with coefficients in } \overline{\mathcal{L}}^{-1} \tag{7.2}$$

7.2. The $\mathbf{G}_4$–flux 4**–form.** To get something more invariant and 'geometric', which behaves as an ordinary flux we may define the following[30] 4–form on the twelve–fold $\mathcal{Y}_{12}$

$$\begin{aligned} \mathbf{G}_4 &= \frac{1}{4\operatorname{Im}\tau}\Big(\overline{G}\wedge\frac{dX}{Y} + G\wedge\frac{d\overline{X}}{\overline{Y}}\Big)\\ &\equiv \frac{1}{2\operatorname{Im}\tau}\Big(\overline{G}\wedge dz + G\wedge d\bar{z}\Big). \end{aligned} \tag{7.3}$$

$\mathbf{G}_4$ is a *bona fide* 4–form on the twelve–fold. Hence it is the natural object from the viewpoint that sees $\mathcal{Y}_{12}$ as the fundamental space. Moreover, it is this 4–form flux which directly compare with M–theory

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28 In a diffent language: this is the metric on the fiber torus for a constant Kähler class of E_x.

29 Indeed: $e = y^{-1/2}dz$ is the unitary 2–bein for the metric along the fiber, since the metric is $ds^2 = |e|^2$.

30 Normalization as in ref. [**31**] eqn.(10.70).

fluxes when we relate F–theory compactifications to M–theory compactifications trough duality [**32**]. See section 10 in chapter **2**.

7.3. Other fields. For the fields which transform trivially under $SL(2,\mathbb{Z})$, we can extend the fields on the $10D$ spacetime $\mathcal{X}$ to the full 12–fold $\mathcal{Y}_{12}$ by just taking the pull–back trough the projection $\pi\colon \mathcal{Y}_{12} \to \mathcal{X}$.

8. Are twelve dimensions real?

It is natural to ask if the 12–dimensional space $\mathcal{Y}_{12}$ is real or just a convenient technical trick. Lacking a proper formulation of F–theory, we have only weak clues on the answer.

F–theory is a (locally) supersymmetric theory with 32 real supercharges. From the classification of spinors in spaces of signature (p,q) (see, *e.g.* [GSSFT] APPENDIX B), we see that in twelve dimensional Minkowski space (signature $(11,1)$) the minimal spinor has 64 real components, and no covariant theory can have just 32 supercharges. So the idea of an underlying 'standard' 12–dimensional theory is certainly unviable.

However, in signature $(10,2)$, Majorana–Weyl spinor exist, and have 32 (real) components. So, a naive idea may be that the underlying $12D$ theory is based on a 'space–time' of signature $(10,2)$, that is with *two* 'times'. Such a theory would be troublesome in regard of fundamental physical principles such as causality, unitarity, the second law of thermodynamics[31]...

However, from some points of view, F–theory *does look like* having two times. We review the original argument by Vafa [**1**].

In the preceding sections we have argued that $SL(2,\mathbb{Z})$ should be a symmetry of any consistent non–perturbative completion of Type IIB. The fundamental string and the $D1$–brane are mapped one into the other by this symmetry. So a proper formulation should threat the two symmetrically. In particular, we could take the $D1$ brane as the basic object for a strong coupling asymptotic expansion (since the interchange of the string with $D1$ is supplemented by $g \leftrightarrow g^{-1}$).

The effective world–sheet theory on $D1$ is the $U(1)$ super–Yang–Mills (16 supercharges) in $d=2$. In flat space, this is just the free multiplet. The vector, which does not propagate local degrees of freedom in $d=2$, decouples, and we remain with the same physical local degrees of freedom as the superstring world–sheet theory. However, this is true in a non–covariant light–cone gauge. In a covariant gauge,

[31] Recall Heddington: "If your theory is contradicted by the experiment, don't worry, the experimentalists are not that smart, they are wrong most of the time. However, if your theory contradicts the second law, it is deadly wrong (no matter its experimental successes).

we have a $2d$ vector together with the Fadeev–Popov (super)ghost of the corresponding $U(1)$ gauge symmetry. The ghost central charge is -3. This shifts the critical dimension by $+2$, so we get $10+2=12$ dimensions. On the other hand, the presence of a gauged symmetry changes the BSRT charge by a term proportional to the $U(1)$ current. Writing the $U(1)$ current as $v_\mu \partial X^\mu$, the requirement of nilpotency implies $v_\mu v^\mu = 0$. The physical operators then correspond to oscillators with $v_\nu \psi^\nu = 0$ which are identified modulo oscillators proportional to the vector v_μ. Thus the BRST cohomology kills the oscillators of a pair of coordinates of *opposite* signature. Then the 12 space must have signature $(9,1)+(1,1)=(10,2)$.

How the theory manages not to be in trouble with causality and the like?

Well, Vafa's analysis, in particular, shows that *at strong coupling* the graviton vertex $\psi_\mu \tilde{\psi}_\nu e^{-\phi-\tilde{\phi}} e^{ip\cdot X}$ has legs only in the first $9+1$ dimensions. That is: *there is no metric field in the fiber directions.* All distances are zero along the fiber. The fact that there is *no* metric in the two 'new' dimensions, makes the question *'what is the signature of the metric'* just meaningless. There are no fiber distances along which to propagate signals, and there is no room for causality paradoxes.

True, for certain purposes (typically for representation theoretical arguments) it is convenient to think of the 12–fold as having signature $(10,2)$. For other other choices may be more convenient. The point is that we introduce a metric along the fiber only as a regularizing device, taking the metric to zero at the end of the computation. Using a Lorentzian or an Euclidean metric to regularize should have no effect on the physical observables (as long as it is a proper regularization). There is no contradiction.

The most convenient way to think about F–theory (to the present limited understanding) is as follows. In F–theory all the ordinary field theoretical degrees of freedom live on branes, that is on submanifold of the total 'spacetime' and do not propagate in the bulk of the geometry. Matter fields (would–be quark, leptons, Higgs,...) live on six dimensional submanifolds (5–branes); gauge fields, having a larger spin, live in two more dimensions, namely on eight dimensional subspace (7–branes). The graviton (and its SUSY partners) having an even larger spin must live on a brane with two more dimension, that is on a ten dimensional subspace $\mathcal{X}$. This is a ten dimensional gravitational brane of some higher dimensional manifold. By definition, along the normal direction to the gravitational brane the graviton, and hence the metric, vanishes.

In conclusion, the theory is really 12–dimensional, but the 12–dimensional geometry is not metric, except along a submanifold. People willing to think of spacetime as something with a metric, will conclude that only ten dimensions are real, but this is not the best point of view.

The geometry of F–theory is deep, beautiful, and more interesting than a mere metric geometry.

REMARK. There are alternative viewpoints in which, roughly speaking, F–theory is 'a theory of $F3$–branes' moving in 12 dimensions much in the same sense that M–theory is a theory of $M2$–branes moving in 11 dimensions, and string theory is a theory of one–branes (namely strings) moving in 10 dimensions. See refs. [**33**]. We will not pursue this approach in these introductory lectures.

9.(∗) ADDENDUM: $\Gamma \neq SL(2,\mathbb{Z})$

What does change in the above discussion if Γ is a proper subgroup of $SL(2,\mathbb{Z})$ (with the special properties in....)?

Not really much. The present discussion is added only to illustrate the astonishing power of the finite volume property, which — just by itself — comes very closed to uniquely define the IR structure of the (unknown) non–pertubative theory.

If the scalars' manifold is $\Gamma\backslash\mathcal{H}$ with Γ a subgroup of the modular group, we have a natural quotient map

$$\Gamma\backslash\mathcal{H} \to SL(2,\mathbb{Z})\backslash\mathcal{H}, \tag{9.1}$$

so we can associate an elliptic fibration $\mathcal{Y}_{12} \to \mathcal{X}$ to a scalars' configuration. The only difference is that now we loose some information. Indeed the space $\Gamma\backslash\mathcal{H}$ is *the moduli space classifying elliptic curves with some extra structure* — the actual structure implied being dependent on the particular group Γ — up to the natural isomorphism. *E.g.* if Γ was trivial (a possibility ruled out by the finite volume property) we would get the moduli space of *marked* elliptic curves. The elliptic curve together with the extra structure corresponding to a subgroup Γ are called *enhanced elliptic curves for* Γ.

The situation is particularly simple is Γ is a Hecke subgroup. The structures are as in the table (cfr. ref. [**9**] , §. 1.5)

Hecke group	structure on the elliptic curve E
$\Gamma_0(N)$	a cyclic subgroup C of E of order N
$\Gamma_1(N)$	a point Q on E of order N
$\Gamma(N)$	a pair of points $(P,Q) \in E$ which generate the N–torsion subgroup $E[N]$ and have[32] $e_N(P,Q) = e^{2\pi i/N}$

So, if (say) $\Gamma = \Gamma_1(5)$ the main change is that instead of having *one* God–given section of the fibration $\mathcal{Y}_{12} \to \mathcal{X}$ we have *five* of them: the one corresponding to O_x and those corresponding to $O_x + k(Q_x - O_x)$, $k = 1, 2, 3, 4$.

In the language of *science fiction:* in the 12–dimensional spacetime $\mathcal{Y}_{12}$ we have *five* $D = 10$ gravitational branes $\mathcal{X}_k$. This means five sectors which are mutually *'gravitationally dark'* (that is, they do not interact gravitationally each to the other). A good plot for a science fiction novel, but not much of a physical theory. Indeed all the evidence points to the fact that in the consistent non–perturbative completion of Type IIB we must have

$$\Gamma = SL(2, \mathbb{Z}). \tag{9.2}$$

10.(∗) ADDENDUM: Galois cohomology of elliptic curves

From an abstract point of view, the proper language to describe many important physical objects and properties in an intrinsically non–perturbative fashion is the *Galois cohomology* [**34**]. This is particularly evident in presence of orbifold planes.

Roughly speaking, assume all the fields to be *real–analytic* (or, even better, holomorphic) in the $10D$ manifold B. The real–analytic functions are an integral domain, and it makes sense to consider its quotient field $\mathbb{F}(B)$. Then a real–analytic F–theory configuration is an elliptic curve over the field $\mathbb{F}(B)$ (over the function field $\mathbb{C}(B)$ in the holomorphic case). However, $\mathbb{F}(B)$ is not algebraically closed, so the invariant $j \in \mathbb{F}(B)$ does not define completely the physical configuration (that is: *The observables are not determined once j is known).* Of course, the Galois group is the measure of how much 'non–algebraically–closed' is a field. For functions fields, the Galois group of a finite extension is just the fundamental group π_1 of the associated finite cover $\tilde{B} \to B$, and hence, to compute the physical observables, we need two ingredients: the j–invariant and the action of the *monodromy* associated to appropriate cover. All phenomenological quantities (the low–energy particle spectrum, gauge group, Yukawa couplings, *etc.*) are determined in this way, and hence have an interpretation in the Galois language.

There are many readable accounts of the material (for the case of elliptic curves), see refs. [**23**][**22**][**35**][**36**] (§§. 19–23 of [**23**] is a nice introduction for *pedestrians*). Here we limit to give a vague flavor of the subject.

$\boxed{\textbf{TO BE WRITTEN}}$

CHAPTER 2

Vacua, BPS configurations, Dualities

In chapter **1** we saw that F–theory looks like a twelve–dimensional theory with an extended spacetime $\mathcal{Y}_{12}$ which is elliptically fibered over the $D = 10$ Riemannian manifold $\mathcal{X}_{10}$ on which the graviton propagates. The fibration has automatically a section σ, and its image is identified with the 'gravitational' space $\mathcal{X}_{10}$.

The twelve–dimensional viewpoint is relevant to the extend that the physical properties of a configuration (as, *e.g.* whether it solves or not the equations of motion, the number supersymmetries it preserves, the spectrum of light modes,...) can be more directly described in terms of the intrinsic[1] geometry of $\mathcal{Y}_{12}$ than in terms of the geometry of the Riemannian submanifold $\mathcal{X}_{10}$. If this is the case, we have an higher unification of the scalar's and gravitational sector of the theory.

In the present chapter we shall see that this is indeed the case.

We start by discussing a particular class of (classical) configurations of F–theory, namely the BPS ones (by this we mean configurations which leave invariant some supersymmetry). A particular relevant subclass are the Poincaré (or AdS) invariant vacua. The relevance of these configurations is that they are likely to be protected against quantum corrections by SUSY, and hence can be used to learn and compute something in F–theory. Closely related to this is the topic of dualities in which certain vacua/BPS configurations of F–theory are mapped to the corresponding configurations of M–theory or the heterotic string.

Dualities are another way to make geometric sense out of the 12–fold $\mathcal{Y}_{12}$.

1. Supersymmetric BPS configuration. Zero flux

We are interested in the supersymmetric configurations of F–theory, that is classical (bosonic) configurations which preserve some unbroken supersymmetry. Most of the equations of motion are then automatically satisfied (and, of course, we impose the ones, if any, which are not[2]). This property follows from the fact that they satisfy some generalized BPS condition.

[1] Note that 'intrinsic' means, in particular, *non metric*.

[2] It follows from the general theory of SUGRA (cfr. [GSSFT] chapter **6**) that we need, at most, to enforce the Einstein equations.

A particular class of BPS configurations are the d–dimensional Poincaré invariant *vacua.* They corresponds to $10D$ gravitational *warped* metrics of the form

$$ds^2 = f\big(y^k\big)^2\, \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta + g_{ij}(y^k)\, dy^i\, dy^j, \tag{1.1}$$

where g_{ij} is a metric in some compact Riemannian manifold X of dimension $10-d$, and all the fields Φ are required to satisfy $\partial_{x^\alpha}\Phi = 0$ and thus are, in particular, time–independent.

We start with the simpler case of zero–flux, $F_5 = G_3 = 0$, (but arbitrary seven brane sources!). In such a background, the SUSY transformations of the fermions become[3]

$$\delta\lambda = \frac{1}{2\,\mathrm{Im}\,\tau}\, \partial\!\!\!/\tau\, \epsilon^* \tag{1.2}$$

$$\delta\psi_\mu = \mathcal{D}_\mu\epsilon = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{i}{2\,\mathrm{Im}\,\tau}\,\partial_\mu \mathrm{Re}\,\tau\right)\epsilon. \tag{1.3}$$

$\mathcal{D}_\mu$ is a covariant derivative whose connection takes values, by definition, in the Lie algebra

$$\mathfrak{hol}(M) \oplus \mathfrak{u}(1)_R, \tag{1.4}$$

where M is the $10D$ gravitational manifold (of signature $(9,1)$ or $(10,0)$ if we look for *Euclidean* solutions) and $\mathfrak{hol}(M)$ is its holonomy Lie algebra generated by the Riemannian curvature (cfr. the Ambrose–Singer theorem, see [GSSFT] chapter **3**).

An unbroken supersymmetry correspond to a spinor ϵ such that

$$\mathcal{D}_\mu\epsilon = 0 \tag{1.5}$$

$$\partial\!\!\!/\bar{\tau}\, \epsilon = 0. \tag{1.6}$$

Eqn.(1.5) just says that the spinor ϵ is *parallel* with respect to the combined connection (1.4). The corresponding integrability condition is

$$0 = [\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon = \Big(\tfrac{1}{4}R_{\mu\nu ab}\gamma^{ab} + \tfrac{i}{2}Q_{\mu\nu}\Big)\epsilon, \tag{1.7}$$

where $Q_{\mu\nu}$ is the $\mathfrak{u}(1)_R$ curvature. Contract both sides with γ^ν and use the first Bianchi identity, $R_{\mu cab}\gamma^{cab} = 0$. We get

$$\gamma^\nu\Big(R_{\nu\mu} + i\,Q_{\nu\mu}\Big)\epsilon = 0. \tag{1.8}$$

There are two possibilities: either the holonomy of the $\mathfrak{u}(1)_R$ summand is trivial (that is, *globally* pure gauge) or it is non–trivial.

We study the two cases separately.

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[3] We write the SUSY transformations in the Iwasawa gauge and in the Caley–rotated basis (in which $U(1)_R$ acts diagonally).

2. Trivial $\mathfrak{u}(1)_R$ holonomy.

The vanishing of the curvature of the $\mathfrak{u}(1)_R$ connection Q, implies

$$0 = dQ \equiv d\left(\frac{1}{\operatorname{Im}\tau}\, d\operatorname{Re}\tau\right) = i\,\frac{d\tau\wedge d\overline{\tau}}{2\,(\operatorname{Im}\tau)^2}, \tag{2.1}$$

and then

$$\partial\!\!\!/\tau\ \partial\!\!\!/\overline{\tau} \equiv g^{\mu\nu}\partial_\mu\tau\,\partial_\nu\overline{\tau} + \gamma^{\mu\nu}\partial_\mu\tau\,\partial_\nu\overline{\tau} = g^{\mu\nu}\partial_\mu\tau\,\partial_\nu\overline{\tau}. \tag{2.2}$$

Thus, the existence of a non–zero fermion ϵ satisfying eqn.(1.6) implies

$$g^{\mu\nu}\partial_\mu\tau\,\partial_\nu\overline{\tau} = 0. \tag{2.3}$$

In the same way, eqn.(1.8) with $Q_{\mu\nu} = 0$ gives

$$0 = g^{\mu\nu}\gamma^\rho R_{\mu\rho}\,\gamma^\sigma R_{\nu\sigma}\,\epsilon = g^{\mu\nu}g^{\rho\sigma}R_{\mu\rho}R_{\nu\sigma}\,\epsilon, \tag{2.4}$$

and $\epsilon \neq 0$ implies $R_{\mu\nu}R^{\mu\nu} = 0$.

2.1. Vacuum configurations. If our BPS configuration is also a *vacuum*, the time–derivative of τ vanishes, and eqn.(2.3) reduces to[4]

$$g^{ij}\partial_i\tau\,\partial_j\bar{\tau} = 0 \quad\Rightarrow\quad d\tau = 0. \tag{2.5}$$

Thus, in this case, SUSY requires τ to be constant. These vacua correspond to the well–known 'perturbative' vacua of the IIB superstring.

Eqn.(1.5) says that M is a pseudo–Riemannian manifold admitting a non–trivial parallel spinor. For a vacuum, M is a *warped* metric of the form (1.1), *i.e.* $M = X \times_{f^2} \mathbb{R}^{1,d-1}$. In this case Poincaré invariance gives

$$R_{\alpha i} = 0, \qquad\qquad R_{\alpha\beta} = \eta_{\alpha\beta}\,F \tag{2.6}$$

and a simple computation

$$F = \frac{1}{f}\left(\dot{g}^{ij}\,\dot{D}_i\partial_j f - \frac{(d-1)}{f}\,\dot{g}^{ij}\partial_i f\partial_j f\right) \tag{2.7}$$

where the dotted quantities are computed using the Riemannian metric g_{ij} on X. Then the integrability condition (1.8) splits as

$$F\,\gamma_\alpha\,\epsilon = 0, \qquad \Rightarrow \qquad F = 0 \tag{2.8}$$

$$R_{ij}\gamma^j\epsilon = 0 \qquad \Rightarrow \qquad R_{ij} = 0. \tag{2.9}$$

$F = 0$ is equivalent to the statement that the function

$$\phi = \begin{cases} f^{(2-d)} & d \neq 2 \\ \log f & d = 2 \end{cases} \tag{2.10}$$

is harmonic, $\dot{\Delta}\phi = 0$, on the compact space X, hence a *constant.*

[4] The present discussion applies *verbatim* also to the non–static *Euclidean* BPS configurations.

Therefore M is a *direct* Riemannian product,

$$\mathbb{R}^{1,d-1} \times X, \tag{2.11}$$

where X is a compact Riemannian manifold admitting non–trivial parallel spinors (and hence automatically Ricci–flat, cfr. eqn.(2.9)).

For complenetess, in the next subsection we briefly summarize the geometry of such manifolds X (for more details see [GSSFT]). The reader may prefer to jump ahead to the time–dependent case (it is a good idea!).

2.2.($\mathbb{J}$) **Riemannian manifolds with parallel spinors.** From the integrability condition, $R_{ij}R^{ij} = 0$, we see that, in *positive signature*, a manifold X admitting parallel spinors is automatically Ricci–flat.

By the Bochner and Cheeger–Gromoll theorems (see ref. [**37**] THEOREM **6.56** and COROLLARY **6.67**), the Ricci–flatness condition implies that the universal cover of X is isometric to $\mathbb{R}^{b_1} \times X'$, where X' is a *compact simply–connected* manifold.

Hence, going to a *finite* covering (if necessary), we may assume, without loss of generality, X to be *simply–connected,* provided we also allow some of the remaining flat coordinates to be (possibly) compactified on a torus T^r.

Then, by de Rham's theorem (see [GSSFT] chapter **3**), X is the direct product of compact, simply–connected, irreducible, Ricci–flat manifolds Y_{n_i},

$$X = Y_{n_1} \times Y_{n_2} \times \cdots Y_{n_s}, \tag{2.12}$$

with[5]

$$\dim_{\mathbb{R}} Y_{n_i} \geq 3. \tag{2.13}$$

Being Ricci–flat and non–flat, the Y_{n_i}, cannot be symmetric spaces[6]; hence their holonomy group $\mathrm{Hol}(Y_{n_i})$ should be in the Berger's list ([GSSFT] chapter **3**). $\mathrm{Hol}(Y_{n_i})$ cannot be $U(n)$ or $Sp(2) \times Sp(2n)$ since these groups are non–compatible with the Ricci–flatness ([GSSFT] chapter **3**).

The list of the possible irreducible holonomy groups for Ricci–flat Riemannian metrics is given in TABLE 2.2.

From Wang's theorem (see, say, [GSSFT] THEOREM **3.**5.1), we know that, in a simply–connected irreducible Riemannian manifold X, the

[5] Indeed, there are no compact simply–connected Ricci–flat manifolds with $\dim X \leq 2$.

[6] We stress that this fact is true *only* in Euclidean signature. In Lorentzian signature there exist Ricci–flat non–flat *symmetric* manifolds. Examples below in the discussion of *non–stationary* BPS configurations of F–theory.

Berger's group	real dimension	name	N_+	N_-
$SO(n)$	n	Ricci–flat Riemannian	0	0
$SU(2m)$	$4m$	Calabi–Yau $2m$–fold	2	0
$Sp(2m)$	$4m$	hyperKähler $2m$–fold	$m+1$	0
$SU(2m+1)$	$4m+2$	Calabi–Yau $(2m+1)$–fold	1	1
G_2	7	G_2–manifold	1	
$Spin(7)$	8	$Spin(7)$–manifold	0	1

TABLE 2.1. *Ricci–flat* holonomy groups, the corresponding *irreducible* manifolds, and the numbers $N_\pm$ of parallel spinors of given $\pm$ chirality. (In the case of G_2, since the manifold has odd dimension, the chirality quantum number does not exist, and we have only one number).

number $N_\pm$ of parallel spinors having chirality[7] ± 1 is related to the holonomy group $\mathrm{Hol}(X)$ as in the last two columns of TABLE 2.2.

Therefore, the condition of some unbroken supersymmetry, eliminates the first row of the table, $\mathrm{Hol}(X) = SO(n)$. Then, by inspection of the table, we see that the inequality (2.13) gets replaced by

$$\dim_{\mathbb{R}} Y_{n_i} \geq 4. \tag{2.14}$$

In these lectures, we shall refer to a Riemannian space X with non–zero parallel spinors as *a Ricci–flat space of special holonomy* (since any Ricci–flat space such that $\mathrm{Hol}(X) \neq SO(\dim X)$ automatically admits such spinors).

2.3. Time–dependent BPS configurations. In the trivial $\mathfrak{u}(1)_R$ holonomy case, the most important difference with respect to the vacuum case is that now eqn.(2.3) does not imply that τ is constant but just that the two vectors $\partial_\mu \mathrm{Re}\,\tau$, $\partial_\mu \mathrm{Im}\,\tau$ are *null.* These two vectors should be proportional by eqn.(2.1).

However, since the connection Q_μ is still *pure gauge*, we can set $Q_\mu = 0$ in eqn.(1.5) as a choice of gauge. Then ϵ is a Levi Civita parallel spinor on the Lorentzian manifold M, whose Ricci curvature automatically satisfies $R_{\mu\nu}R^{\mu\nu} = 0$, but, again, in indefinite signature this condition does *not* implies $R_{\mu\nu} = 0$. Instead it says that, *for all*

[7] For a certain conventional orientation. With the opposite orientation one has $N_+ \leftrightarrow N_-$, of course.

vectors v^ν, the vector $v^\nu R_{\nu\mu}$ must be null. Moreover, *given any two vectors, v^ν and w^ρ, we have*

$$2\big(v^\mu R_{\mu\rho}\, w_\nu R^{\nu\rho}\big)\epsilon=\big(v^\mu R_{\mu\sigma}\gamma^\sigma\, w^\nu R_{\nu\rho}\gamma^\rho+w^\nu R_{\nu\rho}\gamma^\rho\, v^\mu R_{\mu\sigma}\gamma^\sigma\big)\epsilon=0$$
$$2\big(v^\mu R_{\mu\nu}\,\partial^\nu\mathrm{Re}\,\tau\big)\epsilon=\big(v^\mu R_{\mu\sigma}\gamma^\sigma\,(\partial_\rho\mathrm{Re}\,\tau)\gamma^\rho+(\partial_\rho\mathrm{Re}\,\tau)\gamma^\rho\, v^\mu R_{\mu\sigma}\gamma^\sigma\big)\epsilon=0,$$

so $\epsilon\neq 0$ implies that the (co)vectors $\partial_\mu\mathrm{Re}\,\tau$, $\partial_\mu\mathrm{Im}\,\tau$ and $v^\nu R_{\nu\mu}$ all belong to a degenerated *subspace of T^*M.* In signature $(k,1)$, $k\geq 1$, a degenerated subspace has dimension one. So these null vectors are all aligned. Then

$$\boxed{R_{\mu\nu}=\rho\,(\partial_\mu\mathrm{Re}\,\tau)(\partial_\nu\mathrm{Re}\,\tau)}\tag{2.15}$$

for some function ρ. In particular, the scalar curvature vanishes, $R=0$. [In fact, this equation is already implied by the Einstein equations, which state that $R_{\mu\nu}$ is proportional to the energy–momentum tensor of the τ–field $(\partial_\mu\tau\,\partial_\nu\bar\tau)/(\mathrm{Im}\,\tau)^2+(\mu\leftrightarrow\nu)$, together with the fact that $d\tau\wedge d\bar\tau=0$.]

The geometry of the Minkowskian manifolds admitting parallel spinors is not covered in [GSSFT]. Therefore we need to summarize here the basic elements of the theory.

We shall return to the explicit time–dependent BPS configurations (with trivial $\mathfrak{u}(1)_R$ connection) in section 4 below.

3. Holonomy and parallel spinors on (r,s) manifolds

3.1. Holonomy of Lorentzian manifolds.

Definition 3.1. A connected pseudo–Riemannian manifold $(M^{r,s},g)$ of signature (r,s) is called *irreducible* if its holonomy group $\mathrm{Hol}(M^{r,s})$ acts irreducibly on the tangent space $T_pM^{r,s}\sim\mathbb{R}^{r,s}$ (at some reference point[8] p). It is called *indecomposable* (or *weakly irreducible)* if its holonomy group $\mathrm{Hol}(M^{r,s})\subset O(r,s)$ does not leave invariant any proper *non–degenerate* subspace of $T_pM^{r,s}\sim\mathbb{R}^{r,s}$.

In the Riemannian case (positive signature) *indecomposable* $\Leftrightarrow$ *irreducible*. Instead, in the indefinite signature case *irreducible* $\Rightarrow$ *indecomposable*, but the opposite arrow is *false*[9].

[8] The statement is independent of the chosen point p.

[9] A trivial example is worth a thousand expanations: Consider the general Lorentian metric in $2d$. It can be put in the form $ds^2=g(x^+,x^-)\,dx^+\,dx^-$. The (local) holonomy, acting by Lorentz transformations, should leave invariants the one–dimensional subspaces of the tangent bundle generated by $\partial/\partial x^+$ and $\partial/\partial x^-$ which transform into multiples of themselves. Leaving invariant the two light–cone directions, the holonomy does not act irreducibly. However the inner product restricted to each invariant space is identically zero. Thus the invariant subspaces are degenerated, and this action is indecomposable according to our definition.

EXAMPLE. The first non–trivial examples appear in dimension 3. We list them to give a flavor of their structure and, more importantly, of their physical meaning. The Lorentz group in $d=3$ is $SO_0(2,1)\simeq SL(2,\mathbb{R})/\pm\mathbf{1}$. The two subgroups which have indecomposable non–irreducible actions are

$$A^1(\mathbb{R})=\left\{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix},\quad b\in\mathbb{R}\right\}\tag{3.1}$$

$$A^2(\mathbb{R})=\left\{\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix},\quad a\in\mathbb{R}^\times,\ b\in\mathbb{R}\right\}\tag{3.2}$$

Examples of metrics with these holonomy groups are, respectively.

$$A^1(\mathbb{R}):\quad ds^2=2\,dx^-dx^+ + f(x^+)^2\,dy^2\tag{3.3}$$

$$A^2(\mathbb{R}):\quad ds^2=2\,dx^-dx^+ + f(x^+)^2\,dy^2+g(x^+)^2(dx^+)^2.\tag{3.4}$$

So they are what in physics we will like to call *pp–waves* (but only the first one is a *pp*-wave according to the technical definition!). A particular instance of such *pp*–waves are the Cohen–Wallach spaces which are *symmetric.*

In general signature (r,s), the usual de Rham's theorem quoted in §. 2.2 is replaced by the following result of Wu:

THEOREM 3.1 (de Rham, Wu [**38**][**39**]). *Let $(M^{r,s},g)$ be a simply connected complete pseudo–Riemannian manifold. Then $(M^{r,s},g)$ is isometric to the product of a flat space time the product of simply connected complete indecomposable pseudo–Riemannian manifolds.*

Recently the possible holonomies of the indecomposable but not–irreducible Minkowskian manifolds had been classified [**40**][**41**][**42**]. For our future applications (to non–trivial $\mathfrak{u}(1)_R$ connections) we must enter in the details. Sorry about that.

Minkowskian signature, $(n-1,1)$, is 'easy' since a degenerate subspace has dimension 1. The subalgebra of $\mathfrak{so}(n-1,1)$ preserving this subspace, $\big(\mathbb{R}\oplus\mathfrak{so}(n-2)\big)\ltimes\mathbb{R}^{n-2}$, consists of matrices of the form[10]

$$\left\{\begin{pmatrix}0 & a & X^t\\ a & 0 & X^t\\ X & -X & A\end{pmatrix}\ :\ a\in\mathbb{R},\ X\in\mathbb{R}^{n-2},\ A\in\mathfrak{so}(n-2)\right\}.\tag{3.5}$$

and the *indecomposable* subalgebras correspond to putting suitable restrictions on a, X and A. One restriction on A is obvious: it should belong to a Lie subalgebra g of $\mathfrak{so}(n-2)$, where $\mathfrak{g}=\mathfrak{z}\oplus\mathfrak{g}'$ (center $\oplus$ semi–simple part).

[10] Somehow masochistically, we write the matrix in a base for which $\eta_{\alpha\beta}=\mathrm{diag}(-1,1,1,\cdots,1)$ and not, as customary, in a ligh–cone basis. The preserved null subspace is $\mathbb{R}\cdot(1,1,0,\ldots,0)$.

DEFINITION 3.2. A Lorentzian space N is called a *Brinkmann space* if $\mathfrak{hol}(N)$ preserves a *null* vector, that is, it is a subalgebra of the algebra of matrices of the form (3.5) with $a = 0$.

More generally, one shows [**40**][**41**] that there are four types of indecomposable subalgebras of the algebra of matrices of the form (3.5):

(1) $\big(\mathbb{R} \oplus \mathfrak{g}\big) \ltimes \mathbb{R}^{n-2}$;
(2) $\mathfrak{g} \ltimes \mathbb{R}^{n-2}$;
(3) $\big(\mathrm{graph}(\phi) \oplus \mathfrak{g}'\big) \ltimes \mathbb{R}^{n-2}$ with $\phi\colon \zeta \to \mathbb{R}$ linear;
(4) $\big(\mathfrak{g}' \oplus \mathrm{graph}(\psi)\big) \ltimes \mathbb{R}^{r}$, where $0 < r < n-2$, $\mathfrak{g} \subset \mathfrak{so}(r)$ and $\psi\colon \mathfrak{z} \to \mathbb{R}^{n-2-r}$ is linear and surjective.

An indecomposable Lorentzian manifold N is a Brinkmann space iff $\mathfrak{hol}(N)$ is of type (2) or (4).

From the above list we see that the important datum in the holonomy of an indecomposable but not–irreducible Lorentzian holonomy is the Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{so}(n-2)$. Then we use this subalgebra to classify the Brinkmann spaces according the definitions (cfr. ref. [**43**]):

DEFINITION 3.3. An indecomposable Brinkmann n–fold N is said

• *a pp–wave*	if $\mathfrak{g} = 0$ (*i.e.* $\mathrm{Hol}(N) = \mathbb{R}^{n-2}$)
• *to have a $\mathfrak{g}$–flag*	if $\mathfrak{hol}(N)$ is of type (2) or (4) with given $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$
• *to have a Kähler l–flag*	if $\mathfrak{g} \subseteq \mathfrak{u}(l) \oplus \mathfrak{h}$ with $\mathfrak{h} \subseteq \mathfrak{so}\big(n-2(l+1)\big)$
• *to have a non–special Kähler flag*	if $\mathfrak{g} = \mathfrak{u}(1) \oplus \mathfrak{k}$ with $\mathfrak{k} \subseteq \mathfrak{su}(m)$
• *to have a special Kähler flag*	if $\mathfrak{g} \subsetneq \mathfrak{su}(m)$
• *to have a Calabi–Yau flag*	if $\mathfrak{g} = \mathfrak{su}(m)$
• *to be a Brinkmann–Leister space*	if $G = \exp \mathfrak{g}$ is trivial or a product of groups of the form $SU(m)$, $Sp(2l)$, G_2, and $Spin(7)$

where $m = [(n-2)/2]$.

We have the elegant

THEOREM 3.2 (Leister ref. [**42**]). *An indecomposable subalgebra of* $\big(\mathbb{R} \oplus \mathfrak{so}(n-2)\big) \ltimes \mathbb{R}^{n-2}$*, of type (1)–(4) is the holonomy algebra of a Lorentzian manifold N if and only if $G = \exp \mathfrak{g}$ is a product of Riemannian (i.e. Berger's) holonomy groups.*

The proof of this fact is algebraic and hard. We scheck a simple geometrical argument under the <u>additional</u> assumption that the indecomposable Lorenztian manifold N is a *Brinkmann* space. The explicit construction in the argument will be needed below to construct the F–theory elliptic fibration over N. Note that we loose nothing by assuming N to be Brinkmann since:

EXERCISE 3.1. Show that if N is an indecomposable, but not irreducible, Lorentzian manifold with a parallel spinor, then N must be (in particular) a Brinkmann space. [HINT: there is a proof in section 8 below].

We phrase our case of Leister theorem as follows[11]:

THEOREM 3.3. *N a simply–connected, complete, indecomposable, Brinkmann space with no closed light–like geodesics. Let ξ be a parallel vector and η a parallel form on N. ξ is required to* never vanish[12]. *(One has $\eta(\xi) = 0$ by definition). $\mathfrak{hol}(N)$ is of the type (2) or (4) for some Lie subalgebra $\mathfrak{g} \subset \mathfrak{so}(n-2)$ and (in case (2)) map ψ. Then*

1. *The parallel form η defines a codimension 1 foliation, $\mathcal{F}$, of N whose leaves $L \subset \mathcal{F}$ are* totally geodesic submanifolds *of N. Moreover, there is a* surjective *submersion $\rho\colon N \to \mathbb{R}$ whose fibers are the leaves of $\mathcal{F}$.*
2. *The vector ξ, being Killing, defines a one–parameter group of isometries, $I \simeq \mathbb{R}$, which acts* freely *on N and leaves invariant the leaves of $\mathcal{F}$.*
3. *Fix $x \in \mathbb{R}$ a consider the corresponding leaf $L_x \subset \mathcal{F}$. L_x is foliated by one–dimensional submanifolds, namely the orbits of the isometry I, which are* null geodesics.
4. *The orbit space $Z_x = L_x/I$ is a* smooth manifold. *Let $\pi_x\colon L_x \to Z_x$ be the canonical submersion.*
5. *Define a Riemannian metric g_x on Z_x by the rule*
$$g_x(V,W) = (\tilde{V},\tilde{W})_{L_x}, \qquad \forall\, V,W \in TZ_x, \tag{3.6}$$
where $(\cdot,\cdot)_{L_x}$ is the degenerate *pairing on TL_x induced by the Lorentzian metric of N and $\tilde{V}$ is* any *vector field on L_x such that $\pi_{x*}\tilde{V} = V$. The metric g_x is* well–defined.
6. *For each x, (Z_x, g_x) is a* Riemannian manifold *with holonomy $\mathfrak{hol}(Z_x) = \mathfrak{g}$.*
7. *In particular, $G = \exp(\mathfrak{g})$ should be a Riemannian (Berger's) holonomy group.*

PROOF. Let N be indecomposable and Brinkmann with parallel vector $\xi = \xi^\mu \partial_\mu$ and parallel 1–form $\eta = \xi_\mu dx^\mu$. Since ξ is null, $\eta(\xi) =$

[11] Our argument is freely inspired to similar ideas in the theory of *K–contact manifolds*, see the book [**44**] *e.g.* **Theorem 7.1.3**, and, in particular, the foliated geometry of *Sasakian and* 3*–Sasakian* manifolds which have foliations with Kählerian, resp. quaternionic–Kähler, spaces of leaves, cfr. *e.g.* **Theorem 13.3.13**. Here we make the additional assumption (whose validity is physically guaranteed by causality) that the manifold has no closed light–like geodesics in order to be sure that the space of leaves is a smooth manifold, not an orbifold (as it is the case in [**44**]).

[12] This assumption is not really needed. But it is guaranteed in the physical case, and makes things a little nicer.

0. The holonomy algebra $\mathfrak{hol}(N)$ is of type (2) or (4) above for a certain $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z} \subset \mathfrak{so}(n-2)$.

(1) The 1–form η, being parallel, is in particularly closed, so by the Frobenius theorem, it defines a *foliation* $\mathcal{F}$ of codimension 1. Specifically, since N is simply–connected, we have $\xi = dx^+$ for a certain function x^+ (defined up to the addition of a constant) and the leaves $L_x \subset \mathcal{F}$ are its level sets $L_x = \{x^+ = x = \text{const.}\}$. The fact that ξ is parallel implies:

The leaves $L \subset \mathcal{F}$ are totally geodesic submanifolds of N.

In particular, the Levi Civita connection of N, D preserves TL_x,

$$V \in TL_x \Rightarrow D_X V \in TL_x, \quad \forall\, X \in TN, \tag{3.7}$$

and hence D may be identified with an affine connection on L_x which we again denote by D.

The submersion $N \to \mathbb{R}$ given by the value of the global light–cone coordinate x^+ is surjective as a consequence of the fact that $\eta = dx^+$ never vanishes and N is complete.

(2) On the other hand, since $v\xi$ is parallel, it is in particular a Killing vector generating a one–parameter isometry group. Locally, we can define a coordinate along the the orbit, call it x^-, such that $\xi = \partial_{x^-}$. x^- is *not* uniquely defined: we have the freedom of redefining $x^- \to x^- + \phi(x^+, y^i)$. The fact that the translation in x^- is an isometry, and the relation with the 1–form η imply that the metric has (locally) the general form

$$\begin{aligned} ds^2 = 2\,dx^-\,dx^+ + f(x^+, y^i)\,(dx^+)^2 + \\ + h_i(x^+, y^j)\,dy^i\,dx^+ + g_{ij}(x^+, y^k)\,dy^i\,dy^j, \end{aligned} \tag{3.8}$$

with some non–trivial restriction on the coefficient functions[13].

(3) Consider a closed path $\gamma(t)\colon [0,1] \to L_x$, with base point p, and a smooth function $h(t)\colon [0,1] \to \mathbb{R}$, with $h(0) = h(1) = 0$. The

[13] Setting $e^+ = dx^+$, $e^- = dx^- + \frac{1}{2} f\,dx^+ + \frac{1}{2} h_i dy^i$ and $e^a = e^a_i(x^+, y) dy^i$ with $g_{ij}(x^+, y) = e^a_i(x^+, y)\, e_{aj}(x^+, y)$, we get Cartan structure equations of the form

$$\begin{aligned} &de^+ = 0 \\ &de^- + \phi_a \wedge e^a = 0 \\ &de^a - \phi^a \wedge e^+ + \omega^a{}_b \wedge e^b = 0, \end{aligned}$$

where ϕ^a is the $\mathbb{R}^{(n-2)}$ part of the connection and $\omega^a{}_b$ is the $\mathfrak{g}$ part. The curvatures are $R_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega^c{}_b$ and $R^a = d\phi^a + \omega^a{}_b \wedge e^b$.

h–deformed closed path[14]

$$\gamma_h(t) := \exp\Big[h(t)\,\mathcal{L}_\xi\Big]\;\gamma(t), \tag{3.9}$$

has the same base point p.

Let $M_\gamma \in \mathrm{End}(T_pL_x)$ be the monodromy matrix along γ, *i.e.* $M_\gamma \equiv M_\gamma(1)$ where

$$\begin{aligned} &\frac{d}{dt}M_\gamma(t) + \Gamma_{\dot\gamma}M_\gamma(t) = 0\\ &M_\gamma(0) = \mathrm{Id}. \end{aligned} \tag{3.10}$$

One has

$$M_{\gamma_h}(t) = M_\gamma(t), \tag{3.11}$$

because, since ξ is parallel (and hence Killing),

$$\Gamma_{\dot\gamma_h} = \Gamma_{\dot\gamma} + \dot h\,\Gamma_\xi = \Gamma_{\dot\gamma}, \tag{3.12}$$

and $M_\gamma(t)$ and $M_{\gamma_h}(t)$ are solutions to the same differential equation with the same boundary condition.

(4) Since ξ is null, $\eta(\xi) = 0$ and hence $\xi \in TL_x$. Let S_x be the rank 1 sub–bundle of TL_x generated by ξ, and consider the quotient bundle Q_x

$$0 \to S_x \to TL_x \to Q_x \to 0. \tag{3.13}$$

The connection D on the vector bundle $TL_x \to L_x$ induces a connection ∇ on the bundle $Q_x \to L_x$ by the rule

$$\nabla_X\big(V \mod S\big) = D_XV \mod S_x, \qquad X, V \in TL_x \tag{3.14}$$

which is well–defined because ξ is parallel. By construction we have

$$\mathfrak{hol}(\nabla) = \mathfrak{g} \subset \mathfrak{so}(n-2). \tag{3.15}$$

Equivalently, the monodromy M_γ induces a linear map

$$\tilde M_\gamma \colon Q_x \to Q_x, \qquad \text{with } \tilde M_\gamma \in G \equiv \exp\mathfrak{g}. \tag{3.16}$$

(5) The one–parameter group of isometries generated by the Killing vector ξ, I, leaves invariant the leaves of $\mathcal{F}$. Since ξ vanishes nowhere, the action of I is locally free. Its orbits are ligh–like geodesics. If a non–trivial element $i \in I$ had a fix point p, the corresponding null geodesics would be closed, which is impossible by assumption. Hence the action of I is *globally* free, and the quotient space $Z_x = L_x/I$ is *smooth.*

[14] In local coordinates (x^-, y^i) on the leaf L, the path $\gamma(t)$ is $\big(x^-(t),\, y^i(t)\big)$, while the path $\gamma_h(t)$ is $\big(x^-(t) + h(t),\, y^i(t)\big)$. Notice that the deformation is well defined, and has an intrinsic meaning independent of the choice of coordinates. Since we can construct any h–deformation by composing 'small' such deformation, the local analyis in sufficient to extablish the result.

(6) Let $\pi_x\colon L_x \to Z_x$ be the canonical submersion. In step **(3)** above we have shown that two (non necessarily closed!) paths, $\gamma, \gamma' \subset L_x$, which project to the same *closed* path in Z_x, have the same monodromy matrix. So we may speak of the monodromy along a path in Z_x.

The kernel of the push–forward map, $\pi_{x*}TL_x \to TZ_x$, is S_x. Then TZ_x is isomorphic to the space of $\mathcal{L}_\xi$–invariant elements of the quotient bundle Q_x.

The Lorentzian metric g on N induces a positive–definite Riemannian metric g_x on the 'transverse' manifolds Z_x. Let $\tilde{D}$ be the corresponding Levi Civita connection. It coincides $\mod S_x$ with D; thus it is ∇ acting on $\mathcal{L}_\xi$ invariant sections of Q_x. The monodromy of $\tilde{D}$ along a path $\gamma \in Z$ is equal to the monodromy of ∇ along any lift of γ in L_x (we already know that it does not depend on the choice of the lift). This implies that

$$\mathfrak{hol}(Z_x) = \mathfrak{hol}(\nabla) = \mathfrak{g}. \tag{3.17}$$

(7) Since there exists a Riemannian manifold, namely Z_x, with $\mathfrak{hol}(Z_x) = \mathfrak{g}$, $G = \exp \mathfrak{g}$ should be an allowed holonomy group in Riemannian geometry. Unless Z_x has some symmetric factor space (a very peculiar case) it should be a product of groups in the Berger's list. □

In fact explicit metrics are known which realize any possible holononomy group, see ref. [**45**], even requiring N to be a globally hyperbolic Lorentzian manifold with complete Cauchy surfaces[**46**]. They are constructed, essentially, by inverting the above process.

The above results can be rephrased in a simpler way in the special case that N is a Brinkmann $2(m+1)$–fold with a Kähler flag, meaning that it holonomy algebra, $\mathfrak{hol}(N)$ is of type (2) or (4) with $\mathfrak{g} \subseteq \mathfrak{u}(m)$. Indeed

Proposition 3.1. *Let N be an indecomposable Brinkmann $2(m+1)$–fold with a Kähler m–flag (i.e. $\mathfrak{g} \subseteq \mathfrak{u}(m)$). Then:*

1. *N has a canonical symplectic structure invariant under the flow generated by the null vector ξ;*
2. *the momentum map $\mu\colon N \to \mathbb{R}$ is given by $p \mapsto -x^+(p)$;*

(3) *the Kähler m–fold Z_{x^+} is the Marsden–Weinstein–Mayer*[15] symplectic quotient *with Kähler form the induced symplectic form.*

$$Z_{x^+} = \mu^{-1}(x^+)//\mathbb{R}. \tag{3.18}$$

(4) *the Kähler form on Z_{x^+} is the symplectic form induced by the symplectic quotient.*

Proof. On N we can introduce an adapted frame $(e^+, e^-, e^a, e^{\bar{a}})$ with torsionless $\mathfrak{su}(m)\ltimes\mathbb{C}^m$ connection 1–form $(\omega^a{}_b, \phi^a)$. The Lorentzian metric is

$$ds^2 = e^- \otimes e^+ + e^+ \otimes e^- + e^a \otimes e^{\bar{a}} + e^{\bar{a}} \otimes e^a. \tag{3.19}$$

while the structure equations are

$$de^+ = 0, \qquad de^- + \phi^{\bar{a}} \wedge e^a + \phi^a \wedge e^{\bar{a}} = 0 \tag{3.20}$$

$$de^a + \omega^{a\bar{b}} \wedge e^b - \phi^a \wedge e^+, \qquad de^{\bar{a}} + \omega^{\bar{a}b} \wedge e^{\bar{b}} - \phi^{\bar{a}} \wedge e^+. \tag{3.21}$$

then

$$d(e^a \wedge e^{\bar{a}}) = -\big(\phi^a \wedge e^{\bar{a}} + e^a \wedge \phi^{\bar{a}}\big) \wedge e^+ = \tag{3.22}$$

$$= de^- \wedge e^+ = d(e^- \wedge e^+) \tag{3.23}$$

and the 2–form

$$\boxed{\Omega = e^+ \wedge e^- + e^a \wedge e^{\bar{a}}} \tag{3.24}$$

is closed hence *symplectic* (but not parallel!).

The one–form e^+ is parallel, and hence identified with $\eta = dx^+$. e^- is dual to the vector ξ, and hence

$$i_\xi e^- = 1, \; i_\xi e^a = i_\xi e^{\bar{a}} = i_\xi e^+ = 0 \quad \Rightarrow \tag{3.25}$$

$$\Rightarrow \mathcal{L}_\xi \, e^A = 0 \;\text{ for } A = +, -, a, \bar{a}. \tag{3.26}$$

In particular, $\mathcal{L}_\xi \Omega = 0$. Then $d i_\xi \Omega = 0$, and since N may be assumed to be simply connected, $i_\xi \Omega = d\mu_\xi$, where the function μ_ξ is the momentum map of the Hamiltonian flow ω. But, from eqn.(3.25) we see

$$d\mu_\xi = i_\xi \Omega = -e^+ = -dx^+. \tag{3.27}$$

---

[15] For the convenience of the reader, we quote the result we need (see **Theorem 8.4.2** in ref. [**44**]):

Theorem 3.4. *Let (M,ω) be a symplectic manifold with a Hamiltonian action of the Lie group G, and let $\mu\colon M \to \mathfrak{g}^\vee$ denote the corresponding moment map. Suppose further that $\alpha \in \mathfrak{g}^\vee$ is a regular value of μ, and that the action of the isotropy subgroup $G_\alpha \subset G$ is proper on $\mu^{-1}(\alpha)$. Then G_α acts locally free on $\mu^{-1}(\alpha)$, and the quotient $M_\alpha = \mu^{-1}(\alpha)/G_\alpha$ is naturally a symplectic orbifold. If in addition the action of G_α is free on $\mu^{-1}(\alpha)$, then the quotient M_α is a smooth symplectic manifold. Furthermore, if (M,ω) has a compatible Kähler structure and G acts by Kähler automorphisms, then the quotient has a natural Kähler structure.*

Remark. Note that the condition that M is compact is not required.

Then the proposition follows from the Marsden–Weinstein–Mayer theorem quoted in footnote 15 on page 67. □

The reason we went trough the argument is that we gained some useful corollaries: see next subsection. More details in APPENDIX ....

3.2. Parallel forms and spinors in a Brinkmann space. THEOREM 3.3 has the following

COROLLARY 3.1. *Let N be a Brinkmann n–fold as in* THEOREM *3.3. Let P_k be the number of linear independent* parallel k–forms *on N, and $N_\pm$ the number of linear independent* parallel spinors *of the given $\pm$ chirality (for n even, otherwise there is only one number N). Let $P_k(\mathfrak{g})$ and $N_\pm(\mathfrak{g})$ be the analogue quantities for a (simply–connected) Riemannian $(n-2)$–fold Z having holonomy $\mathfrak{hol}(Z) = \mathfrak{g}$. Then*

$$(1) \qquad P_{k+1} = P_k(\mathfrak{g}), \qquad 0 \leq k \leq n-2 \tag{3.28}$$

$$(2) \qquad N_\pm = N_\pm(\mathfrak{g}). \tag{3.29}$$

In particular, a Lorentz manifold N with a non–zero parallel spinor is a Brinkmann–Leister space.

REMARK. The holonomy groups of *irreducible* pseudo–Riemannian manifolds $M^{r,s}$ admitting parallel spinors are classified in ref.[**39**]. They are essentially the 'Wick'–rotated counterparts to those appearing in the Wu theorem for the Riemannian case, see TABLE 2.2. For the case of *indecomposable* but *non–irreducible* Lorentz manifolds admitting parallel spinors the classification was first given by Leister [**42**]. Its classification coincides with part (2) of COROLLARY 3.1 since, as you sproved in EXERCISE 3.1 an indecomposable Lorentz manifold with a parallel spinor is necessarily a Brinkmann space.

PROOF. **(1)** Written in the coframe of footnote 13 of page 64, a k–form has one of the following structures

1. $\omega_{a_1\cdots a_k}\, e^{a_1} \wedge \cdots \wedge e^{a_k}$, **2.** $e^+ \wedge \omega_{a_1\cdots a_{k-1}}\, e^{a_1} \wedge \cdots \wedge e^{a_{k-1}}$

3. $e^- \wedge \omega_{a_1\cdots a_{k-1}}\, e^{a_1} \wedge \cdots \wedge e^{a_{k-1}}$, **4.** $e^- \wedge e^+ \wedge \omega_{a_1\cdots a_{k-2}}\, e^{a_1} \wedge \cdots \wedge e^{a_{k-2}}$.

Such a form may be parallel only if it is invariant under the transformations[16]

$$\begin{aligned} \delta e^- &= v_a\, e^a, && \forall\, v_a \in \mathbb{R}^{n-2} \\ \delta e^a &= -v^a\, e^+ + \Lambda^a{}_b\, e^b && \forall\, \Lambda \in \mathfrak{g}. \end{aligned}$$

Thus e^- can be present only if multiplied by the $(n-2)$ form $e^1 \wedge e^2 \wedge \cdots e^{(n-2)}$. Therefore there are only two parallel forms with the structures **3.**,**4.**, namely

$$e^- \wedge e^1 \wedge \cdots \wedge e^{(n-2)}, \qquad e^+ \wedge e^- \wedge e^1 \wedge \cdots \wedge e^{(n-2)}. \tag{3.30}$$

[16] We write the formulae adapted for a holonomy group of type (2). The trivial modifications for type (4) are left as an EXERCISE.

Next, let ω be a k–form with the structure **1.** One has $\delta_v\omega = -e^+ \wedge i_v\omega$ which can vanish for all v's only if $\omega = 0$. We remain with those of the structure **2.** They are invariant under δ_Λ iff $\omega_{a_1\cdots a_{k-1}}$ is a (constant) invariant antisymmetric tensor for $\mathfrak{g}$, that is a parallel form on a Riemannian manifold Z with $\mathfrak{hol}(Z) = \mathfrak{g}$.

(2) Write the n–dimensional γ matrices as

$$\gamma^\pm = \sigma_\pm \otimes \mathbf{1} \tag{3.31}$$

$$\gamma^k = \sigma_3 \otimes \tilde{\gamma}^k, \qquad k = 1, 2\ldots, n-2, \tag{3.32}$$

where $\tilde{\gamma}_k$ are the Dirac matrices of $Spin(n-2)$, and correspondinly the spinors as $\psi \otimes \tilde{\epsilon}$. Since

$$\delta(\psi \otimes \tilde{\epsilon}) = (\sigma_3\sigma^+\psi) \otimes (v_a\tilde{\gamma}^a\tilde{\epsilon}) + \psi \otimes (\Lambda_{ab}\tilde{\gamma}^{ab}\tilde{\epsilon}) \tag{3.33}$$

we see that $\psi \otimes \tilde{\epsilon}$ is parallel iff has the structure

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \tilde{\epsilon}, \qquad \text{with } \tilde{\epsilon} \text{ an invariant tensor for } \mathfrak{g}. \tag{3.34}$$

□

3.3. Differential forms on Brinkmann spaces. In this subsection, N is a Brinkmann n–fold with a $\mathfrak{g}$–flag. ξ and η are, respectively, the parallel vector and form, $\eta(\xi) = 0$. $\Omega^k(N)$ stands for the space of smooth k–forms on N.

Definition 3.4. A k–form α is said *basic* (or *equivariant*) if

$$i_\xi\alpha = \pounds_\xi\alpha = 0. \tag{3.35}$$

We denote the space of smooth basic forms as $\Omega^k_B(N)$.

Lemma 3.1. *$\Omega^*_B(N) = \oplus_k\Omega^k_B(N)$ is a* subring *(with respect to the $\wedge$ product) and a differential* subcomplex *of $\Omega^*_B(B)$. The corresponding graded cohomology ring is denoted as $H^*_B(N)$.*

Moreover, if $\alpha \in \Omega^k_B(N)$, then $\eta \wedge \alpha \in \Omega^{k+1}_B(N)$.

Proof. We have to show that $\alpha \in \Omega^k_B(N)$ implies $d\alpha \in \Omega^{k+1}_B(N)$ and $\eta \wedge \alpha \in \Omega^{k+1}_B(N)$. Indeed,

$$\pounds_\xi\, d\alpha = d\pounds_\xi\,\alpha = 0, \qquad i_\xi\, d\alpha = (\pounds_\xi - d\, i_\xi)\alpha = 0 \tag{3.36}$$

$$\pounds_\xi(\eta \wedge \alpha) = (\pounds_\xi\,\eta) \wedge \alpha + \eta \wedge \pounds_\xi\,\alpha = 0 \tag{3.37}$$

$$i_\xi(\eta \wedge \alpha) = \eta(\xi)\,\alpha - \eta \wedge i_\xi\alpha = 0. \tag{3.38}$$

□

In the ring $\Omega^*_B(N)$ consider the *differential ideal*

$$\Omega^*_C(N) := \ker\Big(\Omega^*_B(N) \xrightarrow{\eta\wedge\cdot} \Omega^{*+1}_B(N)\Big). \tag{3.39}$$

Again, $\Omega^*_C(N)$ is a differential subcomplex. We denote by $H^*_C(N)$ the corresponding cohomology groups. One has the exact sequence of complexes

$$0 \to \Omega^*_C(N) \to \Omega^*_B(N) \xrightarrow{\eta\wedge\cdot} \Omega^{*+1}_C(N) \to 0. \tag{3.40}$$

Since the flow generated by ξ leaves invariant the leaves of the foliation, $L \subset \mathcal{F}$, the basic complex $\Omega^*_B(L)$ and cohomology groups $H^*_B(L)$ are well–defined. Then

PROPOSITION 3.2. *N, L, and Z as in* THEOREM *3.3. Then*

$$H^*(Z) \simeq H^*(L) \simeq H^*_B(L) \simeq H^{*+1}_C(N). \tag{3.41}$$

PROOF. Let $\pi\colon L \to Z$ be the natural projection. One has $\pi^*\Omega^*(Z) = \Omega^*_B(L)$, essentially by definition. Since there are no closed light–like geodesics, π is a retraction, and then π^* is a homotopy map and hence the identity in cohomology.

Consider a form $\alpha \in \Omega^k_C(N)$. It can be written in the form $\xi\wedge\beta$ for a $\beta \in \Omega^{k-1}_B(N)$ *unique* mod $\Omega^{k-1}_C(N)$. The form $\beta_L := \beta\big|_L \in \Omega^{k-1}_B(L)$ is then *uniquely* defined. Thus we have a chain map

$$\gamma\colon \Omega^k_C(N) \to \Omega^{k-1}_B(L), \qquad \alpha \mapsto \beta_L. \tag{3.42}$$

Let $\iota\colon L \to N$ be the embedding. The following operation is an inverse to γ

$$\gamma^{-1}\colon \Omega^{k-1}_B(L) \to \Omega^k_C(N), \qquad \beta \to \xi \wedge \iota^*\beta. \tag{3.43}$$

Since γ, γ^{-1} are chain maps, they make an isomorphism in cohomology. This completes the proof. □

3.4. (*) The inverse problem. Given an indecomposable Brinkmann space N with a $\mathfrak{g}$–flag, we get a one–parameter family of Riemannian metrics with holonomomy $\mathfrak{g}$, namely the metrics on the quotient manifolds $Z_{x^+} \equiv L_{x^+}/\mathbb{R}$, $x^+ \in \mathbb{R}$. We wish to invert the process, namely starting from a smooth family of metrics on some manifold with having a given holonomy algebra $\mathfrak{g}$ to construct a Brinkmann space N with $\mathfrak{hol}(N) = \mathfrak{g} \ltimes \mathbb{R}^{n-2}$.

As it was to be expected, this is possible only if the family satifies an *integrability condition.*

PROPOSITION 3.3. *Z a smooth manifold equipped with a smooth family g_λ, $\lambda \in \mathbb{R}$ of metrics having $\mathfrak{hol}(g_\lambda) = \mathfrak{g}$ for all λ. Let e^a_λ a smooth family of orthonormal coframes adapted to the holonomy* $\exp\mathfrak{g}$ *[that is,*

$$de^a_\lambda + \omega^{ab}_\lambda \wedge e^b_\lambda = 0, \qquad \textit{with}\ \ \omega^{ab}_\lambda = T^*Z \otimes \mathfrak{g} \quad \forall\,\lambda].$$

Consider the family of tensors on Z

$$T_{\mu\nu\rho}(\lambda) = \Big(\partial_\lambda(\omega^{ab}_\lambda)_\mu\Big)(e^a_\lambda)_\nu(e^b_\lambda)_\rho. \tag{3.44}$$

The family g_λ is called integrable *if on Z there is a smooth family of forms ζ_λ such that*

$$T_{\mu\nu\rho} = R_{\nu\rho\mu}{}^{\sigma}\,(\zeta_\lambda)_\sigma \quad \textit{for all } \lambda. \tag{3.45}$$

Assume the above condition is satisfied. Set

$$E^+ = d\lambda \tag{3.46}$$

$$E^- = d\mu + \zeta_\lambda \tag{3.47}$$

$$E^a = e^a_\lambda. \tag{3.48}$$

Then the metric

$$ds^2 = E^- \otimes E^+ + E^+ \otimes E^- + E^a \otimes E^a \tag{3.49}$$

is a Brinkmann metric with a $\mathfrak{g}$ flag.

Conversely, every Brinkmann metric with $\mathfrak{g}$ flag arises this way from some integrable family of metrics.

Notice that, in particular, eqn.(3.45) requires

$$T_{\mu\nu\rho} + T_{\nu\rho\mu} + T_{\rho\mu\nu} = 0. \tag{3.50}$$

For the aficionados the proof is given in Appendix ....

4. Elliptic *pp*–waves

After this long mathematical interlude, we return to our original problem of determining the BPS time–dependent configurations in the case of *trivial* $\mathfrak{u}(1)_R$ connection (the general case being discussed in section 5 below).

Since M has signature $(9,1)$, at most *one* indecomposable manifold in the de Rham–Wu decompostion is Lorentzian. The other factor spaces have positive signature and hence are *irreducible.* Since they have parallel spinors, they appear in the standard Wang list (see Table 2.2) and are, in particular, Ricci–flat. Since these spaces are already undersood (and play no role), we may focus on the single indefinite metric factor manifold which we call N.

By the de Rham–Wu theorem, N is either flat or the action of $\mathrm{Hol}(N)$ is indecomposable.

The vector $\partial_\mu \mathrm{Re}\,\tau$, being null, belongs to TN. By the Einstein equations, the Ricci curvature of N is proportional to

$$\frac{\partial_\mu\tau\,\partial_\nu\bar\tau}{(\mathrm{Im}\,\tau)^2} \tag{4.1}$$

and hence, if N is flat, τ must be constant, and we get back a boring vacuum configuration.

We may assume, therefore, N to be *indecomposable.*

The indefinite metric *indecomposable* manifold N admits a non–zero parallel spinor ϵ_0. Let $\mathrm{Ann}(\epsilon_0) \subset T^*N$ be the subspace of cotangent vectors v_i such that $v_i\gamma^i\,\epsilon_0 = 0$. SUSY requires $d\tau \in \mathrm{Ann}(\epsilon_0)$, while $d\tau \neq 0$; thus the space $\mathrm{Ann}(\epsilon_0) \subset TN$ must have dimension 1. ϵ_0 is invariant under parallel transport, hence so is the one–dimensional subspace $\mathrm{Ann}(\epsilon_0) \subset TN$. Thus N *should be an indecomposable manifold of signature* $(k,1)$ *which is not irreducible.* Since it has a parallel spinor, N *is a Brinkmann–Leister space.*

Comparing with §. 3.2, we see that the condition $\partial_\mu \mathrm{Re}\,\tau\,\gamma^\mu\epsilon_0 = \partial_\mu \mathrm{Im}\,\tau\,\gamma^\mu\epsilon_0 = 0$ means that the 1–forms $d\mathrm{Re}\,\tau$, $d\mathrm{Im}\,\tau$ are proportional to $\xi = dx^+$ and hence $\tau = \tau(x^+)$ is a function of the 'momentum' x^+.

We summarize the result of the present subsection in the following form:

FACT 4.1. *A zero–flux F–theory BPS configuration with a trivial* $\mathfrak{u}(1)_R$ *holonomy, but a non–constant* τ*, has the following structure:*

The universal cover $\tilde{M}$ *of its* $10D$ *gravitational manifold* M *is isometric to* $N \times X$*, where* N *is a simply connected Brinkmann–Leister space, and* X *is a simply–connected Riemannian Ricci–flat manifold having special holonomy (times, possibly, a flat Euclidean space).*

Restricted to each leaf $L_{x^+} \subset \mathcal{F}$ *of the canonical Brinkmann–Leister foliation, the F–theory elliptic fibration* $\mathcal{Y}_{12} \to \tilde{M}$ *is trivial*

$$\mathcal{Y}_{12}\Big|_{L_{x^+}} = L_{x^+} \times E_{x^+} \tag{4.2}$$

where E_{x^+} *is an elliptic curve depending only on the leaf* $L_{x^+} \subset \mathcal{F}$*. In particular, the elliptic fibration induced on each quotient Riemannian manifold* Z_{x^+} *is trivial.*

The corresponding Weierstrass equation would be

$$Y^2 = X^3 + A(x^+)\,X + B(X^+), \qquad A(x^+)^3 + 27\,B(x^+)^3 \not\equiv 0 \tag{4.3}$$

An elliptic Lorentzian 12–fold $\mathcal{Y}_{12}$ with the structure described in FACT 4.1 will be called (by *extreme* abuse of language) an *elliptic pp–wave.*

EXAMPLE. We give a few elementary examples of elliptic *pp*–waves. The reader may construct as many she wishes by referring to the quoted literature.

(1) Take $\tilde{M} = N \times X_{G_2}$, where X_{G_2} is a compact Riemannian manifold with $\mathrm{Hol}(X_{G_2}) = G_2$, and N is an indecomposable but not irreducible 3–fold. From eqns.(3.1)(3.2), we see that such a manifold *has a parallel spinor if and only if* $\mathrm{Hol}(N) = A^1(\mathbb{R})$, since the group $A^2(\mathbb{R})$ does not leave invariant any spinor.

As an example, take the metric (3.3). The only non–vanishing component of the Ricci tensor is

$$R_{x^+\,x^+} = -\frac{1}{f}\,\partial_{x^+}\partial_{x^+} f. \tag{4.4}$$

and the parallel spinor ϵ_0 is a constant spinor such that

$$\gamma^+\epsilon_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \epsilon_0 = 0. \tag{4.5}$$

Finally, $\tau = \tau(x^+)$ is related to the function $f(x^+)$ appearing in the metric by the Einstein equation

$$-\frac{1}{f}\,\partial^2_{x^+} f = \frac{\partial_{x^+}\tau\,\partial_{x^+}\overline{\tau}}{(\tau+\overline{\tau})^2}. \tag{4.6}$$

(2) Consider $\tilde{M} = N \times CY_3$, where N is an indecomposable but not irreducible manifold of dimension 4, and CY_3 a Calabi–Yau 3–fold. Again, N admits a non–zero parallel spinor if and only if

$$\mathrm{Hol}(N) = A^2(\mathbb{C}) \subset SL(2,\mathbb{C}) \simeq SO_0(3,1). \tag{4.7}$$

As an example of a metric with this holonomy group, take [**37**]

$$ds^2 = 2\,dx^- dx^+ + \omega(y^1,y^2)\,(dx^+)^2 + (dy^1)^2 + (dy^2)^2. \tag{4.8}$$

The only non–vanishing components of the Riemann tensor are $R_{i+j+} = \frac{1}{2}\partial_i\partial_j\omega$ (where $\partial_i \equiv \partial_{y^i}$). Consequently, the only non–vanishing component of the Ricci tensor is (again)

$$R_{++} = \frac{1}{2}\Delta\omega. \tag{4.9}$$

Choosing ω to satisfy $\Delta\omega = 0$ leads to a famous Ricci–flat non–flat metric of holonomy $A^2(\mathbb{C})$. However the Ricci–flat case corresponds to a vacuum (static) configuration and is not of interest here. Instead, we take ω to satisfy $\Delta\omega = 2\,C$, a non–zero constant. Then $\tau(x^+)$ can be found by quadratures

$$|\partial_+\tau|^2 = C\,(\tau+\bar{\tau})^2. \tag{4.10}$$

5. Non trivial $\mathfrak{u}(1)_R$ holonomy

Having wasted enough time in trivialities, we arrive at our first topic. Again, we start with the vacuum configurations.

5.1. Vacuum configuration. *A priori,* the $10D$ manifold M is a *warped* product $X \times_{f^2} \mathbb{R}^{1,k}$ as in eqn.(1.1), while τ depends only on the coordinates y^i of the compact Riemannian manifold X. Again, the general integrability equation (1.8) splits in two conditions

$$F\,\gamma_\alpha\,\epsilon = \gamma^i(R_{ij} + i\,Q_{ij})\epsilon = 0, \tag{5.1}$$

so $F = 0$, f is a constant, and the product is a direct one. We focus on the non–trivial positive signature space X.

Going to a cover, if necessary, we may assume X to be simply connected, hence a direct product of a flat space times a product of *irreducible* manifolds

$$\tilde{X} = \mathbb{R}^k \times X_1 \times X_2 \times \cdots \times X_s. \tag{5.2}$$

If the $U(1)_R$ connection is not pure gauge, eqn.(2.1) implies

$$\partial_i \tau \, \partial^i \bar{\tau} \neq 0. \tag{5.3}$$

Assume ϵ is a non–zero (commuting!) spinor on X satisfying eqn.(1.6), then

$$0 = \epsilon^* \, \partial\!\!\!/ \tau \; \partial\!\!\!/ \bar{\tau} \, \epsilon \equiv (\partial_i \tau \, \partial^i \bar{\tau}) \, \epsilon^* \epsilon + \partial_i \tau \, \partial_j \bar{\tau} \, \epsilon^* \gamma^{ij} \epsilon \tag{5.4}$$

$$\Longrightarrow \quad \epsilon^* \gamma_{ij} \, \epsilon \neq 0, \tag{5.5}$$

that is, the 2–form $\kappa_{ij} = \epsilon^* \gamma_{ij} \, \epsilon$ is non–zero. Since ϵ is $\mathcal{D}$–parallel, so is the 2–form κ. But κ has $U(1)_R$ charge zero, and $\mathcal{D}$–parallel means parallel with respect to the Levi Civita connection of X (valued in $\mathfrak{hol}(X)$). Thus the manifold X has *a non–zero parallel two–form κ.*

We know (see *e.g.* [GSSFT]) that an irreducible Riemannian manifold has a non–zero parallel two–form if and only if it is Kähler[17].

Then, for each irreducible factor space in (5.2), we must have one of the following:

- $\kappa\big|_{X_k} \neq 0$, and X_k is Kähler with Kähler form $\omega^{(k)} = i \, \lambda_k \, \kappa\big|_{X_k}$, where $\lambda_k \in \mathbb{R}^\times$;
- $\kappa\big|_{X_k} = 0$,

(for the flat factor $\mathbb{R}^k$, $\kappa\big|_{X_k}$ is a constant 2–form but it does not need to be non–degenerate; we ignore the irrelevant flat factor from now on). Moreover[18]

$$\kappa = \sum_k \pi_k^* \left(\kappa\big|_{X_k} \right), \tag{5.6}$$

where π_k is projection on the k–th factor in eqn.(5.2).

Consider the identity

$$\kappa_{ij} \, \partial^j \, \bar{\tau} = \partial^j \, \bar{\tau} \, \epsilon^* \gamma_{ij} \, \epsilon = \epsilon^* \gamma_i \, \partial\!\!\!/ \bar{\tau} \, \epsilon - \partial_i \bar{\tau} \, \epsilon^* \epsilon \equiv -(\epsilon^* \epsilon) \, \partial_i \bar{\tau}. \tag{5.7}$$

and restrict it to each irreducible factor space

$$\kappa_{ij}\Big|_{X_k} \partial^j \, \bar{\tau} = -(\epsilon^* \epsilon) \, \partial_i \bar{\tau} \tag{5.8}$$

---

[17] The proof is so short that it fits in a footnote. Consider the negative–definite *symmetric* matrix $L_i^j = \kappa_{ik} g^{kl} \kappa_{lm} g^{mj}$. It is parallel, hence commutes with the holonomy group. But the holonomy group acts irreducibly, so (Shur's lemma) L_i^j is a multiple of the identity. By a change of the normalization of κ, we may assume $L_i^j = -\delta_i^j$ Thus the tensor $I_i^j = \kappa_{ik} g^{kj}$ has the property $I^2 = -1$, and is a almost complex structure. Since this alsmost tensor structure is parallel, it is automatically integrable (by the Nijenhuis theorem) and hence the irreducible space is a *complex manifold.* Since κ_{ij} is antisymmetric, the metric is Hermitian with Kähler form κ. Since κ is parallel, it is in particular closed, and hence the metric is Kähler.

[18] This follows from the Shur's lemma argument of the previous footnote. If you are not happy with that, use the Künneth formula (the factor spaces are simply connected!).

Then we have two possibilities:

- *either* X_k is Kähler with Kähler form $i\,\kappa\big|_{X_k}/(\epsilon^*\epsilon)$ and τ is a holomorphic function of the coordinates of X_k;
- *or* $d\tau\big|_{X_k} = 0$.

Indeed, in the first case, eqn.(5.8) is the Cauchy–Riemann equation defining the holomorphic functions[19], whereas if the conditions are not satisfied we get $d\tau = 0$ as the only solution. Notice that the scalar $\epsilon^*\epsilon$, being parallel, is just a *constant*.

Therefore, X has the structure $K \times Y$, with K a (non neccessarily irreducible) Kähler manifold and Y any (simply connected) manifold. τ is the pull–back of a holomorphic function on K. [Well, this is true only locally; typically there are no global holomorphic functions on K; but τ is not a global field either, it *must have* $SL(2,\mathbb{Z})$ jumps somewhere. We shall make the correct global statement momentarily. Stay tuned.]

From this result and the explicit form of the $\mathfrak{u}(1)_R$ curvature Q eqn.(2.1), we see that Q is the pull–back of a $(1,1)$–form on K. Then the general integrability condition (1.8) splits as

$$\begin{aligned}
Y:&\quad \gamma^b\, R_{ab}\,\epsilon = 0 && \Rightarrow\quad Y \text{ is Ricci–flat of special holonomy}\\
K:&\quad \gamma^{\bar j}\left(R_{i\bar j} + i\,Q_{i\bar j}\right)\epsilon = 0 &&\\
&\quad \gamma^{j}\left(R_{\bar i j} + i\,Q_{\bar i j}\right)\epsilon = 0 && \Rightarrow\quad R_{i\bar j} + iQ_{i\bar j} = 0
\end{aligned}$$

that is, the Ricci form is minus the curvature of the the $\mathfrak{u}(1)_R$. Since the $\mathfrak{u}(1)_R$ connection gauges the Hodge line bundle $\mathcal{L}^{-1}$ (see chapter **1**), while the $\mathfrak{u}(1)$ part of the Levi Civita–Kähler connection gauges[20] the canonical bundle $\mathcal{K}_K$, we get an identification of the two.

Summary:

FACT 5.1. *In absence of fluxes, an F–theory* vacuum <u>*with* $d\tau \neq 0$</u> *has the following structure:*

(1) *The universal cover* $\tilde{M}$ *of its* $10D$ *gravitational manifold* M *is a product of the form*

$$\tilde{M} = \mathbb{R}^{1,k} \times K \times Y \tag{5.9}$$

where

[19] Here a conventional choice of the sign of λ is implied. Of course, both I and $-I$ are complex structures, and the choice of sign corresponds to choosing what we mean by holomorphic *vs. anti*holomorphic.

[20] This statement is trivial: the trace part of the $\mathfrak{u}(m)$ Kähler holonomy, namely $\mathfrak{u}(1)$ is obviously generated by the trace part of the (Riemann) curvature, which is the Ricci curvature.

(a) *K is a simply–connected Kähler manifold which is a product of irreducible complex manifolds K_i with holonomy group strictly*[21] *equal to $U(\dim_{\mathbb{C}} K_i)$;*
(b) *Y is a compact simply–connected Ricci–flat manifold of special holonomy.*

(2) *There is a* complex *manifold $\mathcal{E}$ which is elliptically fibered (with section) over the Kähler space K, such that the* holomorphic *elliptic fibration $\mathcal{E} \to K$ pulls–back to the F–theory elliptic fibration $\mathcal{Y}_{12} \to \tilde{M}$ (that is: $\mathcal{Y}_{12} = \mathbb{R}^{1,k} \times \mathcal{E} \times Y$).*
(3) *The complex manifold $\mathcal{E}$ is* Kählerian[22]
(4) *The Hodge line bundle $\mathcal{L}^{-1} \to K$ (whose fiber over $k \in K$ is $H^0(E_k, \Omega^1_{E_k})$) is* holomorphic *and its curvature is minus the Ricci form of K, so $c_1(\mathcal{L}) = c_1(K)$ and $\mathcal{K}_{\mathcal{E}}$ is trivial.*
(5) *Hence*

$$\boxed{\mathcal{E} \text{ is an elliptic Calabi–Yau space}} \tag{5.10}$$

Thus we see that we get back the old condition (the compactification space is Calabi–Yau) but this time for the total 12–dimensional manifold $\mathcal{Y}_{12}$. Notice that $\mathcal{Y}_{12}$, which is a quotient of $\mathbb{R}^{1,k} \times \mathcal{E} \times Y$, does admit parallel spinors[23], while the gravitational manifold M does not! This well illustrates Cumrun Vafa's idea that it is the 12–fold which is 'intrinsic' not the $10D$ 'gravitational brane'.

All the statements in FACT 5.1 have been proved, except (maybe) for (4) and (5). The claim that $\mathcal{K}_{\mathcal{E}}$ is trivial should be obvious. Those

[21] Indeed, any factor space with holonomy contained in $U(m)$ but not equal to $U(m)$ would be either flat or an irreducible Ricci–flat manifold of special holonomy. Both these factor spaces may be absorbed in $\mathbb{R}^{1,k} \times Y$.

[22] Our terminology is as follows: A *complex manifold* is *Kählerian* if it admits a Kähler metric. A complex Riemannian manifold is Kähler if the given metric is Kähler. Let us proof the claim in the text. We use a technique first introduced in APPENDIX C of [**47**] (se also [**?**]).

Let z be a complex coordinate along the fiber. z is well–defined up to the periodicity, *i.e.* $z \sim z+n+m\tau$, with τ depending *holomorphically* on the coordinates of the Kählerian base K (with Kähler form ω_K). The $(1,1)$ form

$$\omega_{\mathcal{E}} = \lambda\, \pi^* \omega_K + \partial\overline{\partial}\left(\frac{(z-\bar{z})^2}{\tau-\bar{\tau}}\right)$$

is manifestly closed, and positive–definite for λ big enough (recall that K is effectively compact!). One has only to show that it is globally well-defined, namely that it is invariant under $\delta_{n,m}\colon z \to z+n+m\tau$. Indeed, the variation of the function in the second term is

$$\delta_{n,m}\frac{(z-\bar{z})^2}{\tau-\bar{\tau}} = (2mz + m^2\tau) - (2m\bar{\tau} + m^2\bar{\tau}) = \text{harmonic}$$

and hence it is annihilated by the operator $\partial\overline{\partial}$.

[23] *Recto:* it can be equipped wit a CY metric and, with such a *special* metric, admits parallel spinors.

that do not find it so obvious may read this footnote[24]. The *very* pedantic reader may prefer to wait for chapter **n**, where we discuss Kodaira's theory of elliptic fibrations. The implication (3)+(4) $\Rightarrow$ (5) is also well–known, and it will be reviewed in §.6.1 below.

FACT 5.1 is *very* good news. The fact that all manifolds and fibrations are complex analytic (but for spectator ones like $\mathbb{R}^{1,k} \times Y$) grantes us, for free, the immense power and all the tools of complex analytic geometry. In fact, we get much more: all the tremendous insight of Algebraic Geometry. Indeed,

PROPOSITION 5.1. *Let X be a* strict[25] *Calabi–Yau manifold with* $\dim_{\mathbb{C}} X \geq 3$. *Then X is an algebraic (projective) manifold. Conversely any algebraic manifold X with trivial canonical bundle K_X is a Calabi–Yau manifold; it is strict iff $h^{2,0}(X) = 0$ (for* $\dim_{\mathbb{C}} X \geq 3$*).*

In fact, a compact manifold X of strict holonomy $SU(m)$ has Hodge numbers $h^{p,0}(X) = 0$ for $1 \leq p \leq m-1$, as you well know from [GSSFT] (cfr. THEOREM **3.**5.2). Thus, if $m \geq 3$, $h^{2,0} = 0$ and the manifold is algebraic by a well–known corollary to Kodaira's embedding theorem (cfr. *e.g.* THEOREM 3.5 in Ref.[**48**]).

REMARK. In the remaining case, $\dim_{\mathbb{C}} X = 2$, X is a $K3$ surface which is a very well–know object (which may or may not be algebraic). We shall discuss some of his properties in .....

FACT 5.1 is so central in F–theory that, before going to different topics, we pause a while to comment it. Here are two sections of comments, one on the geometry and one on the *physics* of these configurations.

6. Nice subtleties and other geometric wonders

A priori, in FACT 5.1 the term *'Calabi–Yau'*[26] has it *weakest* sense, namely it means a complex manifold X which admits a metric with

[24] Let us refer to the Weierstrass representation of $\mathcal{E}$. The form dX/Y is a $(1,0)$ form along the fiber which transforms as a *meromorphic* section of $\mathcal{L}^{-1} \to K$. Let z_i^α be local coordinates on K in the coordinate patch U_i, with $K = \cup_i U_i$ a sufficiently fine open cover. The $(n,0)$ form $dz_i^1 \wedge dz_i^2 \wedge \cdots \wedge dz_i^n$ transforms (by definition!) as a section of $\mathcal{K}_K^{-1} \simeq \mathcal{L}$. This implies that we can find local changes of trivialization $\psi_i \in \Gamma(U_i, \mathcal{O}^\times)$ and a global meromorphic function ϕ on K such that the local holomorphic $(n+1)$–forms

$$\phi(z)\, \psi_i(z)\, \frac{dX}{Y} \wedge dz_i^1 \wedge dz_i^2 \wedge \cdots \wedge dz_i^n \in \Gamma(U_i, \Omega^n)$$

glue into a global $(n+1)$–form without zeros on K, and hence on $\mathcal{E}$ since it is translational invariant along the fibers. This form is a global trivialization of $\mathcal{K}_{\mathcal{E}}$.

[25] By strict Calabi–Yau we mean a manifold of holonomy $\mathfrak{hol}(X) = \mathfrak{su}(m)$ and not a proper subgroup of $\mathfrak{su}(m)$.

[26] I guess that the fact that a $2m$–fold has holonomy $\subseteq \mathfrak{u}(m)$ (resp. $\subseteq \mathfrak{su}(m)$) if and only it the metric is Kähler (resp. Calabi–Yau $\equiv$ Kähler and Ricci–flat) should be well–known, and is implicitly proven in footnote .... Here I give a quick summary

$\mathfrak{hol}(X) \subseteq \mathfrak{su}(\dim_{\mathbb{C}} X)$. We speak of *strict Calabi–Yau* when the symbol $\subseteq$ may be replaced by $=$.

Then the natural question is: *Can we be more precise?*

Which *kinds* of Calabi–Yau metrics are allowed on $\mathcal{E}$?

I start by reviewing some well–known facts about Calabi–Yau metric, fullfiling my promise to show that (3)+(4) $\Rightarrow$ (5) in FACT 5.1. Again, most of the readers may prefer to jump ahead to §.6.2

6.1.($\mathbb{J}$) **Basic geometric facts about Calabi–Yau's.** We shall devote a specific chapter to the geometry of the relevant complex/algebraic manifolds, expecially to develop the computational tools needed to extract 'experimental' prediction from F–theory (see chapter ...). Here we limit ourselves to the very basic properties we need for the present purposes as well as to make sense out of the F–/M–theory duality in §. below.

I start by making my definitions slighlty more precise:

DEFINITION 6.1. By a *weak Calabi–Yau manifold* we mean a complex manifold which admits a Ricci–flat Kähler metric. A Ricci–flat Kähler metric g is called a *Calabi–Yau metric* (as we already showed in the footnotes, this implies $\mathfrak{hol}(g) \subseteq \mathfrak{su}(\dim_{\mathbb{C}} X)$. A *strict Calabi–Yau metric* is a metric of holonomy $\mathfrak{hol}(g) \equiv \mathfrak{su}(\dim_{\mathbb{C}} X)$. A *strict Calabi–Yau manifold* is a manifold admitting a strict Calabi–Yau metric.

A complex space X is called an *elliptic Calabi–Yau* if it is a Calabi–Yau and there exists a holomorphic fibration $\mathcal{E} \to Z$, on a complex space Z.

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of the ideas. The crucial step is to show that for a Riemannian $2m$–fold X

$$\mathrm{Hol}(X) \subseteq U(m) \Longleftrightarrow X \text{ is Kähler} \quad \text{(hence complex)}.$$

Indeed, by definition, saying that the holonomy $\mathrm{Hol}(X) \subseteq U(m)$ is equivalent to saying that there is a decomposition of the (complexified) tangent bundle $TX \otimes \mathbb{C}$ in irreducible $U(m)$ representations of the form $TX \otimes \mathbb{C} = m \oplus \overline{m}$ which is *invariant under parallel transport.* Then let I be the almost complex structure which is multiplication by i on the subspace m and multiplication by $-i$ on $\overline{m}$. Since the decomposition is invariant under parallel transport so is the almost complex structure I. Thus $\nabla_i I = 0$, which, in particular, means that I is *integrable* to a true complex structure. Hence X is a *complex manifold.*

Let g_{ij} be the metric (which, being $U(m)$ invariant, is obviously Hermitian). The Kähler form $I_i{}^k g_{kj}\, dx^i \wedge dx^j$ is parallel ($\equiv$ covariantly constant), since both tensors g and I are. But a parallel form is, in particular, *closed.* Then g is a Kähler metric.

Conversely, a Kähler m–fold has holonomy group $\subseteq U(1) \times SU(m)$. By the Ambrose–Singer theorem, the corresponding Lie algebra $\mathfrak{hol}(X) = \mathfrak{u}(1) \oplus \mathfrak{su}(m)$ is generated by the Riemann tensor. The trace part, $\mathfrak{u}(1)$ is generated by the trace of the Riemann tensor, *i.e.* the Ricci curvature, and the $\mathfrak{u}(1)$ part of the holonomy vanishes iff $R_{\mu\nu} = 0$.

Finally, let ρ be the Ricci form. $c_1(X) = \rho/2\pi$. So Ricci–flat implies $c_1(X) = 0$.

We notice the

LEMMA 6.1. *If the elliptic fibration $\mathcal{E} \to Z$ has a section, Z is Kählerian.*

Proof. Trivial.

Since the Ricci $(1,1)$–form of a Kähler metric is -2π times a representative of $c_1(X)$, (Kähler Ricci–flat) $\Rightarrow c_1(X) = 0$. The inverse implication, namely that a compact Kählerian manifold X with $c_1(X) = 0$ admits a metric with holonomy $\mathfrak{hol}(g) \subseteq \mathfrak{su}(\dim_{\mathbb{C}} X)$ is the Calabi–Yau theorem (see refs... or [GSSFT] for further discussion). Before we need a further definition:

DEFINITION 6.2. X a Kählerian manifold. By the *Kähler cone* $\mathcal{C}_X$ of X we mean the strictly[27] convex cone in the $\mathbb{R}$–vector space $H^2(X, \mathbb{R})$ of the classes $[\alpha]$ such that there is a *positive definite* Kähler metric on X whose Kähler form ω is cohomologous to $[\alpha]$, *i.e.* $[\omega] = [\alpha]$.

THEOREM 6.1 (Calabi, Yau). *Let X a compact complex manifold, admitting Kähler metrics, with $c_1(X) = 0$. Then there is a* unique *Ricci–flat Kähler metric in each Kähler class on X. The Ricci–flat Kähler metric on X form a smooth family of dimension $h^{1,1}(X)$ isomorphic to the Kähler cone $\mathcal{K}_X$ of X.*

Notice that here we keep the complex structure of X *fixed.* The Ricci–flat metrics will depend also on the complex moduli, of course.

COROLLARY 6.1. *In the statement of* FACT *5.1, (3)+(4) $\Rightarrow$ (5).*

6.2. Restrictions on $\mathfrak{hol}(\mathcal{E})$.

Let us return to our problem, to be more precise about $\mathfrak{hol}(\mathcal{E})$. Let me make a

CLAIM 6.1. *The elliptic Calabi–Yau $\mathcal{E}$ in* FACT *5.1 has $\mathfrak{hol}(\mathcal{E}) \equiv \mathfrak{su}(\dim_{\mathbb{C}} \mathcal{E})$ (that is, it is always* strict *!).*

This is a geometrical result that is, in fact, a physical consistency check. Indeed, any other holonomy group will lead to physical parodoxes and, were they possible, we will be forced to conclude that F–theory is sick. Fortunately, it is *not* the case. The way it happens, geometrically, sounds magics.

As we mentioned above, de Rham's, Cheeger–Gromoll, and Berger theorems imply that a compact *weak* Calabi–Yau space X has a finite cover of the form

$$\mathcal{E} = T^{2k} \times CY_1 \times \cdots \times CY_p \times Hy_{p+1} \times \cdots \times Hy_q, \tag{6.1}$$

[27] A convex cone in $\mathbb{R}^2$ is *strict* if does not contain any full straight line.

where T^{2k} is a flat torus, the CY_k's are irreducible manifolds of strict holonomy $SU(m_k)$ and the Hy_r are irreducible manifolds of strict holonomy $Sp(2l_r)$ (with $l_r \geq 2$). We first show that in the RHS of (6.1) there is just one irreducible factor.

Let p_l be the projection of $\mathcal{E}$ on the l–th factor space in eqn.(6.1) ($l = 1, 2, \dots, n$), ω_l its Kähler form, and m_l its (complex) dimension.

We have a section $\sigma\colon K \to \mathcal{E}$ which is a complex isomorphism of K into its image. The $(1,1)$ forms $\sigma^*\omega_l$, $l = 1, 2, \dots, n$ are *parallel* on K, and $\omega = \sum_l \sigma^*\omega_l$ is a Kähler form on K. Thus

$$\Big(\sum\nolimits_l \sigma^*\omega_l\Big)^{\sum_j m_j - 1} \neq 0, \tag{6.2}$$

which means that for all but one values of the index l, $\sigma^*\omega_l^{m_l} \neq 0$, while for the exceptional one l_0 $\sigma^*\omega_{l_0}^{m_{l_0}-1} \neq 0$. Since the forms ω_l are parallel, we see that K is the direct product of n Kähler manifolds K_l of dimension

$$\dim K_l = \begin{cases} m_l & l = 1,\ 2\dots,\ \widehat{l_0}, \dots,\ n \\ m_{l_0} - 1 & l = l_0 \end{cases} \tag{6.3}$$

moreover for $l \neq l_0$ the manifolds K_l are isometric to the l–th factor space in eqn.(6.1). But K_l is a product of *strict* Kähler spaces (FACT 5.1 (1)(a)) and no space in the RHS of eqn.(6.1) is a *strict* Kähler space. This is a contradiction unless $n = 1$.

Moreover $\mathcal{E}$ and K cannot be flat, if $d\tau \neq 0$. Thus

COROLLARY 6.2. *The Kählerian spaces K and $\mathcal{E}$, defined as in* FACT *5.1.(1)(a), are* irreducible.

It remains to show that $\mathfrak{hol}(\mathcal{E}) \neq \mathfrak{sp}(2l)$ with $l \geq 2$ (in words: it cannot be a hyperKähler manifold, unless it is $K3$). This is a deep fact, related *inter alia* to the theory of integrable models. We state it as a geometrical theorem. Before we give yet another definition.

DEFINITION 6.3. A *holomorphic symplectic manifold* X is a complex $2n$–fold with a $(2,0)$–form Ω such that $\Omega^n \neq 0$ at each point.

EXERCISE 6.1. Show that a *compact* holomorphic symplectic manifold is, in particular, (weak) Calabi–Yau and that any Calabi–Yau metric g has holonomy $\mathfrak{hol}(g) \subseteq \mathfrak{sp}(2n)$. Show the converse too.

THEOREM 6.2 (Matsushita [**49**][**50**]). *Le X be an irreducibly holomorphic symplectic manifold of dimension $2n$ and let $X \to B$ a non–constant morphism of positive fibre dimension onto a Kähler manifold B. Then*

1. *B is of dimension n, projective and satisfies $B_2(B) = \rho(B) = 1$. Moreover K_B^{-1} is ample, i.e. B is Fano.*
2. *Every fiber is complex Lagrangian and, in particular, of dimension n.*

(3) *Every smooth fiber is an n–dimensional complex torus.*

Thus a compact holomorphic symplectic manifold ($\equiv$ a compact hyperKähler) may be fibered with one–dimensional fibers only if $n = 1$, that is if it is a $K3$! Moreover, in this case the base B should be $\mathbb{P}^1$.

It is impossible to overstate the relevance of Matsushita theorem for theoretical physics. It is one of the central results for almost *every* branch of our discipline.

In the next section we describe a couple of the physical meaning in our present context.

7. Physics of the ‘elliptic’ vacua

Most of the present lectures are aimed[28] to extract physics out of the F–theory vacua described by FACT 5.1. Here we limit ourselves to some *very* preliminary comments.

7.1. Supersymmetries. Let us consider a (zero–flux) vacuum as in FACT 5.1. It is supersymmetric. *How many supersymmetries it has?*

The spinors of Type IIB SUGRA on M are direct products of spinors ϵ on K and of spinors ϵ' on $\mathrm{R}^{1,k} \times Y$. For a vacuum configuration as in FACT 5.1 ϵ' must be a *parallel* spinor on $\mathrm{R}^{1,k} \times Y$.

We recall that, on an irreducible Kähler m–fold K, the Dirac spinor bundle, $\mathcal{S}$, is isomorphic to

$$\mathcal{S} \sim \bigoplus_{k=0}^{m} \Omega^k_K \otimes \mathcal{K}_K^{-1/2} \tag{7.1}$$

as $\mathfrak{u}(m)$–associated vector bundles. The complex spinor ϵ is a section of $\mathcal{S} \otimes \mathcal{L}^{1/2}$ while ϵ^* is a section of $\mathcal{S} \otimes \mathcal{L}^{-1/2}$. Since, for the vacuum configurations $\mathcal{L} = \mathcal{K}_K$, we get precisely one $\mathfrak{su}(m) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)_R$ invariant spinor ϵ and one invariant ϵ^* of the same chirality if m is even or of opposite chirality if m is odd.

The chiralities of ϵ and ϵ' should be equal in order the $10D$ spinor to have chirality $+1$. In TABLE 2.2 we list the spaces $\tilde{M}$ which are allowed as bases of the elliptic fibration $\mathcal{Y}_{12} \to \tilde{M}$ for a supersymmetric no–flux vacuum with $d\tau \neq 0$. $\#Q_F$ is the number of unbroken real supercharges. For instance, the third and fourth rows give the SUSY configurations which are invariant under the $4d$ Poincaré symmetry. We have, respectively, 4 and 8 supercharges, which means, respectively, $\mathcal{N} = 1$ and $\mathcal{N} = 2$ SUSY. By comparison, we also listed the M–theory duals. In the table, $\tilde{Z}$ is the manifold so that $\widetilde{\mathcal{Y}}_{12} \simeq \mathbb{R}_{\text{space}} \times \tilde{Z}$. [As a matter of notation, in TABLE 2.2, K_n stands for an *irreducible strict Kähler n*–fold, G for an *irreducible manifold of holonomy G_2*, and CY_n

[28] If the reader has got a different impression, she is excused.

$\tilde{M}$	$\#Q_F$	$\tilde{Z}$	$\#Q_M$
$\mathbb{R}^{1,7} \times \mathbb{P}^1$	16	$\mathbb{R}^{1,6} \times K3$	16
$\mathbb{R}^{1,5} \times K_2$	8	$\mathbb{R}^{1,5} \times CY_3$	8
$\mathbb{R}^{1,3} \times K_3$	4	$\mathbb{R}^{1,2} \times CY_4$	4
$\mathbb{R}^{1,3} \times K3 \times \mathbb{P}^1$	8	$\mathbb{R}^{1,2} \times K3 \times K3$	8
$\mathbb{R}^{1,1} \times K_4$	2	$\mathbb{R}^{1,0} \times CY_5$	2
$\mathbb{R}^{1,1} \times K3 \times K_2$	4	$\mathbb{R}^{1,0} \times K3 \times CY_3$	4
$\mathbb{R}^{1,0} \times G \times \mathbb{P}^1$	2	–	–

TABLE 2.2. F–theory compactification spaces, the number of preserved supercharges and comparison with the corresponding M–theory quantities.

for an *irreducible strict Calabi–Yau n–*fold. $\#Q_M$ is the number of unbroken (real) supercharges for M–theory compactified on the manifold $\tilde{Z}$. Notice that the second and third colums are equal, so the two list of configurations are exactly dual. This duality will be discussed more in detail in section .... below.

Here we see a first miracle connected with Matsushita theorem. If an elliptically fibered hyperKähler 4–fold (real dimension 8) had existed, we could have used it to define a Poincaré invariant F–theory vacuum in $d = 4$ whose M–theory dual would be a $d = 3$ Poincaré vacuum. How many supercharges? From TABLE 2.2 we get 6, which is $\mathcal{N} = 3$ in $d = 3$, which is fine on the M–theory side. But 6 supercharges are $\mathcal{N} = 3/2$ in $d = 4$, so this would be *non–sense* in F–theory. What saves the day is that (compact) hyperKähler 4–folds exist and make perfectly nice M–theory vacua but are never elliptically fibered so they never make dual F–theory vacua.

The same happens with irreducible manifolds of holonomy G_2 and $Spin(7)$. From TABLE 2.2 they are, respectively, M–theory vacua in $d = 4$ with $\#Q_M = 4$ and in $d = 3$ with $\#Q_M = 2$. The putative F–theory duals would have, respectively, $\#Q_F = 4$ in $d = 5$, which is inconsistent since in $d = 5$ the number of supercharges should be divisible by 8, and $\#Q_F = 2$ in $d = 4$ which is absurt. Anyhow these spaces are not allowed by FACT 5.1 and no real paradox emerges.

7.2. Seven branes. The vacua we are considering have $d\tau \neq 0$ and thus a varying axion/dilaton field (with $SL(2,\mathbb{Z})$ jumps). We saw in chapter **1** that, 'perturbatively', the eight–dimensional submanifold on which a D7 brane wraps is characterized by the fact that going along a loop encircling them we get a non trivial monodromy transformation $SL(2,\mathbb{Z})$, namely T. Indeed, by Stokes theorem, this is equivalent to saying that they are δ–like sources of the RR–flux $F_1 \sim dC_1$.

More generally, any eight–dimensional submanifold Z of the $10D$ manifold M such that going along a closed loop around them we get back to the original point with the fields rotated by a non–trivial element of $SL(2,\mathbb{Z})$, should be considered a generalized seven–brane. In particular, O7 planes are seven–branes in this language, but of a very special kind: As discussed in chapter **1**, they correspond to non–trivial $SL(2,\mathbb{Z})$ monodromies which become trivial in $PSL(2,\mathbb{Z})$.

We take these monodromy properties as 'non–perturbative' (and imprecise) definitions of *seven–branes* and *seven–orientifold*.

Since the F–theory vacua described in FACT 5.1 necessarily contain non–trivial $SL(2,\mathbb{Z})$ monodromies (as we shall see momentarily), they do describe BPS configurations of generalized seven branes. These seven–branes span all the 'spectator' dimensions $\mathbb{R}^{1,k}\times Y$ of FACT 5.1, as well as a submanifold of K of real codimension 2. Indeed, the $\mathcal{E}$ fibrations is the pull–back of one over K, and τ depends only on the coordinates of K. Topologically, the seven–branes are characterized by a group homomorphism

$$\ldots\colon \pi_1\Big(M \setminus \bigcup_i Z_i\Big) \longrightarrow SL(2,\mathbb{Z}) \tag{7.2}$$

of the fundamental group π_1 of M minus the branes into the monodromy group. From this we see that the 'seven–branes' (as defined above) should actually have co–dimension 2: branes of codimension 1 would disconnect M while branes of codimension > 2 will not affect the fundamental group.

We wish to say more about these seven–branes, making precise the vague statements around eqn.... in §. **1**....

First of all, physically, our vacuum is a stationary configuration, and hence there can be no force between the branes. By universal convexity principles, this requires (in particular) that they are all branes, and no antibrane is present[29]. *Is this true?*

Of course. All seven branes should have the same orientation. Indeed let γ be a small loop in K and U an open neighborhood containg γ. If $\pi^{-1}U$ contains only smooth fibers, topologically we have $\pi^{-1}(U)\simeq U\times T^2$, and there is no room for a non–trivial monodromy. Thus a non–trivial monodromy arises if we go around points in $k\in K$ such that the fiber $E_k\in\mathcal{E}$ is *singular.* This is equivalent to *The union of all the seven–brane world–volumes*[30] $\bigcup_i Z_i$ *is the locus in* K *where the elliptic fibers of* $\mathcal{E}\to K$ degenerate.

[29] In a certain convention of what we call brane. It is the same as our convention of what is holomorphic *vs.* what is *anti*holomorphic.

[30] Times, of course, the 'spectator' dimensions $\mathbb{R}^{1,k}\times Y$.

But $\mathcal{E}$ is given explicitly in the Weierstrass form. We have alredy learned in chapter **1** that a Weierstrass curve degenerates if and only if its discriminant vanishes. Hence

FACT 7.1. *In a vacua as in* FACT *5.1, the seven branes' world–volume is the locus in* $(\mathbb{R}^{1,k} \times Y \times)$ K *where*

$$\Delta(w) = A^3(w) + 27\, B^2(w) = 0. \tag{7.3}$$

In particular, it is a complex analytic hypersurface *in* K.

Indeed, eqn.(7.3) is a holomorphic defining equation of the said locus, since $A(w)$, $B(w)$ are *holomorphic* sections of $\mathcal{L}^2$ and, respectively, $\mathcal{L}^3$.

Hence the irreducible components of the analytic locus $\Delta = 0$ — which are the single branes — are (complex) codimension 1 *complex submanifolds* of K. Complex submanifolds have a natural orientation, so all branes have the same orientation, and there are no *anti*branes around.

But this is not enough. A vacuum is a fundamental state, that is the lowest energy configuration in its sector. The energy of a brane is given by its tension and so it is proportional to its volume. Then a vacuum should contain only branes which are of *minimal volume.* Here 'minimal' may only mean minimal in its topological class which is specified by the Poincaré dual class in $H^2(K, \mathbb{Z})$. *Are our branes of minimal area?*

Of course. To show this we have to specialize the previous orientation argument for complex submanifolds to the case in which the ambient space, K, is Kähler as (luckily enough) is assured by FACT 5.1. We make a little mathematical digression and then return to the branes.

7.3. Wirtinger theorem. Let K be a Kähler n–fold with Kähler form ω. Its volume is just

$$\mathrm{Vol}(K) = \frac{1}{n!} \int_K \omega^n \tag{7.4}$$

since $\omega^n/n!$, written in real notation, is the standard Riemannian volume form $\sqrt{g}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$. Let $\iota\colon Z \subset K$ be a complex submanifold of dimension $m < n$. With respect the induced metric, Z is still Kähler, with Kähler form $\iota^*\omega$. Then

$$m!\, \mathrm{Vol}(Z) = \int_Z \omega^m = [\omega^m]([Z]) \tag{7.5}$$

where the notation in the RHS means the element $[\omega^m] \in H^{2m}(K, \mathrm{R})$ evaluated on the fundamental cycle $[Z] \in H_{2m}(K, \mathbb{Z})$. Therefore: *the volume of a complex submanifold of a Kähler manifold depends only on its homology class.*

This leads us to suspect that the volume of a complex submanifold is minimal among all smooth submanifolds in its homology class. In fact,

THEOREM 7.1 (Wirtinger). *The volume of a complex analytic submanifold X of a Kähler space K is minimal in its homology class.*

PROOF. Let $\iota\colon Z \to K$ be a *smooth* submanifold of dimension $2m < 2n$. Equip it with the metric induced by the embedding. Let e^a $(a = 1, \dots, 2m)$ an orthormal coframe in T^*Z, and v_a the dual orthonormal frame in TZ. Let ω be the Kähler form of K, and consider the smooth 2–form $\iota^*\omega$ on Z. It can be expanded in the basis e^a. Performing a suitable $SO(2m)$ redefinition of the e^a's, we may assume it is diagonal

$$\iota^*\omega = \sum\nolimits_l \lambda_l\, e_{2l-1} \wedge e_{2l}. \tag{7.6}$$

The (real) skew–eigenvalues λ_l are given by[31]

$$\lambda_l = \omega(\iota_* v_{2l-1}, \iota_* v_{2l}) = (I\, \iota_* v_{2l-1},\, \iota_* v_{2l}) \leq \|\iota_* v_{2l-1}\|\, \|\iota_* v_{2l}\| = 1 \tag{7.7}$$

Notice that we have equaliy iff $\iota_* v_{2l} = I\, \iota_* v_{2l-1}$. The volume form on Z is $\pm e^1 \wedge e^2 \wedge \cdots \wedge e^{2m}$. Thus

$$\frac{1}{m!}\big(\iota^*\omega\big)^m = (\lambda_1\lambda_2\cdots\lambda_m)\, e^1 \wedge e^2 \wedge \cdots \wedge e^{2m} \leq |e^1 \wedge e^2 \wedge \cdots \wedge e^{2m}| \tag{7.8}$$

with equality if and only if $\lambda_l = 1$ for all l and the orientations agree. Integrating

$$\mathrm{Vol}(Z) \geq \frac{1}{m!}\int_Z \omega^m \tag{7.9}$$

with equality if and only if $e^{2l} = I e^{2l-1}$ for all l. This means that $\iota^* I$ is an almost complex structure on Z. Since it is the one induced from K, Z is a complex submanifold. □

COROLLARY 7.1. *Consider the elliptic fibration $\mathcal{E} \to K$ and equip $\mathcal{E}$ with a Calabi–Yau metric. Then the volume of the fiber,* $\mathrm{Vol}(E_k)$*, is the same for all $k \in K$.*

Indeed the fibers are complex submanifolds and belong all to the same homology class.

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[31] Here $(\cdot,\cdot)$ is the pairing in TK given by the Kähler metric, and I is the complex structure on TK (we see K as a smooth manifold with the almost complex structure I). In (7.7) we used the relation of the Kähler form to the Kähler metric, the Schwartz inequality, the fact that I is an unitary operator, and that the $\iota_* v_a$'s are orthonormal vectors.

7.4. Seven branes again. Hence our seven branes minimize volume and tension. Good. But this is certainly not enough to have a sensible physical vacuum. We need at least other two general conditions to be fullfilled.

First of all the vacuum should be a physical state, that is, it must satisfy the Gauss' law for the seven brane fluxes. Let F_1 be the corresponding field–strenght (which is a 1–form). Heuristically, one would expect a 'magnetic' Gauss' law of the standard form

$$\text{“}\; dF_1 = g^{-1} \sum\nolimits_i T_{Z_i} \;\text{”} \tag{7.10}$$

where T_{Z_i} is the 2–current[32] corresponding to the submanifold Z_i on which the i–th brane wraps. But now $g = (\operatorname{Im}\tau)^{-1}$ is spacetime dependent, and the naive equation is not consistent. One would write

$$\text{“}\; d\big(g\, F_1\big) = \sum\nolimits_i T_{Z_i} \;\text{”}\,, \tag{7.11}$$

to get an equality between closed forms. But this has no longer the form of a Bianchi identity with singularities. So only its integrated version makes sense.

But $g\, F_1$ is, up to normalization, just the $\mathfrak{u}(1)_R$–connection. Thus the Gauss' law reduces to

$$\big(\mathfrak{u}(1)_R \text{ curvature}\big) = \lambda \sum\nolimits_i T_{Z_i}. \tag{7.12}$$

In our normalizations, the $\mathfrak{u}(1)_R$ connection gauges the line bundle $\mathcal{L}$. So Gauss' law says

$$\boxed{c_1(\mathcal{L}) = C \sum\nolimits_i [Z_i] \in H^2(K, \mathbb{Z})} \tag{7.13}$$

In chapter **1**, when discussing the finite volume property, we computed the numerical constant C to be $1/12$. *Is this Gauss' law fullfilled?*

Of course. The seven brane locus is given by $\Delta = 0$ and coincides with the divisor (Δ) (counting branes with appropriate *multiplicities*). Then

$$\sum\nolimits_i [Z_i] = c_1(\Delta) = 12\, c_1(\mathcal{L}) \quad \text{since } \Delta \text{ is a section of } \mathcal{L}^{12}. \tag{7.14}$$

So the magnetic Gauss' law is satisfied.

Notice that, in particular, this means that $\sum_i [Z_i]$ *is* 24 *times an integral class in* $H^2(K, \mathbb{Z})$. (24 because the fermions have $U(1)_R$ charge $\pm 1/2$, in my normalization, so $\mathcal{L}^{\pm 1/2}$ should be integral bundles).

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[32] T_{Z_i} is popularly known as 'a δ–function along Z_i'. It is the current ($\equiv$ a form–valued distribution dual to the smooth forms) such that

$$\int_K T_Z \wedge \alpha = \int_Z \alpha,$$

for all *smooth* forms α.

Finally, in order to have a vacuum, the various seven branes should not exert a net force one on the other, otherwise they will move and the configuration would not be static. Physically, this means that the repulsion between the branes due to their equal magnetic F_1 charge is exactly balanced by the gravitational atraction. This is guaranteed by the Einstein equations, or, in the integrated version, by the equality $\mathcal{L} = \mathcal{K}_K$. Thus, *the statement that $\mathcal{E}$ is Calabi–Yau is just the condition that the brane–brane forces are cancelled.*

8. Time–dependent BPS configurations

I will be sketchy, leaving to the reader to fill in the (obvious) details.

As before, $\tilde{M}$ is a product of indecomposable manifolds, all of which are definite (and hence irreducible) but, at most, for one, say N

$$\tilde{M} = N \times \mathbb{R}^k \times X_1 \times \cdots \times X_s \tag{8.1}$$

where X_k are irreducible simply connected Riemannian manifolds.

If τ does not depend on the coordinates of the N factor, which then must be Ricci–flat with a parallel spinor, the condition of having unbroken supersymmetries gives

$$\tilde{M} = N \times \mathbb{R}^k \times Y \times K, \tag{8.2}$$

where Y and K are as in Fact 5.1, and N is a (spectator) Brinkmann–Leister space. The elliptic fibration is the pull–back of an elliptic one over K. The Riemann equations are identically satisfied. This is the boring possibility.

Exercise 8.1. Make the table of all possible numbers of conserved supercharges $\#Q_F$.

Let us consider the case in which τ depends also on the coordinates of N. It is obious that the flat space $\mathbb{R}^k$ plays no role, so we replace $\tilde{M}$ by the reduced product $\tilde{M}_{\text{red.}}$ which is the product of all factor spaces in eqn.(8.1) *but* the flat one. To save print, we shall omit the subscript 'red.' in $\tilde{M}_{\text{red.}}$ from now on.

It is useful to contrast the present truly Lorentzian situation with the effectively Euclidean ones we studied up to now. In positive signature, we exploited the fact that the spinor bilinear $\epsilon^\dagger\epsilon$ is both a neutral scalar and positive–definite to show that a certain 2–form, $\kappa_{\mu\nu}$, was parallel and hence implied a Kähler structure. In Lorentzian signature this is not true any longer: The scalar bilinear is $\bar{\epsilon}\epsilon \equiv \epsilon^\dagger\gamma_0\epsilon$ which is not positive definite (for bosonic spinors !), and then may be identically zero even for $\epsilon \neq 0$. However, the same argument in Lorentzian signature leads to the conclusion that we have *a non–zero* parallel 1–form (or equivalently vector) $v_\mu = \bar{\epsilon}\gamma_\mu\epsilon$ (which is either time–like or light–like). A non–zero parallel vector is, in particular, a Killing vector.

Thus our Lorentzian manifold has a continuous isometry. Moreover, the condition $\bar{\partial}\bar{\tau}\,\epsilon = 0$ implies

$$\mathcal{L}_v \tau = v^\mu \partial_\mu \bar{\tau} = 0, \tag{8.3}$$

i.e. the field τ — and hence all the fields — are invariant under the isometry generated by v which is then a *a symmetry of the physical problem.* Indeed, it is just the bosonic symmetry generated by the anticommutator of the conserved supercharge. In the vacuum case, this bosonic symmetry was part of the unbroken Poincaré symmetry, but in the present context is more subtle.

In the time–dependent case, the parallel vector v^μ cannot be tangent to any factor space X_k in eqn.(8.1), since their holonomy is irreducible. So v is tangent to N. Then N should be indecomposable *but* not–irreducible, and hence $\bar{\epsilon}\gamma_\mu\epsilon$ should be *null*, $v_\mu v^\mu = 0$. Since $\mathfrak{hol}(N)$ admits *null* parallel vectors, N is (in particular)*a Brinkmann space.* Thus we infer

FACT 8.1. *The* $10D$ *gravitational manifold* M*, of* <u>*any*</u> F*–theory configuration (with no fluxes*[33]*) which has some* <u>*unbroken supersymmetry,*</u> *has a universal cover isometric to one of the following two:*

- **a):** $\mathbb{R}^{1,k} \times X$*,* X *a simply–connected Riemannian manifold which is a product of irreducible ones;*
- **b):** $N \times \mathbb{R}^k \times X$*, where* N *is a simply–connected* indecomposable *Brinkmann space and* X *as above.*

In the *definite* signature case, we obtained from the existence of a (trivial in that case) parallel vector and the non–triviality condition $d\tau \wedge d\bar{\tau} \neq 0$ that there was a *non–zero parallel* 2–form $\kappa_{\mu\nu}$ which should, on general grounds, be the Kähler form of some irreducible factor space X_{k_0} which then must be Kähler (and, in particular, complex). Then the two form

$$R_{\mu\nu}{}^{\alpha\beta}\,\kappa_{\alpha\beta} \tag{8.4}$$

(with an appropriate normalization) represents the first Chern class of X_{k_0}. Then we deduced that $(1,1)$ form $c_1(\mathcal{L}) \equiv -c_1(X_{k_0})$, and hence that the elliptic fiber space over X_{k_0} is Calabi–Yau.

In the present context, we will find that there is a *non–zero parallel* 3–form β, satisfying a number of identities, which plays the same role as the Kähler form in the static case. Indeed, the parallel 3–form of the static case (which is a special instance of time–dependence!) is just

$$\beta\Big|_{\text{vacuum case}} = dt \wedge \omega_K, \tag{8.5}$$

where ω_K is the Kähler form of the base of the elliptic fibration K.

[33] This restriction will be relaxed momentarily.

Let us show the above claims, and describe the relative spaces. Notice that the parallel vector $\bar{\epsilon}\gamma_\mu\epsilon$ is real. Thus, from $\not{\partial}\bar{\tau}\,\epsilon = 0$,

$$(\partial^\mu\tau)\,(\bar{\epsilon}\gamma_\mu\epsilon) = (\partial^\mu\bar{\tau})\,(\bar{\epsilon}\gamma_\mu\epsilon) = 0. \tag{8.6}$$

Then, arguing as in the definite case, we get the identities[34]

$$(\bar{\epsilon}\gamma_{\mu\nu\rho}\epsilon)\,\partial^\rho\bar{\tau} = (\partial_\mu\bar{\tau})(\bar{\epsilon}\gamma_\nu\epsilon) - (\partial_\nu\bar{\tau})(\bar{\epsilon}\gamma_\mu\epsilon) \tag{8.7}$$

$$-\,(\bar{\epsilon}\gamma_{\mu\nu\rho}\epsilon)\,\partial^\rho\tau = (\partial_\mu\tau)(\bar{\epsilon}\gamma_\nu\epsilon) - (\partial_\nu\tau)(\bar{\epsilon}\gamma_\mu\epsilon). \tag{8.8}$$

Since we are assuming $d\tau\wedge d\bar{\tau}\neq 0$, the RHS's of these two equations cannot be both zero. Then the 3–form

$$\boxed{\beta_{\mu\nu\rho}\equiv\bar{\epsilon}\gamma_{\mu\nu\rho}\epsilon} \tag{8.9}$$

is *non–zero and parallel* (in particular, closed).

To simplify the analysis, we note the

LEMMA 8.1. *Write $\tilde{M} = N\times R$, where R denotes the Riemannian 'rest' in eqn.*(8.1). *Then either $d\tau\big|_N = 0$ or $d\tau\big|_R = 0$.*

The idea of the proof is in the footnote[35].

We have already solved the (boring) case $d\tau\big|_N = 0$, so it remains to consider the configurations with $d\tau\big|_N \neq 0$ and $d\tau\big|_R = 0$. Then R has a parallel spinor, and *is a product of Ricci–flat spaces of special holonomy.* We are reduced to the study of the geometry of N.

N is a Brinkmann space with a parallel 3–form β satisfying all the above requirements. As in section 3 we denote ξ and η, respectively, the parallel vector and form.

Recalling the various steps (and the notation) we used in the proof of COROLLARY 3.1, we see that β has the structure

$$\beta = e^+\wedge e^a\wedge\left(\psi_{ab}\,e^b + \mu_a\,e^-\right)\qquad\Big[\psi_{ab},\ \mu_a \text{ invariant under } \mathfrak{g}\Big]. \tag{8.10}$$

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[34] The second one is the Hermitian conjugate of the first one.

[35] To make a long story short (possibly at the expense of elegance) let us impose the Einstein equations to the configuration. The indices $\alpha,\beta,\dots$ will refer to the Lorentzian factor N and the indices $a,b,\dots$ to the Euclidean R. Since $R_{\alpha a} = 0$, we must have

$$\partial_\alpha\tau\,\partial_a\bar{\tau} + \partial_\alpha\bar{\tau}\,\partial_a\tau = 0.$$

This equation has two kind of solutions: *i)* either $d\tau$ vanishes when restricted to one of the two spaces N, R, or *ii)* $d\mathrm{Re}\,\tau\big|_N = 0$, $d\mathrm{Im}\,\tau\big|_R = 0$ or *viceversa*. In case *i)*, one factor space is Ricci–flat while the other has a Ricci tensor of rank 2; in the case *ii)*, both Ricci tensors have rank 1. But we have $\beta_{\mu\nu\rho}\partial^\nu\tau\partial^\rho\bar{\tau} = -v_\mu(\partial^\rho\tau\partial_\rho\bar{\tau})\neq 0$. Then, in case *ii)* the Künneth decomposition of the parallel 3–form β must contain a non–zero term of the form $p^*_{(1)}\omega_2^{(1)}\wedge p^*_{(2)}\xi_1^{(2)}$, where the subscript denotes the degree of the form and the (1) and (2) refer, respectively, to the space N and R. Then $\xi_1^{(2)}$ is a parallel 1–fom on R, and there is no such object. Thus we are in case *i)*.

Then eqn.(8.7), $i_{\mathrm{grad}\,\tau}\,\beta = d\tau\wedge\eta$, splits as

$$\partial_-\bar\tau = \mu_a\,\partial^a\bar\tau \qquad\Rightarrow\quad \mu_a\partial^a\bar\tau = 0 \quad \text{cfr. } (8.3) \tag{8.11}$$

$$\partial_a\bar\tau = \psi_{ab}\,\partial^b\bar\tau \tag{8.12}$$

The meaning of these equations is more transparent if we go to the quotient Riemannian manifold Z_{x^+}. By de Rham theorem, we can write it as

$$Z_{x^+} = \mathbb{R}^s \times X^{(1)}_{x^+} \times X^{(2)}_{x^+} \times\cdots\times X^{(r)}_{x^+} \tag{8.13}$$

with $X^{(k)}_{x^+}$ irreducible. Then eqn.(8.12) says that $d\tau\big|_{(k)} \neq 0$ *only if the corresponding factor space* $X^{(k)}_{x^+}$ *is Kähler and, in this case,* τ *is (locally) a holomorphic function and* ψ_{ab} *a Kähler form.* Restricted to the quotient space, Z_{x^+}, the situation is very much the same as in the previous (static) case. At this stage, the dependence of τ on x^+ is not specified; but, of course, it is dictated by the components of the Einstein equations of the form $R_{++} = \cdots$ and $R_{+a} = \cdots$ that we have still to enforce. The x^+ dependence is also restricted by a subtle geometric requirement: In general, the complex structure of the Kähler space $X^{(k_0)}_{x^+}$ will depend on x^+; the explicit dependence on x^+ should be clever enough to maintain $\tau\big|_{x^+}$ holomorphic in each distinct holomorphic structure. The general solution then looks as a combination of the static elliptic ones and of the elliptic *pp*–waves.

Notice that we loose nothing by setting $\mu_a = 0$, as we shall do from now on.

It is convenient to write the integrability condition for the parallel spinor ϵ on N in the original form, eqn.(1.7),

$$\big(R_{\mu\nu ab}\,\gamma^{ab} + 2i\,Q_{\mu\nu}\big)\epsilon = 0, \tag{8.14}$$

so

$$\begin{aligned} 0 &= \bar\epsilon\gamma_\rho\big(R_{\mu\nu ab}\,\gamma^{ab} + 2i\,Q_{\mu\nu}\big)\epsilon = \\ &= R_{\mu\nu}{}^{\sigma\tau}\,\beta_{\rho\sigma\tau} + 2R_{\mu\nu\rho}{}^{\sigma}\,\eta_\sigma + 2i\,Q_{\mu\nu}\,\eta_\rho = \\ &= R_{\mu\nu}{}^{\sigma\tau}\,\beta_{\rho\sigma\tau} + 2i\,Q_{\mu\nu}\,\eta_\rho \quad \text{(since } \eta \text{ is parallel)} \end{aligned} \tag{8.15}$$

With $\mu^a = 0$, this equation reads explicitly (in the frame of eqn.(8.10))

$$R_{\mu\nu}{}^{ab}\,\psi_{ab} + 2i\,Q_{\mu\nu} = 0, \tag{8.16}$$

which, restricted on Z_{x^+}, just says that the $\mathfrak{u}(1)_R$ connection on the line bundle $\mathcal{L}^{-1}$ is the same as the

$$\mathfrak{u}(1) \subset \mathfrak{u}(m) \subset \mathfrak{g} \subset \mathfrak{g} \ltimes \mathbb{R}^{(n-2)} \tag{8.17}$$

projection of the Levi Civita connection for the Brinkmann space N having a Kähler m–flag. Thus, the elliptic fibration $\mathcal{E}\to N$, when restricted to Z_{x^+} becomes holomorphic and $\mathcal{E}\big|_{x^+}/\mathbb{R}$ is a 'Calabi–Yau' (a Kählerian space with trivial canonical bundle). Working on each quotient manifold Z_{x^+}, we can repeat word–for–word the analysis of the stationary case, concluding that one factor (at most[36]) is strictly Kählerian, while all the others (if any) are Ricci–flat with special holonomy. Then, over Z_{x^+}, the F–theory elliptic fibration defines a Calabi–Yau space of equation

$$Y^2 = X^3 + A_{x^+}(y_{x^+})X + B_{x^+}(y_{x^+}) \tag{8.18}$$

where y_{x^+} are holomorphic coordinates in the complex structure at x^+. By pull–back one gets an elliptic fibration on L_{x^+} which smootly glue in an elliptic fibration over N, with the fibre elliptic curve depending non–trivial on the light–like coordinate x^+.

One would expect that the resulting 12–fold $\mathcal{Y}_{12} \to N$ is a Brinkmann space with a $(\mathfrak{su}(m+1) \oplus \mathfrak{s})$–flag (where $\mathfrak{s}$ is a direct sum of $\mathfrak{su}(k)$, $\mathfrak{sp}(2l)$, $\mathfrak{g}_2$ and $\mathfrak{spin}(7)$ Lie algebras), but to extablish this would require the Brinkmann space version of the Calabi–Yau theorem:

QUESTION 8.1. *Let $CY_\lambda \to K_\lambda$ ($\lambda \in \mathbb{R}$) be a smooth family of elliptic Calabi–Yau spaces with section σ_λ, and g_λ a smooth family of Calabi–Yau metrics on CY_λ such that the family of Kähler metrics $\sigma_\lambda^* g_\lambda$ on K_λ is* integrable *in the sense of section 3. Then, **is the original Calabi–Yau family integrable?***

However we must recall that the metric along the fiber has no physical meaning in F–theory, and we usually introduce a metric just as a technical regularization of our computations, taking the limit of zero fiber metric in the final answer. Thus the true physical question is

QUESTION 8.2 (The physical relevant one). *Assume we have a* SUSY *configuration of F–theory with (N, g) a Brinkmann space with a Kähler m–flag and an elliptic fibration $\mathcal{N} \to N$ as described above. Can we construct a Brinkmann metric $\tilde{g}$ on $\mathcal{N}$ with the properties:*

(1) *it induces the original metric g on the section of the fibration;*
(2) *the fibers have volume ϵ;*
(3) *as $\epsilon \to 0$, the metric $\tilde{g}$ flows to a metric with holonomy*

$$\mathfrak{hol}(\tilde{g}) \subseteq \big(\mathfrak{su}(m) \oplus \mathfrak{s}\big) \ltimes \mathbb{R}^{n-2}$$

that is, to a Brinkmann metric with a Calabi–Yau flag?

There is a very strong physical argument in favor of the answer *yes* to QUESTION 8.2, namely the F–theory/M–theory duality that we shall discuss in section 10 below. There we shall show that the answer is indeed YES.

We summarize the results:

[36] If no factor space in Z_{x^+} is strictly Kähler, then the elliptic fibration would depend only on x^+ and we get back the special case of the *elliptic pp–waves* already discussed in sect...

FACT 8.2. *In the absence of flux, a* time–dependent *F–theory* BPS configuration *has the following structure:*

(A) *The universal cover* $\tilde{M}$ *of its* $10D$ *gravitational manifold* M *belongs to one the two types:*

(1) $\tilde{M} = N \times \mathbb{R}^k \times K \times Y$ *where*
   (a) N *is a simply–connected indecomposable Brinkmann–Leister space;*
   (b) K *is an irreducible simply–connected strictly Kähler manifold;*
   (c) Y *is a compact simply–connected Ricci–flat manifold of special holonomy.*

(2) $\tilde{M} = N \times \mathbb{R}^k \times Y$ *where*
   (a) N *is a simply–connected indecomposable Brinkmann* n*–fold, of type (2) or (4), having a* $\mathfrak{g}$*–flag with*

$$\mathfrak{g} = \mathfrak{u}(m) \oplus \mathfrak{s} \subset \mathfrak{so}(n-2) \tag{8.19}$$

$$\mathfrak{s} = \left[\begin{array}{l} \text{a direct sum of } \mathfrak{su}(k),\ \mathfrak{sp}(2l), \\ \mathfrak{g}_2 \text{ and } \mathfrak{spin}(7) \text{ Lie algebras.} \end{array} \right. \tag{8.20}$$

   (b) Y *is a compact simply–connected Ricci–flat manifold of special holonomy.*

(B) *In case (1), the F–theory elliptic* 12*–fold,* $\mathcal{Y}_{12}$ *has the structure*

$$\mathcal{Y}_{12} = N \times \mathbb{R}^k \times \mathcal{E} \times Y, \tag{8.21}$$

where $\mathcal{E}$ *is an elliptic Calabi–Yau, elliptically fibered over the Kähler base* K *(with section). The fibration is holomorphic, and* $c_1(\mathcal{L}) = c_1(K)$*.*

(C) *In case (2), the the F–theory elliptic* 12*–fold,* $\mathcal{Y}_{12}$ *has the structure*

$$\mathcal{Y}_{12} = \mathcal{N} \times \mathbb{R}^k \times Y, \tag{8.22}$$

where $\mathcal{N}$ *is an elliptic Brinkamnn space elliptically fibered over the Brinkmann space* N *(with section). The parallel form* η *of* $\mathcal{N}$ *is the pull back of the one for* N*, and the codimension one foliation* $\mathcal{F}_{\mathcal{N}}$ *of* $\mathcal{N}$ *is the pull–back of the one in* N*. The elliptic fibration is equivariant under* $\pounds_\xi$*. Then there exists an induced elliptic fibration (with section) at the level of the quotient manifolds which takes the form*

$$\mathcal{L}_{x^+}/\mathbb{R} \simeq \mathcal{E}_{x^+} \times Y_{x^+} \to L_{x^+}/\mathbb{R} \simeq K^m_{x^+} \times Y_{x^+}, \tag{8.23}$$

$$\begin{aligned} \text{where: } & \mathcal{E}_{x^+} \text{ is a strict Calabi–Yau } (m+1)\text{–fold,} \\ & K^m_{x^+} \text{ is a Kähler } m\text{–fold,} \\ & Y_{x^+} \text{ is a Ricci–flat manifold of special holonomy} \end{aligned} \tag{8.24}$$

and

$$\mathcal{E}_{x^+} \to K^m_{x^+} \quad \left[\begin{array}{l} \textit{is a holomorphic elliptic fibration} \\ \textit{(with section) over the Kähler base } K^m_{x^+}. \end{array} \right. \tag{8.25}$$

The corresponding Weierstrass hypersurface has the form

$$Y^2 = X^3 + A(x^+, y)\, X + B(x^+, y) \tag{8.26}$$

with $A(x^+, y)\big|_{x^+}$ *and* $B(x^+, y)\big|_{x^+}$, *holomorphic sections, respectively, of* $\mathcal{K}^{-4}_{K^m_{x^+}}$ *and* $\mathcal{K}^{-6}_{K^m_{x^+}}$.

Physically, this result means that in a supersymmetric configuration either all seven branes are at rest in the same Poincaré frame (the definition of which requires a Poincaré symmetry to be present!) or they move all together at the speed of light (which requires a parallel null vector). The solutions of the second kind are — heuristically at least — the limit of those of the first kind for infinite boost.

9. Compactifications of M–theory

We consider M–theory compactified down to $d = 2l-1$ dimensions, that is, M–theory defined on the manifold $\mathbb{R}^{1,2(l-1)} \times X_{2(6-l)}$, with $X_{2(l-6)}$ compact.

In order for the given M–theory configuration to be dual to an F–theory one, we must require the internal manifold $X_{2(6-l)}$ to be *elliptically fibered* with section. To avoid any misunderstanding, we stress that this condition is only required for the duality with F–theory, and it is not needed from the M–theory standpoint.

9.1. Ricci–flat compactifications of M–theory.

As in F–theory case, for the moment we consider the *zero flux* configurations, that is, the 4–form field strenght $F_4 = dC_3$ is set to zero. In this case, the $11D$ equations of motion reduce to $R_{MN} = 0$, and the Riemannian manifold $X_{2(6-l)}$ is *Ricci–flat.*

As in sect. 2.2, by the Bochner and Cheeger–Gromoll theorems, the Ricci–flatness condition implies that the universal cover of $X_{2(6-l)}$ is isometric to $\mathbb{R}^{b_1} \times X'$, where X' is a *compact simply–connected* manifold. Hence, going to a *finite* covering (if necessary), we may assume, without loss of generality, $X_{2(6-l)}$ to be *simply–connected,* provided we also allow some of the remaining $2l-1$ flat coordinates to be (possibly) compactified on a torus T^r. Again, by de Rham's theorem, $X_{2(6-l)}$ is the direct product of compact, simply–connected, irreducible, Ricci–flat manifolds Y_{n_i}. The list of the possible Ricci–flat holonomy groups is given in TABLE 2.2.

9.2. Supersymmetric compactifications. We are especially interested in M–theory backgrounds preserving some supersymmetries.

If the background flux F_4 and the fermions are set to zero, the condition of SUSY invariance reduces to requiring the corresponding spinorial parameter ϵ to be *parallel*,

$$\delta\psi_M = D_M\epsilon = 0. \tag{9.1}$$

From Wang's theorem (see, say, [GSSFT] THEOREM **3.**5.1), we know that, in a simply–connected irreducible manifold Riemannian X, the number $N_\pm$ of parallel spinors having chirality[37] ± 1 is related to the holonomy group $\mathrm{Hol}(X)$ as in the last two columns of TABLE 2.2.

However, we are not interested in *any* supersymmetric M–theory configuration; we are interested in supersymmetric M–theory compactifications which are *dual* to F–theory supersymmetric compactifications to *one more* dimension $d+1$. This requires the internal manifold X to be elliptically fibered. Morever, as discussed in section .... we have the requirement that the supercharges make full representations of the unbroken Poincaré group, and this requires $N_+(X) + N_(X) =$ *even.* As already discussed in §. this means that X has the form

$$X = CY_m \times Y, \tag{9.2}$$

where Y is Ricci–flat with special holonomy and CY_m is an elliptic Calabi–Yau m–fold, elliptically fibered (with section) over a Kähler $(m-1)$–fold K_{m-1}.

The compactification of M–theory to d dimension Minkowski space on the manifold $CY_m \times Y$ is a *bona fide* M–theory vacuum with $\mathcal{N} = 2\big(N_+(X) + N_(X)\big)$. We wish to show that *is it dual* to the compactification of F–theory to $d+1$ flat spacetime dimension on the non–Ricci–flat space $X \times K_{m-1}$ which has $\mathcal{N} = \big(N_+(X) + N_(X)\big)$ SUSY. The 12–dimensional space is then identified with $\mathbb{R}^{1,d} \times X \times CY_m$, and hence gets a 'geometrical reality' in the dual M–theoretic framework.

10. M–theory/F–theory duality

In comparing M–theory and F–theory on the 'same' elliptic Calabi–Yau CY_m we have to recall the fundamental physical difference in the role of this manifold on the two sides of the duality: the graviton propagates on the full manifold CY_m in the M–theory case, while it lives on a real codimension 2 'brane' in the F–theory. The fact that the metric does not propagate in the fiber directions, means that all distances are zero in that direction. Working with a singular metric with distinct points being at zero distance is not convenient[38], so —

[37] For a certain conventional orientation. With the opposite orientation one has $N_+ \leftrightarrow N_-$, of course.

[38] At least not convenient in the present context.

being pragmatic physicists — we shall introduce a 'regularized' metric of size ϵ along the fibers and take $\epsilon \to 0$ at the end.

That this regularization procedure is possible, follows from the Calabi–Yau theorem. Indeed, in §. 7.3 we learned from the Wirtinger theorem that the volume of the fiber E_k is equal to the cohomology invariant $\omega[E_k]$, where ω is the CY_m Kähler form. and the same for all fibers. Consider the following 'regularized' Kähler form

$$\omega_\epsilon \equiv \pi^* \sigma^* \omega + \epsilon\, \omega \qquad \epsilon > 0. \tag{10.1}$$

ω_ϵ obviously belongs to the Kähler cone $\mathcal{K}_{CY_m}$, and under this 'regularized' metric the fibers have volume $\epsilon\, \omega[E]$. Then the Calabi–Yau THEOREM 6.1 guarantees the existence of a (unique) Ricci–flat Kähler metric with a Kähler form cohomologous to ω_ϵ for all $\epsilon > 0$.

In fact, we already encountered a regularized Kähler metric of this form, compare the formula in footnote 22 on page 76 (rescaled by λ^{-1} and the large parameter λ set equal to ϵ^{-1}), namely

$$\omega_\epsilon = \pi^* \sigma^* \omega - i\, \epsilon\, \partial\overline{\partial} \left(\frac{(z-\bar{z})^2}{\operatorname{Im} \tau} \right). \tag{10.2}$$

Of course, ω_ϵ is *not* Ricci–flat: it is an easy corollary to the Bochner and de Rham's theorems that *a compact simply–connected Ricci–flat manifold has a finite isometry group,* while the metric (10.2) has two continuous isometries corresponding to $z \to z + \lambda_1 + \lambda_2 \tau$. However, I

CLAIM 10.1. *(1) The Kähler form ω_ϵ in eqn.*(10.2) *is Ricci–flat to the* leading *order in $\epsilon \to 0$.*
(2) In the same limit, the answer to QUESTION *8.2 is YES.*

Since, physically, $\epsilon = 0$, this leading order result is all we need.

PROOF. **(1)** A Kähler form $\tilde{\omega}$ on a compact complex manifold X with $c_1(X) = 0$ corresponds to a Calabi–Yau (*i.e* Ricci–flat) *metric iff* it satisfies the complex Monge–Amperé equation that we write in the form (see [**51**][**52**])

$$(\tilde{\omega})^m = m!\, \tilde{A}\, (-1)^{m(m-1)/2}\, i^m\, \theta \wedge \overline{\theta} \tag{10.3}$$

where θ is a $(m, 0)$ holomorphic form (unique up to normalization) and $\tilde{A}$ is a real *constant* which measures the relative normalization of the volume forms $\tilde{\omega}^m/m!$ and $(-1)^{m(m-1)/2}\, i^m \theta \wedge \overline{\theta}$. Then the CLAIM (1) is true iff

$$(\omega_\epsilon)^m = \epsilon\, A'\, \theta \wedge \overline{\theta} + O(\epsilon^2) \tag{10.4}$$

for some constant A'. The LHS is equal to

$$\epsilon\, C\, \det[g_{\alpha\bar{\beta}}]\, (\operatorname{Im} \tau)^{-1} \theta \wedge \overline{\theta} + O(\epsilon^2), \tag{10.5}$$

where $g_{\alpha\bar{\beta}}$ is the Kähler metric of the base K_m and C is a combinatoric constant. Thus, to leading order in ϵ, the Monge–Amperé equation is satisfied iff

$$\frac{\det(g_{\alpha\bar{\beta}})}{\operatorname{Im}\tau} = \text{const} \tag{10.6}$$

but this is precisely the condition that the $\mathfrak{u}(1)_R$ curvature is equal to the Ricci form, as implied by the integrability of condition for the parallel spinor (or by the Einstein equations). Thus, the Kähler form (10.2) is a solution to the to the Monge–Amperé equation up to $O(\epsilon^2)$. By uniqueness, in the zero fiber volume limit, all solutions should be equivalent to this one.

(2) (Special case $\mathfrak{s} = 0$). Assume $\mathcal{N}$ be a Brinkmann space with a Kähler flag such that the quotient manifolds satisfy $c_1(Z) = 0$.

A Brinkmann space with a Kähler flag is, in particular, a symplectic manifold. Let Ω be symplectic form, and let Υ be the $(2m+1)$–form closed form on $\mathcal{N}$ corresponding to $\theta \wedge \bar{\theta}$ on Z. Then the flag of $\mathcal{N}$ is actually Calabi–Yau iff

$$\Omega^{m+1} = A' E^- \wedge \Upsilon. \tag{10.7}$$

To leading order in ϵ, we get the same condition as before. □

In the limit $\epsilon \to 0$, the elliptic manifold $\mathbb{R}^{d-1} \times Y_{12-d}$ locally (away from the singular fibers) looks like a 2–torus (with a slowly–varying complex modulus $\tau(x)$) times a 9–dimensional space flat space. In this situation we may invoke the adiabatic argument to perform the usual flat–space dualities *fiber–wise*. The 2–torus is $S^1 \times S^1$, and we take one of the two to be the M–theory circle. In the small radius limit we get weakly coupled Type IIA. Performing T–duality on the second vanishing circle, we get Type IIB in the *decompactification limit* with a space–time depending axion/dilaton $\tau(x)$.

To be concrete, we have M–theory on an elliptic 11–fold $Z_{11} \to B_9$ with the metric corresponding to our ω_ϵ Kähler form. We use the notation: $z = x + \tau y$, with x, y real coordinates periodic of fixed period 1, and $\tau(b) = \tau_1 + i\tau_2$. We note the 'magic' chain of identities:

$$\begin{aligned}
&-\frac{1}{2}\partial\bar{\partial}\frac{(z-\bar{z})^2}{\tau-\bar{\tau}} = \\
&= \frac{dz\wedge d\bar{z}}{\tau-\bar{\tau}} - \frac{z-\bar{z}}{(\tau-\bar{\tau})^2}\big(d\tau\wedge d\bar{z} + dz\wedge d\bar{\tau}\big) + \frac{(z-\bar{z})^2}{(\tau-\bar{\tau})^3}d\tau\wedge d\bar{\tau} \\
&= \frac{1}{\tau-\bar{\tau}}\big(dz\wedge d\bar{z} - y\,d\tau\wedge d\bar{z} - y\,dz\wedge d\bar{\tau} + y^2 d\tau\wedge d\bar{\tau}\big) \\
&= \frac{(dz - y\,d\tau)\wedge(d\bar{z} - y\,d\bar{\tau})}{\tau-\bar{\tau}} \\
&= \frac{(dx + \tau\,dy)\wedge(dx + \bar{\tau}\,dy)}{\tau-\bar{\tau}}
\end{aligned}$$

so that the M–theory metric simplifies drastically to

$$ds^2_M = ds^2_9 + \frac{\epsilon}{\tau_2}\Big((dx+\tau_1 dy)^2 + \tau_2^2\, dy^2\Big) + O(\epsilon^2). \tag{10.8}$$

The above chain of identities are, of course, Wirtinger theorem in operation. Notice that we got exactly the same metric that is used in heuristic treatments of the duality [**53**] but now we know that the metric is exactly Kähler and Calabi–Yau to leading order in ϵ.

The general relation between the M–theory and Type IIA metric is [**54**]

$$ds^2_M = e^{-2\chi/3}\, ds^2_{IIA} + L^2\, e^{4\chi/3}\,(dx + C_1)^2, \tag{10.9}$$

which is supplemented by the lenght relation $l_s = (l_M^3/L)^{1/2}$ and the string coupling formula $g_{IIA} = e^{\chi}(l_M/l_s)^3$. L is a conventional lenght scale which may be fixed to any convenient value by the redefinition $\chi \to \chi + \text{const.}$.

Eqn.(10.9) gives[39]

$$ds^2_{IIA} = \left(\frac{\epsilon}{L^2\,\tau_2}\right)^{1/2}\Big(\epsilon\tau_2\, dy^2 + ds^2_9\Big), \qquad e^{4\chi/3} = \frac{\epsilon}{L^2\,\tau_2} \tag{10.10}$$

$$g_{IIA} = \left(\frac{\epsilon}{l_M^2\,\tau_2}\right)^{3/4}, \qquad C_1 = \tau_1\, dy. \tag{10.11}$$

Type IIA is compactified on the y–circle of length

$$R_{IIA} = (\epsilon^3 \tau_2/L^2)^{1/4}. \tag{10.12}$$

Now perform a T–duality *fiber–wise* along the circle parameterized by y to get a Type IIB configuration 'compactified' on a circle of lenght $R_{IIB} = l_s^2/R_{IIA} = O(\epsilon^{-3/4}) \to \infty$. The R–R Type IIB axion is given by

$$C_0 = (C_1)_y = \tau_1, \tag{10.13}$$

while the string coupling

$$g_{IIB} = \frac{l_s}{R_{IIA}}\, g_{IIA} = \frac{l_s L^{1/2}}{l_M^{3/2}}\,\frac{1}{\tau_2} \equiv \frac{1}{\tau_2}, \tag{10.14}$$

so that, as physically expected

$$C_0 + \frac{i}{g_{IIB}} = \tau. \tag{10.15}$$

The *string frame* dual Type IIB metric is

$$ds^2\Big|_{\text{string frame}} = \left(\frac{\epsilon}{\tau_2\, L^2}\right)^{1/2}\left(ds^2_9 + \frac{l_s^4 L^2}{\epsilon^2} d\tilde{y}^2\right). \tag{10.16}$$

[39] C_1 stands for the Type IIA R–R 1–form field.

To get the corresponding *Einstein frame* metric we have to multiply by $g_{IIB}^{-1/2}$. Then fixing our conventional scale $L^2 = \epsilon$, and rescaling the coordinates as $\tilde{y} \to y \equiv l_s^2 \tilde{y}/\sqrt{\epsilon}$ we get

$$ds^2\Big|_{\text{Einstein frame}} = ds_9^2 + dy^2 \tag{10.17}$$

where now the real flat coordinate y is periodic with period $O(1/\epsilon^{1/2})$, and hence becomes uncompactified in the limit.

This shows that a M–theory vacuum on $\mathbb{R}^{1,2k} \times CY_{5-k}$, with $CY_{5-k} \to K_{4-k}$ an *elliptic Calabi–Yau* (with section) is *dual* to an F–theory vacuum with gravitational $10D$ manifold $\mathbb{R}^{1,2k+1} \times K_{4-k}$ and $\tau = C_0 + i/g$ equal to the period τ of the corresponding elliptic fiber (modulo $SL(2,\mathbb{Z})$ transformations).

Notice that nothing change in the argument if we consider *time–dependent* BPS configuration. If $\mathcal{N}$ is a Brinkmann space[40] with a Calabi–Yau flag, elliptically fibered (with section) over the Brinkmann space N with a Kähler flag:

$$\boxed{M\text{–theory on } \mathcal{N} \quad \xrightarrow{\text{duality}} \quad F\text{–theory on } N \times \mathbb{R}}$$

11. Adding fluxes: General geometry

It is time to add fluxes to the game.

The theory of BPS configurations with generic fluxes is based on the same principles we used above for the fluxless case, but the corresponding geometrical theorems are less powerful (and less elementary).

For the special case of SUSY vacua ($\equiv$ Poincaré invariant compactifications) there are strong *no–go theorems* (that we review in §.) which severely restrict the possibilities. The generic BPS configuration is, of course, rather complicated since it should describe a lot of different BPS objects that exist in the theory.

In this section we discuss the general geometry of the SUSY configurations. For notational simplicity, we shall work in M–theory, but, of course, the geometrical methods are quite general, and can be extended straighforwardly to F–theory, directly of trough the duality with M–theory we discussed in section 10.

[40] Not necessarily indecomposable!!

11.1. General principles. From the no flux case we learned some general physical lessons which apply in full generality. We recall them:

General lesson 11.1. *If the configuration has a non–zero Killing spinor ϵ (namely, a* SUSY *spinorial parameter ϵ which leaves the field configuration invariant, $\delta\psi_\mu = 0$, $\delta\lambda = 0$), then $K_\mu = \bar{\epsilon}\gamma_\mu\epsilon$ is a Killing vector which vanishes nowhere*[41]*; K_μ is either time–like or light–like. All the fields Φ^A are invariant under K, $\pounds_K \Phi^A = 0$.*

This General lesson is just the statement that the anticommutator of two supersymmetries should be a physical symmetry which acts on the metric by an isometry. Since K_μ never vanishes, its action is free, and we may consider the *quotient manifold* as we did in the previous sections.

In fact, in the no–flux case we also found that higher–degree form bilinears in ϵ, of the form[42] should not vanish. From the properties of the Clifford algebra in $\mathbb{R}^{1,10}$ we get General lesson in more detail, assuming we have N linear indipendent Killing spinors ϵ^i, $i = 1, 2, \ldots, N$) which are $11D$ Majorana spinors.

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[41] The assertion that K_μ vanishes nowhere is geometrically trivial in the no–flux case; it requires some work in the general case. We prove the claim using the *continuity method*: let $\mathcal{Z} \subset M$ the locus in which $K_\mu = 0$. Since K_μ is smooth, $\mathcal{Z}$ is closed in M. If we can show that it is also *open*, then (since M is assumed to be *connected*!) $\mathcal{Z}$ must be either the full space M or empty. In the first case $\epsilon = 0$ everywhere, and we have no SUSY. Thus $\mathcal{Z} = \emptyset$. Let us show that $\mathcal{Z}$ is *open*. If $\mathcal{Z}$ is empty, there is nothing to show, so we may assume there is a $p \in \mathcal{Z}$. Consider all time–like and null geodesics $\gamma(\tau)$ passing trough p. Then $\dot{\gamma}^\mu K_\mu = 0$ in p. But

$$\frac{d}{d\tau}(\dot{\gamma}^\nu K_\nu) = \dot{\gamma}^\mu D_\mu(\dot{\gamma}^\nu K_\nu) = \dot{\gamma}^\mu \dot{\gamma}^\nu D_\mu K_\nu = 0 \quad \text{(by the Killing eqn.)}$$

so $\dot{\gamma}^\mu K_\mu = 0$ along the geodesic. But K_μ is time–like or null, and so is $\dot{\gamma}_\mu$; their product may vanish only if $K_\mu = 0$. We conclude that the interiors of the past and future light cones of p are in $\mathcal{Z}$. Consider a point p' in the future of p along a time–like geodesics (close enough to p so that the exponential map is injective). The interior of the past light–cone of p' belongs to $\mathcal{Z}$ and contains a neighborhood of p. Hence $\mathcal{Z}$ is *open*.

[42] In this section, the bilinears are meant to be written in $D = 11$ Majorana conventions, so $\bar{\epsilon}\gamma_{\mu_1\cdots\mu_k}\epsilon$ means $\epsilon^T C \gamma_{\mu_1\cdots\mu_k}\epsilon$, with C the charge–conjugation matrix. In particular,

$$\bar{\epsilon}^i \gamma_{\mu_1\cdot\mu_k}\epsilon^j = \mp(-1)^{(k-1)(k-2)/2}\, \bar{\epsilon}^j \gamma_{\mu_1\cdot\mu_k}\epsilon^i \qquad \begin{array}{c} - \text{ anticommuting} \\ + \text{ commuting.} \end{array}$$

General lesson 11.2. *(1) If the configuration has N Killing spinors, the various bilinear forms have the following properties:*

$X^{ij} = \bar{\epsilon}^i \epsilon^j$	*antisymmetric*	
$K^{ij}_\mu = \bar{\epsilon}^i \Gamma_\mu \epsilon^j$	*symmetric*	*Killing vector,* K^{ii} *time–like or null*
$\Omega^{ij}_{\mu_1\mu_2} = \bar{\epsilon}^i \Gamma_{\mu_1\mu_2} \epsilon^j$	*symmetric*	Ω^{ii} *never vanishing*
$Y^{ij}_{\mu_1\mu_2\mu_3} = \bar{\epsilon}^i \Gamma_{\mu_1\mu_2\mu_3} \epsilon^j$	*antisymmetric*	
$Z^{ij}_{\mu_1\mu_2\mu_3\mu_4} = \bar{\epsilon}^i \Gamma_{\mu_1\mu_2\mu_3\mu_4} \epsilon^j$	*antisymmetric*	
$\Sigma^{ij}_{\mu_1\mu_2\mu_3\mu_4\mu_5} = \bar{\epsilon}^i \Gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \epsilon^j$	*symmetric*	Σ^{ii} *never vanishing*

where (anti)symmetry refers to $i \leftrightarrow j$*.*

(2) All the above forms, Ψ^{ij}*, as well as the* 4*–form field strenght* $F_4 = dC_3$*, are invariant under the isometries generated by the* K^{ij}*,* $\mathcal{L}_{K^{ij}} \Psi^{kl} = 0$*,* $\mathcal{L}_{K^{ij}} F_4 = 0$*.*

For simplicity, we focus on just one Killing spinor ϵ, and suppress the extension indices i, j. They can be restored whenever needed.

From the M–theory SUSY transformation (in the SUGRA approximation)

$$\delta\psi_\mu = D_\mu \epsilon - \frac{1}{288}\, F_{\nu\rho\sigma\tau} \Big(\Gamma_\mu{}^{\nu\rho\sigma\tau} - 8\, \delta_\mu{}^\nu\, \Gamma^{\rho\sigma\tau}\Big)\epsilon, \tag{11.1}$$

we get

$$D_\mu K_\nu = \frac{1}{6} \Omega^{\sigma_1\sigma_2}\, F_{\sigma_1\sigma_2\mu\nu} + \frac{1}{6!} \Sigma^{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5} (*F)_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\mu\nu}, \tag{11.2}$$

form which the Killing equation $D_\mu K_\nu = -D_\nu K_\mu$ is evident. As a matter of notation, we write $\kappa = K_\mu dx^\mu$ for the form, and $K = K^\mu \partial_\mu$ for the corresponding vector.

11.2. Geometry of the forms κ, Ω and Σ. The above forms satisfy a number of relations which expresses the algebraic consistence of the underlying supersymmetry; they may be deduced using Fierz identities [**55**]. They can also inferred by the geometric wisdom we gained from the zero–flux case.

Consider $i_K\Omega$: It is a one–form defined by the spinor ϵ, hence it should be proportional to κ. But $i_K^2 \Omega \equiv 0$, and hence we have a contradiction (in the time–like case, but the formulae should be the same in the null case) unless

$$i_K \Omega = 0. \tag{11.3}$$

Then take the symmetric tensor $\Omega_{\mu\rho} g^{\rho\sigma} \Omega_{\sigma\nu}$. Its eigenvectors are preferred directions defined by ϵ alone. But ϵ defines just one vector, namely K^μ, then we must have

$$\Omega_{\mu\rho}\, g^{\rho\sigma}\, \Omega_{\sigma\nu} = \lambda(g_{\mu\nu}\, K^2 - K_\mu\, K_\mu) \tag{11.4}$$

where we used (11.3). Fierz identities give $\lambda = 1$ [**55**]. The same argument implies that $i_K\Sigma$ must be proportional to $\Omega \wedge \Omega$. A direct computation gives [**55**]

$$i_K\Sigma = \frac{1}{2}\,\Omega \wedge \Omega. \tag{11.5}$$

Again,

$$\frac{1}{4!}\Sigma_{\mu\sigma_1\sigma_2\sigma_3\sigma_4}\, g^{\sigma_1\rho_1} g^{\sigma_2\rho_2} g^{\sigma_3\rho_3} g^{\sigma_4\rho_4} \Sigma_{\rho_1\rho_2\rho_3\rho_4\nu} = 14\, K_\mu\, K_\nu - 4 g_{\mu\nu}\, K^2, \tag{11.6}$$

where only the overall relative normalization needs to be checked.

From eqn.(11.3) and $\pounds_K\Omega = 0$ we get

$$i_K d\Omega = 0 \quad \Rightarrow \quad d\Omega = i_K\Lambda_4, \tag{11.7}$$

for some 4–form Λ_4. Since the background is $\pounds_K$–invariant, from $\pounds_K\Lambda_4 = 0$ we infer

$$d\Lambda_4 = i_K\Lambda_6 \tag{11.8}$$

for some 6–form Λ_6.

In the same vein, from eqns.(11.5)(11.7) we get

$$0 = \pounds_K\Sigma = (di_K + i_k d)\Sigma = \Omega \wedge d\Omega + i_k d\Sigma = \tag{11.9}$$

$$= \Omega \wedge i_K\Lambda_4 + i_k d\Sigma = i_K\Big(d\Sigma + i_K(\Omega \wedge \Lambda_4)\Big) \tag{11.10}$$

$$\Rightarrow \qquad d\Sigma = i_K\Lambda_7 - \Omega \wedge \Lambda_4. \tag{11.11}$$

for some 7–form Λ_7. Now,

$$\begin{aligned} i_K d\Lambda_7 &= -di_K\Lambda_7 = \\ &= -d(\Omega \wedge \Lambda_4) = -(i_K\Lambda_4) \wedge \Lambda_4 - \Omega \wedge i_K\Lambda_6 = \\ &= -i_K\Big(\frac{1}{2}\,\Lambda_4 \wedge \Lambda_4 + \Omega \wedge \Lambda_6\Big) \\ &\Rightarrow \qquad i_K\Big(d\Lambda_7 + \frac{1}{2}\Lambda_4 \wedge \Lambda_4 + \Omega \wedge \Lambda_6\Big) = 0. \end{aligned} \tag{11.12}$$

Now, what is Λ_4? In the game we have only two 4–forms namely $\Omega \wedge \Omega$ and $F_4 = dC_3$, and Λ_4 should be a linear combination of them. However $i_K(\Omega \wedge \Omega) = 0$ so $d\Omega$ should be proportional to $i_K F_4$. Then $\Lambda_6 = 0$.

Then eqn.(11.12), with the identification $\Lambda_7 = *F_4$ becomes the component of the equation of motion for the C_3 field[43]. obtained by

[43] The C_3 equation of motion has a higher curvature correction $-\beta X_8$ due to the mechanism to cancel the anomalies. Then we must have $i_K X_8 = 0$. See discussion in ....

contraction with the Killing vector K. Thus, we learn that

$$i_K\Omega = 0 \tag{11.13}$$

$$i_K\Sigma = \frac{1}{2}\Omega \wedge \Omega \tag{11.14}$$

$$d\Omega = i_K F_4 \tag{11.15}$$

$$d\Sigma = i_K * F_4 - \Omega \wedge F_4. \tag{11.16}$$

To understand the geometric meaning of these relations, we introduce the G–structures.

11.3. *G*–structures. (See [GSSFT] and references therein for further details).

11.3.1. *Definitions.* Let M be a smooth n–fold and $L(M)$ the bundle of linear frames over M. $L(M)$ is a principal fibre bundle with group $GL(n,\mathbb{R})$. By a *G–structure* we mean a differential subbundle P of $L(M)$ with structure group G (which we take to be a *closed* subgroup of $GL(n,\mathbb{R})$).

Since $GL(n,\mathbb{R})$ acts on $L(M)$ on the right, the subgroup G also acts on the right. More or less by construction, the G–structures of M are in one–to–one correspondence with the sections of the quotient bundle $L(M)/G$.

A G–structure P is said to be integrable if there exist local coordinates such that $(\partial_{x^1}, \partial_{x^2}, \cdots \partial_{x^n})$ is (locally) a section of P.

Proposition 11.1 (Kobayashi [**56**]). *Let $\boldsymbol{K}$ be a tensor in the vecor space $\mathbb{R}^n$ and G the group of linear transformations of $\mathbb{R}^n$ leaving $\boldsymbol{K}$ invariant. Let P a G–structure on M and K the tensor field on M defined[44] by $\boldsymbol{K}$ and P. Then*

1. *P is integrable iff there are local coordinates in which the components of K are constant;*
2. *a diffeomorphism $f\colon M \to M$ is an automorphism of P if and only if f leaves K invariant;*
3. *a vector field X is an infinitesimal automorphism of P iff $\mathcal{L}_X K = 0$.*

Remark. Of course, we can generalize the statement to the subgroup $G \subset GL(n,\mathbb{R})$ preserving a *set* of tensor fields $K^{(i)}$ $i = 1, 2, \ldots, L$.

Example. Many 'classical' structures on a manifold $\mathcal{M}$ can be described in terms of G–structures and associated tensors K. I give a *very* non–exaustive list:

[44] By this we mean the following: At each point $x \in M$ choose a frame in T_xM belonging to P. This sets an isomorphism $T_xM \to \mathbb{R}^n$ which extends to the tensor algebra. Let K_x be the image of $\boldsymbol{K}$ under this isomorphism. Since $\boldsymbol{K}$ is G–invariant, K_x is independent of the choices, and defines a tensor field on M.

i): an *orientation* of $\mathcal{M}$ is an $GL^+(n,\mathbb{R})$–structure. The tensor K is the volume form $\varepsilon_{i_1\cdots i_n}$;
ii): a (positive definite) *metric* is an $O(n)$–structure. The tensor K is the metric g_{ij}.
iii): an *almost complex structure* is an $GL(n/2,\mathbb{C})$–structure. The tensor K is the almost complex structure $I_i^{\ j}$;
iv): an *almost Hermitean structure* is an $U(n/2)$–structure;
v): an *almost symplectic structure* is an $Sp(n,\mathbb{R})$–structure. K is a 2–form Ω with $\Omega^{n/2}\neq 0$ everywhere.

Of course, we can combine the different structures. An $SO(n)$ structure is a $O(n)$ structure which is also a $GL^+(n,\mathbb{R})$ structure and the defining tensors are g and ε, with the compatibility condition that the volume form is equal to $\sqrt{g}\, d^nx$. *Ect. ect.*

Since our manifolds $\mathcal{M}$ are always oriented and metric, all our G's will be subgroups of $SO(10,1)$. In fact, they will be the subgroups of $SO(10,1)$ preserving a set of tensor fields $K^{(i)}$ as in PROPOSITION 11.1. In the set of defining tensors we always have the metric tensor g and the orientation ε.

11.3.2. *Intrinsic torsion of a G–structure.* The fact that we have a G–structure means that we can introduce an adapted (co)frame e^a, where a is the index of a suitable representation of G, and a $\mathfrak{g}$–valued connection $\omega^a{}_b$. Then the tensors defining the G–structure take the form

$$\kappa^{(i)}_{a_1a_2\cdots a_k} e^{a_1}e^{a_2}\cdots e^{a_k}, \tag{11.17}$$

with $\kappa^{(i)}_{a_1a_2\cdots a_k}$ G–invariant constant tensors.

The *torsion* of the G–structure is given by

$$\Theta^a = de^a + \omega^a{}_b\wedge e^b. \tag{11.18}$$

The G–structure is said to be *torsion–less* if the torsion vanishes. In some sense, a G–structure is 'natural' precisely if it is torsionless.

EXAMPLE. *i)* a *torsion–less* $GL(n/2,\mathbb{C})$–structure is a *complex structure*, *ii)* a *torsion–less* $U(n/2)$–structure is a *Kähler metric, ii)* a *torsion–less* $Sp(n,\mathbb{R})$ structure is a *symplectic structure*, and *so on.*

The existence of a torsion–less connection for a given G–structure is a deep and quite hard problem, which was solved only recently by quite sophisticate techniques. The *fundamental theorem of differential geometry* states that for the $O(p,q)$–structures (namely metrics of (p,q) signature) there is a *unique* torsion–less connection, namely the Levi–Civita one.

Changing the G–connection, the torsion changes as follows

$$\tilde{\Theta}^a-\Theta^a=\big(\tilde{\omega}_c{}^a{}_b-\omega_c{}^a{}_b\big)e^c\wedge e^b, \qquad \big(\tilde{\omega}_c{}^a{}_b-\omega_c{}^a{}_b\big)\in\mathfrak{g}\otimes T^*\mathcal{M} \tag{11.19}$$

so we can find a new G–connection, $\tilde{\omega}^a{}_b$, which is torsion-less if and only if the the tensor $\Theta^a = \Theta^a{}_{bc} e^a \wedge e^b$ which *a priori* is just an element of $T\mathcal{M} \otimes \wedge^2 T^*\mathcal{M}$ actually *belongs to the subspace* $\mathfrak{g} \otimes T^*\mathcal{M}$. Thus the projection of the torsion Θ^a on the quotient vector space

$$T\mathcal{M} \otimes \wedge^2 T^*\mathcal{M} \Big/ \mathfrak{g} \otimes T^*\mathcal{M}$$

is independent of the choice of G–connection and an *obstruction* to finding a torsion–less G–connection. This projection, which depends only on the G–structure, is called the *intrinsic tension* of the G–structure. In [GSSFT] the intrinsic connection was characterized in terms of the *Spencer cohomology* of the G–structure.

A special instance is when our G–structure is, in particular, a $SO(p,q)$–structure (that is, G is a closed subgroup of $SO(p,q)$). In this case, if a torsion–less G–connection exists, it should coincide with the Levi–Civita one.

More specifically, we shall be interested in G–structures with G the closed subgroup of $SO(p,q)$ leaving invariant a set of *forms* (that is: totally antisymmetric tensors)

$$K^{(i)} = \kappa^{(i)}_{a_1 \cdots a_k}\, e^{a_1} \wedge \cdots \wedge e^{a_k}. \tag{11.20}$$

Then

$$d\big(\kappa^{(i)}_{a_1 \cdots a_k}\, e^{a_1} \wedge \cdots \wedge e^{a_k}\big) = k\, \kappa^{(i)}_{a_1 a_2 \cdots a_k}\, \Theta^{a_1} \wedge e^{a_2} \wedge \cdots \wedge e^{a_k} \tag{11.21}$$

where the RHS depends only on the intrinsic part of the torsion (the non–intrinsic part decouples by the G–invariance of $\kappa^{(i)}_{a_1 \cdots a_k}$). *Conversely,* we may reconstruct uniquely the intrinsic torsion from the expressions $dK^{(i)}$. Thus the amount of information contained in the intrinsic torsion of such a G–structure and in the exterior derivatives of the defining tensor $K^{(i)}$ is the same.

Then,

PROPOSITION 11.2. *Let G be the closed subgroup of $SO(p,q)$ leaving invariant the set of antisymmetric tensors $\boldsymbol{K}^{(i)}$ in $\mathbb{R}^{p,q}$.*

(1) *If such a G–structure has zero intrinsic torsion, then the Levi Civita connection has* holonomy G, *and the defining forms $K^{(i)}$ are* Levi Civita–parallel.

(2) *The* intrinsic torsion *is equal to minus the image of the Levi Civita connection under the natural quotient map*

$$\varrho\colon \mathfrak{so}(p,q) \otimes T^*\mathcal{M} \to \mathfrak{so}(p,q) \otimes T^*\mathcal{M} \Big/ \mathfrak{g} \otimes T^*\mathcal{M}. \tag{11.22}$$

PROOF. **(1)** By assumption, we can find a zero–torsion G–connection. Since $G \subset SO(p,q)$, this torsion–less connection is the (unique) Levi Civita one. The fact that the Levi Civita connection is a G–connection is the same as saying that it has holonomy G. Since the forms $K^{(i)}$ are invariant under the holonomy, they are parallel ([GSSFT] chapter **3**).

(2) Let $\varpi^a{}_b$ be the Levi Civita connection that we write as $\varpi^a{}_b = \omega^a{}_b + \varrho^a{}_b$ with $\omega^a{}_b \in \mathfrak{g} \otimes T^*\mathcal{M}$. Since the Levi Civita connection is torsion–less

$$\Theta^a = de^a + \omega^a{}_b \wedge e^b = de^a + \varpi^a{}_b \wedge e^b - \varrho^a{}_b \wedge e^b = -\varrho^a{}_b \wedge e^b. \quad (11.23)$$

□

11.4. Strategy. The geometric strategy in the flux case is as follows: we use the forms κ, Ω, and Σ (as well X, Y, and Z, if we have more than one Killing spinor) to define a G–structure of the kind discussed at the end of §. 11.3.2. *A priori,* the three forms κ, Ω and Σ define a G–structure with G some closed subgroup of $GL(11, \mathbb{R})$. However, eqns.(11.4)(11.6) imply that $G \subset CO(10, 1)$, that is, the forms κ, Ω and Σ define, in particular, *a conformal structure* which is compatible with the conformal structure defined by the metric.

Here we have two choices: Either we work with the conformal structures (which is probably the most intrinsic way of proceeding), or we redefine our G–structure in such a way of being compatible with the metric structure. The second path is the one followed in the physics literature. Reducing the conformal $CO(10, 1)$–structure to a metric $SO(10, 1)$–structure means specifying a conformal factor of the metric that will appear as a *warp factor* in the equation. So, in the metric language, we end up quite generally in warp products.

In order for our G–structure to be compatible with the metric one, the defining forms should be normalized to have constant norms with respect to the given metric structure (cfr. PROPOSITION 11.1). Then, to get geometric objects which define a G structure which is in particular a metric structure, we must redefine the meaning of our symbols as follows

$$\kappa \to \kappa\big|_{\text{new}} \equiv e^f\,\kappa\big|_{\text{old}} \quad (11.24)$$
$$\Omega \to \Omega\big|_{\text{new}} \equiv e^f\,\Omega\big|_{\text{old}} \quad (11.25)$$
$$\Sigma \to \Sigma\big|_{\text{new}} \equiv e^f\,\Sigma\big|_{\text{old}} \quad (11.26)$$

where $e^{-2f} = -K_\mu K^\mu$. The exterior derivatives of the three structure–defining forms are obtained from eqns.(11.2)(11.15)(11.16)

$$d\kappa = df \wedge \kappa + \frac{2}{3} i_\Omega F_4 + \frac{1}{3} i_\Sigma(*F_4) \quad (11.27)$$
$$d\Omega = df \wedge \Omega + i_\kappa F_4 \quad (11.28)$$
$$d\Sigma = df \wedge \Sigma + i_\kappa(*F_4) - \Omega \wedge F_4, \quad (11.29)$$

where for two forms α, β of degrees k and $l \geq k$ the symbol $i_\alpha\beta$ stands for the form

$$\frac{1}{k!}\,\alpha^{\rho_1\cdots\rho_k}\,\beta_{\rho_1\cdots\rho_k\mu_1\cdots\mu_{l-k}}.$$

We decompose both sides of each equation (11.27)–(11.29) in irreducible representations of G. The components of the flux transforming is certain representations of G will decouple from the equations (11.27)–(11.29) by symmetry reasons. The components which do not decouple then correspond precisely to the intrinsic torsion, by the argument discussed around eqn.(11.21). These components of the flux are then uniquely constructed out of the Levi Civita connection of M by PROPOSITION 11.2.**(2)**.

This method solves the condition for the existence of a supersymmetry in terms of a few free functions. These functions are then fixed by imposing the equations of motion.

11.5. Killing spinors in M–theory. If M is a spin–manifold, it is obvious that in PROPOSITION 11.1, the tensor $\mathbf{K}$ may be replaced by a spinor on $\mathbb{R}^n$. Then a non–zero spinor field, ϵ, will reduce the structure group of the manifold to its isotropy subgroup.

In the case of M–theory, this means a reduction from $Spin(10,1)$ to the isotropy group which is:

- $SU(5)$ if the associated Killing vector K is time–like;
- $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ if K is null.

The proof of this statement is given in ref.[**57**] using the octonionic realization of $Spin(10,1)$. Let us give a simpler motivation: the isotropy group is an algebraic fact which is indepedent of the flux (namely of the intrinsic torsion). Hence it has to coincide with the holonomy group of a zero–flux SUSY configuration with, respectively, $M = \mathbb{R}^{1,0} \times X_{10}$ and N, a Brinkmann–Leister 11–fold. From the explicit classification of sections ...., we know that in the first case we have $\mathfrak{hol}(X_{10}) \subseteq \mathfrak{su}(5)$, while in the second one $\mathfrak{hol}(N) = \mathfrak{g} \ltimes \mathbb{R}^9$ with $\mathfrak{g} \subseteq \mathfrak{spin}(7) \subset \mathfrak{so}(9)$.

Alternatively, we can consider the G–structure where G is the subgroup of $SO(10,1)$ which leaves invariant the three forms K_μ, $\Omega_{\mu_1\mu_2}$ and $\Sigma_{mu_1\mu_2\mu_3\mu_4\mu_5}$. Invariance of K reduces $SO(10,1)$ to $SO(10)$ (time–like case) or $SO(9)$ (null case). In the first case, Ω reduces to a $U(5)$–structure and Σ further to $SU(5)$. In the second case, Ω is trivial and $*\Sigma$ is a four form which reduces to $Spin(7)$.

11.6. K_μ time–like. In this case we may introduce a $SU(5)$–11–bein $(e^0, e^a, e^{\bar a})$ (with $a = 1,2,3,4,5$). The $SU(5)$ invariance implies:

$$ds^2 = -e^0 \otimes e^0 + \sum_a \left(e^a \otimes e^{\bar a} + e^{\bar a} \otimes e^a\right) \tag{11.30}$$

$$K = e^{-f}\, e^0 \tag{11.31}$$

$$\Omega = i \sum_a e^a \wedge e^{\bar a} \tag{11.32}$$

$$\Sigma = A\, e^0 \wedge \Omega \wedge \Omega + B\left(\varepsilon + \bar\varepsilon\right) \tag{11.33}$$

where

$$\varepsilon = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5, \tag{11.34}$$

where in the third line we used $\Omega^2 = 5$ which follows from eqn.(11.4).

Since K acts freely, we can introduce a coordinate t so that $K = \partial_t$. Then f, the frame and the $SU(5)$–connection would be t–indepedent. Notice that the Killing vector K generates an automorphism of the $SU(5)$–structure in the sense of Proposition 11.1.(3).

Moreover, one has

$$i_K e^0 = e^{-f}, \qquad i_K e^a = i_K e^{\bar{a}} = 0. \tag{11.35}$$

Thus, comparing with eqn.(11.16),

$$e^f\, i_K \Sigma = A\, \Omega \wedge \Omega \quad \Rightarrow \quad A = \frac{1}{2}. \tag{11.36}$$

then from $\Sigma^2 = 6$ we get $B = \sqrt{8}$.

We recall from Kähler geometry ([GSSFT] or [**66**]) some formulae which hold for any $U(m)$–structure on a $2m$–fold. In presence of a $U(m)$–structure we may classify differential forms by (p, q) type by expanding in an adpated co–frame. We say that a k–form ψ (with $k \leq m$) is primitive iff $\Omega^{m-k+1} \wedge \psi = 0$, that is, if it is a lowest weight vector in the spin $j = \frac{m-k}{2}$ representation of the Lefshetz $SU(2)$. Let ψ be a *primitive* (p, q) form; then

$$*\,\psi = \frac{i^{p-q}\,(-1)^{k(k-1)/2}}{(m-k)!}\, \Omega^{m-k} \wedge \psi, \qquad k = p + q. \tag{11.37}$$

The flux 4–form F_4 then may be expanded in our (co)frame

$$\begin{aligned} F_4 =& e^0 \wedge G + H = \\ =& e^0 \wedge \big(G_{3,0} + G_{2,1} + G_{1,2} + G_{0,3}\big) + \\ &+ \big(H_{4,0} + H_{3,1} + H_{2,2} + H_{1,3} + H_{0,4}\big), \end{aligned} \tag{11.38}$$

and

$$*_{11} F_4 = *G + e^0 \wedge *H, \tag{11.39}$$

where $*$ in the RHS is the one defined by the $SU(5)$–structure. Now,

$$d(e^{-f}\,\Omega) = i_K F_4 \quad \Rightarrow \quad G = e^f\, d(e^{-f}\,\Omega), \tag{11.40}$$

so the component of the 4–flux which is encoded in the 3–form G on the quotient 10–manifold $M/\mathbb{R}$ is determined by the intrinsic torsion of the $SU(5)$–structure (namely, by the Levi Civita connection) and f (*i.e.* the square of the Killing vector).

In the same way

$$\begin{aligned}
d(e^{-f}\,\Sigma) &= \frac{1}{2}d[e^{-f}\,e^0\wedge\Omega^2]+Bd[e^{-f}(\varepsilon+\bar{\varepsilon})]=\\
&=\frac{1}{2}d(e^f e^0)\wedge(e^{-f}\Omega)^2-e^0\wedge\Omega\wedge d(e^{-f}\Omega)+Bd[e^{-f}(\varepsilon+\bar{\varepsilon})]=\\
&=\frac{1}{2}d(e^f e^0)\wedge(e^{-f}\Omega)^2-e^0\wedge\Omega\wedge e^{-f}G+Bd[e^{-f}(\varepsilon+\bar{\varepsilon})]=\\
&=e^{-f}(i_\kappa *_{11} F_4-\Omega\wedge F_4)=\\
&=e^{-f}(*H-\Omega\wedge e^0\wedge G-\Omega\wedge H).
\end{aligned} \tag{11.41}$$

Thus,

$$*H-\Omega\wedge H=B\,e^f\,d[e^{-f}(\varepsilon+\bar{\varepsilon})]+\frac{1}{2}\,e^{-f}d(e^f\,e^0)\wedge\Omega^2. \tag{11.42}$$

A 4–forms Ψ on a 10–fold with an $U(5)$–structure[45] can be decomposed according to the $SU(2)$ representations (Lefshetz decompostion) as

$$\Psi=\sum_{p+q=4}\psi_{p,q}+\Omega\wedge\sum_{p+q=2}\psi_{p,q}+\Omega^2\,\psi_{0,0} \tag{11.43}$$

with $\psi_{p,q}$ primitive. By the rule in eqn.(11.37)

$$\begin{aligned}
(*\Psi-\Omega\wedge\Psi)=&\sum_{p+q=4}\Big((-1)^p-1\Big)\Omega\wedge\psi_{p,q}+\\
&+\frac{1}{2}\sum_{p+q=2}\Big((-1)^p-2\Big)\Omega^2\wedge\psi_{p,q}-\frac{2}{3}\,\Omega^3\,\psi_{0,0}
\end{aligned} \tag{11.44}$$

and we see that the components of H of types $(4,0)$, $(0,4)$, as well as the *primitive part* of the component of type $(2,2)$, drop out of eqn.(11.42). All other components of H are determined in terms of the intrinsic torsion of the $SU(5)$–structure and f.

Consistency requires that the RHS of eqn.(11.42) does not contain components of type $(1,5)$, $(5,1)$, nor $(3,3)$ components of the form $\Omega\wedge\beta_{2,2}$ with $\beta_{2,2}$ *primitive.* The only terms of type $(3,3)$ in the RHS of eqn.(11.42) come from the last term which has the form $\Omega^2\wedge(\cdots)$ and hence is not the dual of a *primitive* $(2,2)$–form. The terms of type $(5,1)$ come from $Be^f d(e^{-f}\varepsilon)$. The projection into type $(5,1)$ of this expression then should vanish:

$$0=e^f\,i_\varepsilon d(e^{-f}\varepsilon)=-df\,i_\varepsilon\varepsilon+i_\varepsilon d\varepsilon \tag{11.45}$$

$$\Rightarrow\quad df=i_\varepsilon d\varepsilon\equiv\text{component }(5,1)\text{ of the intrinsic torsion} \tag{11.46}$$

in particular, this component of the intrinsic torsion (or, equivalently, of the Levi Civita connection) is *exact.*

[45] In our case, this is the quotient 10–fold $M/\mathbb{R}$ with the $SU(5)$–structure, which, of course, is a special instance of an $U(5)$–structure.

There is a last equation to consider, the one for $d\kappa$, (11.27)[46]

$$\begin{aligned} e^f\, d(e^{-f}\kappa) =& \frac{4}{3}\, e^0\wedge \Lambda G + \frac{4}{3}\,\Lambda H + \frac{1}{6}\Lambda^2 * H + \\ &+ \frac{B}{3}\Big(i_\varepsilon * G + \text{c.c.}\Big) + \frac{B}{3}\, e^0\wedge \Big(i_\varepsilon * H + \text{c.c.}\Big) \end{aligned} \tag{11.47}$$

The *primitive* $(2,2)$ part $H^{\text{pr.}}_{(2,2)}$ satisfies[47] $\Lambda H^{\text{pr.}}_{(2,2)}=0$, $i_\varepsilon * H^{\text{pr.}}_{(2,2)}=0$, and

$$*(\Lambda^2 * H^{\text{pr.}}_{(2,2)}) = *^2\,\Omega^2\wedge H^{\text{pr.}}_{(2,2)}\equiv 0, \tag{11.48}$$

so, again, the component $H^{\text{pr.}}_{(2,2)}$ of the flux decouples from the equation.

Instead, the $(4,0)$ component of H, which decoupled from the previous equations, now is also determined in terms of the intrinsic torsion[48]

$$B\, i_{\bar\varepsilon} * H_{(4,0)} + 4\,\Lambda\, G_{(2,1)} + \text{c.c.} = 6\, d\log f. \tag{11.49}$$

Therefore: *in a* SUSY *configuration, all components of the flux are determined by the intrinsic torsion of the* $SU(5)$ *structure (and hence by the metric) but for the* primitive part *of the* $(2,2)$*–component.*

Remark. There is a simple reason why the primitive part of the $(2,2)$ decouples from the condition of existence of a Killing spinor. Indeed, we have

$$\delta\psi_\mu = D_\mu\epsilon + F^{\nu_1\nu_2\nu_3\nu_4}(a\,\Gamma_{\mu\nu_1\nu_2\nu_3\nu_4} - b\, g_{\mu\nu_1}\,\Gamma_{\nu_2\nu_3\nu_4})\epsilon. \tag{11.50}$$

A spinor ϵ determines an $SU(5)$ structure as follows: we may take it as a Clifford vacuum (a fermionic vacuum) which splits the (complexified) gamma–matrices into creation/annihilation operators. We write Γ_i for the annihilation and $\Gamma_{\bar i}$ for the corresponding creators

$$\Gamma_i\epsilon = 0, \qquad \Gamma_{\bar i}\epsilon \neq 0. \tag{11.51}$$

A metric is compatible if (up to normalization!) is of type $(1,1)$ that is if

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 0, \qquad \Gamma_i\Gamma_{\bar j} + \Gamma_{\bar j}\Gamma_i = g_{i\bar j}. \tag{11.52}$$

Let F be a *primitive* $(2,2)$ form. The fact that it is primitive means (by the Wick theorem for fermionic operators) that we can anticommute all the annihilators in $F^{\nu_1\nu_2\nu_3\nu_4}\Gamma_{\mu\nu_1\nu_2\nu_3\nu_4}$ and $F^{\mu\nu_2\nu_3\nu_4}\Gamma_{\nu_2\nu_3\nu_4}$ to the right. If the type is $(2,2)$ we have at least one annihilator, so these expressions arer automatically zero when applied to ϵ.

---

[46] As in Kähler geometry, we write $\Lambda\alpha$ for $i_\Omega\alpha/2$, where Λ is the adjoint of the operator L acting on forms as $L\alpha = \Omega\wedge\alpha$.

[47] In fact, $H^{\text{pr.}}_{(2,2)}$ is a lowest weight vector of a spin 3/2 representation of $SU(2)$ and Λ is precisely the *lowering* operator of the relevant $SU(2)$. On the other hand, $i_\varepsilon * H^{\text{pr.}}_{(2,2)}$ is a form of type $(-1,4)$ and hence zero.

[48] To get this formula, one uses that $\mathcal{L}_K e^0 = 0$, so $e^{-f} i_{e^0} e^0 = i_K de^0 = -d(i_K e^0) = -de^{-f}$.

EXERCISE 11.1. Extend the above analysis to the case in which the Killing vector K_μ is light–like.

11.7. Equations of motion. The above $SU(5)$–structure analysis completely solves the *geometrical* problem of characterizing the manifolds/flux backgrounds which have Killing spinors (with a time–like Killing vector). It remains the *physical* problem of understanding when such a geometry is a SUSY configuration of F–theory, that is, when such a geometry actually solves the equation of motion.

In the no–flux case, we know that a geometry with a Killing spinor is automatically a solution to the equations if K_μ is time–like (in the null case, we have to enforce only the ++ component of the Einstein equations). This follows from the integrability condition of the parallel spinor condition, $[\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon = 0$, which gives

$$(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R - T_{\mu\nu})\gamma^\nu \epsilon = 0. \tag{11.53}$$

The same argument applies in the general case. For M–theory, the integrability condition reads:

$$\begin{aligned} 0 = &\Big[R_{\mu\nu} - \frac{1}{12}\Big(F_{\mu\sigma_1\sigma_2\sigma_3}F_\nu{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{12}g_{\mu\nu}\,F^2\Big)\Big]\Gamma^\nu\epsilon \\ &- \frac{1}{6\cdot 3!}\Big(*\Big[d*F + \frac{1}{2}F\wedge F\Big]\Big)_{\sigma_1\sigma_2\sigma_3}\Big(\Gamma_\mu{}^{\sigma_1\sigma_2\sigma_3} - 6\,\delta_\mu^{\sigma_1}\,\Gamma^{\sigma_2\sigma_3}\Big)\epsilon \\ &- \frac{1}{6!}\,(dF)_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}\Big(\Gamma_\mu{}^{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5} - 10\,\delta_\mu^{\sigma_1}\,\Gamma^{\sigma_2\sigma_3\sigma_4\sigma_5}\Big)\epsilon, \end{aligned} \tag{11.54}$$

from which we see that, if the flux F_4 satisfies the Bianchi identity $dF_4 = 0$ and the *contracted* component of the equation of motion

$$i_K\Big(d*F_4 + \frac{1}{2}F_4\wedge F_4\Big) = 0, \tag{11.55}$$

then it also satisfied the Einstein equations if K_μ is time–like (otherwise we have to enforce just the ++ component). Moreover, from eqn.(11.12) we also know that eqn.(11.55) is automatically satisfied.

Thus one has to worry only of the other components of the C_3 equations of motion

$$d*F_4 + \frac{1}{2}F_4\wedge F_4 + \beta\,X_8 = 0, \tag{11.56}$$

where X_8 is a four curvature term induced by the brane anomaly cancellation

$$X_8 = \frac{1}{192}(p_1^2 - 4\,p_2) \tag{11.57}$$

$$p_1 = -\frac{1}{8\pi^2}\mathrm{tr}\, R^2 \tag{11.58}$$

$$p_2 = -\frac{1}{64\pi^4}\mathrm{tr}\, R^4 + \frac{1}{128\pi^4}(\mathrm{tr}\, R^2)^2 \tag{11.59}$$

$$\beta = \frac{2\pi}{T_5}. \tag{11.60}$$

Notice that consistecy with eqn.(11.55) requires $i_K X_8 = 0$.

Writing, as before, $F_4 = e^0 \wedge G + H$, the equations of motion become

$$0 = d * G + d(e^0 \wedge *H) + e^0 \wedge G \wedge H + \frac{1}{2} H \wedge H + \beta X_8. \tag{11.61}$$

The equation splits in two: the terms containing e^0 should cancel by themselves. Indeed, this is automatic by eqn.(11.12). We remain with

$$d * G + \frac{1}{2} H \wedge H + \beta X_8 = 0, \tag{11.62}$$

which, in view of eqn.(11.40), may be written as

$$d * \big(e^f d(e^{-f}\Omega)\big) + \frac{1}{2} H \wedge H + \beta X_8 = 0. \tag{11.63}$$

Eqn.(11.63) is the only equation we have to enforce explicitly (in the time–like case, which includes, in particular, all the vacuum configurations).

11.8. Modifications for F–theory. In F–theory (or in Type IIB SUGRA) the SUSY parameters ϵ^i are two $10D$ Majorana–Weyl spinors of the *same* chirality. For Majorana–Weyl spinor of the same chirality the bilinears $\bar{\chi}\,\Gamma_{\mu_1\mu_2\cdots\mu_n}\psi \equiv \chi^T C\,\Gamma_{\mu_1\mu_2\cdots\mu_n}\psi$ have the following properties (for commuting spinors !!)

$$\bar{\chi}\,\Gamma_{\mu_1\mu_2\cdots\mu_n}\psi = 0 \quad \text{if } n \text{ is even} \tag{11.64}$$

$$\bar{\chi}\,\Gamma_{\mu_1\mu_2\cdots\mu_{2k+1}}\psi = (-1)^k\,\bar{\psi}\,\Gamma_{\mu_1\mu_2\cdots\mu_{2k+1}}\chi. \tag{11.65}$$

Writing ϵ as a single complex Weyl spinor, we may define a Killing vector $K_\mu = \bar{\epsilon}\,\Gamma_\mu \epsilon$, which has zero $U(1)_R$–charge, a R–charge $+1$ three–form

$$\Phi_{\mu\nu\rho} = \epsilon^T\,C\,\Gamma_{\mu\nu\rho}\epsilon \tag{11.66}$$

and a neutral self–dual 5–form

$$\Sigma_{\mu_1\cdots\mu_5} = \bar{\epsilon}\,\Gamma_{\mu_1\cdots\mu_5}\epsilon. \tag{11.67}$$

Again, the corresponding G–structure depends on whether K_μ is time–like or null. The relevant G–structures correspond to the groups [?]:

- $Spin(7) \ltimes \mathbb{R}^8$ or $SU(4) \ltimes \mathbb{R}^8$ for the null case;

- G_2 in the time–like case.

The groups are those predicted by the general arguments developped so far.

Additional geometric details may be found in refs. [**?**].

12. An example: conformal Calabi–Yau 4–folds

I must resist the temptation of entering into the details of all possible geometries arising from the above G–structures. The subject of this introductory course is F–theory and not the spinorial geometry of Lorentzian manifolds with G–structures (which deserves a course by its own). The interested reader may consult the huge literature. Here I limit myself to apply the general geometric methods we discussed above in a simple example. In the next section, we will see that this simple example is in fact the only possibility relevant to 'F–theory phenomenology', that is the most general vacuum configuration which may lead to a four dimensional effective MSSM model.

From the M–theory point of view, we need to look in the compactifications down to 3 dimensions, since they are potentially *dual* to F–theory compactifications to four dimensions, as we saw in the fluxless case. From the general analysis of section 11 we know that the metric of a SUSY configuration with a time–like Killing spinor must have the form

$$ds^2 = -e^{-2f} dt^2 + \cdots, \tag{12.1}$$

so, if we ask 3–dimensional Poincaré symmetry, we must have a wrapped product

$$M = X \times_{e^{-2f}} \mathbb{R}^{1,2} \tag{12.2}$$

with metric

$$ds^2 = g_{\alpha\beta}(y)\, dy^\alpha\, dy^\beta + e^{-2f(y)}\; \eta_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{12.3}$$

and X compact. The $3d$ Poincaré symmetry also requires the $SU(5)$–structure to reduce to a $Spin(7)$– or a $SU(4)$–structure. The first case, however, cannot be dual to an F–theory solution (cfr. §. 7.1), so it is not interesting for us.

The simplest possibility is that the $SU(4)$–structure has no torsion. But then, by PROPOSITION 11.2.**(1)**, X is Calabi–Yau, and the Einstein equations set the flux to zero.

The second simplest possibility is to consider, instead of a torsion–less $SU(4)$–structure, a *torsion–less* $(\mathbb{R}^\times \times SU(4))$–structure[49]. Let us

[49] As we discussed in section .... from a geometric standpoint the natural structure is the $\mathbb{R}^\times \cdot G$–structure rather than the G–structure (where G is the little group of the spinor). Hence, for M–theory with a time–like Killing vector, geometrically one would naturally work with a $\mathbb{R}^\times \cdot SU(5)$–structure rather than with $SU(5)$–structure. Here we use the geometrically obvious as an *ansatz*.

introduce a $(\mathbb{R}^\times \times SU(4))$–adapted (co)frame $(e^a, e^{\bar a})$ $(a = 1,2,3,4)$. The $(\mathbb{R}^\times \times SU(4))$–structure is torsion–less if

$$de^a + \omega^a{}_b \wedge e^b = \eta \wedge e^a, \qquad \omega^a{}_b \in \mathfrak{su}(4) \otimes T^*X, \tag{12.4}$$

for some η.

Then, from the $11D$ point of view, we have a $SU(5)$–frame $e^0, e^a, e^{\bar a}$ with

$$e^0 = e^{-f(y)}\, dt, \qquad \sqrt{2}\, e^5 = e^{-f(y)}\,(dx^1 + i\, dx^2), \tag{12.5}$$

$$\Omega = i\, e^5 \wedge e^{\bar 5} + \omega \tag{12.6}$$

$$\varepsilon = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \tag{12.7}$$

$$d\varepsilon = 4\eta \wedge \varepsilon + e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge de^5 = (4\,\eta - df) \wedge \varepsilon \tag{12.8}$$

and so by eqn.(11.46),

$$df = i_\varepsilon \varepsilon = 4\,\eta - df \quad \Rightarrow \eta = \frac{1}{2} df. \tag{12.9}$$

which means that $g_{\alpha\beta} = e^{f(y)} \tilde g_{\alpha\beta}$, with $\tilde g_{\alpha\beta}$ a Calabi–Yau metric[50]. Then X is a Calabi–Yau manifold. Let ϕ be the holomorphic $(4,0)$. One has

$$e^{-f}\, \varepsilon = \frac{1}{\sqrt{2}} (dx^1 + i\, dx^2) \wedge \phi. \tag{12.10}$$

Hence, from eqn.(11.42)

$$*H - \Omega \wedge H = B\, e^f \Big(d(e^{-f} \varepsilon) + \text{c.c.} \Big) + \frac{1}{2}\, e^{-f} d(e^f\, e^0) \wedge \Omega^2 = 0. \tag{12.11}$$

Moreover

$$G = e^f d(e^{-f}\Omega) = -3i\, df \wedge e^5 \wedge e^{\bar 5} \tag{12.12}$$

and

$$df + \frac{1}{3}\, e^f\, i_\Omega\, d(e^{-f}\Omega) = 0, \tag{12.13}$$

which, in view, of eqn.(11.49) means that the $(4,0)$, $(0,4)$ components of the internal flux H vanish.

Notice that $\mathbb{R}^\times \times SU(5)$ is not one of the *holonomy groups* in the *torsion–less* classification of Merkulov and Schwachhöfer [**67**]. Then the holonomy of such a connection should be $SU(5)$. Indeed,

Proposition 12.1. *Let $G \subseteq U(m) \subset SO(2m)$. On the $2m$–fold M, consider a torsion–less $(\mathbb{R}^\times \times G)$–structure. Its curvature takes values in G and the $(\mathbb{R}^\times \times G)$–structure is locally conformal to an (obviously metric) G–structure.*

Proof. In an adapted frame, one has $de^a + \omega^a{}_b \wedge e^b = \eta \wedge e^a$. The proposition follows iff $\mathrm{Re}\,\eta$ is closed. Write $\Omega = i e^a \wedge e^{\bar a}$. Then

$$d\Omega = (\eta + \bar\eta) \wedge \Omega \quad \Rightarrow \quad d(\eta + \bar\eta) \wedge \Omega = 0 \quad \Rightarrow \quad d(\eta + \bar\eta) = 0.$$

□

[50] Compare the Proposition in the previous footnote.

Comparing with the general theory developped in the previous section, we conclude:

FACT 12.1. *A* torsion–less $(\mathbb{R}^\times \times SU(4))$*–structure on the compact* 8*–fold* X *corresponds to a Poincaré invariant compactification of* M*–theory to three dimensions with a metric*

$$ds^2 = e^{-2f(y)}\, \eta_{\mu\nu} dx^\mu dx^\nu + e^{f(y)}\, \tilde{g}_{\alpha\beta}(y)\, dy^\alpha\, dy^\beta, \tag{12.14}$$

$$d\tilde{s}^2 = \tilde{g}_{\alpha\beta}(y)\, dy^\alpha\, dy^\beta \;\; \textit{Calabi–Yau metric on } X, \tag{12.15}$$

and a 4*–form flux* F_4

$$\begin{aligned} F_4 &= -3i\, e^0 \wedge e^5 \wedge e^{\bar{5}} \wedge df + H^{\text{pr.}}_{(2,2)} = \\ &= -dt \wedge dx^1 \wedge dx^2 \wedge d(e^{-3f}) + H^{\text{pr.}}_{(2,2)}, \end{aligned} \tag{12.16}$$

provided the function f *and the internal flux* $H^{\text{pr.}}_{(2,2)}$ *satisfy the constraint in eqn.*(11.63) *or, explicitly,*

$$d * (e^{3f} de^{-3f}) + \frac{1}{2} H^{\text{pr.}}_{(2,2)} \wedge H^{\text{pr.}}_{(2,2)} + \beta\, X_8 = 0 \qquad \textit{on } X \tag{12.17}$$

In this way we have reproduce the celebrated vacuum solution originally found by Becker & Becker [**58**].

This is a very nice solution. Although we have a non–trivial flux, X is still a Calabi–Yau *manifold*. The metric is only in the conformal class of a Kähler Ricci–flat metric, but X is still a complex (in fact algebraic) manifold with $c_1(X) = 0$. All the deep complex analytic and algebraic techniques are still available, and many relevant aspects depend on X as a complex manifold, more than on a metric.

In some sense, the message is to de–mitize the *flux vacua:* we saw that the *physics* of SUSY implies an $(\mathbb{R}^\times \times SU(5))$–structure. Both flux–less and flux vacua are described by a *torsion–less* such G–structure, and the theory is exactly the same in the two cases.

13. Duality with an F–theory compactification to $4D$

By duality we expect that the BB solution leads to a flux $\mathcal{N} = 1$ $4D$ vacuum for F–theory.

In fact, we cannot expect that any primitive $(2,2)$–form G can be a background with an F–theory dual. The geometry of G needs to have the 'right' interplay with the elliptic fibration.

Let θ^i be a basis of integral harmonic forms along the fibers, and χ be an integral two form generating the two–dimensional cohomology of the fibers. A four form G on the elliptic manifold has an expansion

$$G = g + p \wedge \chi + \sum_i H_i \wedge \theta^i. \tag{13.1}$$

where g, p and H_i are 4–, 2–, and 3–forms on the basis B of the fibration. Since G is integral, so are g and p. Since G is primitive of type $(2,2)$, it is self–dual, and the forms g and p are related by duality. In the limit of size $\epsilon \to 0$ of the fiber the duality reads

$$g_{i_1\cdots i_4} = \frac{\epsilon}{2}\, \epsilon_{i_1\ldots i_4 k_1 k_2}\, g^{k_1 j_1} g^{k_2 j_2}\, p_{j_1 j_2} + O(\epsilon^2), \tag{13.2}$$

which is not compatible with integrality unless $g = p = 0$. Then we remain with a pair of 3–forms H_i on the base B which transform as a doublet under $SL(2,\mathbb{R})$. In this way we recover the results of section ... of chapter **1**.

What about the space–time component $d(e^{-3f}) \wedge dt \wedge dx^1 \wedge dx^2$ of the F_4 flux (cfr. eqn.(12.16))? The only purpose of this component is to solve the equation of motion (12.17) in presence of a non–zero internal flux $H^{\text{pr.}}_{(2,2)}$. In §. of chapter **1** we saw that this equation becomes in Type IIB SUGRA the non–linear Bianchi identity of the (anti)self–dual 5–form F_5. Thus, this component of F_4 becomes the component of F_5 which is proportional to the volume form of $\mathbb{R}^{1,3}$ which is needed to solve the Bianchi identity (or, equivalently to guarantee that $dC_4 + \frac{1}{2}\epsilon_{ij} B^i \wedge H^j$ is (anti)self–dual.

It remains to see the geometry of the F–theory configuration which emerges from the above solution of M–theory trough the chain of dualities in sect.

The metric in eqn.(10.8) gets replaced by

$$\begin{aligned} ds^2_M &= e^{-2f}\, \eta_{\mu\nu} dx^\mu dx^\nu + e^f\; ds^2_B + \\ &\quad + e^f\, \frac{\epsilon}{\tau_2}\Big((dx + \tau_1 dy)^2 + \tau_2^2\, dy^2\Big) + O(\epsilon^2) \end{aligned} \tag{13.3}$$

where ds^2_B is the Kähler metric on the base 3–fold B of the elliptic Calabi–Yau X. All the manipulations from eqn.(10.8) to eqn.(10.17) remain valid with the volume of the fiber ϵ replaced by the conformal modified one $e^f(y)\epsilon$ (which now depends on the point $y \in B\,!$). Then we can read the new metric from eqns.(10.16)(10.17)

$$ds^2\Big|_{\text{string frame}} = \left(\frac{e^f}{\tau_2}\right)^{1/2} \left(e^{-2f}\, \eta_{\mu\nu}\, dx^\mu dx^\nu + e^f\, ds^2_B + \frac{l_s^4}{\epsilon} e^{-2f}\, d\tilde{y}^2\right) \tag{13.4}$$

$$ds^2\Big|_{\text{Einstein frame}} = e^{-2f}\big(\eta_{\mu\nu}\, dx^\mu dx^\nu + dy^2\big) + e^f\, ds^2_M \tag{13.5}$$

where $y = l_s^2 y/\sqrt{\epsilon}$ is the uncompactified coordinate (in the limit $\epsilon \to \infty$).

Notice that the metric corresponds to a warped product $B \times_{e^{-2f}} \mathbb{R}^{1,3}$ which is Poincaré invariant in the $4d$ sense. This looks almost a miracle, and it is a consequence of the interplay between the G–structures which govern the geometry of M– and F–theory.

13.1. The G–structure viewpoint.

$\boxed{\textbf{TO BE WRITTEN}}$

14. No–go theorems

Above we discussed a simple example of how from the G–structure description we may easily deduce non–trivial flux SUSY configurations of F–theory. However, we did not attempted a full classification of all possible BPS solutions, we just limited ourselves to the very simplest possibility. Hence we may worry of having lost interesting vacua by focusing on the simplest possible geometries. The answer is NO. This follows from a general *no–go theorem* due to Giddings, Kachru and Polchinski [**59**]. This theorem, under very mild assumptions, rules out Poincaré invariant compactifications of F–theory to four dimensions which are more general than those we obtained in section 13 above. *It should be emphasized that supersymmetry is <u>not</u> an condition of the theorem, so the result applies to most non–*SUSY *compactifications as well.*

To have Poincaré symmetry in $4D$: *i)* The metric must be a warped product that we may always write in the convenient form

$$ds^2_{10} = e^{2A(y)}\,\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}\tilde{g}_{mn}\,dy^m dy^n, \tag{14.1}$$

where y^n are coordinates in a compact real 6–fold X; *ii)* The 3–form flux should be purely internal, and *iii)* the 5–form flux must have the form

$$F_5 = (1+*)d\alpha\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3 \tag{14.2}$$

for some function α on X.

The Einstein equations may be written (reversing traces)

$$R_{MN} = T_{MN} - \frac{1}{8}g_{MN}T \tag{14.3}$$

where the energy–momentum tensor, T_{MN}, is the sum of two pieces: the contribution of the supergravity fields, T^{sugra}_{MN} and the contribution of the localized objects (branes and the like) T^{loc}_{MN}.

The non–compact part of the Einstein equation are

$$\begin{aligned} R_{\mu\nu} = -g_{\mu\nu}\left(\frac{G_{mnp}\,\overline{G}^{mnp}}{48\,\mathrm{Im}\,\tau} + \frac{1}{4}e^{-8A}\partial_m\alpha\,\partial^m\alpha\right) + \\ + \left(T^{\text{loc}}_{\mu\nu} - \frac{1}{8}g_{\mu\nu}\,T^{\text{loc}}\right). \end{aligned} \tag{14.4}$$

The Ricci tensor is given by the usual formula for warped products (ref. [**37**] §. 9.J) which we used already many times

$$R_{\mu\nu} = -\eta_{\mu\nu}\,e^{4A}\,\tilde{\nabla}^2 A \tag{14.5}$$

(tilded quantities are computed using the metric $\tilde{g}$). Comparing with eqn.(14.4), we get

$$\begin{aligned}\tilde{\nabla}^2 e^{2A} = e^{2A}\,\frac{G_{mnp}\,\overline{G}^{mnp}}{12\,\mathrm{Im}\,\tau} + e^{-6A}\Big[\partial_m\alpha\,\partial^m\alpha + (\partial_m e^{4A})\,(\partial^m e^{4A})\Big] + \\ + \frac{1}{2}\,e^{2A}(T^m_m - T^\mu_\mu)^{\mathrm{loc}}.\end{aligned} \tag{14.6}$$

The integral of the LHS over the compact manifold X vanishes. The supergravity sources (the first line of the RHS) are positive definite. So, in absence of localized sources ('defects' from the SUGRA viewpoint) the fluxes must vanish and the warp factor should be a constant (hence M is a direct product $\mathbb{R}^{3,1}\times X$). This is a 'boring' fluxless vacuum.

To get a vacuum with a non trivial flux the local source $(T^m_m - T^\mu_\mu)^{\mathrm{loc}}$ must be *negative*. This is possible in superstring theory. Consider, for instance, a spacetime filling p–brane wrapped on a $(p-3)$–cycle $\Sigma_{p-3}\subset X$. In this case (neglecting higher orders in α' and the fluxes along the brane itself)

$$S\Big|_{\mathrm{loc}\ \Sigma_{p-3}} = -\int\limits_{\mathbb{R}^{3,1}\times\Sigma_{p-3}} d^{p+1}z\,T_p\,\sqrt{-g} + \mu_p \int\limits_{\mathbb{R}^{3,1}\times\Sigma_{p-3}} C_{p+1}, \tag{14.7}$$

where, in the *Einstein frame*, the tension is

$$T_p = |\mu_p|\,e^{(p-3)\phi/4}. \tag{14.8}$$

Now,

$$T^{\mathrm{loc}}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\,\frac{\delta S\big|_{\mathrm{loc}\ \Sigma_{p-3}}}{\delta g^{\mu\nu}} = -T_p\,e^{2A}\,\eta_{\mu\nu}\,T_{\Sigma_{p-3}} \tag{14.9}$$

$$T^{\mathrm{loc}}_{mn} \equiv -\frac{2}{\sqrt{-g}}\,\frac{\delta S\big|_{\mathrm{loc}\ \Sigma_{p-3}}}{\delta g^{mn}} = -T_p\,\Pi_{mn}\,T_{\Sigma_{p-3}} \tag{14.10}$$

where $T_{\Sigma_{p-3}}$ is the δ–current of the submanifold Σ_{p-1} and Π the orthogonal projection $TX\to T\Sigma_{p-1}$. Then

$$(T^m_m - T^\mu_\mu)^{\mathrm{loc}} = (4-(p-3))\,T_p\,T_{\Sigma_{p-3}} = (7-p)\,T_p\,T_{\Sigma_{p-3}}. \tag{14.11}$$

Thus, in presence of localized sources with $p<7$, some of them must have *negative* tension in order to have a Poincaré invariant compactifications.

Notice that the 'elliptic' SUSY vacua we found in sections ... evade the constraint since the only non–trivial localized sources are seven branes (with *positive* tension), which decouple from the above equations, as does the gradient of the complex scalar τ.

In fact this is true only to the leading order in α'. Already at order α'^2 there is a Chern–Simons correction of the form

$$\begin{aligned}-\mu_3 \int\limits_{\mathbb{R}^{3,1}\times\Sigma_{p-3}} C_4\wedge\frac{p_1(R)}{48} &= \\ &= \frac{\mu_7}{96}(2\pi\alpha')^2 \int\limits_{\mathbb{R}^{3,1}\times\Sigma_{p-3}} C_4\wedge \mathrm{Tr}\big(R\wedge R\big)\end{aligned} \tag{14.12}$$

corresponding to the fact that the topologically non–trivial fields on a seven brane induce a 3–brane charge along the brane which then behaves, in the present respect, much as a $D3$ brane. In Type IIB we expect that all localized sources, which may carry a 3–brane charge density ϱ_3^{loc}, satisfies the local BPS bound

$$\frac{1}{4}(T^m_m - T^\mu_\mu)^{\mathrm{loc}} \geq T_3\,\varrho_3^{\mathrm{loc}}, \tag{14.13}$$

which is the local analog of the global BPS bound $E\geq T_3Q_3$ which follows from the $(2,0)$ SUSY algebra representations in flat space. Of course this bound may be violated by some sort of local sources, but at the price of physical plausibility of the theory and also of the idea of *protected quantities* that may be safely computed. All reasonable sources satisfy the bound (14.13) [**59**], and given its direct physical meaning, we shall *assume it.* To be more precise, we define effective 3–brane ϱ_3^{loc} trough the Bianchi identities/equations of motion of the self–dual 5–form

$$dF_5 - H_3\wedge F_3 = 2\,T_3\,\varrho_3^{\mathrm{loc}}. \tag{14.14}$$

All terms appearing in the equations of motion for the self–dual flux (including all possible α' or quantum corrections) are included in the definition of ϱ_3; the 'protected' nature of the BPS quantities than requires that the Einstein equations are correspondingly corrected in such a way that the effective (local) energy–momentum tensor will continue to satisfy the BPS bound (14.13).

Using the Poincaré invarian ansatz (14.2), the Bianchi identity becomes[51]

$$\tilde{\nabla}^2\alpha = i\frac{e^{2A}}{12\,\mathrm{Im}\,\tau}G_{mnp}\big(*_6\overline{G}^{mnp}\big) + 2\,e^{-6A}\,\partial_m\alpha\,\partial^m e^{4A} + 2e^{2A}T_3\varrho_3. \tag{14.15}$$

Taking the difference of this equation and the Einstein equations (14.6), we get

$$\begin{aligned}\tilde{\nabla}^2(e^{4A}-\alpha) =& \frac{e^{2A}}{6\,\mathrm{Im}\tau}\,|i\,G_3 - *_6G_3|^2 + e^{-6A}\,\big|\partial(e^{4A}-\alpha)\big|^2 + \\ &+ 2e^{-2A}\left[\frac{1}{4}(T^m_m - T^\mu_\mu)^{\mathrm{loc}} - T_3\,\varrho_3^{\mathrm{loc}}\right].\end{aligned} \tag{14.16}$$

[51] $*_6$ stands for the Hodge operator in X.

The integral over the compact space X of the LHS vanishes, whereas the RHS is the sum of positive contributions, which then should vanish separately. We find three necessary conditions:

- The 3–form flux G_3 satisfies the duality condition
$$i\, G_3 = *_6 G_3; \tag{14.17}$$
- the warp factor e^{4A} and the 5–form flux potential α are identified;
- the local BPS inequality (14.13) is saturated.

We stress that these three conditions are obtained by assuming a Poincaré invariant compactification to $4d$, *without requiring any unbroken supersymmetry.*

If we *do* require an unbroken SUSY, we have, in addition, a G–structure (which, in the present case, is a $(T_{\mathbb{C}} \times SU(3))$–structure). In particular, we can reinterprete the duality condition (14.17) in terms of $SU(3)$ representations (cfr. eqn. in §.)
$$G_3 \in \overline{\mathbf{6}} \oplus \mathbf{1} \oplus \overline{\mathbf{3}} \tag{14.18}$$
corresponding, respectively, to *primitive* $(2,1)$–forms on X, $(0,3)$ forms, and non–primitive forms $\Omega \wedge \phi$, with ϕ of type $(0,1)$. Only the first is compatible with the $(T_{\mathbb{C}} \times SU(3))$–structure[52]. Thus we find the condition we used in section... The second condition, relating the warp factor to the five–flux potential is universal, and so should coincide with the one for our previous explicit solution.

We remain with the last condition, the saturation of the BPS bound for the sources. Surely enough in F–theory there are many BPS objects which saturate the bound: space–time filling $D3$–branes and $O3$–planes, higher dimensional branes... There is no shortage of possibilities. However, even if from the point of view of physical mechanisms we may have a lot of choices, *geometrically* they are all equivalent. Let us return back to FACT 12.1: Once the geometry is worked out, all the physics is encoded in the only equation of motion we need to impose, namely eqn.(12.17). In that equation X_8 is a higher α' correction, related to anomaly cancellation, which corresponds *dually* to the induced ϱ_3 charge of the seven–branes. To describe the geometry of the supersymmetric solution, we had no need to specifiy X_8: its presence only affects the warp factor, namely the conformal factor in the internal metric. Clearly, we may replace X_8 by the complete correction to the flux equation of motion; after performing the duality transformation to F–theory this full correction is precisely what we called $2T_3\varrho_3$ before. Its value (and thus the presence of arbitrary objects saturating the BPS bound (14.13)) will afect only the explicit conformal factor, but not the conformal Calabi–Yau condition. Therefore

[52] For an alternative discussion in another language, see [**60**].

General lesson 14.1. *Under physically sound assumptions, for all superPoincaré invariant compactifications, $M = X \times_{e^{2A}} \mathbb{R}^{3,1}$, of F–theory the metric on X is conformal to the induced Kähler metric on the basis of an elliptically fibered Calabi–Yau 4–fold (with section) in the limit of zero fiber volume. The complex scalar field $j(\tau) = j(C_0 + ie^{-\phi})$ is equal to the j–invariant of the fiber elliptic curve. The 3–flux G is a* primitive *$(2,1)$–form on X.*

Thus the geometry is essentially unique (morally speaking: the number of Calabi–Yau 4–folds is quite huge!). Since the physical predictions depend on the geometry, we are back in business. Moreover, the power of complex analytic and algebro–geometric methods are still at our disposal.

15. Global constraints on supersymmetric vacua

In studying the M–theory flux compactifications to 3 dimensions we ended up with the equation (12.17) which, after duality to a four–dimensional F–theory compactification, becomes eqn.(14.15). These equations may be seen as topological constraints.

More generally, the M–theory equation of motion, in presence of $M2$–branes with world–volumes $V_i \subset M_{11}$, is

$$d * F_4 = \frac{1}{2} F_4 \wedge F_4 + X_8 + \sum_i T_{V_i}. \tag{15.1}$$

If $M_{11} = X \times_{e^f} \mathbb{R}^{2,1}$, integrating eqn.(15.1) over the compact space X gives

$$\frac{1}{2} \int_X F_4 \wedge F_4 + N_{M2} = - \int_X X_8 \tag{15.2}$$

One has [**4**][**61**]

$$X_8 = \frac{1}{192}(p_1^2 - 4\, p_4) \tag{15.3}$$

where the Pontryagin classes p_k are given by

$$p_1 = -\frac{1}{2\,(2\pi)^4} \operatorname{tr} R^2 \tag{15.4}$$

$$p_2 = -\frac{1}{4\,(2\pi)^4} \operatorname{tr} R^4 + \frac{1}{8\,(2\pi)^4} \big(\operatorname{tr} R^2\big)^2. \tag{15.5}$$

For a complex 4–fold X, seen as a real 8–fold $X_{\mathbb{R}}$, the Pontryagin classes $p_k(X_{\mathbb{R}})$ are ([**62**], THEOREM 4.5.1)

$$\Big(1 - p_1(X_{\mathbb{R}}) + p_2(X_{\mathbb{R}})\Big) = \\ = \Big(1{+}c_1(X){+}c_2(X){+}c_3(X){+}c_4(X)\Big)\Big(1{-}c_1(X){+}c_2(X){-}c_3(X){+}c_4(X)\Big). \tag{15.6}$$

For a Calabi–Yau 4–fold X, $c_1(X) = 0$ and

$$p_1(X_{\mathbb{R}}) = -2\, c_2(X) \tag{15.7}$$

$$p_2(X_{\mathbb{R}}) = 2\, c_4(X) + c_2(X)^2. \tag{15.8}$$

Then

$$p_1^2(X_{\mathbb{R}}) - 4\, p_2(X_{\mathbb{R}}) = \\ = 4\, c_2(X)^2 - 4\big(2\, c_4(X) + c_2(X)^2\big) = -8\, c_4(X). \tag{15.9}$$

and

$$\int\limits_X X_8 = \frac{1}{192}\int\limits_X (p_1^2 - 4\, p_2^2) = -\frac{1}{24}\int\limits_X c_4(X) = -\frac{1}{24}\,\chi(X), \tag{15.10}$$

where $\chi(X)$ is the Euler characteristic of X. Then eqn.(15.2) becomes

$$\frac{1}{2}\int_X F_4 \wedge F_4 + N_{M2} = \frac{1}{24}\,\chi(X). \tag{15.11}$$

This equation is called the *tadpole condition.*

By duality, we can find an analogous condition on the F–theory (see [**63**] for a detailed check of the duality). To perform the duality transformation, we write the M–theory 4–flux in terms of 3 fluxes as in section ....

$$G_4 = H_3 \wedge dx + F_3 \wedge dy \tag{15.12}$$

where dx and dy are forms of unit period in the torus fiber. Then

$$\int\limits_B F_3 \wedge H_3 + N_{D3} = \frac{\chi(X_4)}{24}. \tag{15.13}$$

The physical meaning of this equation is that an elliptic Calabi–Yau X_4 (with section) describes a configuration with seven branes (whose homology class is given by $12\, c_1(\mathcal{L})$). Since $R \wedge R$ is non–trivial on the world–volume of these seven branes, they have an induces 3–brane charge as in section 14; this induced $D3$ charge is also a source of F_5–flux. The total flux on the compact space B should be zero. The LHS of eqn.(15.13) is the flux generated by the Chern–Simons coupling in the SUGRA Lagrangian and by the $D3$–branes. Then $-\chi(X_4)/24$ should be the integrated 3–brane charge induced by the curvature on all the seven branes[53]. This induced charge is, of course, completely determined by the geometry of the elliptic CY 4–fold X_4.

The LHS of eqn.(15.13) is an *integer* (since the fluxes are integral by the Dirac quantization). So should be the RHS. *A priori* the RHS may be non–integral. In fact we know Calabi–Yau manifolds for which the Euler characteristic is a multiple of 6 but not[54] of 24. Hence, in these cases, we get a contradition, or — more precisely — a topological constraint on the Calabi–Yau's that may appear in a compactification. This is an actual constraint in the M–theory. In F–theory the situation is simpler.

PROPOSITION 15.1 (See refs. [**64**][**65**]). *X_4 an* elliptic *Calabi–Yau 4–fold with section. Then*

$$\boxed{72 \,\big|\, \chi(X_4)} \tag{15.14}$$

So the RHS of eqn.(15.13) is integral (actually 3 times an integer) and there is no topological obstruction to compactify F– (or M–) theory on such an elliptic Calabi–Yau. (But, in general, one has a non–zero number N_{D3} of space–time filling three–branes around).

We shall go trough all the details of the argument, since it is very typical of the kind of gymnastics one does all the time when extracting 'phenomenological' consequences out of F–theory.

PROOF. We write X_4 as the zero locus of a Weierstrass *homogeneous* cubic polynomial

$$s = Z\,Y^2 - X^3 - A\,X\,Z^2 - B\,Z^3 \tag{15.15}$$

in the *total space* of the projectivized vector bundle

$$\mathbb{P}\big(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}\big) \longrightarrow B \tag{15.16}$$

with homogeneous coordinates along the fiber $(X : Y : Z)$ (this is just the statement that X and Y are, respectively, sections of $\mathcal{L}^2$ and $\mathcal{L}^3$, see chapter **1**).

Let S be the tautological sub–bundle of the natural bundle

$$\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O} \to \mathbb{P}\big(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}\big), \tag{15.17}$$

[53] See also the discussion in reference [**59**], as well [**64**].

[54] Consider the easiest possible CY 4–fold: namely a hypersurface of degree 6 in $\mathbb{P}^5$. An elementary application of the Lefshetz hyperplane theorem [**66**] plus the Griffiths residue theorem [**29**], gives the Hodge numbers

$$\begin{aligned} &h^{0,0} = h^{1,1} = h^{3,3} = h^{4,4} = h^{4,0} = h^{0,4} = 1 && h^{2,2} = 1752 \\ &h^{3,1} = h^{1,3} = 426 && \text{all others} = 0, \end{aligned}$$

so that

$$\chi = 6 + 2(426) + 1752 = 2610 = 2 \cdot 3^2 \cdot 5 \cdot 29,$$

which is divisible by 6 but not by 12.

and let $x = c_1(S^*)$. We are precisely in the set–up which leads to the definition of the Chern classes *á la* Grothendieck (compare §. 20 of ref. [**19**]). Then, *by the very definition of the Chern classes* (cfr. eqn.(20.6) of ref. [**19**])

$$\begin{aligned} 0 = x^3 + c_1\big(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}\big)\, x^2 + c_2\big(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}\big)\, x + c_3\big(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}\big) = \\ = x\big(x + 2\, c_1(\mathcal{L})\big)\big(x + 3\, c_1(\mathcal{L})\big) \end{aligned} \tag{15.18}$$

where in the second line we used the Whitney product formula (cfr. eqn.(20.10.3) of [**19**]).

s in eqn.(15.15) is a section of $(S^*)^3 \otimes \mathcal{L}^6$. Essentially by definition, this is also the normal bundle[55] $N_{X_4/W}$ of X_4 in the total space of the bundle $W \equiv \mathbb{P}\big(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}\big)$. Hence $c_1\big((S^*)^3 \otimes \mathcal{L}^6\big) = 3\,x + 6\, c_1(\mathcal{L})$ is the Poincaré dual of the fundamental cycle of X_4 in $W \equiv \mathbb{P}\big(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}\big)$ and

$$\int_{X_4} \alpha = 3 \int_W \big(x + 2\, c_1(\mathcal{L})\big) \wedge \alpha, \qquad \forall\, \alpha \in H^8(W). \tag{15.19}$$

Thus, restricted to classes on X_4, the relation (15.18) simplifies to

$$x^2 = -3\, x\, c_1(\mathcal{L}). \tag{15.20}$$

Moreover, since $N_{X_4/W} = (S^*)^3 \otimes \mathcal{L}^6$, we have the exact sequence of vector bundles (over[56] X_4)

$$0 \to TX_4 \to T\mathbb{P}\big(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}\big) \to (S^*)^3 \otimes \mathcal{L}^6 \to 0. \tag{15.21}$$

On the other hand, as bundles over $\mathbb{P}\big(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}\big)$, we have

$$\begin{aligned} 0 \to S^* \otimes S \to S^* \otimes \Big(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}\Big) \to \\ \to T\mathbb{P}\big(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}\big) \to TB \to 0. \end{aligned} \tag{15.22}$$

Let C the total Chern class of TB

$$C = 1 + c_1(B) + c_2(B) + c_3(B), \tag{15.23}$$

and $\hat{C}$ the total Chern class of $TW \equiv T\mathbb{P}\big(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}\big)$. From the exact sequence (15.22) and the Whitney product formula, we have

$$\hat{C} = C \cdot \big(1 + x + 2\, c_1(\mathcal{L})\big)\big(1 + x + 3\, c_1(\mathcal{L})\big)\big(1 + x\big). \tag{15.24}$$

[55] In the complex analytic sense!

[56] $\mathcal{L}$ means actually the pull–back of the bundle $\mathcal{L} \to B$ to the space $\mathbb{P}\big(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}\big)$ *via* the canonical projection. All bundles are then restricted to X_4, that is pulled–back *via* the inclusion map. Everywhere we use the *naturality* of the Chern classes under pulling–back. These precisations hold for the subsequent formluae too.

Applying the same formula to the sequence (15.21) we get

$$1+\sum_{k=1}^{4} c_k(X_4) = \hat{C}\Big/\Big(1+3\,x+6\,c_1(\mathcal{L})\Big) \tag{15.25}$$

Finally, the condition that X_4 is Calabi–Yau is equivalent to $c_1(\mathcal{L}) = c_1(B)$ as we discussed in section 5. This condition can be recovered from eqns.(15.25)(15.24)(15.23)(15.20). Indeed,

$$\begin{aligned} c_1(X_4) = \hat{c}_1 - 3\,x - 6\,c_1(\mathcal{L}) &= \\ = c_1(B) + 3\,x + 5\,c_1(\mathcal{L}) - 3\,x - 6\,c_1(\mathcal{L}) &= \\ = c_1(B) - c_1(\mathcal{L}), & \end{aligned} \tag{15.26}$$

so

$$c_1(X_4) = 0 \quad \Longleftrightarrow \quad c_1(B) = c_1(\mathcal{L}). \tag{15.27}$$

Eqns.(15.25)(15.24)(15.23) give (with $c_k \equiv c_k(B)$)

$$c_4(X_4) = \sum_{k=0}^{4} c_k \left.\frac{(1+x+2\,c_1)(1+x+3\,c_1)(1+x)}{(1+2\,x+6\,c_1)}\right|_{8-2k \text{ form}}. \tag{15.28}$$

Expanding the expression in the RHS and using the relation (15.20), one finds the coefficients of c_k

coeff. c_3	$-c_1$
coeff. c_2	$4\,x\,c_1 + 12\,c_1^2$
coeff. c_1	$-72\,c_1^3 - 24\,x\,c_1^2$
coeff. c_0	$432\,c_1^4 + 144\,x\,c_1^3$

The forms $c_3 c_1$, $c_2 c_1^2$, c_1^4 vanish since they are the pull–back of eight form on B which has only three (complex) dimensions. Finally,

$$c_4(X_4) = 120\,x\,c_1^3 + 4\,x\,c_1\,c_2 \tag{15.29}$$

The Euler characteristic of X_4 is then equal to

$$\chi(X_4) \equiv \int_{X_4} c_4(X_4) = 4\int_F x \int_B \big(c_1\,c_2 + 30\,c_1^3\big), \tag{15.30}$$

where F is the homology class of a generic fiber. By the Wirtinger theorem, $\int_F x$ is the degree of the hypersurface which is 3. Thus

$$\chi(X_4) = 12\int_B c_1\,c_2 + 360\int_B c_1^3. \tag{15.31}$$

Now we have to compute the integral $\int_B c_1\,c_2$ where B is the base of an elliptic Calabi–Yau. I claim

LEMMA 15.1. *Let $\pi\colon X_4 \to B$ be a compact* irreducible *elliptic Calabi–Yau 4–fold (with a section). Then*

$$\boxed{\int_B c_1\, c_2 = 24} \tag{15.32}$$

Proof of the lemma. It is basically a consequence of the geometric wonders we discussed in section 6.

First of all, if the Calabi–Yau 4–fold is irreducible (that is, its holonomy is $SU(4)$ and not a subgroup), it is algebraic. Then $B \subset X_4$ is also *algebraic* by virtue of Chow's theorem[57]. Then THEOREM 20.2.2 of ref. [**62**] states

$$\text{Arithmetic genus of } B \equiv$$
$$\equiv \sum_{q=0}^{3} (-1)^q \, \dim H^q(B, \mathcal{O}_B) = \int\limits_B \mathrm{Tod}(B). \tag{15.33}$$

I claim that the arithmetic genus of B is 1. This again follows from the wonders of section 6. There (or in [GSSFT]) it is shown that on a strict Calabi–Yau X_4,

$$\dim H^0(X_4, \Omega^p) = \begin{cases} 1 & p = 0, 4 \\ 0 & \text{otherwise.} \end{cases} \tag{15.34}$$

Assume on B there is a (non–zero) holomorphic $(p, 0)$–form ϕ. Then $\pi^*\phi$ is a non–zero $(p, 0)$–form on X_4; since there are none for $p \neq 0$,

$$\dim H^0(B, \Omega^1) = \dim H^0(B, \Omega^2) = \dim H^0(B, \Omega^3) = 0. \tag{15.35}$$

B is Kähler, and hence $\dim H^p(B, \mathcal{O}) = \dim H^0(B, \Omega^p)$ (by the symmetry of the Hodge diamond [**66**]). Then

$$\text{Arithmetic genus of } B = \dim H^0(B, \mathcal{O}) = 1, \tag{15.36}$$

as claimed. Let us compute the RHS of eqn.(15.33), $\int_B \mathrm{Td}(B)$. The 6–form component of Tod, T_3, may be read in the table on page 14 of ref. [**62**]: $T_3 = \frac{1}{24} c_2 c_1$. Hence eqn.(15.33) gives

$$\frac{1}{24} \int_B c_1\, c_2 \equiv \int_B \mathrm{Tod}(B) = 1, \tag{15.37}$$

which is the lemma.

Conclusion of the proof of the proposition. Using the LEMMA, eqn.(15.31) becomes

$$\boxed{\chi(X_4) = 72 \left(4 + 5 \int_B c_1^3 \right)} \tag{15.38}$$

□

[57] See pag. 167 of ref. [**66**].

REMARK. From eqn.(15.31), we see that

$$\begin{matrix}\text{induced}\\ \text{3–brane charge}\end{matrix} = -\frac{\chi(X_4)}{24} = -\frac{1}{2}\int_B c_1 c_2 - 15\int_B c_1^3. \tag{15.39}$$

The usual anomaly in–flow arguments [**68**][**69**] give a induced $(p-4)$–brane charge on a p–brane of world–volume V equal to[58] $\frac{1}{48}p_1(V)$. Thus

$$\begin{matrix}\text{3–brane charge}\\ \text{of the 7–branes}\end{matrix} = \tag{15.40}$$

$$= \frac{1}{48}\sum_i \int\limits_{V_i} p_1 \qquad \text{(anomaly inflow)} \tag{15.41}$$

$$\equiv -\frac{1}{24}\sum_i \int\limits_{V_i} c_2 \qquad \text{(by definition of } p_1,\ \text{eqn.(15.6))} \tag{15.42}$$

$$= -\frac{1}{2}\int_B c_1\, c_2 \qquad \Big(12\, c_1 \text{ is Poincaré dual to } \sum_i V_i\Big) \tag{15.43}$$

so the first term in the RHS of eqn.(15.39) is precisely the 3–brane charge of the seven branes which are described by the given Calabi–Yau 4–fold X_4. What about the second term?

Well, the seven branes world–volumes (or, rather, the discriminant locus $\Delta = \sum_i V_i$) is *not* smooth. Even if the individual irreducible components are smooth, these components intersect along real codimension 2 submanifolds $\Sigma_{ij} = V_i \cap V_j$, and three such components intersect along real codimension 4 submanifolds $p_{ijk} = V_i \cap V_j \cap V_k$. Physically these submanifold appear as the world volume of, respectively, some kind of 5–branes and, respectively, 3–branes.

$144\, c_1^2$ and $1728\, c_1^3$ are, respectively, the Poincaré duals of the self–intersection and triple self–intersection of the discriminant locus Δ which are, in some 'abstract' sense, 5–branes and 3–branes with their own induced 3–brane charges. The second term in the RHS of eqn.(15.39) measures the combined effect of all these lower dimensional branes.

[58] As before, p_k are the Pontryagin classes.

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