Trieste Lectures on Wall–Crossing Invariants*

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Abstract

We give a pedagogical introduction to wall–crossing for $\mathcal{N}=2$ theories in both two and four dimensions from the point of view of the quantities whose BPS–chamber invariance implies the wall–crossing formula. The basic such invariant is the conjugacy class of the quantum monodromy, which may be thought of as a generalization of the Coxeter element in the Weyl group of a Lie algebra.

The relationships with singularity theory, quiver and algebras representation theory, topological strings, (quantum) cluster algebras, the Thermodynamical Bethe Ansatz, $2d$ CFT's, and Number Theory are outlined.

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Contents

1 Introduction

The first explicit *Wall–Crossing Formula* (WCF) for the change in the number of BPS states across a wall of marginal stability was obtained in 1992 in the context of the two–dimensional $\mathcal{N} = 2$ models in reference [1]. The purpose of that paper was more general and ambitious than just to get a wall–crossing formula: It was to classify and characterize all the $\mathcal{N} = 2$ models which are consistent as non–perturbative quantum field theories (without making reference to specific Lagrangian realizations, which may not exist). Such a program includes, in particular, the characterization of the possible spectra of BPS states, and hence of their allowed changes under continuous deformations of the theory, which is what we mean by a *Wall–Crossing–Formula*.

The $4d$ WCF was firstly formulated, in the context of some related mathematical problems, by Kontsevitch and Soibelman in ref. [2] and Joyce and Song in ref. [3], and then proved (or interpreted) as the physical $4d$ $\mathcal{N} = 2$ WCF in many different contexts and viewpoints in refs. [4–12].

The purpose of the present lectures is to discuss both the $2d$ and $4d$ cases in the spirit of the original program, that is, with emphasis on the wall–crossing *invariants*. In fact, as it will be clear, the change in the number of BPS states (with given quantum numbers) across a wall may be regarded as the result of the adjoint action on some 'arithmetic' group WC. The group WC may be characterized trough the set of its *invariants*. These invariants encode physical information which *do not* jump across the walls of marginal stability; this invariant physical information is, in a sense, more fundamental and intrinsic that the chamber-dependent BPS spectrum. In particular, given the invariants, it is elementary to construct the group WC (and its action on the BPS multiplicities) and hence to deduce the WCF.

It should be emphasized that, so stated, the problem is much more difficult than just deducing a WCF. *E.g.* the WCF for $2d$ was obtained in full generality in ref. [1], but the general program was never completed even in two–dimension.

In its simplest form, the problem may be stated as follows. Let us start with a $\mathcal{N} = 2$ superconformal theory (SCFT) and deform it away from criticality by adding some relevant coupling and/or switching on the vacuum expectation of some fields (*e.g.* Coulomb branch parameters). The deformed theory will flow in the IR to a non–conformal supersymmetric theory which typically possesses non–trivial BPS states — indeed, one of the consequences of the theory is that if we start from a *non–trivial* SCFT in the UV we must *necessarily* have some BPS–saturated massive state in the physical spectrum of the IR theory. The detailed BPS spectrum will depend on the particular deformation we consider. One asks which characteristics of the BPS spectra are 'universal', that is independent of the specific deformation. Clearly, these universal characteristics should correspond to (possibly very subtle) properties of the parent UV superconformal theory. However, it is not difficult to extend the analysis to theories which are just asymptotically free in the UV, where new phenomena appear (see ref. [1] for the $2d$ case).

For pedagogical purposes, we start the analysis in $2d$ and then go to the $4d$ case (and also limit ourselves to a special class of 'simple' $4d$ theories). From an abstract point of view, one may consider the $4d$ WCF as a special instance of the $2d$ one. Indeed, we may consider a $4d$ theory as a $2d$ one with infinitely many fields; we shall see that the WCF formula will have the same abstract form in both $4d$ and $2d$ (and there are mixed ones [8, 13]). In particular, the general structure of the WCF is the same in both dimensions.

A second reason for discussing the four–dimensional and two–dimensional is that from the theory there emerge deep, beautiful and unexpected connection between $4d$ and $2d$ which are also very useful for the actual computations.

2 Two–dimensions revisited

2.1 Set–up

Although the results are independent of the Lagrangian formulation of the theory[1], to keep the technicalities to the minimum, we use the language of the Landau–Ginzburg (LG) models. Thus we have a $2d$ $(2, 2)$ supersymmetric theory with action of the

[1] Indeed they depend only on the fact that we may twist a $(2, 2)$ supersymmetric theory in two different ways *tolopogical* and, respectively, *anti–topological*, the two being interchanged by an anti–linear involution (*i.e.* PCT). In the general case, the superpotential is replaced by the deformations of the theory which are non–trivial in the topologically twisted theory, its Hermitian conjugate by those non–trivial in the anti–topological twisted model, while by D–terms we mean everything which is both topologically and anti–topological trivial.

general form

$$S = D\text{–terms} + \int d^2 z \, d^2\theta \, W(X_i) + \text{H.c.}, \tag{2.1}$$

where X_i are chiral superfields (not necessarily elementary, we only require them not to be of the form $\overline{D}^2 V$ for some V) and $W(X_i)$ is a holomorphic function (the *superpotential*). By 'D–terms' we mean any supersymmetric interaction which cannot be written as an integral in $d^2\theta$ (or $d\overline{\theta}^2$). A basic result in supersymmetry is the *non–renormalization* of the superpotential W: Under the renormalization group flow (RG) the D–terms will evolve in some intricate way but W would remain the same[2].

2.1.1 The superconformal limit

Firstly, let us assume the above model is *conformal invariant* (and hence superconformal) at the full quantum level. From the two–dimensional $\mathcal{N} = 2$ superconformal algebra we know that such a theory must have, in particular, an axial $U(1)$ symmetry rotating the two supercharges. We normalize the corresponding charge q in such a way that the supercharges Q_α have $q = \pm\frac{1}{2}$. Since the superpotential W is not corrected by quantum effects, a *necessary condition* for the model to be superconformal is that $W(X_i)$ is invariant under this R–symmetry. Since $d^2\theta$ has charge -1, this requires

$$W(\lambda^{q_i} X_i) = \lambda \, W(X_i) \qquad \forall \, \lambda \in \mathbb{C}^*, \tag{2.2}$$

where $q_i > 0$ is the R–charge of X_i. [One consequence of the $(2,2)$ superconformal algebra is that, for a chiral primary operator, the dimension and the R–charge are equal, and this implies that the q_i are positive]. A superpotential satisfying equation (2.2) for positive real weights q_i is called *quasihomogeneous*.

Conversely, let us start with a theory of the form (2.1) with a quasi–homogeneous superpotential $W(X_i)$, and assume it has a UV fixed point. This UV fixed point should correspond to a $\mathcal{N} = 2$ superconformal theory such that the X_i are *chiral primaries*. Indeed, at the conformal point W should be marginal, and hence have dimension 1. Then eqn.(2.2) implies that the chiral fields X_i have equal dimension and R–charge, and this property is a characterization of the chiral *primaries*. The

[2] Again, the non–renormaliztion theorem is actually the *quantum* topological invariance of the twisyed theory.

chiral primaries ϕ_i have the following remarkable properties [16–18]:

(Chiral ring) the chiral primary operators ϕ_i form an associative *commutative* $\mathbb{C}$–algebra with unit 1 (the identity operator), which we denote as $\mathcal{R}$ and call the *chiral ring*. $\mathcal{R}$ has a multiplication table $\phi_i \cdot \phi_j = c_{ij}{}^k \, \phi_k$ for certain complex numbers $c_{ij}{}^k = c_{ji}{}^k$ (in particular, the operator product $\phi_i(z) \, \phi_j(z')$ is regular as $z \to z'$);

(Spectral flow) we have the isomorphism of $\mathcal{R}$–modules

$$\mathcal{R} \simeq \mathcal{H}_{E=0}, \tag{2.3}$$

where $\mathcal{H}_{E=0}$ is the space of states of zero–energy ($\equiv$ the supersymmetric states). In particular $\dim_{\mathbb{C}} \mathcal{R}$ is equal to the number of SUSY vacua;

(Central charge) the Virasoro central charge is

$$\hat{c} \equiv \frac{c}{3} = \max_{\phi \in \mathcal{R}} \, q(\phi). \tag{2.4}$$

These properties (together with PCT) imply that the $U(1)$ R–charge of the supersymmetric vacuum corresponding to the chiral operator ϕ under the isomorphism (2.3) is [18]

$$q(\phi) - \hat{c}/2. \tag{2.5}$$

They also imply

$$\begin{aligned} \dim \mathcal{H}_{E=0} = 1 \;\Leftrightarrow\; \text{the only chiral primary is the identity} \;\Leftrightarrow \\ \Leftrightarrow\; \hat{c} = 0 \;\Leftrightarrow\; \text{the SCFT is trivial.} \end{aligned}$$

From these facts, it is clear that the chiral ring $\mathcal{R}$ and the $U(1)$ charges $q(\phi_i)$ are the basic physical invariants characterizing the superconformal theory.

It should be stressed that the first two properties **(Chiral ring)** and **(Spectral flow)** do not really depend on the theory being critical [18]: they really reflect the topological nature of the chiral sector of *any* $(2,2)$ supersymmetric theory, critical or not. In particular, the chiral ring $\mathcal{R}$ must be independent of the D–terms, and hence totally determined by the superpotential $W(X_i)$. By the non–renormalization theorem, it can be determined at the classical level, getting

$$\mathcal{R} \simeq \mathbb{C}[X_i]/\langle \partial_i W \rangle, \tag{2.6}$$

where $\langle \partial_i W \rangle$ stands for the Jacobian ideal of W (*i.e.* the set of the polynomials of the form $\sum_i P_i(X_j)\, \partial W/\partial X_i$, for $P_i \in \mathbb{C}[X_i]$). Notice that the classical vacua are precisely the configuration $X_i = X_i^\alpha$ where $\partial_i W(X^\alpha) = 0$, that is the affine variety associated with the Jacobian ideal, classical vacua $= V(\langle \partial_i W \rangle)$.

In this form the properties **(Chiral ring)** and **(Spectral flow)** hold in general, even for massive theories. The LG model has a mass–gap precisely when the vacuum variety $V(\langle \partial_i W \rangle)$ is *reduced* (that is, $\mathcal{R}$ has no nilpotent elements) and of pure dimension zero (*i.e.* it is a set of isolated points). Let $|X^\alpha\rangle$ be the quantum vacuum corresponding[3] to the α–th classical vacuum configuration, $X_i = X_i^\alpha$. Then the isomorphism (2.3) is just

$$\mathbb{C}[X_i]/\langle \partial_i W \rangle \to \mathcal{H}_{E=0} \tag{2.7}$$

$$P(X_i) \mapsto \sum_\alpha P(X_i^\alpha)\, |X^\alpha\rangle, \tag{2.8}$$

(in other words, the vector space $\mathcal{H}_{E=0}$ is the linear space of regular functions on $V(\langle \partial_i W \rangle)$; this is true in general).

2.1.2 The massive theory

From the previous discussion it follows that, in the present language, a deformation away from criticality amounts essentially to adding to the superpotential $W(X_i)$ monomials of total $U(1)$ charge less than 1

$$W(X_i) \to W(X_i) + \sum_{\vec{n}\cdot\vec{q}<1} t_{\vec{n}}\; X_1^{n_1} X_2^{n_2} \cdots X_\ell^{n_\ell}, \tag{2.9}$$

in fact, the condition of charge less than 1 is precisely the condition for the corresponding perturbation to be relevant in the IR, *i.e.* of dimension less than 2. Notice

[3] That we have a one–to–one correspondence between classical and quantum vacua is guaranteed by the Witten index, and — in a stronger sense — by the spectral–flow isomorphism.

that any D–term operator has dimension at least 2, so it cannot be IR relevant.

By adding suitable relevant couplings $t_{\vec{n}}$ (in particular, mass parameters) we get a massive theory. This is equivalent to ask that the critical points of W, *i.e.* the solutions to the equations $\partial W/\partial X_i = 0$ are *isolated* and of multiplicity one. The number of these critical points, *i.e.* the number of supersymmetric vacua is equal to the Witten index of the UV superconformal theory, $m \equiv \dim \mathcal{R}$, and (in particular) larger than 1 (since the UV theory is non–trivial).

The Hilbert space $\mathcal{H}$ of the theory quantized on the line $\mathbb{R}$ splits into several superselected sectors: The condition of finite energy requires the field configuration to approach a vacuum as $x \to \pm\infty$, and the multiplicity of vacua corresponds to a multiplicity of possible boundary conditions at infinity. We write $\mathcal{H}_{\alpha\beta}$ for the space of states approaching the α–th vacuum as $x \to -\infty$ and the the β–th vacuum as $x \to +\infty$. Clearly the fundamental state in the diagonal sector $\mathcal{H}_{\alpha\alpha}$ is the α–th vacuum. However, for $\alpha \neq \beta$, the sector $\mathcal{H}_{\alpha\beta}$ cannot contain any zero–energy state; thus its fundamental state must have some positive energy $E_{\alpha\beta} > 0$. All other states in that sector will have energies $E \geq E_{\alpha\beta}$.

Since nothing depends on the D–terms, let us compute the $E_{\alpha\beta}$ using conventional D–terms, that is $\int d^4\theta\, K(X_i, \overline{X}_j)$ where K is the potential of some Kähler metric $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$. The classical energy of a x–dependent configuration $X_i(x)$ in the (α, β) sector is

$$\begin{aligned}
&\int_{-\infty}^{+\infty} dx \Big\{ g_{i\bar{j}} \partial_x X^i\, \partial_x \overline{X}^j + g^{i\bar{j}}\, \partial_i W\, \partial_{\bar{j}} \overline{W} \Big\} \equiv \\
&\qquad = \int_{-\infty}^{+\infty} dx\, g^{i\bar{j}} \Big(g_{i\bar{k}}\, \partial_x \overline{X}^k - e^{i\theta}\, \partial_i W \Big) \overline{\Big(g_{j\bar{l}}\, \partial_x \overline{X}^l - e^{i\theta}\, \partial_i W \Big)} + \\
&\qquad\qquad + 2\,\mathrm{Re} \Big\{ e^{i\theta} \Big(W(x = +\infty) - W(x = -\infty) \Big) \Big\} \\
&\qquad \geq 2 \Big| W(X_\beta) - W(X_\alpha) \Big|,
\end{aligned} \tag{2.10}$$

with equality if and only if

$$\bullet \qquad e^{i\theta}\Big(W(X_\beta) - W(X_\alpha)\Big) \in \mathbb{R}_+ \tag{2.11}$$

$$\bullet \qquad g_{i\bar{k}}\, \partial_x \overline{X}^k = e^{i\theta}\, \partial_i W \qquad \forall x \in \mathbb{R}. \tag{2.12}$$

A configuration satisfying the second condition is annihilated by two linear combinations of the supercharges namely $Q_\alpha + e^{-i\theta}\,\overline{Q}_\alpha$ and hence is a (classical) BPS configuration. Thus we interprete eqn.(2.10) as the BPS inequality and identify the central charge in the superselected sector $\mathcal{H}_{\alpha\beta}$ as

$$Z_{\alpha\beta} = 2\Big(W(X_\beta) - W(X_\alpha)\Big). \tag{2.13}$$

The discussion was classical, but the non–renormalization of W and supersymmetry promote everything to fully quantum equalities.

In the W plane the description of a BPS configuration is very easy. One has

$$\partial_x\Big(e^{i\theta}\, W(X_i)\Big) = e^{i\theta}\, \partial_i W\, \partial_x X^i = g^{i\bar{j}}\, \partial_i W\, \partial_{\bar{j}} \overline{W} \geq 0, \tag{2.14}$$

so the image of the BPS state in the W plane is a straight line of slope $e^{-i\theta}$ connecting the critical values $W(X_\alpha)$ and $W(X_\beta)$. A BPS state in the sector (α, β) is just an inverse image of this segment under the map $X_i \mapsto W(X_i)$ which is a connected curve starting and ending at the corresponding classical vacua. This description, in particular, shows that the number of (α, β) BPS states *is indepedent of the D–terms.*

2.2 Counting BPS states

Let $N_{\alpha\beta}$ be the number of BPS states in the (α, β) sector. We wish to compute the $N_{\alpha\beta}$'s and relate them to the invariants of the parent UV SCFT.

The $N_{\alpha\beta}$'s are *non–negative integers*. Hence they must be *locally constant functions* of the continuous deformation parameters $t_{\vec{n}}$. At first sight, one would expect them to be also *globally* constant in deformation space, since the possibility that new fundamental states appear in the sector $\mathcal{H}_{\alpha\beta}$ under *mild* (that is, *soft*) deformations of the theory looks physically unsound, violating Dirac's *physical continuity* principle. (*Mild*, in particular, means that the perturbations preserve the sectors $\mathcal{H}_{\alpha\beta}$, while smoothly changing the value of the corresponding central charge $Z_{\alpha\beta}$).

However, the previous description of the Hilbert space sector $\mathcal{H}_{\alpha\beta}$ *implies* that the numbers $N_{\alpha\beta}$ *cannot be globally constant* as functions of the complex couplings $t_{\vec{n}}$ (unless $N_{\alpha\beta} \not\equiv 0$ which we shall show momentarily is also excluded). This follows from a simple symmetry argument that we now review (since it will apply also in $4d$).

2.2.1 $\mathbb{Z}_h$–invariant BPS chambers

The Hagg–Lopuszanski–Sohnius theorem [19] states that the central charge operator Z commutes with all the continuous symmetries of the S–matrix. In $\mathcal{N} = 2$ it must also commute with all simple non–Abelian discrete symmetries (since they have no no–trivial representation of dimension 1). However, some *discrete Abelian* symmetry may still act non–trivially on Z (*e.g.* PCT acts as $Z \mapsto -Z$). Any such group may be written as a product of $\mathbb{Z}_h$'s.

If we may deform the UV SCFT in such a way that the deformed theory has a $\mathbb{Z}_h$ symmetry acting on the central charge as $Z \mapsto e^{2\pi i\ell/h}\, Z$, the BPS spectrum $\{N_{\alpha,\beta}\}$ must also be $\mathbb{Z}_h$ invariant. Starting with a given SCFT we may easily construct two different deformation which have different cyclic symmetries, in such a way that no set of BPS multiplicities $\{N_{\alpha,\beta}\}$ is consistent with both symmetries. Since the two theories are obtained one from the other by a variation of the marginal couplings $t_{\vec{n}}$, we conclude that the $\{N_{\alpha,\beta}\}$ should jump somewhere in coupling space.

Example. We illustrate the above symmetry argument in an easy example[4].

Consider then the critical model defined by the homogeneous superpotential $W(X) = X^{m+1}/(m+1)$ (the A_m minimal SCFT), and focus on the following two

[4] This example was, historically (1991), the first motivation for the development of WCF.

marginal deformations of it[5]:

$$1)\quad W(X) = \frac{X^{m+1}}{m+1} - t\,X \tag{2.15}$$

$$2)\quad W(X) = \frac{t^{(m+1)/2}}{2^m\,(m+1)}\,T_{m+1}(t^{-1/2}X) = \frac{X^{m+1}}{m+1} + \sum_{k=1}^{[(m+1)/2]} c_k\,t^k\,X^{m+1-2k} \tag{2.16}$$

where $T_n(z)$ is the n–th Chebyshev polynomial of the first kind: $T_n(x) = \cos(n\,y)$ with $x = \cos y$. These two massive deformations preserve different discrete subgroups of the UV $U(1)$ R–symmetry: 1) the $\mathbb{Z}_m$ symmetry $X \mapsto e^{2\pi i/m}\,X$, and 2) the $\mathbb{Z}_2$ symmetry $X \mapsto -X$. In the model 1) the critical values in the W–plane are at the vertices of a regular m–gon

$$W_\alpha \equiv W(X^\alpha) = -\frac{m}{m+1}\,t^{1+1/m}\,e^{2\pi i\alpha/m}, \qquad \alpha = 1, 2, \dots, \alpha. \tag{2.17}$$

and the $\mathbb{Z}_m$ symmetry implies

$$N_{\alpha+\gamma,\beta+\gamma} = N_{\alpha,\beta} \tag{2.18}$$

(where the indices are identified mod m). In the model 2) the classical vacua $W'(X^\alpha) = 0$ are given by

$$\frac{t^{m/2}}{2^m}\,\frac{\sin((m+1)y^\alpha)}{\sin y^\alpha} = 0 \;\Rightarrow\; X^\alpha = t^{1/2}\,\cos\!\left(\frac{\pi\,\alpha}{m+1}\right), \qquad \alpha = 1, 2, \dots, m,$$

while the critical values are

$$W_\alpha = \frac{t^{(m+1)/2}}{2^m\,(m+1)}(-1)^\alpha. \tag{2.19}$$

In particular, $Z_{\alpha\beta} = 0$ for $\alpha = \beta \mod 2$. Then, in these sectors, the BPS states must have zero energy, hence be vacua, which is absurd because the boundary conditions break translational invariance. Therefore, in the sectors $\mathcal{H}_{\alpha\beta}$ with $\alpha \neq \beta$, we must

[5] These two particular $\mathcal{N} = 2$ massive theories may be both explicitly solved in terms of (two different families of) Painlevé transcendent (see [20]). The explicit solutions confirm the expectations from the symmetry argument.

have

$$N_{\alpha,\beta} = 0 \quad \text{for } \alpha \neq \beta \text{ and } \alpha \equiv \beta \mod 2. \tag{2.20}$$

Eqns.(2.18)(2.20) are incompatible (take *e.g.* $m = 3$) unless $N_{\alpha\beta} \equiv 0$ (which is impossible).

Therefore, the BPS multiplicities $N_{\alpha\beta}$ must jump *somewhere in coupling constant space.* The jumping locus should have real codimension 1, they are the *walls of marginal stability.* The WCF formula describe the exact jumps of the integers $N_{\alpha,\beta}$ when we cross the wall.

2.3 Refining the problem

In order to get a nice theory, and in particular to define WCF invariants which may be of use to classify the possible UV critical behaviors, we have to refine our problem and replace the non–negative integers $N_{\alpha,\beta}$ with *signed* multiplicities $\mu_{\alpha,\beta}$ such that $|\mu_{\alpha,\beta}| = N_{\alpha,\beta}$. Indeed, there exist pairs of models having the same set of absolute BPS multiplicities $\{N_{\alpha,\beta}\}$ but quite different UV physics (*e.g.* one model may have a minimal $\mathcal{N} = 2$ SCFT as UV limit while the other one is asymptotically free[6]). Thus the signs of $\mu_{\alpha,\beta}$ include crucial physical informations.

The reason the BPS multiplicities carry a *sign* is that the correct way of counting BPS states in a SUSY theory is trough the appropriate *supersymmetry indices.* The one appropriate to count BPS states in $2d$ is[7] [21]

$$Q_{\alpha,\beta}(T) = \lim_{L\to\infty} \frac{\beta}{2L} \, \mathrm{Tr}_{(\alpha,\beta)}\Big[(-1)^F \, F \, e^{-\beta H}\Big], \tag{2.21}$$

(to define the RHS we quantize the model at finite temperature $T = \beta^{-1}$ in a segment of length L with boundary conditions at the two ends corresponding, respectively, to the supersymmetric vacua $|\beta\rangle$ and $|\alpha\rangle$, and then take the infinite volume limit

[6] A simple example is the LG model with superpotential $W(X) = X^4 - \lambda X$ *versus* the LG model $W(X) = e^X - \lambda\, e^{-2X}$, where the field X takes values on the cylinder $X \sim X + 2\pi i$. Both models have a $\mathbb{Z}_3$ R–symmetry permuting the vacua and acting on the central charge. Both have one BPS soliton between each pair of vacua, but they are certainly different theories (the Hilbert space of the second model has winding sectors while the first one has not them).

[7] F is the Fermi number operator. The equation below are valid for the standard definition of F, such that the eigenvalues of $(-1)^F$ are real in all sectors $\mathcal{H}_{\alpha,\beta}$.

$L \to \infty$; the factor $1/L$ is needed to cancel the trivial infinite volume factor from the translational invariance of the b.c.).

Notice that PCT gives $Q_{\beta,\alpha} = -Q_{\alpha,\beta}$.

$Q_{\alpha,\beta}$ receives contributions only from BPS states. The deep reason behind this property is the independence of this index from the couplings entering in the D–terms; the energies of non–BPS states *do* depend on these couplings (and we may always go to a limit where all these states have infinite energies and thus decouple). Naively, only single–particle BPS states contribute to the index $Q_{\alpha,\beta}(T)$, but a careful computation shows that all *multi*–BPS states (with the appropriate quantum numbers) contribute to the the index $Q_{\alpha,\beta}(T)$. In particular,

$$N_{\alpha,\beta} \equiv 0 \quad \Rightarrow \quad Q_{\alpha,\beta}(T) \equiv 0. \tag{2.22}$$

In the zero temperature limit, $\beta \to \infty$, only the fundamental state of the sector $\mathcal{H}_{\alpha,\beta}$ contributes to the index[8]

$$Q_{\alpha,\beta}(T \to 0) \approx -\frac{\mu_{\alpha,\beta}}{2\pi} \left(|Z_{\alpha,\beta}|\, T^{-1}\right) K_1(|Z_{\alpha,\beta}|\, T^{-1}), \tag{2.23}$$

for a real integer $\mu_{\alpha,\beta}$ (with sign) which satisfies

$$\mu_{\alpha,\beta} + \mu_{\beta,\alpha} = 0. \tag{2.24}$$

It is clear that $N_{\alpha,\beta} = |\mu_{\alpha,\beta}|$; the sign of $\mu_{\alpha,\beta}$ is however physically relevant. We are free to redefine the sign of the α–th vacuum, with the effect $\{\mu_{\alpha,\beta}\} \to \{s_\alpha\, \mu_{\alpha,\beta}\, s_\beta\}$ where $s_\alpha = \pm 1$; thus it is the cohomology class of the $\mathbb{Z}_2$ 1–cocycle $\{\mathrm{sign}\, \mu_{\alpha,\beta}\}$ which carries invariant physical information.

We claim the following

Fact. ('Zamolodchikov theorem' [20, 21]) *Let $\lambda(T)$ be the largest eigenvalue of the Hermitian $m \times m$ matrix $i\, Q_{\alpha,\beta}(T)$ as a function of the temperature. Then*

1. *$\lambda(T)$ is monotonically increasing;*

[8] $K_1(\cdot)$ is the usual Bessel function.

2. the CFT central charge of the UV fixed point[9] *is*

$$\hat{c}_{UV} = 2 \lim_{T\to\infty} \lambda(T). \tag{2.25}$$

Combining with eqn.(2.22), we see that if the theory has no BPS–saturated states, $Q_{\alpha\beta} \equiv 0$ and $\hat{c}_{UV} = 0$, as claimed above.

It follows from the quantum action principle that $Q_{\alpha,\beta}(T)$ is a smooth function of the couplings $t_{\vec{n}}$. Then we may have discontinuity of the multiplicities $\mu_{\alpha,\beta}$ only at loci in coupling space where the single BPS states of may mix with multi–BPS states, *i.e.* places where multi–BPS states may give a contribution to the $T \to 0$ limit of $Q_{\alpha,\beta}(T)$ of the same order as the single BPS state of central charge $Z_{\alpha,\beta}$. Then both the energy and the central charge of the two configuration should match

$$Z_{\alpha,\beta} = Z_{\alpha,\alpha_1} + Z_{\alpha_1,\alpha_2} + \cdots + Z_{\alpha_\ell,\beta} \tag{2.26}$$

$$|Z_{\alpha,\beta}| = |Z_{\alpha,\alpha_1}| + |Z_{\alpha_1,\alpha_2}| + \cdots + |Z_{\alpha_\ell,\beta}| \tag{2.27}$$

which may be both satisfied if and only if the BPS phases $\theta_{\alpha,\beta} = -\arg Z_{\alpha,\beta}$ are all equal

$$\theta_{\alpha,\beta} = \theta_{\alpha,\alpha_1} = \theta_{\alpha_1,\alpha_2} = \cdots = \theta_{\alpha_\ell,\beta}. \tag{2.28}$$

Such a locus is called *a wall of marginal stability* and corresponding jump in the μ's is the *wall–crossing phenomenon*. [Generically, only two–particle BPS states degenerate with single–particle BPS states; however one may construct non–generic deformations in which higher multi–particle BPS states enter in the WCF].

We do not prove the above claims at this point since they are a special case of other relations we shall show below from a different viewpoint, which is more convenient in view of the generalization to $4d$. See section 3.

2.3.1 The quiver

The BPS multiplicities $\mu_{\alpha\,\beta}$ are conveniently represented by a *quiver*. The nodes of the quiver are labelled by the supersymmetric vacua of the model. If $\mu_{\alpha\,\beta} > 0$ we

[9] If the theory is asymptotically free, replace by the $\hat{c}_{UV}$ of the corresponding free theory.

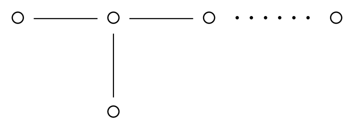
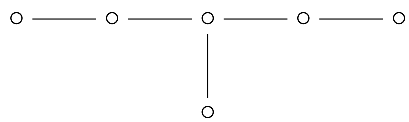
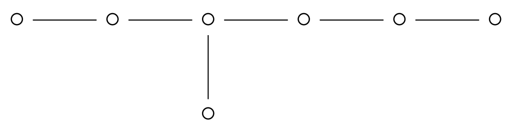
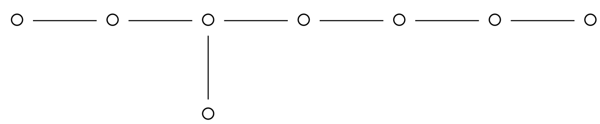
Name	$W_G(X_1, X_2, X_3)$	Dynkin diagram/Resolution graph
A_n	$X_1^2 + X_2^2 + X_3^{n+1}$	
D_n	$X_1^2 + X_2^2\,X_3 + X_3^{n-1}$	
E_6	$X_1^2 + X_2^3 + X_3^4$	
E_7	$X_1^2 + X_2^3 + X_2\,X_3^3$	
E_8	$X_1^2 + X_2^3 + X_3^5$	

Table 1: $\mathcal{N} = 2$ minimal SCFT $\equiv$ Du Val (*a.k.a.* minimal) singularities of hypersurfaces in $\mathbb{C}^3$. The fields X, Y, Z, entering quadratically in W are massive and may be integrated away (that is, they may be ignorated).

draw $\mu_{\alpha,\beta}$ arrows from the node α to the node β; if $\mu_{\alpha,\beta} < 0$ we draw $|\mu_{\alpha,\beta}|$ arrows from the node β to the node α. The resulting quiver is always 2–acyclic[10].

As we have remarked above, the signs of the BPS multiplicities $\mu_{\alpha,\beta}$ are defined only up to a $\mathbb{Z}_2$ coboundary, $\mu_{\alpha,\beta} \to s_\alpha\,\mu_{\alpha,\beta}\,s_\beta$. So the association of a quiver to a massive $(2,2)$ model is partly conventional. However, as we shall illustrate in the examples, one may fix these conventions in a canonical way, using some extra structure of the problem (which is more transparent from the $4d$ perspective) obtaining a better behaved correspondence.

[10] A quiver is 2–acyclic if it has no closed cycle of length ≤ 2, that is, if it has no loops (arrows starting and ending in the same node) and if all the arrows between any two nodes point in the same direction.

2.3.2 Examples: perturbed ADE minimal models [16, 17]

As a preparation to the $4d$ case, we consider the minimal $\mathcal{N} = 2$ SCFT deformed by (relevant) perturbations. The $\mathcal{N} = 2$ minimal models follow an ADE classification scheme, *i.e.* they are in one–to–one correspondence with the finite–dimensional simply–connected simply–laced Lie groups, see table 1.

The A_m minimal model: a special $\mathbb{Z}_2$ deformation

As mentioned above, the A_m minimal model may be realized as the Landau–Ginzurg (LG) model with superpotential $W(X) = X^{m+1}$. Let us deform it to to the Chebyshev superpotential $W(X) = \text{const.}\, T_{m+1}(X)$. Writing $X(s) = \cos y(s)$ for the BPS configuration, the BPS equations (2.14) require $W(X(s)) \equiv \cos[(m+1)y(s)]$ to be real and bounded in absolute value by 1; so, for all $s \in \mathbb{R}$, $y(s) \in \mathbb{R}$. Then a BPS configuration which starts at $s = -\infty$ from the α vacuum should end up to the vacua $\alpha \pm 1$ at $s = +\infty$ (as it is easy to check by solving explicitly the BPS equations[11]). Comparing with the last column of table 1, we see that the graph underlying[12] the quiver of the massive Chebyshev model is precisely the *Dynkin graph* of the Lie algebra A_m labelling its UV critical fixed point.

We choose conventions in such a way that the quiver is oriented in such a way that each node of the quiver is either a source (all arrows going out) or a sink (all arrows going in). This is the physically natural convention; *e.g.* the orientation of the arrows correlates with the direction of the forward (resp. backward) BPS gradient flow (2.12). Nothing depends (of course) of this convention.

General massive deformations of ADE minimal SCFT

More generally, for the minimal SCFT associated to the Lie group G (of ADE type) there is a 'simpler' massive deformation whose BPS quiver $\mathcal{Q}_G$ is the Dynkin diagram of the corresponding Lie algebra, again oriented in such a way that we have only sources and sinks[13]. Since we shall need this fact in $4d$, we sketch a purely geometric proof of this fact. It follows from Milnor's theorem.

[11] Or by solving the tt^* equations (they were solved explicitly, for this particular class of models, in sect. 8 of ref. [20]).

[12] The graph underlying a quiver is the unoriented graph obtained by forgetting the orientations of the arrows.

[13] This is always possible, since Dynkin graphs of finite Lie algebras are trees, and hence bipartite graphs.

Milnor theorem [22, 23]. *Let M_G ($G = ADE$) be the hypersurface in $\mathbb{C}^3$ of equation*

$$W_G(X_1, X_2, X_3) + \sum_{\vec{n}\cdot\vec{q}<1} t_{\vec{n}} X_1^{n_1}\, X_2^{n_2}\, X_3^{n_3} \equiv W(X_1, X_2, X_3) = \text{const.} \tag{2.29}$$

where the leading polynomial $W_G(X_1, X_2, X_3)$ is as in Table 1. Then $\dim H_2(M_G, \mathbb{Z}) = \operatorname{rank} G$*, and we may choose the generators $\gamma_i \in H_2(M_G, \mathbb{Z})$ in such a way that their intersection matrix is equal to* minus *the Cartan matrix C_G of the Lie algebra G,*

$$\gamma_i \cdot \gamma_j = -(C_G)_{ij}. \tag{2.30}$$

Then, consider two critical points (classical vacua), X^α and X^β, with critical value of the superpotential W_α and W_β, respectively. Let $M_{\alpha,\beta} \subset \mathbb{C}^3$ be the hypersurface of equation $W(X, Y, Z) = (W_\alpha + W_\beta)/2$. Consider the locus $S_\alpha \subset M_{\alpha,\beta}$ of points P in this hypersurface such that there exists a solution to the equation

$$g_{i\bar{j}} \partial_s \overline{X}^j = e^{i\theta_{\alpha,\beta}}\, \partial_i W \qquad \text{where } e^{-i\theta_{\alpha,\beta}} = \text{phase of } (W_\alpha - W_\beta), \tag{2.31}$$

starting at $s = 0$ from that point P and approaching the α–th classical vacuum as $s \to -\infty$. $S_\alpha \subset M_{\alpha,\beta}$ is a 2–sphere which represents a class in $H_2(M_G, \mathbb{Z})$ with[14] $[S_\alpha] \cdot [S_\alpha] = -2$. By construction, $|\mu_{\alpha,\beta}| = [S_\alpha] \cdot [S_\beta]$. On the other hand, comparing with the Milnor's theorem, we see that the cycle $[S_\alpha]$ is identified with a root of the Lie algebra of G, and hence the BPS multiplicities — whiuch are equal to the inner products of the corresponding roots — may be read directly from the Dynkin diagram of G.

However, as we saw above for the deformed A_3 model, the numbers $|\mu_{\alpha,\beta}|$ must jump somewhere in coupling constant space even for the (perturbed) ADE minimal theories. *How this happens?*

The Milnor theorem states that the 2–cycles vanishing at the classical vacua, $\gamma_\alpha \simeq [S_\alpha]$ form a basis of the root lattice Γ_G of G and $|\mu_{\alpha,\beta}| = |\gamma_\alpha \cdot \gamma_\beta|$. But it is not

[14] This follows from the following facts: 1) S_α is a holomorphic Π^1 embedded in the complex hypersurface. S_α may be contracted (indeed the BPS solution contracts its to a point, the α–th classical vacuum. Hence, by the adjunction formula, $-2 \leq [S_\alpha]^2 \leq -1$. Since its contraction leaves a singularity, $[S_\alpha]^2 = -2$ by the Castelnuovo criterion [24].

necessarily true that this basis corresponds to the set of *simple roots* of G. In some region in coupling constant space (as in the Chebyshev example) this is true, and the BPS quiver is just (an oriented version of) the Dynkin diagram of G. In other regions, we may get a different basis of Γ_G.

We may go from a basis of Γ_G to any other one by a sequence of elementary basis–mutations, in which we change just one element of the basis at the time. The Weyl group Weyl(G) acts transitively on the set of roots; and Weyl(G) is generated by the simple reflections. Therefore, the changes of bases in the root lattice are obtained by repeated applications of basic modifications, in which we change just one element of the basis, say γ_α, by replacing it with its image under an elementary reflection[15] $s_\beta \in \text{Weyl}(G)$, *i.e.*

$$\begin{aligned} &\gamma_\alpha \to \gamma'_\alpha = \gamma_\alpha - (\gamma_\alpha \cdot \gamma_\beta)\gamma_\beta \\ &\gamma_\delta \to \gamma'_\delta = \gamma_\delta \qquad \delta \neq \alpha. \end{aligned} \tag{2.32}$$

The corresponding change of the BPS multiplicities is

$$|\mu'_{\alpha,\delta}| = \gamma'_\alpha \cdot \gamma'_\delta = \gamma_\alpha \cdot \gamma_\delta - (\gamma_\alpha \cdot \gamma_\beta)(\gamma_\beta \cdot \gamma_\delta) = |\mu_{\alpha,\delta}| - |\mu_{\alpha,\beta}| \cdot |\mu_{\beta,\delta}| \tag{2.33}$$

which *must be* precisely the wall crossing formula [1] at the wall of marginally stability where

$$|Z_{\alpha,\delta}| = |Z_{\alpha,\beta}| + |Z_{\beta,\delta}| \tag{2.34}$$

(we shall prove this *physically* below).

Thus we see that the BPS chambers should, in this class of models, be identified with the Weyl chambers, and the walls of marginaly stability with the Weyl walls.

2.4 The Coxeter element of the quiver

At a *generic* value of the marginal couplings $t_{\vec{n}}$, the BPS quiver of a perturbed ADE model, looks quite different from a Dynkin diagram (however, it is still a graphical representation of the same Cartan matrix, albeit written in a different basis). One may ask whether there is some *invariant* of the set of quivers one obtains by repeated crossing at the various walls (this is called a *mutation–class* of quivers). According

[15] The action on $H_2(M, \mathbb{Z})$ of this change of basis is called a Picard–Lefshetz transformation [23].

to our general phylosophy such an invariant would be a property of the original undeformed $\mathcal{N}=2$ SCFT.

In fact, the identification of BPS walls and Weyl walls makes it clear that — for the deformed ADE models — the *wall–crossing invariants* are the same objects as the *Weyl chambers invariants.* These last invariants are well–known, and we may borrow them from the standard Lie algebra textbooks. The basic result is the following (cfr. Bourbaki, [25] §.**V**.6, PROPOSITION 1):

Proposition. *The conjugacy class in* Weyl(G) *of the Coxeter element* **Cox** *is independent of the ordered Weyl chamber.*

More generally, in the mathematics literature there is a definition of the Coxeter matrix, **Cox**, for a larger class of quivers, see, *e.g.*, **Definition III.3.14** and **Proposition VII.4.7** in ref. [26]. In these cases, **Cox** is always an *integral* $m \times m$ matrix, where m is the number of nodes in the quiver. The conjugacy class of **Cox** is again a quiver mutation–class invariant.

Below we shall introduce a generalized **Cox** for all two–dimensional $(2,2)$ theories, and an even more exoteric extension for the $4d$ $\mathcal{N}=2$ theories. In all cases, the requirement that **Cox** is invariant (up to conjugacy) determines the wall–crossing formula, which is precisely the (unique) jump in the BPS multiplicities which keeps **Cox** invariant. Thus, just as in the ADE examples, we have in full generality[16]

$$\boxed{\mathbf{Cox}\ \text{invariant up to conjugacy} \quad \Longleftrightarrow \quad \text{WCF}}$$

Thus, the (conjugacy class of) **Cox** is the fundamental invariant characterizing the UV SCFT fixed point. As always, physicists and mathematicians adopt opposite sign conventions for the same objects, so in physics we use the invariant[17] $\mathbb{M} := -\mathbf{Cox}$, which is called *the quantum monodromy.*

$\mathbb{M}$ may seem a rather unlikely SCFT invariant. We are more accustomed to conformal invariants like the central charge c, the conformal dimensions and $U(1)$

[16] We shall present general definitions, proofs, and further examples momentarily.

[17] In reality, in most of the physical applications we know only the *adjoint* action of $\mathbb{M}$ on the quantum operator–algebra. Since **Cox** and $-$**Cox** have the same adjoint action, we typically identify them.

charges of primary operators h_α and q_α, or the operator product coefficients $C_{\alpha\beta\gamma}$, *etc.* So we may ask: *How we extract the <u>conventional</u> SCFT invariants out of* $\mathbb{M}$*?*

The obvious conjugacy invariants of the $m \times m$ matrix $\mathbb{M} = -\mathbf{Cox}$ are its eigenvalues λ_i. In the particular case of the Dynkin quiver of G, $\mathcal{Q}_G$, ($G = ADE$) we know that the eigenvalues are [25]

$$\lambda_i = \exp\left\{2\pi i\left(\frac{\ell_i(G)}{h(G)} - \frac{1}{2}\right)\right\}, \qquad \left| \begin{array}{l} h(G) = \text{Coxeter number of } G, \\ \ell_i(G) = \text{exponents of } G. \end{array} \right. \tag{2.35}$$

It is trivial to show that the UV $U(1)$ charges of the elements ϕ_i of the chiral ring $\mathcal{R}_G$ are given by $(\ell_i(G) - 1)/h(G)$, $i = 1, \dots, \text{rank}(G)$. For instance, for A_m we have $W = X^{m+1}$, so $q(X) = 1/(m+1)$ and the chiral primaries are X^k, for $k = 0, 1, 2, \dots, m-1$, which have charge $k/(m+1)$. Then, by spectral flow (cfr. eqn.(2.5)), the $U(1)$ charges of the RR–vacua are

$$q(|\phi_i\rangle) = q(\phi_i) - \frac{\hat{c}}{2} = \frac{\ell_i(G)}{h(G)} - \frac{1}{2} \tag{2.36}$$

so, up conjugacy, we have the equality

$$\mathbb{M}(\mathcal{Q}_G) = e^{2\pi i R}\Big|_{\mathcal{H}_{E=0}}, \tag{2.37}$$

where R is the $U(1)$ symmetry generator in the UV SCFT. The RHS of eqn.(2.37) encodes the charges and conformal dimensions of the chiral primary fields which are the usual invariants of an $\mathcal{N} = 2$ SCFT. The above formula allows to pass from the superconformal invariant given by the conjugacy class of $\mathbb{M}$ to the standard ones.

2.5 The WC group

The above situation may be rephrased in a form which applies to any $\mathcal{N} = 2$ two–dimensional model. To each value of the BPS angle $\theta \in \{\theta_{\alpha,\beta}\}$ (which are *not* supposed to be necessarily distinct) we associate an element of $SL(m, \mathbb{Z})$

$$g(\theta)_{\alpha,\beta} = \delta_{\alpha,\beta} - \sum_{\theta_{\alpha,\beta}=\theta} \mu_{\alpha,\beta} \tag{2.38}$$

where in the RHS we sum over the BPS multiplicities of *all* BPS states with the *given* BPS phase θ. The subgroup of $SL(m,\mathbb{Z})$ generated by all such matrices $g(\theta)$ is the WC group.

We define the quantum monodromy matrix

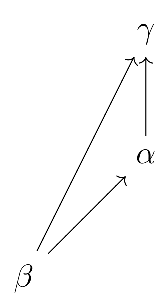
$$\mathbb{M} = T \prod_{\theta=0}^{2\pi} g(\theta), \tag{2.39}$$

where the symbol T means that the product of matrices is ordered in increasing 'time', where we identify the BPS angle θ with periodic time.

Our previous discussion is equivalent to the following **Fact**:

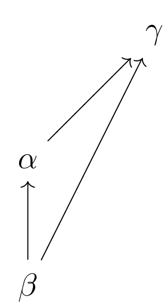
Fact. *The conjugacy class of* $\mathbb{M}$ *is a wall–crossing invariant, and* conversely *the WCF is just the condition of invariance of its conjugacy class. Its eigenvalues are equal to* $\exp(2\pi i q_\alpha)$ *with* $\{q_\alpha\}$ *the* $U(1)$ *charges of the supersymmetric vacua in the UV fixed theory.*

Let us check that we get the previous 'Weyl reflection'–like WCF formula for the particular case of a wall where just two BPS states get aligned (see figure)



$g(\theta_{\alpha\gamma})\, g(\theta_{\beta\gamma})\, g(\theta_{\beta\alpha})$

BPS geometry *before* the wall–crossing



$g(\theta_{\beta\alpha})\, g(\theta_{\beta\gamma})\, g(\theta_{\alpha\gamma})$

BPS geometry *after* the wall–crossing

Equating the elements of $SL(2,m)$ before and after the wall–crossing which are written below the corresponding picture (to avoid cluttering, we write $(1,2,3)$ instead of (α,β,γ))

$$\begin{pmatrix}1 & 0 & -\mu_{13}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -\mu_{23}\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -\mu_{21} & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=$$
$$=\begin{pmatrix}1 & 0 & 0\\ -\widetilde{\mu}_{21} & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -\widetilde{\mu}_{23}\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -\widetilde{\mu}_{13}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{2.40}$$

which gives $\mu_{13}=\widetilde{\mu}_{13}$, $\mu_{21}=\widetilde{\mu}_{21}$, and

$$\mu_{23}=\widetilde{\mu}_{23}-\widetilde{\mu}_{21}\,\widetilde{\mu}_{13}, \tag{2.41}$$

which is our WCF (now with signs). Notice, however, that the formulation given in eqns.(2.38)(2.39) and the **Fact** is more general that this example, since it holds also for the (non–generic) case in which many (even *infinitely* many) BPS states align at the wall.

2.6 The three problems

From eqn.(2.39) and the **Fact**, we see that the *integral* $SL(m,\mathbb{Z})$ matrix $\mathbb{M}\equiv T\prod_{\theta=0}^{2\pi} g(\theta)$ satisfies

$$\det\big[z-\mathbb{M}\big]=\prod_{\text{UV Ramond vacua}}\Big(z-e^{2\pi i\,q_\alpha}\Big), \tag{2.42}$$

where the LHS is determined by two discrete IR data:

IR1. the BPS multiplicities $\mu_{\alpha,\beta}\in\mathbb{Z}$;

IR2. a (cyclic) ordering of the BPS phase $\prec$.

Armed with eqn.(2.42) we may study three problems:

- **Direct problem.** We are given the UV SCFT, and in particular the $U(1)$ charges q_α. We ask for the set of possible BPS spectra $(\{\mu_{\alpha,\beta}\},\prec)$ that we can get by deforming the superconformal theory by relevant operators. In this problem, (2.42) is seen as an equation for the IR unknown $(\{\mu_{\alpha,\beta}\}, \prec)$;

- **Inverse problem.** We are given the BPS data $(\{\mu_{\alpha,\beta}\}, \prec)$ of a massive $\mathcal{N}=2$ theory. We determine the UV SCFT fixed point. In this case we see eqn.(2.42) as an equation for the unknowns q_α;
- **The classification problem [1].** We consider both $(\{\mu_{\alpha,\beta}\}, \prec)$ and q_α as unknowns, and look for the most general consistent $\mathcal{N}=2$ theory. The equation (2.42) says that the monodromy, $\mathbb{M} \in SL(m, \mathbb{Z})$, has a spectrum belonging to the unit circle $|\lambda_\alpha|^2 = 1$, and this gives a (set of) Diophantine equations for the *integral* entries of $\mathbb{M}$, generalizing the famous Markoff equation for $m = 3$ [1].

Both in two and four dimensions, we are interested in all these three problems. For what is known for the $2d$ classification problem see ref. [1].

3 Proofs: R–Twisting

Until now we have made many claims and suggested many natural correspondences, but we have given no proofs. It is time to fill the gap. We shall give an argument which works (essentially) also in $4d$. It also shed light why *physically* a relation as equation (2.42) should be true.

Clearly, it is enough to show the **Fact** stated in §. 2.5. It is a matter of connecting the IR (zero temperature) quantities $(\mu_{\alpha,\beta}, \prec)$ with the UV (infinite temperature) charges q_α.

In supersymmetry one usually works as follows: one writes the quantity one is interested to study as a protected index, independent of some class of parameters, and then compute it in some convenient limit of these parameters.

In the present case we have two possibilities: we can use the BPS counting index $Q_{\alpha,\beta}(T)$, and study its $T \to \infty$ limit. This is the way the WCF was obtained in the '90s [1, 21]. It is mathematically a very rigorous argument, but not physically illuminating. Its $4d$ counterpart is even more technically sophisticated (it can be done, in fact it was done by Gaiotto, Moore and Neitzke [4, 5] in the QFT case, and generalized to supergravity in ref. [27]). The second possibility is to devise an index computing our UV data, and then look for a deformation of it that localizes on the IR configurations we wish to capture.

3.1 The index I_k

We are interested in the eigenvalues of $\exp(2\pi i R)$ on the Ramond vacua of the UV SCFT. We claim that for all $k \in \mathbb{Z}$ the following objects

$$I_k = \mathrm{Tr}\Big[e^{2\pi i k\, R}\,(-1)^{(k-1)F}\, e^{-\beta H}\Big] \tag{3.1}$$

where $\mathrm{Tr}[\cdots]$ means the trace over the Hilbert space of the UV SCFT quantized on a circle (Ramond–Ramond sector) are SUSY *indices*. In fact the supercharges have $U(1)$ charges $\pm\frac{1}{2}$, therefore $\forall\, k \in \mathbb{Z}$, the operators $e^{2\pi i k R}(-1)^{kF}$ generate a symmetry *commuting* with all supercharges. I_k is then a standard twisted Witten index [28]. Only the Ramond vacua contribute to I_k, so

$$I_k = \sum_\alpha e^{2\pi i k\, q_\alpha}, \tag{3.2}$$

encode our UV data ($U(1)$ charges of chiral primaries).

Our claim is equivalent to the equality [14]

$$I_k = \mathrm{tr}\left(T \prod_{\theta=0}^{2\pi k} g(\theta)\right), \tag{3.3}$$

where $g(\theta) \in SL(2,\mathbb{Z})$ is as in §. 2.5 and $\mathrm{tr}(\cdots)$ stands for the trace of $m \times m$ matrices.

3.1.1 An illuminating remark

As the notation T for the operator–ordering in θ suggests, and as the arguments of section 3.3 would make precise, the angle θ is interpreted as a kind of periodic Euclidean time. Then the RHS of eqn.(3.3) may be interpreted as a kind of quantum partition function provided:

- our quantum system has a finite–dimensional Hilbert space $\mathcal{H} \simeq \mathbb{C}^m$, on which it acs an algebra of operators isomorphic to the matrix algebra $\mathbb{M}(m, \mathbb{C})$;

- we have the identification

$$\left(T \prod_{\theta=\theta_i}^{\theta_f} g(\theta) \right)_{\alpha\,\beta} \;=\; \langle \alpha, \theta_f \,|\, \beta, \theta_i \rangle \tag{3.4}$$

  where in the RHS we have the quantum evolution amplitude from the initial time θ_i to the final one θ_f (both sides of the equation are $m \times m$ matrices).

Now, for the LG models at hand, $\mathbb{C}^m$ is precisely the quantum Hilbert space of the *topological* field theory obtained by twisting it. We expect that this will remain true in the general case — including the four–dimensional one — namely that *the relevant quantum monodromy* $\mathbb{M}$ *will be realized as (a time–ordered product) of quantum operators acting on the Hilbert space of the corresponding topological quantum theory.* However, the amplitude itself is not the topological one (as it is obvious from the fact that it is explicitly time–dependent).

For other proofs of the identity (3.3) see refs. [1, 29].

3.2 A time–dependent supersymmetry

Since I_k is independent of the length of the circle L on which we quantize the theory, we may take $L \to 0$, *i.e.* the index I_k may be equivalently computed in the dimensional reduced to $1d$ theory, *i.e.* in SQM. The computation in $2d$ dimension is essentially the same (see the paper [14]), while the $1d$ notation allows to write more compact expressions.

The index I_k is represented by a path–integral with the boundary conditions

$$X_i(\beta) = e^{2\pi i k q_i}\, X_i(0) \tag{3.5}$$

$$\psi_i(\beta) = e^{2\pi i k q_i}\, \psi_i(0), \tag{3.6}$$

under which the critical quasi–homogeneous superpotential is periodic. Using the periodicity of W, we may rewrite the action in the form

$$\int_0^\beta dt\Big(\dot{X}_i + \overline{\partial_i W}\Big)\Big(\dot{X}_i^* + \partial_i W\Big) + \text{fermions.} \tag{3.7}$$

We may also rewrite the fields as

$$X_i(t) = e^{2\pi i k q_i t/\beta}\, Y_i(t) \tag{3.8}$$

$$\psi_i(t) = e^{2\pi i k q_i t/\beta}\, \chi_i(t) \tag{3.9}$$

where $Y_i(t)$ and $\chi_i(t)$ are strictly periodic. In the new variables, the bosonic action becomes

$$\int_0^\beta \sum_i \Big|\dot{Y}_i + \tfrac{2\pi i k q_i}{\beta} Y_i + e^{-2\pi i k t/\beta}\, \overline{\partial_i W(Y_i)}\Big|^2\, dt. \tag{3.10}$$

Even though the original path-integral was defined in the conformal theory, where W is quasi-homogeneous, *the path-integral we have ended up makes sense even when W is deformed by relevant terms away from the quasi-homogeneous limit.* We thus consider the above action for arbitrary deformed $W(Y_i)$[18].

The resulting path integral is an invariant *index*. This follows from the fact that it has still a supersymmetry, albeit not of the standard kind[19]: Our path integral is invariant under a time–dependent generalization of the Parisi–Sourlas supersymmetry [30–33].

Indeed, writing

$$h_i = \dot{Y}_i + \tfrac{2\pi i k q_i}{\beta} Y_i + e^{-2\pi i k t/\beta}\, \overline{\partial_i W(Y_i)} \tag{3.11}$$

[18]Rewriting this action in terms of X^i, ψ^i gives the usual LG Lagrangian with two differences: The superpotential W is now time-dependent, and we have an additional term given by $[-\int dt\, e^{i\alpha}\frac{\partial W}{\partial t} +$ *c.c.*].

[19] The system is not invariant under time translations, so the square of a supersymmetric transformation is not a translation in time.

the bosonic action (3.10) becomes simply

$$S_B = \int_0^\beta \sum_i |h_i|^2 \, dt, \tag{3.12}$$

while the fermionic action is

$$S_F = \int_0^\beta \begin{pmatrix} \tilde{\chi}_j & \tilde{\chi}_{\bar{j}} \end{pmatrix} \begin{pmatrix} \frac{\delta h_{\bar{j}}}{\delta Y_i} & \frac{\delta h_{\bar{j}}}{\delta Y_{\bar{i}}} \\ \frac{\delta h_j}{\delta Y_i} & \frac{\delta h_j}{\delta Y_{\bar{i}}} \end{pmatrix} \begin{pmatrix} \chi_i \\ \chi_{\bar{i}} \end{pmatrix} \, dt. \tag{3.13}$$

The action $S = S_B + S_F$ is invariant under the supersymmetry

$$\delta Y_i = \chi_i \, \epsilon \qquad \delta \overline{Y}_{\bar{i}} = \chi_{\bar{i}} \, \epsilon \tag{3.14}$$

$$\delta \tilde{\chi}_j = h_j \, \epsilon \qquad \delta \tilde{\chi}_{\bar{j}} = h^*_{\bar{j}} \, \epsilon \tag{3.15}$$

$$\delta \chi_j = 0 \qquad \delta \chi_{\bar{j}} = 0. \tag{3.16}$$

where h_j (resp. $h^*_{\bar{j}}$) stands for the RHS of eqn.(3.11) (resp. its complex conjugate).

It is elementary to show that the above path–integral — which is the Witten index for this peculiar *time–dependent* supersymmetric theory — is invariant under the continuous deformations of the superpotential which do not change its leading behavior at infinity.

3.3 Computing the path–integral

To compute the path integral, we replace $W(Y_i)$ by $\lambda \, W(Y_i)$ and exploit its independence from the continuous parameter λ.

In the limit $\lambda \to 0$, corresponding to the extreme UV regime, our path integral reproduces the original definition of the index I_k in the UV SCFT.

The limit $\lambda \to \infty$ corresponds to the IR limit and should give the same answer for the index I_k. The bosonic action is rewritten as

$$\int_0^\beta \lambda^2 \left| \frac{1}{\lambda}\frac{dY_i}{dt} + \tfrac{2\pi i k q_i}{\beta\lambda} Y_i + e^{-2\pi i k t/\beta}\, \overline{\partial_i W(Y_i)} \right|^2 dt, \tag{3.17}$$

and, in the limit $\lambda \to \infty$, the path–integral should be saturated by the configurations satisfying

$$\frac{1}{\lambda}\frac{dY_i}{dt} + \tfrac{2\pi i k q_i}{\beta\lambda} Y_i + e^{-2\pi i k t/\beta}\, \overline{\partial_i W(Y_i)} = 0. \tag{3.18}$$

Let us consider the (Euclidean) transition amplitudes

$$\langle Y_i', t_0 + t \,|\, Y_i, t_0 - t\rangle = \int_{Y_i\; t_0+t}^{Y_i',\; t_0-t} [d(\text{fields})]\, e^{-S_E} \tag{3.19}$$

from a configuration Y_i at time $t_0 - t$ to a configuration Y_i' at $t_0 + t$ defined by the above path integral.

As $\lambda \to \infty$ the RHS of eqn.(3.19) gets saturated by the solutions to eqn.(3.18) having the right boundary conditions. We set $\tau = 2\pi k\lambda(t - t_0)/\beta$, $\hat{\theta} = 2\pi k t_0/\beta$, and $\mu = \beta/(2\pi k)$. Then

$$\frac{dY_i}{d\tau} + i\frac{q_i}{\lambda} Y_i + \mu\, e^{-i(\tau/\lambda + \hat{\theta})}\, \overline{\partial_i W(Y_i)} = 0. \tag{3.20}$$

As $\lambda \to \infty$, this equation becomes the one describing a BPS soliton of phase $e^{i\hat{\theta}}$. Writing $Y_i(\tau) = e^{-iq_i\tau/\lambda} X_i(\tau)$,

$$\frac{dX_i}{d\tau} + \mu e^{-i\hat{\theta}} \sum_q e^{i(q-1)\tau/\lambda}\, \overline{\partial_i W_q(X_i)} = 0. \tag{3.21}$$

One looks for an asymptotic solution as $\lambda \to \infty$. If t in the LHS of (3.19) is much smaller than β, so that $\tau/\lambda \ll 1$ everywhere along the paths contributing to the path integral in the RHS of eqn.(3.19), the $O(\lambda^{-1})$ corrections to the BPS soliton equation (3.20) remain small as the fields go from one vacuum to the other. Measured in units of τ, the small time $2t$ becomes of order $O(\lambda)$, and hence infinitely long as $\lambda \to \infty$, so there is plenty of τ time to complete the transition from one asymptotic vacuum to the other one interpolated by the given BPS soliton. Indeed, there is plenty of rescaled time to accomodate a *chain* of BPS kinks, corresponding to multiple jumps from one vacuum to the next one, provided all the involved solitons have the (same) appropriate BPS phase. The saturating configuration differs from a classical vacuum by a quantity of order $O(e^{-M\tau}) = O(e^{-C\lambda(t-t_0)})$. Therefore, as $\lambda \to \infty$ the saturating configuration is a vacuum, *except* for a time region of size $O(1/\lambda)$ (measured in 'physical' time t) around the special times t_0 at which $e^{i\hat{\theta}} = e^{2\pi i k t_0/\beta}$ is the phase of a BPS soliton.

In conclusion, in the limit $\lambda \to \infty$, for almost all times t, t', with $t - t' \ll \beta$, one may effectively replace the quantum amplitude (3.19) with a finite $m \times m$ matrix (where m is the number of supersymmetric vacua)

$$\langle Y_i, t' | Y_i, t\rangle \to (g_{t',t})^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} - (\mu_{t',t})^{\alpha}{}_{\beta} \tag{3.22}$$

where the *integer* $-(\mu_{t',t})^{\alpha}{}_{\beta}$ counts *with signs* the number of different BPS kink *chains* connecting vacuum β to vacuum α though a sequence of intermediate vacua of the form

$$\beta \equiv \beta_0 \to \beta_1 \to \beta_2 \to \cdots \to \beta_m \equiv \alpha, \tag{3.23}$$

where each vacuum transition $\beta_i \to \beta_{i+1}$ is triggered by a BPS state of the appropriate phase which is monotonically increasing with 'time'. The *signs* arise because we are computing a (generalized) Witten index which is an integer, but may be a positive or negative (for the sign rule, see ref. [14]).

If the time interval $t' - t$ is small enough that the angular sector

$$\left(e^{2\pi i k t/\beta},\ e^{2\pi i k t'/\beta}\right) \tag{3.24}$$

contains only one BPS phase $e^{i\theta}$ we have

$$\left(g_{t',t}\right)^{\alpha}{}_{\beta} = \left(g(\theta)\right)^{\alpha}{}_{\beta}. \tag{3.25}$$

where $g(\theta) \in SL(m, \mathbb{Z})$ is as in eqn.(2.38), and the BPS phase θ gets effectively identified with Euclidean time up to a finite rescaling

$$\theta = \frac{2\pi k}{\beta}\, t. \tag{3.26}$$

A period in time, β, corresponds to a BPS angle $2\pi k$. Hence the full path integral

$$I_k = \mathrm{Tr}\langle Y, \beta | Y, 0\rangle = \mathrm{tr}\left(T \prod_{\theta=0}^{2\pi k} g(\theta) \right), \tag{3.27}$$

which is our claim. More precise derivations in references [1, 14, 29].

The RHS of the above equation is deformation–invariant and hence *does not jump* across a wall. However the various factors in the trace get reordered (because the BPS phase order changes) and the BPS multiplicities μ should jump to keep the full expression invariant. See §. 2.5.

The above argument is based on two main ideas:

1. twisting the original theory by the action of the R–symmetry in a time–dependent fashion;
2. the identification of time and BPS phase. Thus, the time–ordered prescription T, which is automatically implemented by the path integral, reproduces the order in increasing BPS phase of the BPS group elements $g(\theta)$ which is the tricky aspect in the WCF.

The same R–twisting strategy works in $d = 4$, at least formally. Before going to $4d$ we pause a while to discuss some mathematical technology we will be useful for analyzing the explicit $4d$ examples and which helps to clarify some deep, beautiful, and unexpected correspondence between the $2d$ and $4d$ theories.

4 Interlude: equivalence and products of quivers

4.1 Equivalence classes of 2–acyclic quivers

In the mathematics literature on (2–acyclic) quivers there is a notion of equivalence *(mutation–equivalence)*. When interpreted as BPS quivers, two mutation–equivalent quivers, are in particular wall–crossing equivalent, *i.e.* they arise from the same UV theory. The inverse statement is not true: mutation–equivalence is (generally) *finer* than $2d$ wall–crossing equivalence. However, the result of the mathematical theory are helpful in organizing the physical computation, and are crucial in making the connection between the $2d$ and $4d$ corresponding models. In the context of quiver gauge theory the mutation–equivalence of quivers is known as *Seiberg duality.*

One defines a *basic mutation* $m_\alpha(\mathcal{Q})$ of the quiver $\mathcal{Q}$ at the α–th vertex by performing the following two operations [34, 35]:

1. reverse all arrows incident with the vertex α;
2. for all vertices $\beta \neq \gamma$ distinct from α, modify the numbers of arrows between β and γ as shown in the box

$\mathcal{Q}$	$m_\alpha(\mathcal{Q})$
$\begin{tikzcd} \beta \arrow[rr, "r"] \arrow[dr, "s"'] & & \gamma \\ & \alpha \arrow[ur, "t"'] & \end{tikzcd}$	$\begin{tikzcd} \beta \arrow[rr, "r+st"] & & j \arrow[dl, "\gamma"] \\ & \alpha \arrow[ul, "s"] & \end{tikzcd}$
$\begin{tikzcd} \beta \arrow[rr, "r"] & & j \arrow[dl, "\gamma"] \\ & \alpha \arrow[ul, "s"] & \end{tikzcd}$	$\begin{tikzcd} \beta \arrow[rr, "r-st"] \arrow[dr, "s"'] & & \gamma \\ & \alpha \arrow[ur, "t"'] & \end{tikzcd}$

where r, s, t are non-negative integers, and an arrow $\beta \xrightarrow{l} \gamma$ with $l \geq 0$ means that l arrows go from β to γ while an arrow $i \xrightarrow{l} j$ with $l \leq 0$ means $|l|$ arrows going in the opposite direction.

Notice that the definition implies that m_α is an involution:

$$(m_\alpha)^2 = \text{identity}. \tag{4.1}$$

Two quivers are said to be in the *same mutation–class* (or mutation–equivalent) if one can be transformed into the other by a finite sequence of such quiver mutations. It is natural to identify quivers which differ only by a relabelling of the nodes. So a permutation of the node labels $\alpha = 1, 2, \ldots, m$ may be also seen as a quiver–mutation.

From the diagrams in the box, we see that a basic quiver mutation is, for a BPS quiver, a special case of a wall–crossing transformation, in which the node α crosses once all the possible marginal stability walls (in a direction defined by the orientation of the arrows).

Therefore, any wall–crossing invariant should be, in particular, a quiver mutation–invariant. This is why the theory of the latter invariants comes to help. The Coxeter element (and its variants) is an example.

4.1.1 A remark on the $\mathbb{Z}_2$ cocyle

As we have noticed above, the BPS quiver $\mathcal{Q}$ of a given $\mathcal{N}=2$ model is unique only up to $\mu_{\alpha,\beta} \to s_\alpha\, \mu_{\alpha,\beta}\, s_\beta$. In particular, if the underlying un–oriented graph has no closed loops — as it is the case for the BPS quivers of any deformed ADE minimal models — all orientations should be equivalent. Indeed, the corresponding quivers are in the same mutation–equivalence class, see **Proposition 9.2** in the second paper of ref. [34].

4.2 Products of 2–acyclic quivers

Given two (2–acyclic) quivers $\mathcal{Q}_1$ and $\mathcal{Q}_2$ we may form a new quiver by taking their 'product'. In fact, in the math literature one finds many different notions of product of quivers [35]. For our present purposes (in $4d$) two products are relevant the *square* one, $\mathcal{Q}_1 \,\square\, \mathcal{Q}_2$, and the *triangle* one $\mathcal{Q}_1 \boxtimes \mathcal{Q}_2$. The two products are in the same mutation class.

We may give a physical interpretation to these quiver products [15]. Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be the BPS quivers of the $\mathcal{N}=2$ specified by the superpotential $W_1(X_i)$ and $W_2(Y_j)$, respectively. The square product corresponds to the BPS quiver of the theory with decoupled superpotential $W(X_i) + e^{i\alpha}\, W(Y_j)$ where the phase $e^{i\alpha}$ is chosen general enough for no BPS state of the $W_1(X_i)$ model to be aligned with a BPS state of a the $W_2(Y_j)$ model. The canonical square product comes with a particular sign convention. *E.g.* if we have two ADE Dynkin quivers — which we denote by the same symbol as the corresponding Lie group — the quiver $G \,\square\, G'$ is oriented in such a way that each row/column of nodes sink and sources alternate, while the arrow form a closed cycle around each 'plaquete' (see figure 1 for the $A_m \,\square\, A_n$) case.

Notice that $G' \,\square\, G = (G \,\square\, G')^\vee$, where $\vee$ denotes the dual quiver (the one obtained by reversing all arrows).

$$
\begin{tikzcd}
1,1 \arrow[d] & 2,1 \arrow[l] \arrow[r] & 3,1 \arrow[d] & 4,1 \arrow[l] \arrow[r] & \cdots\cdots \arrow[r] & m,1 \arrow[d] \\
1,2 \arrow[r] & 2,2 \arrow[u] \arrow[d] & 3,2 \arrow[l] \arrow[r] & 4,2 \arrow[u] \arrow[d] & \cdots\cdots \arrow[l] & m,2 \arrow[l] \\
1,3 \arrow[u] \arrow[d, dotted, no head] & 2,3 \arrow[l] \arrow[r] \arrow[d, dotted, no head] & 3,3 \arrow[u] \arrow[d, dotted, no head] & 4,3 \arrow[l] \arrow[r] \arrow[d, dotted, no head] & \cdots\cdots \arrow[r] & m,3 \arrow[u] \arrow[d, dotted, no head] \\
1,n \arrow[r] & 2,n & 3,n \arrow[l] \arrow[r] & 4,n \arrow[r, dotted, no head] & {} & m,n \arrow[l]
\end{tikzcd}
$$

Figure 1: The $A_m \,\square\, A_n$ quiver.

The triangle product has a similar definition but with the phase $e^{i\alpha}$ chosen in such a way of getting suitable alignements between the BPS states of the two sectors of the decoupled model. For $A_m \boxtimes A_n$, see figure 2.

Thus, from the $2d$ QFT viewpoint, square and triangle products are obtained one from the other by (repeated) wall–crossing.

5 $4d$: a simple set–up [6, 15]

We start from M-theory on flat space

$$\mathbb{R}^4 \times \mathbb{C}^2_{x,y} \times (\mathbb{C}_z \times \mathbb{R}_p)$$

(subscripts give the coordinates we will use on the space), with an $M5$-brane wrapped on the locus

$$\mathbb{R}^4 \times \Sigma \times \{z = 0, p = 0\}$$

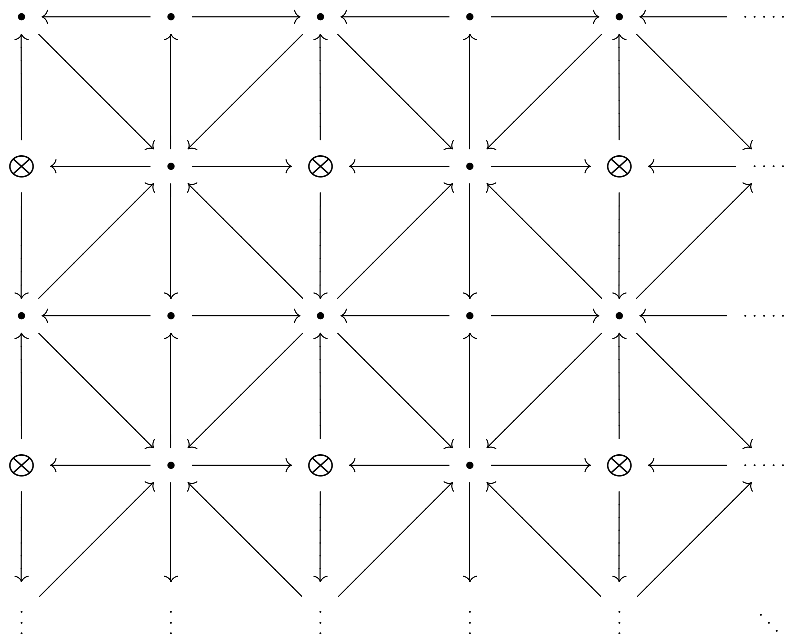


Figure 2: The $A_m \boxtimes A_n$ quiver. Mutation at the nodes $\otimes$ gives back $A_m \square A_n$.

where Σ is a (non-compact) Riemann surface

$$\Sigma = \{f(x,y) = 0\} \subset \mathbb{C}^2_{x,y}.$$

This gives an $\mathcal{N} = 2$ theory in the last $\mathbb{R}^4$ [36, 37], where Σ is the Seiberg-Witten curve, and $\lambda = ydx$ is the Seiberg-Witten differential.

We compactify on two circles, thus replacing $\mathbb{R}^4$ by

$$\mathbb{R}^4 \rightsquigarrow \mathbb{R}^2 \times S^1 \times S^1,$$

and further modify the geometry as follows. Let g be some symmetry of $\mathbb{C}^2_{x,y}$ preserving Σ (thus g induces a symmetry of the $\mathcal{N} = 2$ theory in $\mathbb{R}^4$.) As we go around the first circle, we make a twist of $\mathbb{C}^2_{x,y}$ by g. We write the resulting $11d$ space $\mathcal{M}_{11}$

and $M5$–brane world–volume as

$$\begin{aligned} \mathcal{M}_{11}\colon &\quad \mathbb{R}^2 \times \mathbf{S}^1 \times \big(S^1 \times_g \mathbb{C}^2_{x,y}\big) \times (\mathbb{C}_z \times \mathbb{R}_p) && (5.1)\\ M5\colon &\quad \mathbb{R}^2 \times \mathbf{S}^1 \times (S^1 \times_g \Sigma) \times \{z=0, p=0\}, && (5.2) \end{aligned}$$

where $S^1 \times_g$ stands for the mapping torus of g.

Let us view $\mathbf{S}^1$ as the small M-theory circle. Then, reducing to Type IIA, we get the $10d$ space and $D4$ brane world–volume

$$\begin{aligned} \mathcal{M}_{10}\colon &\quad \mathbb{R}^2 \times \big(S^1 \times_g \mathbb{C}^2_{x,y}\big) \times (\mathbb{C}_z \times \mathbb{R}_p) \equiv \big(\mathbb{R}^2 \times \mathbb{C}_z\big) \times K && (5.3)\\ D4\colon &\quad \mathbb{R}^2 \times (S^1 \times_g \Sigma) \times \{z=0, p=0\} = \big(\mathbb{R}^2 \times \{z=0\}\big) \times L. && (5.4) \end{aligned}$$

Where

$$\begin{aligned} K &= \big(S^1 \times_g \mathbb{C}^2_{x,y}\big) \times \mathbb{R}_p && (5.5)\\ L &= \big(S^1 \times_g \Sigma\big) \subset K. && (5.6) \end{aligned}$$

We wish do indentify K with a Calabi–Yau 3–fold, and $L \subset K$ with a Lagrangian submanifold. However we do not yet specify g or the Calabi-Yau structure on K.

Next, we now consider the topological A model on K, with a brane on L.

By the general properties of the topological strings, the open topological partition function may be written as a trace in the Hilbert space of Chern–Simons theory quantized on Σ,

$$Z^{\text{open}}_{\text{top}}(K, L) = \mathrm{Tr}_{\text{CS}}\big[M_g\big] \tag{5.7}$$

where the operator implementing the g–twist has the form

$$M_g = T\big(\prod_\alpha O^\alpha\big) \tag{5.8}$$

with the O^α the operators induced by the instanton corrections in the given g–twisted geometry.

By gluing together two such geometries we see that

$$M_{gg'} = M_g \, M_{g'} \tag{5.9}$$
$$M_{\mathrm{id}} = \mathbf{1}. \tag{5.10}$$

Then assume g has finite order,

$$g^r = 1.$$

we get

$$M_g^r = 1. \tag{5.11}$$

5.1 $\mathcal{N} = 2$ SCFT: first examples

Let us specialize to the case where Σ is *singular* and our theory in $\mathbb{R}^4$ is actually an $\mathcal{N} = 2$ SCFT. g is chosen to be an appropriate element of the R-symmetry group of the SCFT, or, when there are extra symmetries, a certain fractional power of it.

The simplest class of examples are the model with (singular) SW curves

$$f(x, y) = y^m - x^n \tag{5.12}$$

which correspond to $4d$ SCFTs. The case $(m, n) = (2, 3)$ and its generalization to $(2, n)$ are the original SCFTs studied by Argyres-Douglas [38]. In this case, following [39] we assign R-charges to the coordinates (x, y), in such a way that f is homogeneous (to get a symmetry) and $dx \wedge dy$ has charge 1 (to have a canonically normalized R–symmetry). We take $g = \exp(2\pi i R)$ (*i.e.* the $U(1)$ monodromy), or explicitly,

$$g \colon (x, y) \mapsto (\omega^m x, \omega^n y), \qquad \text{with } \omega^{m+n} = 1. \tag{5.13}$$

It is easy to see that the order r of g is precisely

$$r = \frac{(m + n)}{\gcd(m, n)}. \tag{5.14}$$

In Table 2 we list the orders r of the R–monodromies $g = \exp(2\pi i R)$ of all the singular SW curves $f(x, y) = 0$ where $f(x, y)$ is an ADE canonical singularities (of

singularity	$f(x,y)$	$r =$order g
A_{n-1}	$y^2 - x^n$	$\begin{cases} n & n \text{ odd} \\ n/2 & n \text{ even} \end{cases}$
D_n	$x^{n-1} + xy^2$	$\begin{cases} n & n \text{ odd} \\ n/2 & n \text{ even} \end{cases}$
E_6	$x^3 + y^4$	7
E_7	$x^3 + xy^3$	5
E_8	$x^3 + y^5$	8

Table 2: The order of the R–monodromy for the ADE Argyres–Douglas theories.

course, they are the same functions we used as UV superpotentials in the $2d$ minimal models). It is easy to show (for details see ref.) that the dimensions and R–charges of the SCFT operators are all integral multiples of $1/r$.

For defineteness, let us return to the example (5.12). Writing $\zeta = e^{p+i\varphi}$, K may be seen as a $\mathbb{C}^2$–bundle over $\mathbb{C}^\times_\zeta$, *locally* identified with $\mathbb{C}^\times_\zeta \times \mathbb{C}^2_{x,y}$, with the transition function

$$(\zeta, x, y) \sim (e^{2\pi i}\zeta, \omega^m x, \omega^n y).$$

Now we can specify the Calabi-Yau structure of K. We choose local complex coordinates to be

$$(w_1 = x + \overline{y},\ w_2 = y - \overline{x},\ \zeta)$$

with the holomorphic 3-form

$$\Omega = d\zeta \wedge dw_1 \wedge dw_2$$

and Kähler form

$$k = i\frac{d\zeta \wedge d\overline{\zeta}}{\zeta\overline{\zeta}} + i\, dw_i \wedge d\overline{w}_i.$$

Note that even though w_i are not global coordinates, Ω and k is globally defined (indeed $dw_i \wedge d\overline{w}_i = dx \wedge dy + d\bar{x} \wedge d\bar{y}$ which is invariant under g.)

One can check directly that our brane, given locally by

$$L = \Sigma \times \{|\zeta| = 1\} \subset \mathbb{C}^2_{x,y} \times \mathbb{C}^\times_\zeta,$$

is Lagrangian as required.

5.2 Deforming away from the SCFT point

The curve Σ is singular. We deform the theory away from the conformal point, replacing Σ with the smooth curve

$$\tilde{\Sigma} = \{y^m - x^n + \sum_{0 \le k < n,\, 0 \le l < m} c_{k,l} x^k y^l = 0\} \subset \mathbb{C}^2_{x,y}. \tag{5.15}$$

In the four-dimensional language $c_{k,l}$ are parameters which move the theory away from the conformal point (Coulomb branch vevs and/or mass deformations).

Naively this deformation would not be allowed: $\tilde{\Sigma}$ is not g-invariant, precisely because the R–symmetry is only present at the conformal point. The construction we used in $2d$ suggests a way around this difficulty: make a 'time'–dependent R–twist, *i.e.* replace f by

$$\tilde{f} = y^m - x^n + \sum_{0 \le k < n,\, 0 \le l < m} \zeta^{\frac{mn - km - ln}{m+n}} c_{k,l} x^k y^l.$$

The brane $L = \{\tilde{f} = 0\}$ is now nonsingular. It is convenient to change variables to

$$\tilde{x} = \zeta^{\frac{m}{n+m}} x, \qquad \tilde{y} = \zeta^{\frac{n}{n+m}} y.$$

The new $\tilde{x}, \tilde{y}$ are globally defined, and

$$\tilde{f}(\tilde{x}, \tilde{y}) = \zeta^{nm}(\tilde{y}^m - \tilde{x}^n + \sum_{0 \le k < n,\, 0 \le l < m} c_{k,l} \tilde{x}^k \tilde{y}^l).$$

So at any fixed ζ, L looks complex-analytically like a copy of the deformed Seiberg-Witten curve $\tilde{\Sigma}$ from (5.15). Moreover, at fixed ζ the Kähler form k restricts to

$$-i\, k = dw_i \wedge d\overline{w}_i = \zeta\, d\tilde{x} \wedge d\tilde{y} - \text{c.c.}$$

The BPS states correspond to holomorphic curves $\mathcal{C} \subset K$ ending on L — where "holomorphic" refers to the complex structure on K, in which w_1, w_2, ζ are complex coordinates. Such a holomorphic curve necessarily sits at some fixed $\zeta = e^{it}$, has boundary on $\tilde{\Sigma}$, and has[20]

$$\int_{\mathcal{C}} k = i\,\zeta \int_{\mathcal{C}} d\tilde{x} \wedge d\tilde{y} = i\,\zeta\ Z > 0, \tag{5.16}$$

where Z is the BPS central charge. We thus see that the phase of the corresponding BPS charge correlates with the phase of ζ, i.e. the choice of point θ_1 on S^1. Thus, the situation is exactly parallel to that in $2d$ R–twisting.

As before, let us label the various holomorphic curves $\mathcal{C}$ by the index α; they sit at various $\zeta_\alpha = e^{it_\alpha}$. The topological partition function is [15]

$$Z_{top}^{open}(K, L) = \mathrm{Tr}\, M_g$$

where

$$M_g = T\Big(\prod_\alpha O_\alpha(\gamma, s)(t_\alpha)\Big). \tag{5.17}$$

The RHS is a time–ordered product of contributions from the $4d$ BPS states. Since time and BPS phase are correlated (just as in $2d$ R–twisting) this, again, is order in BPS phase.

It remains to describe the operators $\mathcal{O}_\alpha$ which are operators acting on the Hilbert space of the $SL(1,\mathbb{C})$ Chern–Simons theory quantized on $\widetilde{\Sigma}$.

Remark. Notice that the $SL(1,\mathbb{C})$ Chern–Simons theory is precisely the TFT on whose Hilbert space our monodromy operators act, in perfect analogy with the $2d$ case.

The operator algebra is as follows. We have the charge lattice $\Gamma \in H_1(\widetilde{\Sigma}, \mathbb{Z})$ endowed with a skew–symmetric pairing $\langle \gamma, \gamma' \rangle$, namely the intersection. For each element $\gamma \in \Gamma$ we have an operator X_γ, corresponding to the $SL(1,\mathbb{C})$ holonomy along the cycle γ. The quantization of Chern–Simons produces the quantum torus

[20] To get this equation we use the fact that $dw_1 \wedge dw_2\big|_{\mathcal{C}} = 0$.

algebra associated with the lattive Γ, $\mathbb{T}_\Gamma$:

$$X_\gamma\, X_{\gamma'} = q^{\langle\gamma,\gamma'\rangle}\, X_{\gamma'}\, X_\gamma \tag{5.18}$$
$$X_{\gamma+\gamma'} = N[X_\gamma\, X_{\gamma'}] \equiv q^{-\langle\gamma,\gamma'\rangle}\, X_\gamma\, X_{\gamma'}, \tag{5.19}$$

where $q = e^{-\hbar}$ and $N[\cdots]$ denotes 'normal order'.

The standard results of topological string theory then gives

$$\mathcal{O}_\gamma = \Psi(q^{s_\gamma}\, X_\gamma; q)^{\Omega(\gamma, s_\gamma)} \ \in \mathbb{T}_\Gamma^\times \tag{5.20}$$

where (as in $2d$)

$$\Omega(\gamma, s_\gamma) = \begin{cases} \text{signed 'number' of BPS stateswith charge } \gamma, \\ \text{spin } s_\gamma, \text{ and BPS phase } \exp(i\theta_\gamma) \equiv Z_\gamma/|Z_\gamma| \end{cases}, \tag{5.21}$$

and

$$\Psi(X; q) = \prod_{n=0}^{\infty} (1 - q^{n+1/2} X) \qquad \text{\textit{'quantum dilogarithm'}}. \tag{5.22}$$

Putting everything together, the quantum monodromy $\mathbb{M}(q)$, which was a finite matrix in $2d$, is now a quantum operator in a separable Hilbert space,

$$\mathbb{M}(q) = T \prod\nolimits_{\theta_\gamma} \Psi(q^{s_\gamma}\, X_\gamma; q)^{\Omega(\gamma, s_\gamma)} \ \in \mathbb{T}_\Gamma^\times, \tag{5.23}$$

but has otherwise the same abstract structure. Formally, we just passed from $SL(m, \mathbb{Z})$ to a (subgroup of) $SL(\infty, \mathbb{Z})$.

As in the case of the $2d$ R–twisting, the conjugacy class of the quantum monodromy $\mathbb{M}(q)$ is independent of the (relevant/marginal) deformations of the theory, and hence a *wall–crossing invariant*. In fact, the (refined) Kontsevich–Soilbelmann (KS) wall–crossing formula for $d = 4$ $\mathcal{N} = 2$ field theory [2] is equivalent to this statement

$$\boxed{\text{KS WCF} \quad \Longleftrightarrow \quad \left| \begin{matrix} \text{the conjugacy class of } M(q) \text{ in } \mathbb{T}_\Gamma^\times \\ \text{is a chamber–mutation invariant} \end{matrix} \right.} \tag{5.24}$$

In this way R–twisting gives us a physical argument for the KS WCF.

Therefore, the basic wall–crossing invariants we wish to study are

$$\mathrm{Tr}_{\mathrm{CS}}\, \mathbb{M}(q)^{\ell}, \tag{5.25}$$

where *a priori* $\ell \in \mathbb{Z}$ but we shall show that in many models we get good invariants also taking certain fractional values of ℓ of the form $\ell = n/h$ where h is the Coxeter number of some symmetry group acting on the non–singular SW curve $\widetilde{\Sigma}$.

According to our general phylosophy, the invariants $\mathrm{Tr}_{\mathrm{CS}}\, \mathbb{M}(q)^{\ell}$ *should be seen as property of the parent UV SCFT.*

5.2.1 A word of caution

Contrary to the $2d$ case, in $4d$ there is no known precise construction of a time–dependent topological supersymmetry. Thus, while when Σ is singular our brane L is Lagrangian, this is no longer the case after the deformation. Then, while the argument for the invariance of the conjugacy class of $\mathbb{M}(q)$ is still valid (since it depends only on the IR asymptotics of the path integral[21]), it is no longer true that what we are computing is a protected index, as it is obvious from the fact that its depends explicitly on q, that is on $\hbar$.

Nevertheless, the claim that the invariants (5.25) probe properties of the UV SCFT rests on their universality under all its deformations, not on their nature of indices. This is a lucky turn: we get as WC invariants interesting *functions* (with remarkable modular properties) rather than boring *numbers*.

The drawback is that it is less clear how to interprete this invariants in the parent SCFT. However, we shall see that they have rather magical properties, and encode many physical properties of the critical theories, as well as yet to be descovered wonders.

[21] Besides, there are by now many proofs of the KS WCF, which is *verbatim* the invariance of $\mathbb{M}(q)$.

6 The three problems in $d = 4$

What is our quantum monodromy $\mathbb{M}(q)$ good for?

Again, as in $d = 2$ we may think of three different problems.

Direct problem. For a few cases, as for the perturbed A_n model the BPS spectrum is known (in some 'canonical' chamber, corresponding to the 'Chebyshev' chamber of $d = 2$) [40]. We may use eqn.(5.23) to determine the properties of the UV theory.

Inverse problem. For many models, including the ADE Argyres–Douglas (and the more general to be discussed momentarily) we know the order of the monodromy r. Explicitly $\mathbb{M}(q)^r = 1$ reads

$$\Big(T \prod\nolimits_\gamma \Psi(X_\gamma;\, q)^{\Omega(\gamma)}\Big)^r = \text{Identity on } \mathbb{T}_\Gamma^\times \text{ for all } q \in \mathbb{C}^\times \tag{6.1}$$

which is quite a remarkable and strong equation. In the inverse problem the unknowns are the BPS multiplicities $\Omega(\gamma)$ (and BPS phase–order). Solving this operator equation we get the BPS spectrum (up to WCF mutation equivalence) out of the totally trivial UV datum r. Using this technique we have confirmed the previously known BPS spectra and computed (infinitely many) new ones;

Classification. Find all *integers* $\{r, \Omega(\gamma)_{\gamma\in\Gamma}\}$ such that the above identity holds for all $q \in \mathbb{C}^\times$. In particular, it must hold as $q^{1/2} \to \pm 1$ (classical limits); already in this limit — and even restricting oneself to the very simplest situations — the classification problem seems to be related to many deep theories such as the Bloch group in Number Theory [41], the Nahm conjectures in RCFT [42], *ect.*

Before going to discuss the above problems, it is better to enlarge both our supply of (simple) examples and the set of objects to be studied. We do this in the next two subsections.

6.1 A more general class of models

The previous ADE AD theories may be seen as the result of the compactification of Type IIB on the local CY

$$f(x, y) + u^2 + v^2 = 0. \tag{6.2}$$

We may consider the compactification on more general hypersufaces

$$W(x, y, u, v) = 0 \tag{6.3}$$

If the 'superpotential' W is quasi–homogeneous, $W(\lambda^{q_i} x_i) = \lambda\, W(x_i)$, $(\forall \lambda \in \mathbb{C})$ the hypersurface is singular. It corresponds to a singularity at finite distance in CY moduli space iff $\hat{c} := 4 - 2\sum_{i=1}^{4} q_i < 2$ [43], where $\hat{c}$ is the central charge of the $2d$ model defined by the superpotential W. Thus, the condition is $\sum_{i=1}^{4} q_i > 1$.

The ADE singularities, eqn.(6.2) automatically satisfies the condition. A more general class of solutions to this condition are

$$W(x, y, u, v) = W_G(x, y) + W_{G'}(u, v) \tag{6.4}$$

where $G = ADE$ and $W_G(x, y)$ is the canonical singularity associated to the given simply–laced Lie algebra as in the second column of Table 2. We call the corresponding $4d$ field theories the (G, G') $\mathcal{N} = 2$ SCFT, the previous ADE models being (G, A_1). As we already saw, for the (A_{n-1}, A_{m-1}) model, $x^n - y^m + u^2 + v^2$ the order r of the quantum monodromy is

$$r = (n + m)/\gcd(n, m), \tag{6.5}$$

while in the general case we have

$$r = \begin{cases} \frac{1}{4}\frac{h(G)+h(G')}{\gcd(h(G)/2, h(G')/2)} & G, G' = A_1, D_{2n}, E_7, E_8 \\ \frac{h(G)+h(G')}{\gcd(h(G), h(G'))} & \text{otherwise,} \end{cases} \tag{6.6}$$

where $h(G)$ denotes the Coxeter number of the Lie algebra G.

6.2 The fractional monodromy $\mathbb{Y}(q)$

Assume we may deform the UV singular curve Σ_{sing} to a smooth curve (or, more generally, a smooth local 3–CY) while preserving some discrete $\mathbb{Z}_h$ R–symmetry. This assumption selects a class of deformations, and hence a class of compatible

BPS chambers, and the objects we costruct will be wall–crossing invariant only with respect to wall–crossing between these $\mathbb{Z}_h$–symmetric chambers. Nevertheless these restricted invariants will turn out to be very interesting.

The symmetry $\mathbb{Z}_h$ is required to act on the central charge as $Z \to e^{2\pi i/h} Z$ and on the quantum torus algebra (*i.e.* on the algebra of CS quantum operators) $\mathbb{T}_\Gamma$ by some operator U $X_\gamma \to U\, X_\gamma\, U^{-1}$ with $U^k = 1$.

For instance, for the smooth deformation of the (A_{h-1}, A_{n-1}) curve,

$$y^h + x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0, \tag{6.7}$$

we may consider the symmetry

$$y \to e^{2\pi i/h} y, \quad x \to x \qquad \lambda \equiv y\, dx \to e^{2\pi i/h}\, \lambda. \tag{6.8}$$

Then the element of the quantum KS group $\mathbb{T}_\Gamma^\times$ associated to the angular sector (θ', θ),

$$KS(\theta', \theta) = T \prod_{\theta_{\mathrm{BPS}}=\theta}^{\theta'} \Psi(X_\gamma; q)^{\Omega(\gamma)}, \tag{6.9}$$

has the $\mathbb{Z}_h$ symmetry property

$$KS(\theta' + 2\pi/h,\, \theta + 2\pi/k) = U\, KS(\theta',\, \theta)\, U^{-1}. \tag{6.10}$$

Taking the product of all KS elements for the angular sectors $(2\pi(k+1)/h, 2\pi k/h,$

$$\mathbb{M}(q) = \big(U^{-1}\, KS(2\pi/h, 0)\big)^h \tag{6.11}$$

The operator

$$\mathbb{Y}(q) = U^{-1}\, KS(2\pi/h, 0) \tag{6.12}$$

is what we mean by the quantum $1/h$–*monodromy* since

$$\mathbb{M}(q) = \mathbb{Y}(q)^h. \tag{6.13}$$

6.3 Back to the three problems

For the (simple[22]) class of (G, G') theories the seemingly impossibly hard 'Diophantine' equation[23]

$$\Big(T_{\prec} \prod\nolimits_{\gamma} \Psi(X_\gamma;\, q)^{\Omega(\gamma)}\Big)^r = \text{Identity on } \mathbb{T}_\Gamma^\times \text{ for all } q \in \mathbb{C}^\times \tag{6.14}$$

may be put in a 'canonical' form using the machinery of *cluster algebras* [34,35]. Not only this technology allows to manage the problem, but also uncovers some beautiful and totally unexpected connections with the corresponding $d = 2$ BPS/SCFT problems (see ref. [15] for a detailed discussion).

For simplicity, I illustrate the ideas in the language of the **inverse** problem. I limit myself to the (G, G') models (where $G, G' = ADE$) which are the only ones for which the WC invariants are understood in some detail.

In the **inverse** problem we are given the order r of the quantum monodromy $\mathbb{M}(q)$ (which, in fact, we know for all (G, G') models, see eqn.(6.6)) and we wish to compute the BPS multiplicities[24] $\Omega(\gamma)$ (for which no direct computation exists in the literature, but for the (A_m, A_1) models in some special BPS chamber [40]) as well as their phase–order $\prec$.

In practice, we may approach the formidable looking equation (6.14) in two steps:

1. find a particular solution;
2. argue that it is the only solution up to KS WCF.

Cluster algebras give a particular solution and suggests it is the unique one (up to WC equivalence).

[22] These $\mathcal{N} = 2$ theories are simple in what they have in the spectrum only BPS hypermultiplets and no BPS vector multiplets.

[23] We write $T_{\prec}$ to make explicit the dependence of the phase–order T from the cyclic ordering $\prec$ of the BPS phases.

[24] For this class of models, we may omit the datum of the *spin* of the BPS states since they are all 'scalar' hypermultiplets.

7 Cluster algebras

7.1 Quivers again

To the quantum torus algebra $\mathbb{T}_\Gamma$ defined by the lattice Γ with skew–symmetric pairing $\langle \cdot, \cdots \rangle \in \mathbb{Z}$ one naturally associates a 2–acyclic quiver Q_Γ by the rules

- to each generator of Γ, γ_i, $(i = 1, 2, \dots, \ell)$ associate a node of Q_Γ;
- if $\langle \gamma_i, \gamma_j \rangle > 0$ we draw $\langle \gamma_i, \gamma_j \rangle$ arrows from node i to node j;
- if $\langle \gamma_i, \gamma_j \rangle < 0$ we draw $|\langle \gamma_i, \gamma_j \rangle|$ arrows from node j to node i

In other words, one sets

$$B_{ij} = -B_{ji} \equiv \langle \gamma_i, \gamma_j \rangle \qquad \text{exchange matrix of the quiver } Q_\Gamma \tag{7.1}$$

For the particular case of an ADE Argyres–Douglas model, (G, A_1), the quiver Q_Γ is just the Dynkin quiver, that is the Dynkin diagram with arrows oriented in such a way that the nodes are, alternatively, sinks and sources. Needless to say, it is the *same* as the BPS quiver of the $2d$ $\mathcal{N} = 2$ model with the superpotential W equal to the defining equation of the Seiberg–Witten curve.

For the general (G, G') model the quiver (in some 'generic' chamber) is equal to the *square product* of the G, G' Dynkin quivers

$$G \,\square\, G' = (G' \,\square\, G)^\vee \qquad B^\vee = -B, \tag{7.2}$$

that is, again, the BPS quiver of the $2d$ LG model with superpotential the defining equation of the local CY 3–fold

$$W(x, y, u, v) = W_G(x, y) + W_{G'}(u, v) = 0. \tag{7.3}$$

To each node of Q_Γ we associate the corresponding generator of the quantum torus algebra $\mathbb{T}_\Gamma$

$$X_i \equiv X_{\gamma_i}. \tag{7.4}$$

7.2 Quantum mutations

A *quantum mutation* in $\mathbb{T}_\Gamma$ is an (ordered) sequence of **elementary** mutations [44].

The *elementary quantum mutation*, $\mathcal{Q}_k$, at the k–th node of Q_Γ is the composition of **two** transformations:

1. a *basic quiver mutation* m_k at the k–th vertex (as introduced in the context of $2d$ wall–crossing). As we discussed in the context of the Weyl group, m_k is identified with a identified with a change of the (root) lattice Γ. This change of basis in Γ corresponds to choosing a different set of generators of the algebra $\mathbb{T}_\Gamma$ according to the rule

$$X_i \to X_i' = q^{-\langle\gamma_i,\gamma_k\rangle\,[\langle\gamma_i,\gamma_k\rangle]_+/2}\, X_i\,(-X_k)^{[\langle\gamma_i,\gamma_k\rangle]_+} \qquad i\neq k \tag{7.5}$$

$$X_k \to X_k' = X_k^{-1} \qquad \text{where } [a]_+ \equiv \max\{a,0\}. \tag{7.6}$$

   Notice that m_k^2 is the identity on the quiver, but a non–trivial transformation on the set of generators of $\mathbb{T}_\Gamma$

$$m_k^2\colon X_i \mapsto (-1)^{\langle\gamma,\gamma_k\rangle}\, q^{-\langle\gamma_i,\gamma_k\rangle^2/2}\, X_i X_k^{\langle\gamma_i,\gamma_k\rangle}. \tag{7.7}$$

   Remark. m_k is not an automorphism of the algebra $\mathbb{T}_\Gamma$; a composition of m_k's is an algebra automorphism iff it is the identity on the underlying quiver since in this case it leaves invariant the commutation relations.

2. the adjoint action on $\mathbb{T}_\Gamma$ of the quantum dilogarithm[25] of $X_k \equiv X_{\gamma_k}$

$$X_\gamma \mapsto \Psi(X_k;q)^{-1}\, X_\gamma\, \Psi(X_k;q). \tag{7.8}$$

Thus, explicitly,

$$\begin{aligned}&\mathcal{Q}_k(X_\gamma) = \mathrm{Ad}[\Psi(X_k;q)^{-1}]\, m_k(X_\gamma) \equiv m_k\Big(\Psi(X_k^{-1};q)^{-1}\, X_\gamma\, \Psi(X_k^{-1};q)\Big)\\ &\text{i.e.}\quad \mathcal{Q}_k = m_k \circ \mathrm{Ad}[\Psi(X_k^{-1};q)].\end{aligned} \tag{7.9}$$

[25] In ref. [15] one uses a different sign convention for the argument of the quantum dilogarithm. The two conventions are related by a different choice of the sign of the square root $\sqrt{q}$. This explains the 'strange' sign in eqn.(7.5).

For our present purposes, it is important that the elementary quantum mutations are *involutions*[26] of $\mathbb{T}_\Gamma$

$$\mathcal{Q}_k^2 = \text{identity on } \mathbb{T}_\Gamma, \tag{7.10}$$

and hence the elementary mutations $\mathcal{Q}_k$ may be seen as the analogue of the 'elementary reflections' s_k generating the Weyl group Weyl(G) for the $2d$ deformed ADE models, cfr. §. 2.3.2.

From eqn.(7.9) one gets for a generic quantum mutation[27]

$$\overrightarrow{\prod_s} \mathcal{Q}_{k_s} = \overrightarrow{\prod_s} m_{k_s} \circ \mathrm{Ad}\left[\overrightarrow{\prod_s} \Psi\left(\left(\overrightarrow{\prod_{\ell\leq s}} m_{k_\ell}\right)^{-1}(X_{k_s});\ q\right)^{-1}\right], \tag{7.11}$$

and hence

$$\overrightarrow{\prod_s} \mathcal{Q}_{k_s}(X_\gamma) = \overrightarrow{\prod_s} m_{k_s}\big(\boldsymbol{U}^{-1}\, X_\gamma\, \boldsymbol{U}\big), \tag{7.12}$$

where $\boldsymbol{U} \in \mathbb{T}_\Gamma$ is a product of the form $\prod_s \Psi(X_{\gamma_s}; q)$'s for certain $\gamma_s \in \Gamma$. If, moreover, the quiver mutation $\overrightarrow{\prod_s} m_{k_s}$ acts as the identity on the underlying quiver, it can be represented as the adjoint action of some operator $\boldsymbol{V} \in \mathbb{T}_\Gamma$ and eqn.(7.12) reduces to

$$X_\gamma \mapsto \boldsymbol{V}\,\boldsymbol{U}^{-1}\, X_\gamma\, \boldsymbol{U}\,\boldsymbol{V}^{-1}. \tag{7.13}$$

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[26] PROOF: One has

$$\mathcal{Q}_k^2(X_\gamma) = m_k^2\Big(\Psi(X_k)^{-1}\Psi(X_k^{-1})^{-1} X_\gamma \Psi(X_k^{-1})\Psi(X_k)\Big) = m_k^2\Big(\Theta(-X_k;q)^{-1}\, X_\gamma\, \Theta(-X_k;q)\Big),$$

where $\Theta(z;q) = \sum_{m\in\mathbb{Z}} z^m q^{m^2/2}$ and we used Jacobi triple product. From the identity

$$X_\gamma\, \Theta(X_k;-q) = q^{-\langle\gamma,\gamma_k\rangle^2/2}\, \Theta(X_k;q)\, (-X_k)^{-\langle\gamma,\gamma_k\rangle}\, X_\gamma$$

and using eqn.(7.7) one gets

$$\mathcal{Q}_k^2(X_\gamma) = (-1)^{\langle\gamma,\gamma_k\rangle}\, q^{-\langle\gamma,\gamma_k\rangle^2/2}\, m_k^2\big(X_k^{-\langle\gamma,\gamma_k\rangle}\, X_\gamma\big) = q^{-\langle\gamma,\gamma_k\rangle}\, X_k^{-\langle\gamma,\gamma_k\rangle}\, X_\gamma\, X_k^{\langle\gamma,\gamma_k\rangle} \equiv X_\gamma.$$

[27] Here and below the over–arrow means that the non–commuting elementary mutations $\mathcal{Q}_k$ are supposed to be suitably ordered; an over–arrow in the opposite direction stands for the opposite order of the operators.

7.2.1 Classical limits

As $q^{1/2} \to \pm 1$, the algebra $\mathbb{T}_\Gamma$ becomes classical (*i.e.* commutative), in fact the algebra of functions on the character torus of Γ, and correspondingly the quantum mutations reduces to the **classical** ones (more widely studied in the mathematical literature) which are rational maps.

7.3 Cluster mutations *vs.* our 'Diophantine' equation

As shown in eqn.(7.12), a quantum cluster mutation

$$\boldsymbol{M} = \overrightarrow{\prod} \mathcal{Q}_k \tag{7.14}$$

may be written as the corresponding base change in $\mathbb{T}_\Gamma$ times the adjoint action of a product of Ψ's with arguments X_{γ_s} (for certain $\gamma_s \in \Gamma$)

$$\boldsymbol{M} = \left(\overrightarrow{\prod} m_k\right) \circ \prod_s \mathrm{Ad}^{-1}\big[\Psi(X_{\gamma_s}; q)\big]. \tag{7.15}$$

Then $\boldsymbol{M}$ is a solution to our 'Diophantine' problem, eqn.(6.14) — namely, it is a candidate quantum monodromy $\mathbb{M}(q)$ for the given $\mathcal{N} = 2$ model with UV order r — if and only if the following *three* conditions are fulfilled:

M1 (triviality of the underlying quiver–mutation): the quantum cluster–mutation $\boldsymbol{M}$ has to be written as a product of adjoint actions of $\Psi(X_\gamma; q)^{\pm 1}$, since the action of the quantum monodromy $\mathbb{M}(q)$, eqn.(5.23), on the operator algebra $\mathbb{T}_\Gamma$ is just an (ordered) product of adjoint actions of quantum operators of the form $\Psi(X_\gamma; q)^{\pm 1}$. In other words, the corresponding product of base changes/Seiberg dualities, $\overrightarrow{\prod} m_k$ (taken in the same order), must be the identity on $\mathbb{T}_\Gamma$;

M2 (BPS phase ordering): $\mathbb{M}(q)$ is not just any product of quantum dilogarithms, it is a T–ordered product. So, if we wish to reinteprete a quantum cluster–mutation $\boldsymbol{M}$ satisfying **M1** as a quantum monodromy, there must exist a consistent assignement of BPS phase $\theta(\gamma_s)$ for each charge vector $\gamma_s \in \Gamma$

appearing in the product

$$\boldsymbol{M} = \mathrm{Ad}^{-1}\Bigg[T \prod_s \Psi\big(X_{\gamma_s}; q\big)\Bigg] \tag{7.16}$$

such that[28]:

1. the cyclic order of the operators $\Psi(X_{\gamma_s}; q)$ in the RHS of eqn.(7.16) corresponds to the cyclic order $\prec$ of the corresponding phases $\theta(\gamma_s)$;
2. one has
$$\theta(n\,\gamma_s) = \begin{cases} \theta(\gamma_s) & \mod 2\pi \quad n > 0 \\ \theta(\gamma_s) + \pi & \mod 2\pi \quad n < 0. \end{cases} \tag{7.17}$$
In particular, a given factor $\Psi(X_{\gamma_s}; q)$ cannot appear more than once at different BPS angles;
3. assume $\gamma_{s_1} \prec \gamma_{s_2}$. Then, whenever we have a relation of the form
$$n_3\,\gamma_{s_3} = n_1\,\gamma_{s_1} + n_2\,\gamma_{s_2}, \qquad n_i \in \mathbb{N}_+, \tag{7.18}$$
we must have
$$\theta(\gamma_{s_1}) \prec \theta(\gamma_{s_3}) \prec \theta(\gamma_{s_2}). \tag{7.19}$$

M3 (finite order): $\boldsymbol{M}^r = \mathbf{1}$ on $\mathbb{T}_\Gamma$ for the given order $r \in N$, while $\boldsymbol{M}^k \neq \mathbf{1}$ for $1 \leq k < r$.

Remark. In practice, one finds solutions to conditions **M1** and **M3** and then checks if the solution also satisfies **M2**. Indeed, **M2** is needed to eliminate some spurious trivial solutions to conditions **M1**, **M3**. *E.g.* if $\boldsymbol{M}$ is a solution to **M1**, **M3** and $\ell \in \mathbb{N}$ is coprime to r, $\boldsymbol{M}^\ell$ is another solution. However it cannot be consistently interpreted as the quantum monodromy because the same BPS state of charge γ_s repeats itself ℓ times at ℓ different angles. The only consistent interpretation is that $\boldsymbol{M}^\ell$ is the product of the Kontsevich–Soibelman group elements in the angular sector $(0, 2\pi\,\ell)$, that is the ℓ–th power of the quantum monodromy.

[28] **Important remark:** two phase–orderings differing only by the inversion of two factors $\Psi(X_{\gamma_1})$, $\Psi(X_{\gamma_2})$ for commuting (that is, mutually local) elements X_{γ_1}, X_{γ_2} of $\mathbb{T}_\Gamma$ are considered to be equivalent.

7.4 Cluster–mutations and $\mathbb{Z}_h$–symmetric chambers

Furthermore, assume there is a cluster quantum mutation $\boldsymbol{Y} = \prod_s \mathcal{Q}_{k_s}$ such that

$$\boldsymbol{M} = \boldsymbol{Y}^h, \qquad h \in \mathbb{N}, \tag{7.20}$$

and, moreover, that the corresponding quiver–mutation $\prod_s m_{k_s}$ acts as the identity on the underlying quiver, so that it may be written as the adjoint action of some $U \in \mathbb{T}_\Gamma$. Then $\boldsymbol{Y}$ is naturally identified with a $1/h$–fractional monodromy. Indeed, the quantum mutation $\boldsymbol{Y}$ may be written as

$$\boldsymbol{Y}(X_\gamma) = \mathbb{Y}^{-1}\, X_\gamma\, \mathbb{Y}, \tag{7.21}$$

where (cfr. eqn.(7.15))

$$\begin{aligned} \mathbb{Y} &= (\text{a base change in } \mathbb{T}_\Gamma) \times (\text{ordered product of } \Psi(X_\gamma; q)^{\pm 1}) \\ &= \qquad\qquad U^{-1} \qquad\qquad \times \qquad\qquad KS(2\pi/h, 0), \end{aligned} \tag{7.22}$$

and moreover that which has precisely the form predicted for the fractional monodromy in presence of a $\mathbb{Z}_h$ unbroken symmetry acting on the central charge Z.

In presence of a candidate $\frac{1}{h}$–fractional monodromy $\boldsymbol{Y}$, to check **M2** it is enough to check the action of U and the order in the wedge $(0, 2\pi/h)$.

The cluster algebras are quite smart mathematical structures. Informally speaking, they combine in a cleaver way the $2d$ and the $4d$ wall–crossing transformations associated to the same 2–acyclic quiver, getting objects with nicer algebraic (and physical) properties. It seems that the cluster algebras *know* when there exists special BPS chambers with additional symmetries.

8 Relation to the Thermodynamical Bethe Ansatz

For the (simple) class of models we are considering, *i.e.* the (G, G') theories, the corresponding quivers $G \,\square\, G'$ are *simply–laced*, that is, two nodes are connected at most by *one* arrow.

The quantum mutations of the (quantum) cluster algebras associated to a simple–laced quiver are particularly simple. Indeed the following holds:

Lemma. *If Q_Γ is simply–laced (i.e. $|B_{ij}| \leq 1$) $\boldsymbol{M}^r = \mathbf{1}$ iff the corresponding* <u>*classical*</u> *cluster mutation $\boldsymbol{M}_{\text{class}} \equiv \boldsymbol{M}\big|_{q\to 1}$, has order r.*

Proof. See ref. [15].

Then, for the models we are studying, we may replace the quantum mutations by their classical counterpart. Equivalently, we may work with the *classical* rather than the quantum cluster algebras. This is a good new, since the mathematical literature discusses prevalently the classical algebras.

For the classical cluster algebra associated to the simply–laced quiver $G \,\Box\, G'$ (G, G' a pair of ADE Dynkin diagrams) there is a nice theorem, proven in the general case by Bernard Keller [45], which originally was formulated as a conjecture about two–dimensional quantum field theory by Zamolodchikov [46]. Of course, the R–twisting technique we are discussing is strong enough to give a proof (shorter than the mathematical one) of this theorem. Anyhow, to make a long story short, we simply state it as a theorem and refer to Keller for a proof. Special cases of this result where established by many people. A (perhaps incomplete) list includes [47].

Theorem. (Zamolodchikov [46], Keller [45]) *Let E (resp. O) be the* involutive *cluster mutation of the classical cluster algebra with quiver $G \,\Box\, G'$ equal o the product of all the elementary mutations $\mathcal{Q}_\alpha$ at the even (resp. odd) nodes of $G \,\Box\, G'$,*

$$E = \prod_{\textit{even nodes}} (\mathcal{Q}_\alpha)_{\text{class}} \tag{8.1}$$

$$O = \prod_{\textit{odd nodes}} (\mathcal{Q}_\alpha)_{\text{class}}, \tag{8.2}$$

(the classical mutation are identified with rational map; note that E, O are well defined since the elementary mutation at sites of the same parity commute).

Consider the sequence of rational functions $\{Y_{k,a}(s)\}_{s\in\mathbb{Z}}$ defined by the recursion relations

$$Y_{k,a}(s) = \begin{cases} O \cdot Y_{k,a}(s-1) & s \text{ odd} \qquad k = 1, \dots, \operatorname{rank} G \\ E \cdot Y_{k,a}(s-1) & s \text{ even} \qquad a = 1, \dots, \operatorname{rank} G' \end{cases} \tag{8.3}$$

are a solution to Zamolodchikov TBA Y–system associated to the pair of ADE

Dynkin graphs (G, G')

$$Y_{k,a}(s+1)\, Y_{k,a}(s-1) = \frac{\prod_{j\neq k}\big(1+Y_{j,a}(s)\big)^{-C_{kj}}}{\prod_{b\neq a}\big(1+Y_{k,b}(s)^{-1}\big)^{-C'_{kj}}} \tag{8.4}$$

where the matrices C_{kj} *and* $C'_{a,b}$ *are, respectively, the Cartan matrices of the Lie algebra* G *and* G'*.*

The mutation $\boldsymbol{Y}_{\text{class}} = E\,O$ *has order*

$$r = \begin{cases} \frac{1}{2}\Big(h(G)+h(G')\Big) & G, G' = A_1, D_{2n}, E_7, E_8 \\ h(G)+h(G') & \text{otherwise.} \end{cases} \tag{8.5}$$

(That is, the r*–fold reiteration of the rational map* $\boldsymbol{Y}_{\text{class}}$ *is the identity map).*

Remark 1. The Y–system corresponds to the functional equations satisfied (in the complex rapidity plane) by the solution to the TBA integral equations with kernel defined by the purely elastic S–matrix of the *integrable* two dimensional model associated to the given pair of Dynkin diagrams [46]. The Y-system, together with the regularity conditions in the complex rapidity plane uniquely determine the solutions to the integral equation [48].

Gaiotto, Moore and Neitzke [4] show that the hyperKähler geometry of the σ–model obtained by reducing the $\mathcal{N}=2$ $4d$ theory to $3d$ is determined by solutions to TBA having the same functional equations. Thus, from that viewpoint it is obvious that the classical monodromy of the $\mathcal{N}=2$ model is the same as the one defined by the corresponding Y–system.

Remark 2. Notice that $\boldsymbol{Y}_{\text{class}}$ is constructed just in the same way as the Coxeter element **Cox**. For simplicity, let us consider the single Dynkin diagram case, (G, A_1). One has

$$\mathbf{Cox}(G) = \prod_{\text{even nodes}} s_\alpha \prod_{\text{odd nodes}} s_\alpha \qquad\qquad s_\alpha^2 = \text{identity} \tag{8.6}$$

$$\boldsymbol{Y}_{\text{class}} = \prod_{\text{even nodes}} \mathcal{Q}_\alpha \prod_{\text{odd nodes}} \mathcal{Q}_\alpha \qquad\qquad \mathcal{Q}_\alpha^2 = \text{identity}, \tag{8.7}$$

where s_α are the simple reflections in the Weyl group of G. Notice that both s_α,

s_β and, respectively, $\mathcal{Q}_\alpha$, $\mathcal{Q}_\beta$ commute for α, β of the same parity. The structural analogy of the two expressions is manifest.

8.1 Solution to the inverse problem

Since all our quivers are simply–laced, by the lemma, the order to of any quantum cluster mutation $\boldsymbol{L}$ is the same as that of its classical counterpart $\boldsymbol{L}_{\text{class}}$ (which is just a rational map). Combining with the theorem, we see that — for all pairs (G, G') — the quantum cluster mutation

$$\boldsymbol{M} = \boldsymbol{Y}^{h(G')}, \tag{8.8}$$

(where $\boldsymbol{Y}$ is the quantum version of $\boldsymbol{Y}_{\text{class}}$ defined in the theorem) has precisely the order r predicted by the UV singular geometry, see eqn.(6.6). Then we have the equality

$$\boldsymbol{M}^r = \text{Identity} \qquad \text{in } \mathrm{End}(\mathbb{T}_\Gamma), \tag{8.9}$$

where we mean that the *adjoint action* of the LHS is the identity in $\mathbb{T}_\Gamma$ (which means that the operator $\boldsymbol{Y}^r$ is central in $\mathbb{T}_\Gamma$, hence it acts as a c–number in each irreducible representation of the quantum torus algebra).

Comparing with the discussion following eqn.(7.14), we see that $\boldsymbol{M}$ in eqn.(8.8) is a solution to 'Diophantine' problem, eqn.(6.14), $\Leftrightarrow$ $\boldsymbol{M}$ may be written in the form $\mathrm{Ad} \prod \Psi(X_\gamma; q)^{\pm 1}$, that is, if and only if $\Leftrightarrow$ the associated quiver mutation is the **identity**.

We have observed before that the quiver mutation is a special case of the $2d$ wall–crossing (together with a peculiar sign convention). It is then not surprising to learn that (see the original paper [15] for the details of the combinatorics)

$$\boldsymbol{Y} = \mathbf{Cox}(G') \cdot \mathrm{Ad}\Big(\text{a product of } \Psi(X_\gamma; q)\text{'s}\Big) \tag{8.10}$$

where the quiver mutation is precisely the Coxeter element of the Lie algebra G', $\mathbf{Cox}(G')$. Then, by the very definition of the Coxeter number,

$$\mathbf{Cox}(G')^{h(G')} = \text{Identity}, \tag{8.11}$$

while $\mathbf{Cox}(G')^\ell \neq 1$ for all $1 \leq \ell < h(G')$. Then $\boldsymbol{M}$ is a solution to conditions **M1**, **M3** of section 7.3. We shall check in some examples below that the phase–ordering **M2** also works.

Conclusions:

1. the quantum cluster mutation $\boldsymbol{M}$ is the same as quantum monodromy $\mathbb{M}(q)$ of the (G, G') model. (*Same* meaning that the two have the same adjoint action on $\mathbb{T}_\Gamma$);
2. the quantum cluster mutation $\boldsymbol{Y}$ is the $\frac{1}{h(G')}$–fractional monodromy of the (G, G') model. Therefore:
   (a) the $\mathcal{N} = 2$ theories obtained by compactifying Type IIB on a local (G, G') 3–CY geometry have $\mathbb{Z}_{h(G')}$–symmetric BPS chambers;
   (b) the operator U implementing the $\mathbb{Z}_{h(G')}$ symmetry on the Hilbert space is $\mathbf{Cox}(G')^{-1}$.

8.1.1 The BPS spectrum $\Omega(\gamma)$

Now we are ready to state the solution to the **inverse** problem for (G, G'): Since we have an explicit formula for $\mathbb{M}(q)$, namely

$$\mathbb{M}(q) = \Big(\prod_{\text{even}} \mathcal{Q}_\alpha \prod_{\text{odd}} \mathcal{Q}_\alpha\Big)^{h(G')}, \tag{8.12}$$

to extract the BPS multiplicities we have just to expand out the products, rewriting the RHS in the standard form

$$T \prod\nolimits_\gamma \Psi(X_\gamma;\, q)^{\Omega(\gamma)}. \tag{8.13}$$

Doing the exercise, one finds agreement with direct computation of BPS spectrum for (A_m, A_1). For the general (G, G') case, one finds[29] that *in some* $\mathbb{Z}_{h(G')}$*–symmetric*

[29] The following statements are based (rather than on proof) on very strong evidence, including explicit computations in many classes of models (see ref. [15]).

BPS chamber, to each node of the G Dynkin diagram there are associated BPS states in one–to–one correspondence with the roots of the Lie algebra G′. In these special chambers, the total number of BPS half–hypermultiplets is

$$\#\,\mathrm{BPS} = \mathrm{rank}\,G \cdot \mathrm{rank}\,G' \cdot h(G') = \mathrm{rank}\,G \cdot \#\{\text{roots of } G'\}. \tag{8.14}$$

Let α_a, $a = 1, 2, \dots, \mathrm{rank}(G)$, (resp. $\widetilde{\alpha}_i$, $i = 1, 2, \dots, \mathrm{rank}(G')$) be the simple roots of the Lie algebra G (resp. G'). We write $\alpha_a \,\square\, \widetilde{\alpha}_i$ for the generator of the charge lattice $\Gamma_{G\square G'}$ corresponding to the (a, i) node of the quiver $G \,\square\, G'$. Then the BPS multiplicities in the $\mathbb{Z}_h(G')$–symmetric chamber are

$$\Omega\Big(\sum\nolimits_{a,i} c_{a,i}\, \alpha_a \,\square\, \widetilde{\alpha}_i\Big) = \begin{cases} 1 & \text{if } c_{a,i} = \delta_{a,a_0}\, c_i \text{ and } \sum_i c_i\, \widetilde{\alpha}_i \text{ is a root of } G' \\ 0 & \text{otherwise.} \end{cases} \tag{8.15}$$

Remark. This result rests heavily on the following result about Weyl groups: **Theorem V.§ 6.1.(ii)**, Exercise V.§ 6.1), and **Proposition VI.§ 1.33** of ref. [25].

8.1.2 Examples

★ **The (A_2, A_1) model.** In the $\mathbb{Z}_2$ chamber, we have $2\times 2 = 4$ BPS $\frac{1}{2}$–hypermultiplets, or two BPS hypermultiplets, of charges $\pm\alpha_1$, $\pm\alpha_2$, that is, in terms of eletric/magnetic charges

$$(e, m) = \pm(1, 0),\ \pm(0, 1). \tag{8.16}$$

Notice that $\mathbb{Z}_2$ acts on the BPS states as PCT, namely $Z \mapsto -Z$. In fact, in this chamber we may have a full $\mathbb{Z}_4$ symmetry acting transitively on the BPS hapf–hypermultiplets: Consider, say, the smooth SW curve $y^2 - x^3 - x = 0$ which is invariant under the $\mathbb{Z}_4$ symmetry

$$y \mapsto -i\,y, \quad x \mapsto -x, \qquad\qquad y\,dx \mapsto i\,y\,dx \tag{8.17}$$

under which the central charge $Z \mapsto i\,Z$. In particular, we see that (up to a conventional choice of origin) the BPS phases are $\theta_{\mathrm{BPS}} = k\pi/2$ (in agreement with the explicit computations [40]).

Correspondingly, there exists a cluster mutation $\boldsymbol{J}$ such that $\boldsymbol{M} = \boldsymbol{J}^4$ which may be written in the form

$$(\text{quiver mutation}) \times (\text{a single } \Psi(\cdot; q), \tag{8.18}$$

meaning that there is precisely one BPS state with phase in each angular sector $(\theta, \theta + \frac{\pi}{2})$. In fact

$$\boldsymbol{J} = \Psi(X_2; q)\, \mathfrak{F}^{-1} \equiv \mathfrak{F}^{-1}\, \Psi(X_1^{-1}; q), \tag{8.19}$$

where $\mathfrak{F}$ is the 'Fourier' transform[30] on $\mathbb{T}_{A_2}$ quiver–mutation defined by its adjoint action on $\mathbb{T}_{A_2}$

$$\mathfrak{F}^{-1}\, X_1\, \mathfrak{F} = X_2^{-1}, \qquad\qquad \mathfrak{F}^{-1}\, X_2\, \mathfrak{F} = X_1. \tag{8.20}$$

★ **<u>The (A_{2n}, A_1) models.</u>** Again, the $\mathbb{Z}_2$ chamber may be promoted to a $\mathbb{Z}_4$ chamber by considering, say, the smooth SW curve

$$y^2 - x^{2n+1} - x = 0 \tag{8.21}$$

$$y \mapsto -i\, y, \quad x \mapsto -x, \qquad y\, dx \mapsto i\, y\, dx. \tag{8.22}$$

We have $2n \times 2$ BPS half–hypermultiplets whose charges are $\pm$ the generators of the lattice Γ associated with the nodes of the A_{2n} Dynkin diagram. Again the BPS phases are as in the (A_2, A_1) case, see [40].

★ **<u>The (A_1, A_2) model.</u>** For the $\mathbb{Z}_3$ chamber, corresponding to the quiver (A_1, A_2) we have 1×6 half–hypers, or three hypermultiplets permuted by $\mathbb{Z}_3$. They have charges

$$\pm(1, 0), \quad \pm(1, 1), \quad \pm(0, 1). \tag{8.23}$$

This model has just two distinct BPS chambers corresponding, respectively, to the quivers $A_2 \square A_1$ and $A_1 \square A_2$. Then the above $\mathbb{Z}_3$ chamber should contain the $\mathbb{Z}_6$–symmetric one corresponding to the smooth $\mathbb{Z}_6$ SW curve $y^2 + x^3 + 1 = 0$. In fact, it is easy to see that there is a (generalized) cluster mutation $\boldsymbol{H}$ such that $\boldsymbol{M} = \boldsymbol{H}^6$, of the form $\boldsymbol{H} = U\, \Psi(X_1; q)$, with $U^6 = 1$.

[30] If the quantum torus algebra $\mathbb{T}_{A_2}$ is realized as the Weyl algebra $X_1 = e^{ix}$, $X_2 = e^{ip}$, $q = e^{i\hbar}$, and the canonical variables x, p are realized on $L^2(\mathbb{R})$, this is the usual Fourier transform.

8.2 'Level–rank' duality

The local 3–CY geometry

$$W_G(x,y) + W_{G'}(u,v) = 0 \tag{8.24}$$

is manifestly symmetric under $G \leftrightarrow G'$. However, $\mathbb{M} \equiv \boldsymbol{Y}^{h(G')}$ is NOT manifestly invariant.

The reason why the situation looks asymmetric in G and G' is because the particular quivers $G \,\square\, G'$ and $G' \,\square\, G$ correspond to two *different* BPS chambers with, respectively, $\mathbb{Z}_{h(G')}$ and $\mathbb{Z}_{h(G)}$ symmetry (in the same quiver mutation–class we have also other quivers corresponding to BPS chambers with other symmetry properties, *e.g.* the triangle–product quivers). For instance, as we saw in the examples of §. 8.1.2, the $A_2 \,\square\, A_1$ quiver corresponds to a chamber with just 2 BPS states, while the $A_1 \,\square\, A_2$ chamber has 3 BPS states.

However, the quantum monodromy should be invariant **up to conjugacy** under WC and hence also under $G \leftrightarrow G'$. *Is this true?*

Yes, it is. Let us recall that $G' \,\square\, G = (G \,\square\, G')^\vee$. Then,

$$\boldsymbol{Y}_{\text{class}} = E\,O, \qquad \boldsymbol{Y}^\vee_{\text{class}} = O\,E = \boldsymbol{Y}^{-1}_{\text{class}} \tag{8.25}$$

which, together with the periodicity $\boldsymbol{Y}^{h(G)+h(G')} = 1$, gives

$$\begin{aligned} \Rightarrow \quad \boldsymbol{M}_{G' \square G} = (\boldsymbol{Y}^\vee)^{h(G)} = \boldsymbol{Y}^{-h(G)} = \boldsymbol{Y}^{h(G)+h(G')-h(G)} = \\ = \boldsymbol{Y}^{h(G')} = \boldsymbol{M}_{G \square G'} \end{aligned} \tag{8.26}$$

Example. The relation $\boldsymbol{M}_{(A_2,A_1)} = \boldsymbol{M}_{(A_1,A_2)}$ (or, more precisely, the corresponding relation for the *half*–monodromies in the $\mathbb{Z}_2$ and $\mathbb{Z}_6$ chambers) gives the famous **pentagonal identity** for the quantum dilogarithm.

The general relation $M_{(G,G')} = M_{(G',G)}$ then gives higher quantum dilogarithm identities. Taking the classical limit, $q \to 1$, one gets the relations in the Bloch group of Number Theory.

Even more interesting is the interpretation of the $G \leftrightarrow G'$ duality from the point of view of the connection to $2d$ RCFT's (that is the topic of the next section). As we shall see, in that context the obvious geometric symmetry $G \leftrightarrow G'$ becomes the celebrated *level–rank duality*.

8.3 More fun with the (A_2, A_1) model

In these notes we have stressed the analogy between the quantum monodromy $\mathbb{M}(q)$ and the Coxeter element in the Weyl group of a Lie algebra. One may wonder whether there is a more direct connection. In particular: *For the single Dynkin graph models,* (G, A_1)*, is there any relationship between* $\mathbb{M}_G$ *and* $\mathbf{Cox}(G)$*?*

At first, one may think that such a relation should be necessarily subtle since $\mathbf{Cox}(G)$ has order $h(G)$, while $\mathbb{M}_G$ has order $h(G) + h(A_1) \equiv h(G) + 2$. In fact, the relation exists and it is quite elementary. For simplicity, we illustrate it in the special case of $G = A_2$.

In this case it is convenient to introduce a sequence of operators X_k, $k \in \mathbb{Z}$ defined by the 2–terms recursion relation

$$X_{k+1}\, X_{k-1} = 1 - q^{1/2}\, X_k, \qquad k \in \mathbb{Z} \tag{8.27}$$

together with the initial condition that X_1, X_2 are the original generators of $\mathbb{T}_\Gamma$ associated with the nodes of the A_2 quiver

$$\textcircled{1} \longleftarrow \textcircled{2}. \tag{8.28}$$

One checks that these operators satisfy the commutation relations and the property

$$X_{k+1}\, X_k = q\, X_k\, X_{k+1}. \tag{8.29}$$

Notice that the recursion relation (8.27) is just the quantum version of the A_2 Y–system: it reduces to it in the classical limit $q^{1/2} \to -1$, while for general q, the quantum X_k is obtained from the classical one, $\boldsymbol{Y}^{k-1}_{\text{class}}(X_1)$, by taking the same Laurent polynomial in X_1, X_2 with the *normal order* prescription for the non–commuting operators. Therefore one has

$$X_{k+1} = \mathbb{M}^{-1}\, X_k\, \mathbb{M}. \tag{8.30}$$

In particular, the operator sequence is periodic[31] of period 5: $X_{k+5} \equiv X_k$.

Imagine that we may 'omit' the 1 in eqn.(8.27). Rescaling $X_k \to -q^{1/2}\, X_k$, we reduce to the recursion relation

$$X_{k+1} = X_k\, X_{k-1}^{-1}. \tag{8.31}$$

The operator sequence is then

$$\cdots,\ X_1,\ X_2,\ X_2 X_1^{-1},\ q^{-1}\, X_1^{-1},\ q^{-1}\, X_2^{-1},\ X_2^{-1} X_1,\ X_1,\ X_2,\ \cdots \tag{8.32}$$

so $X_{k+6} = X_k$, which is precisely the same periodicity as $-\mathbf{Cox}(A_2)$. Moreover, as $q \to 1$, we may rewrite eqn.(8.31) as

$$\begin{pmatrix} \log X_{k+1} \\ \log X_k \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \log X_k \\ \log X_{k-1} \end{pmatrix}, \tag{8.33}$$

and the 2×2 matrix in the RHS *is* $-\mathbf{Cox}(A_2)$.

The above 'forget the 1 trick' may seems just a silly game or a curiosity. In fact it is not. It corresponds to the *tropical* realization of the cluster–algebras [34, 35]. For its physical (deep!!) meaning see the detailed discussion by Gaiotto, Moore and Neitzke, ref. [9].

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[31] PROOF: $\zeta \equiv -q^{1/2}$

$$\begin{aligned} X_{k+5} &= (1+\zeta X_{k+4}) X_{k+3}^{-1} = X_{k+3}^{-1} + \zeta X_{k+4} X_{k+3}^{-1} = \\ &= X_{k+3}^{-1} + \zeta(1+\zeta X_{k+3}) X_{k+2}^{-1} X_{k+3}^{-1} = \\ &= X_{k+3}^{-1} + \zeta(1+\zeta X_{k+3}) \zeta^{-2} X_{k+3}^{-1} X_{k+2}^{-1} = \\ &= X_{k+3}^{-1} + \zeta^{-1} X_{k+3}^{-1} X_{k+2}^{-1} + X_{k+2}^{-1} = \\ &= X_{k+3}^{-1} (\zeta X_{k+2} + 1) \zeta^{-1} X_{k+2}^{-1} + X_{k+2}^{-1} = \\ &= X_{k+1} (1+\zeta X_{k+2})^{-1} (\zeta X_{k+2} + 1) \zeta^{-1} X_{k+2}^{-1} + X_{k+2}^{-1} = \\ &= (\zeta^{-1} X_{k+1} + 1) X_{k+2}^{-1} = X_{k+2}^{-1} (1 + \zeta X_{k+1}) = \\ &= X_k. \end{aligned}$$

9 Connections with $2d$ RCFT's

9.1 The parafermion characters

Our original problem was to study the WCF invariants

$$\operatorname{Tr}\mathbb{Y}^k \equiv \operatorname{Tr}\mathbb{M}(q)^{k/h} \tag{9.1}$$

These invariants have the form of q–series. From the Euler series expansion

$$\Psi(z;q) = \sum_{n\geq 0} \frac{q^{n^2/2}(-z)^n}{(q)_n} \qquad \text{where } (q)_n = \prod_{k=1}^{n}(1-q^k), \tag{9.2}$$

wee see that our invariant must have the general structure

$$\sum_{\boldsymbol{m}\in\mathbb{N}^\ell} \frac{q^{\frac{1}{2}\boldsymbol{m}\cdot A\cdot\boldsymbol{m}+B\cdot\boldsymbol{m}+C}}{\prod_{i=1}^{\ell}(q)_{m_i}^{a_i}} \tag{9.3}$$

for certain $\ell\times\ell$ matrix A, vector B, integral vector a_i, and constant C. ℓ is an integer which depends on the specific model, the exponent k, and (potentially) the particular BPS chamber we use to define $\mathbb{Y}$.

The characters of a RCFT are q–series (where $q = e^{2\pi i\tau}$) with special modular properties, having *typically* the form of eqn.(9.3) (in fact, often they may be written in many ways in that form using q–hypergeometric identities).

In the simplest case, $k = 1$ and the single–Dynkin graph models (G, A_1), one has [15]

$$\operatorname{Tr}\mathbb{Y} = \sum_{\boldsymbol{m}\in\mathbb{N}^\ell} \frac{q^{\frac{1}{2}\boldsymbol{m}\cdot A\cdot\boldsymbol{m}+B\cdot\boldsymbol{m}}}{(q)_{m_1}\,(q)_{m_2}\,\cdots\,(q)_{m_\ell}} \tag{9.4}$$

where

$$A = \frac{1}{2}\,C_G \text{ the Cartan matrix of } G, \tag{9.5}$$

while the matrix B depends on the particular definition of trace in $\mathbb{T}_\Gamma$; for the standard one $B = 0$.

More generally [15],

$$\mathrm{Tr}\big[\mathbb{Y}\prod_\alpha X_{\gamma_\alpha}\big] = \sum_{\boldsymbol{m}\in\mathbb{N}^\ell} \frac{q^{\frac{1}{2}\boldsymbol{m}\cdot A\cdot\boldsymbol{m}+B\cdot\boldsymbol{m}+C}}{(q)_{m_1}\,(q)_{m_1}\,\cdots\,(q)_{m_\ell}} \tag{9.6}$$

where B and C depend on the operator insertions $\prod_\alpha X_{\gamma_\alpha}$ and $A = C_G/2$.

The q–series in eqn.(9.6) are exactly the *characters of the generalized parafermions* [49], *i.e.* the coset RCFT

$$(\widehat{G}\,)_2/U(1)^\ell, \qquad \ell = \mathrm{rank}\,G. \tag{9.7}$$

This equality is quite remarkable. It is even more remarkable when we realize that the UV fixed point of the massive integrable $2d$ theory whose TBA is associated to the Dynkin pair (G, A_1) $(G = ADE)$ is precisely the parafermionic theory $(\widehat{G}\,)_2/U(1)^\ell$ (see ref. [42] and references therein)!!

One expects that this correspondence between integrable massive $2d$ theories, the characters of their UV RCFT fixed points, and the wall–crossing invariants of the corresponding $\mathcal{N} = 2$ four–dimensional theories will generalize to the (G, G') models. This leads to the **conjecture**

$$\mathrm{Tr}\big[\mathbb{Y}\prod_\alpha X_{\gamma_\alpha}\big]\Big|_{(G,G')\ \text{model}} = \left[\begin{array}{l}\text{linear combination of characters}\\ \text{of the UV RCFT corresponding to}\\ \text{the integrable massive } (G,G') \text{ model}\end{array}\right. \tag{9.8}$$

$$= \sum_{\boldsymbol{m}\in\mathbb{N}^\ell} \frac{q^{\frac{1}{2}\boldsymbol{m}\cdot A\cdot\boldsymbol{m}+B\cdot\boldsymbol{m}+C}}{(q)_{m_1}\,(q)_{m_1}\,\cdots\,(q)_{m_\ell}} \tag{9.9}$$

where

$$\ell = \mathrm{rank}\,G \times \mathrm{rank}\,G' \qquad \boldsymbol{A = C_G \otimes C_{G'}^{-1}}. \tag{9.10}$$

Here is a list of the evidence for the conjecture (see ref. [15] for details):

- ✓ explicit computation for $(G; A_1)$

- ✓ limit $q \to 1$
- ✓ recursion relations for different operator insertions (which, in the classical limit $q \to 1$ reproduce the Verlinde algebra of the corresponding RCFT);
- ✓ symmetries and central charges
- ✓ relation with Nahm's conjectures in $2d$ (modular properties, Number–Theoretical aspects) [42].

9.2 Nahm's conjectures [42]

Particularly deep are the relations with the Nahm conjectures which state that a q–series of the form (9.9) have good $SL(2,\mathbb{Z})$ modular properties *precisely* if it arises from a solution of the TBA equations of some $2d$ integrable model in the way that our wall–crossing invariants $\mathrm{Tr}\,\mathbb{Y}$ arises from the corresponding Y–system. Moreover, in the classical limit $q \equiv e^{-\hbar} \to 1$ the quantum dilogarithms $\Psi(\cdot)$ get replaced by $\exp[\mathrm{Li}_2(\cdot)/\hbar]$ where $\mathrm{Li}_2(\cdot)$ is the classical dilogarithm. The relation $\mathbb{M}^r$ =Identity becomes a relation between values of the classical dilogarithm, which has an interpretation in Number Theory in terms of torsion elements of the Bloch group (more precisely, of the algebraic K–theory group $K_3(\mathbb{C})$. In this context, the WCF reproduces the relations in $(K_3(\mathbb{C}))_{\mathrm{tors}}$. See ref. [42] for a discussion.

9.3 Level–rank duality

Let us apply the above conjecture to the (G, A_m) model in the BPS chamber $G \,\square\, A_m$. We expect $\mathrm{Tr}[\mathbb{Y}(\cdots)]$ to give the characters of the RCFT coset theory

$$\widehat{G}_{m+1}/U(1)^{\mathrm{rank}\,G}. \tag{9.11}$$

Then consider the wall–crossing between the $\mathbb{Z}_{m+1}$– and the $\mathbb{Z}_{n+1}$–symmetric BPS chambers of the (A_n, A_m) model, *i.e.*

$$A_n \,\square\, A_m \;\longleftrightarrow\; A_m \,\square\, A_n. \tag{9.12}$$

Applying the rule in eqn.(9.11) we get the following duality of coset models

$$\widehat{SU(n+1)}_{m+1}/U(1)^n \leftrightarrow \widehat{SU(m+1)}_{n+1}/U(1)^m \tag{9.13}$$

which is precisely the usual rank–level duality [50].

In particular, for $n = 2, m = 1$ this implies the usual pentagonal identity for the quantum dilogarithm [51]. The general rank–duality gives additional quantum dilogaritm identities, whose classical counterpart are central both in Number Theory (Bloch group) as in the theory of *motives*.

9.4 Prediction of new RCFT dualities

We have seen in §. 9.3 that the identification of local CY geometries $(G, G') \leftrightarrow (G', G)$ gives, under the correspondence with RCFT's, the level–rank duality. There are other similar geometric identifications (which often may be interpreted as cluster–algebras equivalences [15]) which, likewise, we expect to correspond to new RCFT dualities of the 'level–rank'–type.

In particular, we have the following geometric identifications which correspond to isomorphisms for the Grassmanian cluster–algebras (see [15]):

isomorphism	RCFT coset duality
$(A_2, A_2) \leftrightarrow (D_4, A_1)$	$\widehat{\mathbf{SU(3)}}_{\mathbf{3}}/\mathbf{U(1)^2} \leftrightarrow \widehat{\mathbf{SO(8)}}_{\mathbf{2}}/\mathbf{U(1)^4}$
$(A_3, A_2) \leftrightarrow (E_6, A_1)$	$\widehat{\mathbf{SU(4)}}_{\mathbf{3}}/\mathbf{U(1)^3} \leftrightarrow (\widehat{\mathbf{E_6}})_{\mathbf{2}}/\mathbf{U(1)^6}$
$(A_4, A_2) \leftrightarrow (E_8, A_1)$	$\widehat{\mathbf{SU(5)}}_{\mathbf{3}}/\mathbf{U(1)^4} \leftrightarrow (\widehat{\mathbf{E_8}})_{\mathbf{2}}/\mathbf{U(1)^8}$

9.5 General invariants $\mathrm{Tr}\,\mathbb{Y}^k$

The story for general k's is only partly understood. The evidence is that (for the (G, G') models) one gets again characters of some RCFT, possibly non–unitary, up to some trivial factors $1/\eta(q)^s$, that is modulo free decoupled $2d$ sectors.

For the simplest model (A_1, A_1) (which corresponds geometrically to compactification of Type IIB on the conifold) we just get the characters of a free fermion (which is, of course, a special case of a parafermion).

For the next simplest model, (A_2, A_1), varying k one gets the full family of the $(*, 5)$ minimal models (note that 5 is the order of $\mathbb{Y}$ according to the Zamolodchikov–Keller theorem):

$$(p, p') = (2, 5),\ (3, 5),\ (4, 5) \tag{9.14}$$

for, respectively, $k = 2, 3$ and 1.

9.6 Rogers–Ramanujan identities and generalization

One interesting aspect is that $\operatorname{Tr}\mathbb{M}(q)^{\ell}$, viewed as a q–series, has a very different form when written in different BPS chambers. We are guaranteed by the WCF, however, that these very different looking expressions are actually equal. In this way from the WCF one proves a large supply of q–series (and q–products) identities the simplest of which (for the (A_2, A_1) model) are the celebrated Rogers–Ramanujan identities.

In this way one proves many remarkable identities with deep number–theoretical meaning some of which were previously known, but many where not previously listed in the tables of q–series/q–products identities.

As it is well–known, q–series identities of the Rogers–Ramanujan type may be used to put in different form the analytic expression for the characters of the RCFT. Physically this has the following interpretation: a given RCFT may be deformed to a massive *integrable* QFT theory in more than one way. For each such integrable deformation we have a TBA and a preferred basis in the Hilbert space which may be used to compute the partition functions and their UV limit – which are the RCFT characters. Different deformations gives quite different analytic expressions, but of the same UV invariants.

This, of course, is the same idea of our wall–crossing invariants, as relevant–deformation independent data which then must be properties of the parent UV theory. Therefore, we may see the Rogers–Ramanujan story as a $\mathcal{N} = 0$ wall–crossing formula (in the integrable QFT set–up).

10 Conclusion

Wall–crossing invariants are a deep and beautiful subject. If you wish to learn more details, read the paper `arXiv:1006.3435`.

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