Mathematical Issues in Quasi-static (Globally and Locally Minimizing) and Dynamic Fracture Evolution

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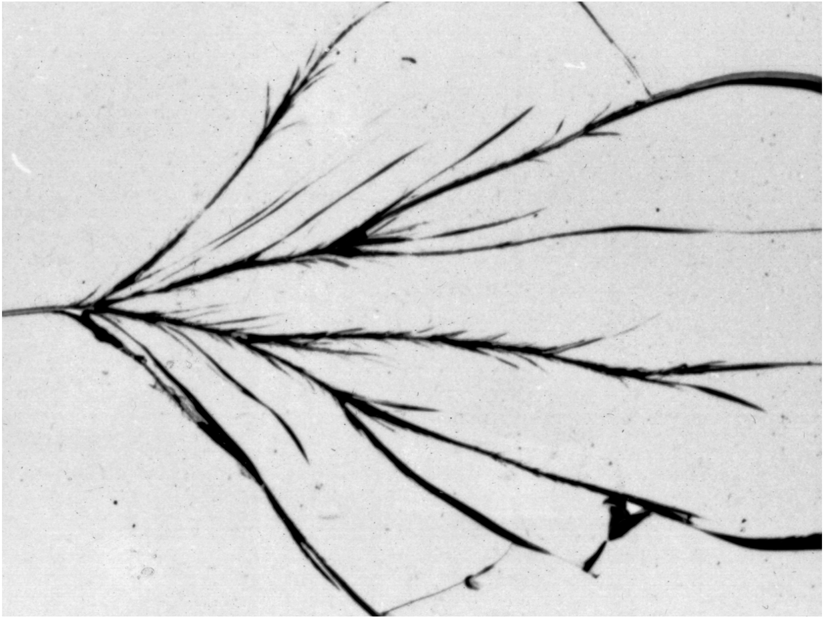


Outline

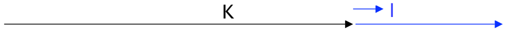
1. Globally Minimizing Quasi-static (Griffith)
    - Definition
    - Discrete-time minimization procedure: $u_n(t)$
    - Passage to the limit $u(t)$
    - Proving properties of u
        - global unilateral minimality
        - energy balance
    - What goes wrong with cohesive
2. Locally Minimizing Quasi-static
3. Dynamics (Griffith and cohesive)

Caveats

- Focus on existence
- Only concern is with models that predict the crack path
- Emphasis is on issues that I have thought about
- Issues that will be discussed are common to scalar and vector-valued functions, and we will assume scalar throughout
- Similarly, will generally assume (strict) convexity of the elastic energy density (calculations will be with $\frac{1}{2}|\nabla u|^2$)

Griffith's criterion

The starting point for predicting crack growth is Griffith's criterion (1920). Griffith considered a pre-existing crack K with a potential future path (here in blue).



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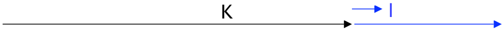

For a crack increment of length l, $E_{el}(l)$ is the elastic energy of the corresponding elastic equilibrium (subject to a Dirichlet condition g or loads). The criterion states that the crack can only grow if the rate of decrease of elastic energy as l increases is large enough, i.e.,

$$-\frac{dE_{el}(l)}{dl} \quad \begin{array}{ll} < G_c & \text{the crack can not run} \\ = G_c & \text{the crack can run} \\ > G_c & \text{the crack is unstable,} \end{array}$$

and the crack should never be unstable.

The static problem

Formulated by Ambrosio & Braides (1995): If u minimizes

$$v \mapsto \frac{1}{2}\int_\Omega |\nabla v|^2 + \mathcal{H}^1(S_v)$$

over $v \in SBV_g(\Omega)$, then the crack $K := S_u$ is stable (taking $G_c = 1$). The reason is that each increment in length l cannot reduce the energy, i.e.,

$$E(l) + l \geq E(0),$$

or

$$-\frac{E(l) - E(0)}{l} \leq 1.$$

Minimizing movements: for discrete times $t_i = \frac{i}{n}\, T$, $u_n(t_i)$ minimizes

$$v \mapsto \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{\Delta t}\|v - u_n(t_{i-1})\|^2_{L^2} + \mathcal{H}^1(S_v \setminus \bigcup_{j<i} S_{u_n(t_j)})$$

over $v \in SBV_g(\Omega)$. Then take the limit as the time step goes to zero, to get $t \mapsto u(t)$.

Globally Minimizing Quasi-static formulation

Globally minimizing evolutions (Francfort & Marigo *JMPS* '98, modified by Dal Maso & Toader *ARMA* '02; Mielke):
Based on the *total* energy

$$E(u, K) := E_{el}(u) + \mathcal{H}^{N-1}(K),$$

Definition

Given Dirichlet data $t \mapsto g(t)$, $t \mapsto (u(t), K(t))$ is a (globally minimizing) quasi-static evolution if:

1. (u, K) is unilaterally minimal at each time:
   for each t, if (w, Γ) is such that $K(t) \subset \Gamma$, then
   $E(u(t), K(t)) \leq E(w, \Gamma)$ (both subject to Dirichlet data $g(t)$)
   $\Rightarrow$ u is an elastic equilibrium at each time
2. Energy balance $E(u(t), K(t)) = E(u(0), K(0))+$ work done by varying g between times 0 and t

The solution satisfies Griffith's criterion if $t \mapsto \mathcal{H}^1(K(t))$ is continuous.

Discrete-time procedure for existence

- For a problem on the time interval $[0, T]$, set $t_n^i := i/n\ T$
- Define $u_n(t_n^i)$ to be a minimizer of $v \mapsto E_{el}(v) + \mathcal{H}^{N-1}(S_v \setminus K_n(t_n^{i-1}))$ subject to Dirichlet data $g(t_n^i)$, where $K_n(t_n^i)) := \cup_{j \leq i} S_{u_n(t_n^j)}$.
- Extend u_n by, e.g., $u_n(t) := u_n(t_n^i)$ for $t \in [t_n^i, t_n^{i+1})$
- Take a countable dense set $D \subset [0, T]$, and for a diagonal subsequence, $u_n(t)$ converges for all $t \in D$, define $u(t)$ to be the limit. Extend u to $[0, T]$ by, e.g., continuity from below. Can define $K(t) := \cup_{\tau \in D, \tau \leq t} S_{u(\tau)}$.

What minimality does $u(t)$ have? $u_n(t_n^i)$ has a minimality property with respect to the previous $u_n(t_n^{i-1})$, but there is no "previous" time for $u(t)$. $u(t)$ can only inherit the unilateral minimality, with respect to "crack" increases:
$u_n(t_n^i)$ minimizes

$$v \mapsto E_{el}(v) + \mathcal{H}^{N-1}(S_v \setminus K_n(t_n^{i-1}))$$

among v with the same boundary data, or

$$E_{el}(u_n(t_n^i)) + \mathcal{H}^{N-1}(S_{u_n(t_n^i)} \setminus K_n(t_n^{i-1})) \leq E_{el}(v) + \mathcal{H}^{N-1}(S_v \setminus K_n(t_n^{i-1}))$$

for all v. In particular, and more simply,

$$E_{el}(u_n(t_n^i)) \leq E_{el}(v) + \mathcal{H}^{N-1}(S_v \setminus S_{u_n(t_n^i)})$$

for all v. This might be inherited by $u(t)$:

$$E_{el}(u(t)) \leq E_{el}(v) + \mathcal{H}^{N-1}(S_v \setminus S_{u(t)})$$

for all v with the same boundary data.

Unilateral minimality

Question: If $u_n \stackrel{SBV}{\rightharpoonup} u$ and

$$E_{el}(u_n) \leq E_{el}(v) + \mathcal{H}^{N-1}(S_v \setminus S_{u_n})$$

for all v with the same boundary data, does it follow that

$$E_{el}(u) \leq E_{el}(v) + \mathcal{H}^{N-1}(S_v \setminus S_u)$$

for all v with the same boundary data?

Why it's not obvious: Neumann sieve and higher dimension.

Issue is turning test functions for u into test functions for u_n.

For now, we assume this minimality (i.e., the ability to alter test functions).

Consequences:

- strong convergence $\nabla u_n \to \nabla u$
- unilateral (global) minimality w.r.t. $K(t)$
- energy balance

Strong convergence: (strict convexity of the elastic energy density)

$$\int_\Omega |\nabla u|^2 \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2$$

so only issue is whether

$$\int_\Omega |\nabla u|^2 < \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2.$$

This is ruled out by using u as a test function for u, and altering it for u_n, together with the unilateral minimality of u_n.

Energy balance: two steps

1. energy balance with $\lim_{n\to\infty} \mathcal{H}^{N-1}(K_n(t))$
2. energy balance with $\mathcal{H}^{N-1}(K(t))$

$$(\Rightarrow \mathcal{H}^{N-1}(K(t)) = \lim_{n\to\infty} \mathcal{H}^{N-1}(K_n(t))).$$

Precisely, we want to show

$$\frac{1}{2}\int_\Omega |\nabla u(t)|^2 + \mathcal{H}^{N-1}(K(t)) = \frac{1}{2}\int_\Omega |\nabla u(0)|^2 + \mathcal{H}^{N-1}(K(0)) + \int_0^t \int_\Omega \nabla u(\tau)\cdot\nabla \dot{g}(\tau)\,dx\,d\tau$$

for every $t \in [0, T]$.

Discrete version

By considering $u_n(t_n^i) + (g(t_n^{i+1}) - g(t_n^i))$ as a competitor for $u_n(t_n^{i+1})$, we get

$$\frac{1}{2}\int_\Omega |\nabla u_n(t_n^{i+1})|^2 + \mathcal{H}^{N-1}(K_n(t_n^{i+1})) \leq \frac{1}{2}\int_\Omega |\nabla u_n(t_n^i)|^2 + \mathcal{H}^{N-1}(K_n(t_n^i))$$
$$+ \int_\Omega \nabla u_n(t_n^i) \cdot \left(\nabla g(t_n^{i+1}) - \nabla g(t_n^i)\right)$$
$$+ \frac{1}{2}\int_\Omega |\nabla g(t_n^{i+1}) - \nabla g(t_n^i)|^2.$$

Summing from $i = 0$ to $j - 1$, we get

$$\frac{1}{2}\int_\Omega |\nabla u_n(t_n^j)|^2 + \mathcal{H}^{N-1}(K_n(t_n^j)) \leq \frac{1}{2}\int_\Omega |\nabla u(0)|^2 + \mathcal{H}^{N-1}(K(0))$$
$$+ \Sigma_{i=0}^{j-1} \int_\Omega \nabla u_n(t_n^i) \cdot \left(\nabla g(t_n^{i+1}) - \nabla g(t_n^i)\right)$$
$$+ \Sigma_{i=0}^{j-1} \frac{1}{2}\int_\Omega |\nabla g(t_n^{i+1}) - \nabla g(t_n^i)|^2.$$

Taking the limit (considering t_n^j independent of n and j), we get (using regularity of g)

$$\frac{1}{2}\int_\Omega |\nabla u(t)|^2 + \limsup_{n\to\infty} \mathcal{H}^{N-1}(K_n(t_n^j)) \leq \frac{1}{2}\int_\Omega |\nabla u(0)|^2 + \mathcal{H}^{N-1}(K(0)) + \int_0^t \int_\Omega \nabla u(\tau)\cdot\nabla \dot{g}(\tau)\,dx\,d\tau.$$

Similarly, taking $u_n(t_n^{i+1}) - (g(t_n^{i+1}) - g(t_n^i))$ to be a test function for $u_n(t_n^i)$, we get

$$\frac{1}{2}\int_\Omega |\nabla u(t)|^2 + \liminf_{n\to\infty} \mathcal{H}^{N-1}(K_n(t_n^j)) \geq \frac{1}{2}\int_\Omega |\nabla u(0)|^2 + \mathcal{H}^{N-1}(K(0)) + \int_0^t \int_\Omega \nabla u(\tau)\cdot\nabla \dot{g}(\tau)\,dx\,d\tau.$$

Hence $\lim_{n\to\infty} \mathcal{H}^{N-1}(K_n(t_n^j))$ exists, and there is energy balance with this limit.

On the other hand, from the minimality of $(u(s), K(s))$ for each s, consider $s < t$ and $(u(t) + g(s) - g(t), K(t))$ a competitor for $(u(s), K(s))$. Then

$$\frac{1}{2}\int_\Omega |\nabla u(t)|^2 + \mathcal{H}^{N-1}(K(t)) \geq \frac{1}{2}\int_\Omega |\nabla u(s)|^2 + \mathcal{H}^{N-1}(K(s)) \\ - \int_\Omega \nabla u(t) \cdot \nabla(g(s) - g(t)) - \frac{1}{2}\int_\Omega |\nabla(g(s) - g(t))|^2.$$

Summing over the discrete times and taking the limit, this gives $\mathcal{H}^{N-1}(K(t)) = \lim_{n\to\infty} \mathcal{H}^{N-1}(K_n(t))$, and the energy balance we sought.

Next, Jump Transfer.

BV and sets of finite perimeter

Definition

The *measure theoretic interior* of a set $E \subset \Omega$ is the set

$$\left\{x \in \Omega : \lim_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} = 1\right\};$$

the *measure theoretic exterior* of a set of finite perimeter $E \subset \Omega$ is the set

$$\left\{x \in \Omega : \lim_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} = 0\right\};$$

and the *measure theoretic boundary* of E, $\partial_* E$, is the set of points in Ω that are neither in the measure theoretic interior nor the measure theoretic exterior. That is,

$$\partial_* E := \left\{x \in \Omega : \limsup_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} > 0 \text{ and } \limsup_{r \to 0} \frac{|B(x,r) \setminus E|}{|B(x,r)|} > 0\right\}.$$

For $E \subset \Omega$ with finite perimeter ($\chi_E \in BV$), we have

$$|\partial E| := |D\chi_E| = \mathcal{H}^{N-1} \lfloor \partial_* E$$

and

$$\int_E \mathrm{div}\phi \mathrm{dx} = \int_{\partial_* \mathrm{E}} \phi \cdot \nu_{\mathrm{E}} \, \mathrm{d}\mathcal{H}^{\mathrm{N}-1}$$

for all $\phi \in C_0^1(\Omega, \mathbb{R}^N)$.

Definition

The *reduced boundary* of E in Ω, $\partial^* E$, is the set of points in $\partial_* E$ that are (or can be) Lebesgue points for ν_E. That is,

$$\lim_{r \to 0} \fint_{B(x,r) \cap \partial_* E} \nu_E \, d\mathcal{H}^{N-1} = \nu_E(x).$$

For $x \in \partial^* E$, define $H^- := \{y \in \mathbb{R}^N : y \cdot \nu_E(x) < 0\}$ and $E_r := \{y \in \mathbb{R}^N : x + ry \in E\}$. Then

Theorem (Blow-up at reduced boundary)

$$\chi_{E_r} \to \chi_{H^-}$$

in $L^1_{loc}(\mathbb{R}^N)$ *as* $r \to 0$. *Furthermore,*

$$\mathcal{H}^{N-1}(B \cap \partial_* E_r) \to \mathcal{H}^{N-1}(B \cap \partial H^-)$$

for all ball $B \subset \mathbb{R}^N$.

Connection to BV: jump sets and coarea....

In what follows, for a given $u \in BV$ and $t \in \mathbb{R}$, define

$$E_t := \{x \in \Omega : u(x) > t\}.$$

Definition

We define the upper and lower values of u at x by

$$u^+(x) := \sup\left\{ t : \limsup_{r\to 0^+} \frac{|E_t \cap B(x,r)|}{|B(x,r)|} > 0 \right\}$$

and

$$u^-(x) := \inf\left\{ t : \limsup_{r\to 0^+} \frac{|E_t^c \cap B(x,r)|}{|B(x,r)|} > 0 \right\}.$$

The jump of u is $[u](x) := u^+(x) - u^-(x)$, and the jump set of u is then defined by $S_u := \{x \in \Omega : [u](x) > 0\}$.

Definition

We define $D_J u := Du \lfloor S_u$ and we say $u \in SBV(\Omega)$ if the singular part of Du is $D_J u$.

S_u and $\partial_* E_t$

Proposition

Let $u \in BV(\Omega)$, let $D \subset \mathbb{R}$ be dense, and recall that for each $t \in \mathbb{R}$, we define

$$E_t := \{x \in \Omega : u(x) > t\}.$$

Then

$$S_u = \bigcup_{\substack{t_1, t_2 \in D \\ t_1 < t_2}} (\partial_* E_{t_1} \cap \partial_* E_{t_2}).$$

First, recall

$$\partial_* E := \left\{ x \in \Omega : \limsup_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} > 0 \text{ and } \limsup_{r \to 0} \frac{|B(x,r) \setminus E|}{|B(x,r)|} > 0 \right\}.$$

Proof part 1.

Let $x \in \partial_* E_{t_1} \cap \partial_* E_{t_2}$ with $t_1 < t_2$. Then using the definitions of $u^-(x)$ and $u^+(x)$ we have

$$x \in \partial_* E_{t_2} \Rightarrow \limsup_{r\to 0^+} \frac{|E_{t_2} \cap B(x,r)|}{|B(x,r)|} > 0 \iff u^+(x) \geq t_2$$

and also

$$x \in \partial_* E_{t_1} \Rightarrow \limsup_{r\to 0^+} \frac{|E_{t_1}^c \cap B(x,r)|}{|B(x,r)|} > 0 \iff u^-(x) \leq t_1,$$

so $x \in S_u$. □

Proof part 2.

Next, suppose $x \in S_u$. Then we can choose $t_1, t_2 \in D$ so that $u^-(x) < t_1 < t_2 < u^+(x)$. Then $E_{t_1} \supset E_{t_2}$ and

$$\limsup_{r\to 0^+} \frac{|E_{t_2} \cap B(x,r)|}{|B(x,r)|} > 0 \text{ and } \limsup_{r\to 0^+} \frac{|E_{t_1}^c \cap B(x,r)|}{|B(x,r)|} > 0$$

$$\Rightarrow x \in \partial_* E_{t_2} \text{ (since } E_{t_1}^c \subset E_{t_2}^c) \text{ and } x \in \partial_* E_{t_1} \text{ (since } E_{t_2} \subset E_{t_1})$$

$$\iff x \in \partial_* E_{t_1} \cap \partial_* E_{t_2}.$$

□

Corollary

Note that as a consequence, we have that if $x \in S_u$, then

$$u^+(x) = \sup\{t : x \in \partial_* E_t\}$$

and

$$u^-(x) = \inf\{t : x \in \partial_* E_t\}.$$

This does not hold for general $x \notin S_u$, as can be seen by considering u constant near x, so that x is in none of the $\partial_* E_t$.

Given a countable dense set D, define the "reduced jump set"

$$S_D^*(u) := S_u \setminus \left(\bigcup_{t \in D} [\partial_* E_t \setminus \partial^* E_t] \right), \tag{1}$$

which has the property that $\mathcal{H}^{N-1}(S_u \setminus S_D^*(u)) = 0$ since $\mathcal{H}^{N-1}(\partial_* E_t \setminus \partial^* E_t) = 0$ for each $t \in D$.

Proposition

For $x \in S^*_D(u)$ and $t \in D \cap (u^-(x), u^+(x))$, $\nu_t(x)$ is independent of t.

Proof.

We have that $x \in \partial^* E_t$ for all $t \in D \cap (u^-(x), u^+(x))$, as well as that $E_{t_1} \supset E_{t_2}$ if $t_1 < t_2$. But if $t_1, t_2 \in D \cap (u^-(x), u^+(x))$, then, by Lemma blowing up E_{t_1} converges to $H_{\nu_{t_1}(x)}$ and blowing up E_{t_2} converges to $H_{\nu_{t_2}(x)}$. But since $E_{t_1} \supset E_{t_2}$, it follows that the same inclusion holds for the limiting half-planes. But, since they are half-planes, they must be equal, and therefore $\nu_{t_1}(x) = \nu_{t_2}(x)$. □

Note that even for $t \in (t_1, t_2) \setminus D$, we have that the blow-up of E_t converges to the same half-plane.

Theorem (Coarea)

If $u \in BV(\Omega)$, *then*

$$|Du|(\Omega) = \int_{\mathbb{R}} \mathcal{H}^{N-1}(\partial_* E_t)dt. \tag{2}$$

Furthermore, from this it follows that for every Borel set S,

$$|Du|(S) = \int_{\mathbb{R}} \mathcal{H}^{N-1}(S \cap \partial_* E_t)dt. \tag{3}$$

Therefore,

$$|D_J u|(\Omega) = \int_{\mathbb{R}} \mathcal{H}^{N-1}(S_u \cap \partial_* E_t)dt$$

and, for $u \in SBV(\Omega)$,

$$\int |\nabla u| = |D_{ac} u|(\Omega) = |Du|(\Omega \setminus S_u) = \int_{\mathbb{R}} \mathcal{H}^{N-1}((\partial_* E_t) \setminus S_u)dt.$$

Two pictures of *SBV*

Consider minimizers of

$$\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(S_u)$$

1. Usual picture: S_u is a countable union of closed pieces of smooth curves (rectifiability), off of which u is harmonic

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1. Usual picture: S_u is a countable union of closed pieces of smooth curves (rectifiability), off of which u is harmonic
2. Extreme Coarea:

$$\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(S_u) = \int_{\mathbb{R}} \int_{\partial_* E_t} (|\nabla u| \chi_{S_u^c} + \frac{1}{[u]} \chi_{S_u}) d\mathcal{H}^{N-1} dt$$

$|\nabla u|$ is the density of different $\partial_* E_t$, so off of S_u, there is a repulsion of different $\partial_* E_t$, but if two intersect ($\partial_* E_{t_1} \cap \partial_* E_{t_2} \neq \emptyset$), this intersection attracts more $\partial_* E_t$.

Jump Transfer: modification of test functions

Given $u_n \stackrel{SBV}{\rightharpoonup} u$ (keys: equi-integrability of $|\nabla u_n|$ and strong convergence $u_n \to u$ in L^1) and a test function v for u, we want to modify it, creating v_n such that

$$\mathcal{H}^{N-1}(S_{v_n} \setminus S_{u_n}) \to \mathcal{H}^{N-1}(S_v \setminus S_u)$$

while $\nabla v_n \to \nabla v$.

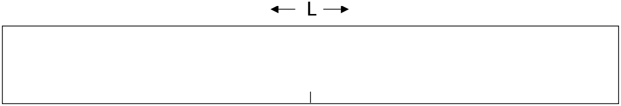
(Blackboard)

Problems with cohesive

(Blackboard)

Connection to Griffith

The solution $u(t)$ with $K(t) := \cup_{\tau \leq t} S_{u(\tau)}$ satisfies Griffith's criterion if $t \mapsto \mathcal{H}^{N-1}(K(t))$ is continuous.
Problem:



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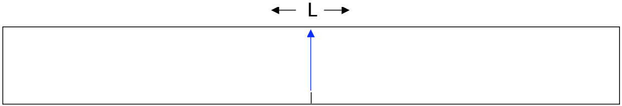

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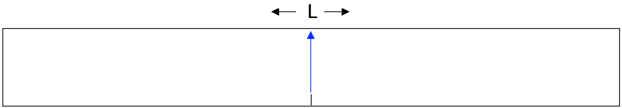

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Note the connection to local vs. global minimality – the initial crack was a strict local minimizer and was stable in the sense of Griffith.

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True Griffith quasi-static evolution

Definition

Given $g(t)$, the pair $(u(t), K(t))$ is a Griffith evolution if:

- $(u(0), K(0))$ is unilaterally stable (e.g., a local minimizer), subject to $g(0)$
- $(u(t), K(t))$ is unilaterally stable, subject to $g(t)$
- Energy inequality:

  $$E(u(t_2), K(t_2)) - E(u(t_1), K(t_1)) \leq \int_{t_1}^{t_2} \int_{\Omega} \nabla u \cdot \nabla \dot{g} dxdt$$

  for every $t_1 \leq t_2$.
- If $u(t^-) \neq u(t^+)$, then $u(t^+)$ is accessible from $u(t^-)$ (there exists a continuously growing crack from $K(t^-)$ to $K(t^+)$ along which the energy is nonincreasing).

Existence: open

Problem with local minimization

Proving existence for models based on local minimality have met with difficulties (e.g., limits of minimizing movements – Dal Maso & Toader M3AS '02)

The convergence we have is due to SBV compactness of $u_n(t)$, which gives (suppressing t):

$$\begin{cases} \nabla u_n & \rightharpoonup & \nabla u \text{ in } L^1(\Omega); \\ [u_n]\nu_n \mathcal{H}^{N-1} \lfloor S_{u_n} & \overset{*}{\rightharpoonup} & [u]\nu \mathcal{H}^{N-1} \lfloor S_u \text{ as measures;} \\ u_n & \to & u \text{ in } L^1(\Omega); \text{ and} \\ u_n & \overset{*}{\rightharpoonup} & u \text{ in } L^\infty(\Omega). \end{cases}$$

It is easy to find examples of $u_n \overset{SBV}{\rightharpoonup} u$ with (u_n, S_{u_n}) unilateral local minimizers of E, but (u, S_u) is not.

Start with u that minimizes the elastic energy given S_u, but the pair is not a local minimizer: (Blackboard)

There is a fix, ε-stability, which implies local minimality.

Definition (ε-accessible)

(v, C) is ε-accessible from (u, K) if there exists a continuous function $\phi:[0,1]\to SBV(\Omega)$ such that $\phi(0) = u$, $\phi(1) = v$, $E(v, C) < E(u, K)$, and

$$\sup_{\tau_1<\tau_2} [E(\phi(\tau_2), K_\phi(\tau_2)) - E(\phi(\tau_1), K_\phi(\tau_1))] < \varepsilon.$$

Here, $K_\phi(\tau) := \cup_{s\leq\tau} S_{\phi(s)}$ and $K_\phi(1) = C$. Such a path to v is called an ε-slide.

We then have the corresponding definition of stability:

Definition (ε-stability)

u is ε-stable if there does not exist an ε-accessible v from u.

We also define $\bar{\varepsilon}$-accessibility, where the last inequality is not strict. Also, unilateral accessibility/stability is as before.

ε-stable evolutions

Definition

Given $g(t)$, the pair $(u(t), K(t))$ is an ε-stable evolution if:

- $(u(0), K(0))$ is ε-stable, subject to $g(0)$ (which implies local minimality)
- $(u(t), K(t))$ is unilaterally ε-stable, subject to $g(t)$
- Energy inequality:

  $$E(u(t_2), K(t_2)) - E(u(t_1), K(t_1)) \leq \int_{t_1}^{t_2} \int_\Omega \nabla u \cdot \nabla \dot{g} \, dx dt$$

  for every $t_1 \leq t_2$.
- If $u(t^-) \neq u(t^+)$, then $u(t^+)$ is $\bar{\varepsilon}$-accessible from $u(t^-)$ and has lower energy than all states that are ε-accessible from $u(t^-)$.

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Main difficulty: for the discrete-time problem, there is no (apparent) energy release rate, unlike in quasi-static evolution. There, a crack increment results in an *immediate* stored elastic energy decrease, which can be compared with the cost of the increment. With dynamics, there is no immediate effect. What would a discrete-time model be?

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There are two settings where dynamic models can be naturally defined: phase-field dynamic Griffith fracture, and cohesive dynamic fracture.

Computational phase-field model

(with Bourdin and Richardson)

Based on Ambrosio-Tortorelli approximation of the static energy:

$$E_\varepsilon(u,v) = \tfrac{1}{2}\int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2 dx + \varepsilon \int_\Omega |\nabla v|^2 dx + \frac{1}{4\varepsilon}\int_\Omega (1-v)^2 dx$$

Γ-converges to

$$E(u) = \tfrac{1}{2}\int_\Omega |\nabla u|^2 dx + \mathcal{H}^{N-1}(S_u)$$

defined on SBV. The stiffness of the material is given by $\eta_\varepsilon + v^2$.

Advantage of phase-field approach: Now, when the crack is advanced ($v \searrow$), there is an *immediate* decrease in stored elastic energy, even if u does not jump to the new equilibrium.

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Given $u(x,0), u_t(x,0)$ and boundary conditions

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$$u_{tt} - \operatorname{div}(A\nabla u) = 0$$

with $A(x,0) := \eta_\varepsilon + v(x,0)^2$, to find $u(x,\Delta t)$

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The idea is simply that the displacement u is following dynamics, with the weakening field v playing exactly the same role as in quasi-static, i.e., Griffith (assuming that alternate minimization works).

Phase-field model (continuous time)

(with Ortner and Süli)

Ortner: Previous algorithm balances energy!?

In fact, can take $\Delta t \searrow 0$, get existence of (u, v) such that

1. $$u_{tt} - \mathrm{div}(A_\varepsilon(v)\nabla u) = 0$$

   with initial conditions
   (need to add arbitrarily small dissipation: $\delta \nabla \dot{u}$)
2. Total energy (kinetic + potential + dissipated) is balanced
3. $v(\cdot, t)$ is the minimizer of $v \mapsto E_\varepsilon(u(x,t), v)$ over $v \leq v(\cdot, t)$.

But, what happens when $\varepsilon \searrow 0$? What is the sharp-interface model? v disappears, what happens to condition 3?

Dynamic fracture model (sharp interface)

(u, K) is a Maximal Dissipation (MD) solution if:

1. u is a solution of the wave equation on $\Omega \setminus K$, i.c., etc.

$$\int_0^\infty (u_t, \phi_t) - (\nabla u, \nabla \phi) = 0$$

   $\forall \phi \in H_0^1((0, \infty); SBV_K)$ $(S(\phi(t)) \subset K(t)\ \forall t)$

2. (u, K) balances energy
3. $\forall T$, if a pair (w, L) satisfies 1 and 2, with $K(t) \subset L(t)\ \forall t \in [0, T]$, then $K(t) = L(t)$ for all $t \in [0, T]$

Just energy balance + maximal dissipation (w.r.t. set inclusion)

Expect to work with other dissipations, e.g., damage (with Garroni – different model for dynamic damage, but seems equivalent...)

Connection to quasi-static models?

Quasi-static model

Francfort & Marigo, modified by Dal Maso & Toader; Mielke:

(u, K) is a solution if:

1. (u, K) is unilaterally minimal at each time:
   for each t, if (w, L) is such that $K(t) \subset L$, then
   $E(u(t), K(t)) \leq E(w, L)$
   $\Rightarrow$ u is an elastic equilibrium (global minimizer) at each time
2. Energy balance (stored elastic + dissipation + work)

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Alternative:

i) $u(t)$ is in equilibrium for every t

ii) Energy balance (stored elastic + dissipation + work)

iii) $\forall T$, if (w, L) satisfies i) and ii), and $K(t) \subset L(t)$ $\forall t \in [0, T]$, then $K(t) = L(t)$ $\forall t \in [0, T]$.

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Easy to see red $\Rightarrow$ blue, plus gives a selection criterion by choosing largest dissipation. ii) and iii) are general, i) is PDE for the evolution of u.

Cohesive fracture

The stored energy for cohesive fracture is of the form

$$E(u) := \int_{\Omega} W(\nabla u)dx + \int_{S_u} \psi([u])d\mathcal{H}^{N-1}$$

where $\psi(0) = 0$, $\lim_{x\to\infty} \psi(x) = G_c = 1$, ψ concave (and typically odd).
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These are all naturally defined for cohesive fracture, e.g., in 1-D. And they are all equivalent if $\phi'(0) < \infty$. All give:

- wave equation off of S_u, force balance on S_u ($u_x = \psi'([u])$)
- energy balance (elastic + kinetic + fracture)
- crack opens at $(x_0, t_0) \iff u_x(x_0, t_0) = \psi'(0)$ and u_x is increasing at (x_0, t_0) if no crack were allowed (decreasing if negative)

Existence for Sharp Interface Griffith? First step: solvability of wave equations for arbitrary growing cracks

(with Dal Maso) Given $t \mapsto K(t)$ with K increasing and $K(T) < \infty$, we want solutions to weak versions of

$$\ddot{u}(t) - \Delta u(t) - \gamma \Delta \dot{u}(t) = f(t)$$

on $\Omega \setminus K(t)$, with a zero Neumann condition on $\partial\Omega \cup K(t)$.

What weak versions? First we see how existence works...

We define u_n^i for $i = -1, 0, ..., n$ inductively by the following: First,

$$u_n^0 := u^{(0)}, \quad u_n^{-1} := u^{(0)} - \tau_n u^{(1)}; \tag{4}$$

then, for $i = 0, 1, \ldots, n-1$, the function u_n^{i+1} is the minimizer in $V_{t_n^{i+1}}$ of

$$u \mapsto \left\| \frac{u - u_n^i}{\tau_n} - \frac{u_n^i - u_n^{i-1}}{\tau_n} \right\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \frac{\gamma}{\tau_n} \|\nabla u - \nabla u_n^i\|_{L^2}^2 - 2\langle f_n^i, u\rangle_{L^2}$$

where

$$f_n^i := \frac{1}{\tau_n} \int_{t_n^{i-1}}^{t_n^i} f(t)dt \tag{5}$$

and

$$V_t := GSBV_2^2(\Omega, K(t)) :=$$
$$\{v \in GSBV(\Omega) \cap L^2(\Omega) : \nabla v \in L^2(\Omega; \mathbb{R}^N), S_v \subset K(t)\}.$$

It follows that we have

$$\left\langle \frac{u_n^{i+1} - u_n^i}{\tau_n} - \frac{u_n^i - u_n^{i-1}}{\tau_n}, \frac{\phi}{\tau_n} \right\rangle_{L^2} + \langle \nabla u_n^{i+1}, \nabla \phi \rangle_{L^2} + \frac{\gamma}{\tau_n} \langle \nabla u_n^{i+1} - \nabla u_n^i, \nabla \phi \rangle_{L^2} = \langle f_n^i, \phi \rangle_{L^2} \tag{6}$$

for every $\phi \in V_{t_n^{i+1}}$. We can take $\phi = u_n^{i+1} - u_n^i$, and we get

$$\left\| \frac{u_n^{i+1} - u_n^i}{\tau_n} \right\|_{L^2}^2 - \left\langle \frac{u_n^{i+1} - u_n^i}{\tau_n}, \frac{u_n^i - u_n^{i-1}}{\tau_n} \right\rangle_{L^2} + \| \nabla u_n^{i+1} \|_{L^2}^2 - \langle \nabla u_n^{i+1}, \nabla u_n^i \rangle_{L^2} + \frac{\gamma}{\tau_n} \| \nabla u_n^{i+1} - \nabla u_n^i \|_{L^2}^2 = \langle f_n^i, u_n^{i+1} - u_n^i \rangle_{L^2}.$$

Using the fact that $\|a\|^2 - \langle a, b\rangle = \frac{1}{2}\|a\|^2 + \frac{1}{2}\|a-b\|^2 - \frac{1}{2}\|b\|^2$, we can then write

$$\left\|\frac{u_n^{i+1}-u_n^i}{\tau_n}\right\|_H^2 + \left\|\frac{u_n^{i+1}-u_n^i}{\tau_n} - \frac{u_n^i-u_n^{i-1}}{\tau_n}\right\|_H^2 + \|\nabla u_n^{i+1}\|_{L^2}^2$$

$$+\|\nabla u_n^{i+1} - \nabla u_n^i\|_{L^2}^2 + \tfrac{2\gamma}{\tau_n}\|\nabla u_n^{i+1} - \nabla u_n^i\|_{L^2}^2$$

$$= \left\|\frac{u_n^i-u_n^{i-1}}{\tau_n}\right\|_H^2 + \|\nabla u_n^i\|_{L^2}^2 + 2\langle f_n^i, u_n^{i+1}-u_n^i\rangle_H. \tag{7}$$

Summing from $i = 0$ to j and using the initial data, we get

$$\left\| \frac{u_n^{j+1} - u_n^j}{\tau_n} \right\|_H^2 + \|\nabla u_n^{j+1}\|_{L^2}^2 + \sum_{i=0}^{j} \left\| \frac{u_n^{i+1} - u_n^i}{\tau_n} - \frac{u_n^i - u_n^{i-1}}{\tau_n} \right\|_H^2$$

$$+ \sum_{i=0}^{j} \|\nabla u_n^{i+1} - \nabla u_n^i\|_{L^2}^2 + \frac{2\gamma}{\tau_n} \sum_{i=0}^{j} \|\nabla u_n^{i+1} - \nabla u_n^i\|_{L^2}^2$$

$$= \|u^{(1)}\|_H^2 + \|\nabla u^{(0)}\|_{L^2}^2 + 2 \sum_{i=0}^{j} \langle f_n^i, u_n^{i+1} - u_n^i \rangle_H .$$

We now define $u_n, \tilde{u}_n, v_n : [0, T] \to V$ for $t \in (t_n^i, t_n^{i+1}]$ by

$$u_n(t) := u_n^i + (t - t_n^i)\frac{u_n^{i+1} - u_n^i}{\tau_n}, \tag{8}$$

$$\tilde{u}_n(t) := u_n^{i+1}, \qquad f_n(t) := f_n^i, \tag{9}$$

$$v_n(t) := \frac{u_n^i - u_n^{i-1}}{\tau_n} + \frac{t - t_n^i}{\tau_n}\left[\frac{u_n^{i+1} - u_n^i}{\tau_n} - \frac{u_n^i - u_n^{i-1}}{\tau_n}\right]. \tag{10}$$

Rewriting the previous sum, for every $t \in (t_n^j, t_n^{j+1})$ we now have

$$\|\dot{u}_n(t)\|_H^2 + \|\nabla u_n(t_n^{j+1})\|_{L^2}^2 + \tau_n \int_0^{t_n^{j+1}} \|\dot{v}_n(t)\|_H^2\, dt + \tau_n \int_0^{t_n^{j+1}} \|\nabla \dot{u}_n(t)\|_{L^2}^2 dt$$
$$+2\gamma \int_0^{t_n^{j+1}} \|\nabla \dot{u}_n(t)\|_{L^2}^2\, dt = \|u^{(1)}\|_H^2 + \|\nabla u^{(0)}\|_{L^2}^2 + 2\int_0^{t_n^{j+1}} \langle f_n(t), \dot{u}_n(t)\rangle_H\, dt.$$

We then have that

$$
\begin{aligned}
&\nabla u_n(t) \text{ and } \nabla \tilde{u}_n(t) \text{ are bounded in } L^2 \text{ uniformly in } t \text{ and } n, &(11)\\
&\gamma \nabla \dot{u}_n \text{ is bounded in } L^2(0,T;L^2) \text{ uniformly in } n, &(12)\\
&\dot{u}_n(t) \text{ and } v_n(t) \text{ are bounded in } H \text{ uniformly in } t \text{ and } n. &(13)
\end{aligned}
$$

We note that (13) together with the fact that $u^{(0)} \in H$ implies that u_n is bounded in H uniformly in t and n. This together with (11) gives

$$u_n(t) \text{ is bounded in } V \text{ uniformly in } t \text{ and } n.$$

Furthermore, using (8), (9), and (10) in (6) gives that for all $t \in (t_n^i, t_n^{i+1})$,

$$\langle \dot{v}_n(t), \phi \rangle_H + \langle \nabla \tilde{u}_n(t) + \gamma \nabla \dot{u}_n(t), \nabla \phi \rangle_{L^2} = \langle f_n(t), \phi \rangle_H$$

for every $\phi \in V_{t_n^{i+1}}$. This gives that for $t \in (t_n^i, t_n^{i+1})$, $\|\dot{v}_n(t)\|^*_{t_n^{i+1}} \le c$.

We then get

$$u_n \text{ is bounded in } H^1(0,T;V) \text{ and in } W^{1,\infty}(0,T;H), \tag{14}$$

$$v_n \text{ is bounded in } L^\infty(0,T;H), \tag{15}$$

$$v_n \text{ is bounded in } W^{1,\infty}(s,T;V_s^*) \text{ for every } s \in [0,T]. \tag{16}$$

We then show that $u_n, \tilde{u}_n \rightharpoonup u$, $v_n \rightharpoonup \dot{u}$, and u is a solution to...

Definition

We say that u is a weak solution of the wave equation on the crack domain $t \mapsto \Omega \setminus K(t)$ if

$$u \in L^\infty(0, T; V) \cap W^{1,\infty}(0, T; H) \cap W^{2,\infty}(s, T; V_s^*) \text{ for all } s \in [0, T],$$
$$u(t) \in V_t \text{ for a.e. } t \in [0, T],$$

and

$$\langle \ddot{u}(t), \phi \rangle_t^* + \langle \nabla u(t) + \gamma \nabla \dot{u}(t), \nabla \phi \rangle_{L^2} = \langle f(t), \phi \rangle_H$$

for every $\phi \in V_t$.

and

Theorem

For fixed $t \mapsto K(t)$ defined on $[0, T]$ such that $K(t_1) \subset K(t_2)$ if $t_1 < t_2$ and $\mathcal{H}^{N-1}(K(T)) < \infty$, given $u^{(0)} \in V_0$ and $u^{(1)} \in H$, there exists a weak solution of the wave equation.

Uniqueness?

We can also solve:

$$\langle \ddot{u}(t), \phi \rangle_t^* + \langle \nabla u(t) + \gamma \nabla \dot{u}(t), \nabla \phi \rangle_{L^2} = 0$$

for every $\phi \in V_t$.

Solutions here balance energy, *not including* the fracture energy. So, crack growth is impossible if the total energy is conserved. This comes from the fact that $\nabla \dot{u}(t) \in L^2$, so $\dot{u}(t) \in V_t$. Without the dissipation, in general $\nabla \dot{u}(t) \notin L^2$.

Expect: (guess:) if u_γ is the solution to this problem, then $\lim_{\gamma \to 0} u_\gamma =: u$ also balances elastic + kinetic energy, but solves the PDE with $\gamma = 0$. This implies non-uniqueness, if there exists a solution with elastic + kinetic energy decreasing as the crack grows.

A flaw in the model: really, should have that $K(t)$ is the *crack set* for $u(t)$, i.e., the minimal set such that $S_{u(\tau)} \subset K(t)$ for all $\tau \leq t$. In fact, given K, we can solve the wave equation as we just did, and get u, and then reduce K as necessary, getting the crack set for u, K^*.

Question: does u solve the wave equation on the cracking domain $t \mapsto \Omega \setminus K^*(t)$?

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Question: does u solve the wave equation on the cracking domain $t \mapsto \Omega \setminus K^*(t)$?

Yes, since $u(t) \in V_t^*$ and $V_t^* \subset V_t$, so the appropriate test functions for K^* are test functions for K, and u solves the weak wave equation w.r.t. these test functions.

Questions:

1. Are solutions of these models consistent with Griffith's criterion?
2. Is there stronger regularity of solutions than (is provable) for quasi-static? (Yes...)
3. Are any of these dynamic models the limit of the phase-field models? (Perhaps in principle and some situations, but probably not always true)
4. What is the quasi-static limit of the phase-field dynamic model? (Probably not phase-field quasi-static global minimizer, except when the dissipation is continuous in time)
5. What is the quasi-static limit of the sharp-interface dynamic model? (Probably not quasi-static global minimizer)

Dynamic cohesive fracture

(with V. Slastikov)
All of our models conserve energy, where the total energy is

$$\int_\Omega u_t^2 dx + \int_\Omega |\nabla u|^2 dx + \int_K \psi([u]) d\mathcal{H}^{N-1}.$$

$S(t) \subset \mathbb{R}$ will be a set of possible discontinuity points for u at time $t \in (0, T)$, and we define $S := \{(x,t) \in \mathbb{R} \times (0,T) : x \in S(t)\}$, which we require to be closed, $\Omega_S := [\mathbb{R} \times (0,T)] \setminus S$, and $H^1_S := H^1(\Omega_S)$.

Definition

We say that $u \in H^1_S$ is a constrained Force Balance solution if ψ' is Lipschitz and u satisfies

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \Omega_S \\ u_x = \psi'([u]) & \text{on } S. \end{cases}$$

Definition

We say that $u \in H^1_S$ is a constrained Stationary Action solution if ψ is Lipschitz and u satisfies

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \Omega_S \\ u_x \in \partial\psi([u]) & \text{on } S, \\ E(t) = E(0) & \text{for all } t, \end{cases}$$

where ∂ denotes subdifferential and E is the total energy.

The point of the regularity is that $\partial\psi(0) = [-\alpha, \alpha]$ for $\alpha := \psi'(0^+)$.

Of course, if ψ is smooth, then $\alpha = 0$ (since ψ is even) and this definition is equivalent to the previous one. In fact, the subdifferential inclusion just means that

$$u_x(x,t) = \psi'([u](x,t)) \text{ if } [u](x,t) \neq 0,$$

and $|u_x(x,t)| \leq \psi'(0^+)$ otherwise.

Derivation of Stationary Action solution

For simplicity, we suppose that $S(t) = \{0\}$ for all $t \in (0, T)$. We define the action to be

$$A(u) = \int_0^T \left(\frac{1}{2}\|u_t\|^2 - \frac{1}{2}\|u_x\|^2 - \psi([u](0,t)) \right) dt,$$

where the norms are L^2 in space. We consider

$$t \mapsto A(u + \lambda v),$$

where $v(x,0) = v(x,T) = \frac{\partial}{\partial t} v(x,t)|_{t=0} = 0$ and $v \in C^1(\mathbb{R} \setminus \{0\} \times [0,T])$.

$$0 \in \int_0^T \Big[\int_{-\infty}^0 (u_t v_t - u_x v_x) dx$$

$$+ \int_0^\infty (u_t v_t - u_x v_x) dx - \partial\psi([u](0,t))[v](0,t) \Big] dt.$$

Assuming sufficient regularity, after integration by parts we get

$$0 \in \int_0^T \left[\int_{-\infty}^0 (-u_{tt}v + u_{xx}v)dx + \int_0^\infty (-u_{tt}v - u_{xx}v)dx \right] dt +$$

$$+ \int_0^T \Big[u_x(0^+,t)v(0^+,t) - u_x(0^-,t)v(0^-,t) - \partial\psi([u](0,t))[v](0,t) \Big] dt.$$

This gives the Stationary Action model, considering first $v(0,t) = 0$, then $[v](0,t) = 0$, and then general v.

Definition

We say that $u \in H^1_S$ is a constrained Maximal Dissipation solution if it satisfies

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \Omega_S \\ u_x \in \partial\psi([u]) & \text{on } S \\ E(t) = E(0) & \text{for all } t, \end{cases} \tag{17}$$

and in addition we have the maximal dissipation condition, namely, that if w also satisfies (17), and is such that for some $\bar{t} \in [0, T)$:

- $w = u$ on $\Omega \times [0, \bar{t}]$ (and $w_t = u_t$ if $\bar{t} = 0$)
- for some $\varepsilon > 0$, $[w](0, t) > (<)\ 0$ on $(\bar{t}, \bar{t} + \varepsilon)$,

then $[u](0, t) \geq (\leq)\ [w](0, t)$ on $\Omega \times (\bar{t}, \bar{t} + \varepsilon)$.

All definitions are equivalent of ψ' is continuous.

Definition

We say that u is an unconstrained Maximal Dissipation solution if it is in H^1_S for some S as defined above, and it satisfies

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \Omega_S \\ u_x \in \partial\psi([u]) & \text{for all } x \in \mathbb{R} \text{ and a.e. } t \\ E(t) = E(0) & \text{for all } t > 0, \end{cases} \tag{18}$$

and in addition we have the maximal dissipation condition, namely, that if w also satisfies (18), and is such that for some $\bar{t} \in (0, T)$:

- $w = u$ on $\Omega \times (0, \bar{t}]$ (and $w_t = u_t$ if $\bar{t} = 0$)
- for some $\varepsilon > 0$, $[w](0, t) > (<)\ 0$ on $(\bar{t}, \bar{t} + \varepsilon)$,

then $[u](0, t) \geq (\leq)\ [w](0, t)$ on $\Omega \times (\bar{t}, \bar{t} + \varepsilon)$.

Existence

First, we give an example of a solution to the constrained problem, which has an opening crack. We construct a solution of the form

$$u(x,t) = \begin{cases} v(x+t) & \text{in } \mathbb{R}_+ \times (0,T) \\ v^*(x-t) & \text{in } \mathbb{R}_- \times (0,T). \end{cases}$$

Our u will be odd in space at all times, so it is enough to construct only v, noting that $[u](0,t) = 2v(0,t) \geq 0$ by construction. First, define

$$g(x) := \int_0^x \frac{1}{\psi'(2s)} ds$$

so that $g'(x) = \frac{1}{\psi'(2x)}$, and we allow g to take the value ∞. Since $\psi' \geq 0$ on $[0,\infty)$, g is monotonic. Define v to be the inverse of g. Then,

$$u_x(0^+,t) = v'(t) = \frac{1}{g'(v(t))} = \psi'(2v(t)) = \psi'([u](0,t)).$$

Lemma

If ψ' is Lipschitz, then given $u_0 \in H^1(\mathbb{R})$ and $u_1 \in L^2(\mathbb{R})$, there exists a unique solution of the constrained Force Balance problem.

Proof.

$$u_0^{\pm}(x) = \begin{cases} u_0(\pm x) & \text{in } \mathbb{R}_+ \\ u_0(\mp x) & \text{in } \mathbb{R}_-, \end{cases}$$

$$u_1^{\pm}(x) = \begin{cases} u_1(\pm x) & \text{in } \mathbb{R}_+ \\ u_1(\mp x) & \text{in } \mathbb{R}_-. \end{cases}$$

$u^+(x,t)$ and $u^-(x,t)$ are given by

$$u^{\pm}(x,t) = \frac{1}{2}\left(u_0^{\pm}(x+t) + u_0^{\pm}(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} u_1^{\pm}(s)\,ds \tag{19}$$

$$\mp \begin{cases} \int_0^{t\mp x} \psi'([u](s))\,ds & \text{for } t \mp x > 0 \\ 0 & \text{for } t \mp x \leq 0. \end{cases} \tag{20}$$

□

Continued.

Define $u(x,t)$ by

$$u(x,t) = \begin{cases} u^+(x,t) & \text{for } x > 0 \\ u^-(x,t) & \text{for } x < 0. \end{cases} \tag{21}$$

One can show that the jump $[u](0,t)$ must satisfy

$$[u](0,t) = u_0(t) - u_0(-t) + \int_0^t u_1(s)\,ds - \int_{-t}^0 u_1(s)\,ds - 2\int_0^t \psi'([u](0,s))\,ds, \tag{22}$$

or more concisely, the jump v satisfies

$$v'(t) = v_0(t) - 2\psi'(v(t)),$$

where v_0 is the derivative of the sum of the first four terms on the right hand side of (22). This can be solved for v uniquely, since ψ' is Lipschitz. We then also have $\psi'(v(t))$, and so we can write an explicit formula for u with this Neumann condition at $x = 0$. (Also, energy balance).

$\psi'(0) = \alpha < \infty$

Theorem

Given $u_0 \in H^1(\mathbb{R})$, $u_1 \in L^2(\mathbb{R})$, and ψ with $\psi'(0^\pm) = \pm\alpha$, α finite, there exists a Stationary Action solution with jump constraint at $x = 0$, with $u(\cdot, 0) = u_0$ and $u_t(\cdot, 0) = u_1$.

Proof.

We consider ψ_n that are smooth, even, equal to ψ outside of $(-1/n, 1/n)$, and such that $\lim_{n\to\infty} \max \psi'_n = \alpha$. Then, by the previous Lemma, there exists a unique u^n. By the energy balance, there exists $u \in H^1_S$ such that $u^n \rightharpoonup u$ in H^1_S (up to a subsequence). Then $[u^n] \to [u]$ in $L^2(0, T)$. Can then show u is a solution, and balances energy (but not uniqueness). □

Theorem

Given N points $x_1, \ldots, x_N$, and given $u_0 \in H^1(\mathbb{R})$, $u_1 \in L^2(\mathbb{R})$, and ψ as above, there exists a solution of the weak cohesive wave equation with jump constraint $S(t) = \{x_1, \ldots, x_N\}$, with $u(\cdot, 0) = u_0$ and $u_t(\cdot, 0) = u_1$.

Proof.

$N = 2$, $x_1 = 0$, and $x_2 = 2$

Step 1: u_1^0 is a solution that can only jump at $x = 0$, and u_1^2 is the solution that can only jump at $x = 2$. From finite speed of propagation, $u_1^0(1, t) = u_1^2(1, t)$ for all $t \in (0, 1)$. Define u on $[\mathbb{R} \setminus \{0, 2\}] \times (0, 1]$ by

$$u(x,t) := \begin{cases} u_1^0(x,t) & \text{for } x \in (-\infty, 1] \\ u_1^2(x,t) & \text{for } x \in [1, \infty) \end{cases}$$

gives a solution on $[\mathbb{R} \setminus \{0, 2\}] \times (0, 1]$.

Step 2: Repeat this procedure with "initial" data $u(x, 1)$, getting a solution on $[\mathbb{R} \setminus \{0, 2\}] \times (1, 2]$, etc.

□

$\psi'(0) = \infty$

Theorem

Let $u_0 \in H^1(\mathbb{R})$ and $u_1 \in L^2(\mathbb{R})$ be given, each with finitely many singularities (or locally finitely many), that is, there exist $x_1, \dots, x_n \in \mathbb{R}$ such that for every neighborhood N of $\{x_1, \dots, x_n\}$, $u_0 \in W^{1,\infty}(\mathbb{R} \setminus N)$ and $u_1 \in L^\infty(\mathbb{R} \setminus N)$. For ψ as above, there exists $u \in H^1_S$ satisfying the wave equation such that u satisfies the Maximal Dissipation condition with $S(t) = \{0\}$ (with an extension to $S(t)$ being a locally finite set of points, as in the previous theorem).

Proof.

Suppose w satisfies the wave equation, with the same initial data as u, and $[w](0,t) > 0$ on $(0, \varepsilon)$ for some $\varepsilon > 0$. Find $\{u_n\}$, corresponding to ψ_n, with $\psi_n \nearrow \psi$ uniformly, ψ_n' Lipschitz, $\psi_n' = \psi'$ outside $(-\frac{1}{n}, \frac{1}{n})$, $\psi_n' \leq \psi'$ on $(0, \frac{1}{n})$, and such that $[u_n](0,0) = 0$ and u_n have the same initial data as u. Since $\psi_n \to \psi$ uniformly and $\psi_n' \to \psi'$ uniformly outside of 0, $u_n \rightharpoonup u$ in H^1_S to some function u. Can show that $[u](0,t) \geq [w](0,t)$ on $(0, \varepsilon)$, using monotonicity of ψ_n. □