SISSA ISAS

SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES
I-34014 Trieste ITALY - Via Beirut 4 - Tel. [+]39-40-37871 - Telex:460269 SISSA I - Fax: [+]39-40-3787528

Fall 2006 - Entrance Examination: Condensed Matter

Solve at least one of the following problems. Write out solutions clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem, and question clearly. Do not write your name on the problem sheet, but use extra envelope.

Problem 1: electron motion in a circular ring

A single free electron is constrained to move on a circular ring, of radius R. Its space coordinate, $\mathbf{R} = R(cos\theta, sin\theta, 0)$, is thus fully specified by a single angle θ describing its position on the ring, and the electron behaves like a rotor.

1. Write down the electron's Hamiltonian (do not worry too much about numerical factors).
2. Solve Schroedinger's equation, and find the spectrum of eigenvalues and eigenvectors.

Now turn on a magnetic field B perpendicular to the ring. If $\Phi = B\pi R^2$ is its flux through the ring, and $\Phi_0 = hc/e$ is the unit flux quantum, let B vary so that $\phi = \Phi/\Phi_0$ runs from 0 to 1.

3. Write down the new electron Hamiltonian including the field (neglect Zeeman coupling of the field to the electron spin) [Hint: $\mathbf{A} = \frac{\mathbf{B}\times\mathbf{R}}{2}$]. From Schroedinger's equation find the new spectrum as a function of ϕ. Comment on what happens when ϕ= 1/2, 1, and larger than 1.
4. Now consider two electrons instead of one (still neglecting Zeeman coupling), and take B such that ϕ= 1/2. Keeping into account spin and the Pauli principle, write down the possible degenerate ground states that the two electrons can form if assumed to be hypotetically non-interacting.
5. Recalling finally that the two electrons repel, specify (without doing any actual calculation) what their unique ground state will be. (Hint: use analogy with Hund's first rule of atomic physics).

Problem 2: Dynamics of a particle with a random force

Consider a one dimensional particle that moves according to the following iterative process:

$$x_{n+1} = x_n + \Delta \cdot f_{x_n} + \sqrt{\Delta} \cdot \eta_n^{(\alpha)} \tag{1}$$

where Δ is the "time step" of the iteration process (ignore the fact that its physical dimension is not time, but rather time^2/mass), $n \geq 0$ is an integer (the iteration step), x_n is the coordinate of the particle at step n, $f_{x_n} = -Kx_n$ is a harmonic force of spring constant K acting on the particle, and finally $\eta_n^{(\alpha)}$ is a random term acting at "time" n: the label $\alpha = 1, 2, \cdots M$ identifies several possible realizations of such a random term.

1. Write the position of the particle $x_N^{(\alpha)}$ after N steps with the initial condition $x_{n=0}^{(\alpha)} = x_0$, in terms of the random terms $\eta_n^{(\alpha)}$, for a given realization α. What is the maximum value Δ_{max} of Δ such that the iterative process is not unstable (i.e., the x_N grow wildly as N grows)?

2. Let us assume that the different realizations of the random term η_n are uncorrelated, have zero average and variance $2D$, namely:

$$\frac{1}{M}\sum_\alpha \eta_n^{(\alpha)} = 0 \tag{2}$$

$$\frac{1}{M}\sum_\alpha \eta_n^{(\alpha)}\eta_m^{(\alpha)} = 2D\delta_{n,m}\ , \tag{3}$$

   where $\delta_{n,m} = 1$ for $n = m$ and $\delta_{n,m} = 0$ for $n \neq m$. Consider the position $x_N^{(\alpha)}$ and its square $(x_N^{(\alpha)})^2$ and average them over the M realizations of the disorder, to compute $\langle x_N \rangle$ and $\langle x_N^2 \rangle$, as a function of the initial condition x_0, the variance D, the spring constant K and the "time" step Δ.

3. Suppose now that Δ is infinitesimally small. What happens for large $N \to \infty$ to $\langle x_N \rangle$ and $\langle x_N^2 \rangle$? If this process has to represent, for $N \to \infty$, the equilibrium state of the oscillator at temperature T, what must be the value of the variance D? [Hint: Use classical equipartition of the potential energy of the oscillator.]

Problem 3: A particle on a the surface of a sphere

Consider a particle with mass m and charge -e constrained onto the surface of a neutralizing, uniformly charged, sphere.

1. Calculate the energy levels of the system.
2. Calculate the electrical polarizability of the ground state
3. Discuss one possible strategy for evaluating numerically the energy levels in a finite electric field.

Problem 4: Earth Energy Balance

From a global thermodynamical point of view the Earth is and open system where energy is received from the Sun, and eventually converted into heat and re-emitted as radiation into space.

Both Sun and Earth can to a good approximation be considered as black bodies, adsorbing all impinging radiation and re-emitting it according to the Stefan-Boltzmann law which states that the emitted power per unit area is $W = \sigma T^4$ where T is the black body temperature and $\sigma = \frac{2\pi^5 k_B^4}{15c^2h^3} = 5.67 \times 10^{-8} W m^{-2} K^{-4}$

1. Calculate the solid angle under which the Sun is viewed from Earth knowing that the Sun radius is 6.96×10^8m and the average Sun-Earth distance is 1.5×10^{11} m.

2. Neglecting human or geothermic energy contributions, estimate the Sun surface temperature from the global energy balance of Earth, that is assuming that the total power adsorbed from the Sun is equal to the power radiated by the Earth. Take the average Earth temperature equal to 14 ^oC=287 K and compare your estimate with the measured Sun temperature (5870 K).

3. Calculate the total energy adsorbed by the Earth from the Sun in one year and compare it with the world energy consumption in 2003 which was 123×10^{12} KWh/yr and with the estimated consumption in 2010 (149×10^{12} KWh/yr). Was the neglect of such contributions in the previous point a good approximation ? Earth circumference is 40000 Km.

   Optional:

   A model system realizing the black body emission is a cavity of volume V containing electromagnetic radiation in equilibrium with the cavity walls at temperature T and with a small hole whose emission is measured.

4. Calculate the black body radiation energy density $u = \int h\nu\rho(\nu)d\nu$, where $\rho(\nu)$ is the density of photons with frequency between ν and $\nu + d\nu$. [Hint: $\int_0^\infty \frac{x^3}{exp(x)-1}dx = \frac{\pi^4}{15}$]

5. Show that for a black body the emission power per unit area is $w = \frac{c}{4}u$ where u is the black body radiation energy density.

6. Obtain from the previous results the expression of the Stefan-Boltzmann constant σ in terms of the fundamental constants k_B, c, h.