SISSA ISAS

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October 2011 - Entrance Examination: Condensed Matter

Solve at least one of the following problems. Write out solutions clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem, and question clearly. Do not write your name on the problem sheet, but use extra envelope.

Problem 1: A quantum particle in a double well potential

A particle of mass m is constrained to move along a straight line under the action of a 1D potential:

$$V(x) = \frac{1}{2}\left(x^2 - x_0^2\right)^2$$

1. Show that, classically, a minimum energy exists below which the particle can only execute a periodic motion for $x < 0$ or $x > 0$, without crossing the $x = 0$ boundary, whereas for larger energies the particle always crosses such boundary.

2. Write down the time-independent Schrödinger equation for this system, and show that *all* its eigenfunctions are different from zero for *both* positive and negative values of x.

3. Using a sensible approximation, calculate the first few energy levels of the system in the limit $m \to \infty$.

4. Show that in the $m \to \infty$ limit a number of eigenfunctions become degenerate in pairs and that linear combinations of the two partners of each pair can be constructed, such that they *approximately* vanish for either $x > 0$ or $x < 0$. Explain what does the word *approximately* mean in this case.

5. Provide a rough estimate of the number of such degenerate pairs, as a function of m.

Problem 2: Two spin-dipolar-coupled electrons

The relative distance between two electrons of spin $S = 1/2$ is kept fixed and equal to $\mathbf{d}$. The two electrons interact among each other only via a magnetic dipolar exchange:

$$H = \frac{\mu_B}{4\pi} \frac{d^2\, \mathbf{S}_1 \cdot \mathbf{S}_2 - 3\, (\mathbf{S}_1 \cdot \mathbf{d})\, (\mathbf{S}_2 \cdot \mathbf{d})}{d^5}, \tag{1}$$

where $\mathbf{S}_{1(2)}$ are the spin operators of the electrons.

1. Calculate the spin eigenfunctions and eigenvalues of the two-electron system.

2. Assume a spin quantization axis parallel to $\mathbf{d}$. With respect to it, the two electrons have initially opposite spins. Determine the time-evolution of such an initial state under the action of the exchange Eq. (1).

3. Assume now that, while the modulus $d =\mid \mathbf{d} \mid$ is still kept fixed, the direction of $\mathbf{d}$ can change according to the Hamiltonian of a free rotor

   $$H_d = \frac{1}{2I}\, \mathbf{L} \cdot \mathbf{L}, \tag{2}$$

   where I is the moment of inertia and $\mathbf{L}$ the angular momentum operator. How will the solution of the question (1) above be modified?

Problem 3: An electron in a magnetic field

Consider an electron with mass m and charge $-e$ ($e > 0$) that moves freely in a box of volume $V = L_x L_y L_z$. The electron wavefunctions obey periodic boundary conditions $\psi(x + L_x, y, z) = \psi(x, y, z)$ and similar equations for the other directions.

1. Calculate the allowed energies of the electron as a function of L_x, L_y and L_z. Is the ground state degenerate? Which is the degeneracy of the first excited state?

2. Write the eigenfunctions of the free electron Hamiltonian. Is the number of allowed states finite or infinite? Describe what happens in the limit $L_x \to \infty$, $L_y \to \infty$, $L_z \to \infty$.

3. Write the total energy of the free electron as $E = E_\perp + E_z$, where $E_\perp$ and E_z are the energies associated to the motion in the xy plane and along z respectively. At fixed E_z, count how many states are available when the total energy is between E_z and $E_z + E_\perp$ as a function of $E_\perp$. Consider only the case in which L_x and L_y are very large.

We now put the box in a uniform magnetic field $\mathbf{B} = (0, 0, B_z)$ parallel to the z axis and described through the vector potential $\mathbf{A} = (0, xB_z, 0)$. Neglecting the electron spin, the Hamiltonian of the electron is given by:

$$H = \frac{1}{2m}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2 \tag{3}$$

where $\mathbf{p}$ is the electron momentum and c is the speed of light.

4. Show that the motion along z remains free, while in the xy plane the Hamiltonian becomes similar to that of an harmonic oscillator. Find the eigenvalues and eigenvectors of H. Hint: Try a solution in the form $\psi(x, y, z) = \alpha(x)\beta(y)\gamma(z)$, where $\beta(y) = \frac{1}{\sqrt{L_y}} e^{ik_y y}$. Neglect boundary effects on the harmonic oscillation.

5. Find the degeneracy of the eigenfunctions at fixed E_z and $E_\perp$. Hint: You can use the fact that the center of oscillation must be between 0 and L_x.

6. Using the answer given at point [3.], compute, at fixed E_z, the number of states available to the free electron between the energy $E = E_z$ and the energy $E = E_z + (E_{\perp,0} + E_{\perp,1})/2$, where $E_{\perp,0}$ and $E_{\perp,1}$ are the two lowest allowed values of $E_\perp$ when $B_z > 0$. Compare with the degeneracy of $E_\perp$ found at previous point.