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October 2012 - Entrance Examination: Condensed Matter

Solve at least one of the following problems. Write out solutions clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem, and question clearly. Do not write your name on the problem sheet, but use extra envelope.

Problem 1: Current in a one dimensional chain

Consider a model consisting of an infinite one dimensional chain of sites i, with nearest neighbor spacing a. Each site carries a single orbital $|i\rangle$ and one electron. Assuming orbitals on different sites to be orthonormal, $\langle i|j\rangle = \delta_{ij}$ and a Hamiltonian consisting of a hopping matrix element $-|t|$ between nearest-neighbor sites:

$$H = -|t| \sum_i \left(|i\rangle\langle i+1| + |i+1\rangle\langle i| \right) \ .$$

1) Write down the (standard) one-electron wavefunction $|k\rangle$ that is a translationally invariant linear combination of all sites $|i\rangle$ and which possesses a wavevector k.

2) Calculate the (standard) energy ϵ_k of that wavefunction $|k\rangle$.

3) Show that the electrical current carried by the wavefunction $|k\rangle$ is proportional to k, and give the proportionality constant, if the electron charge is $-|e|$.

Assuming non-interacting electrons and $T = 0$, fill up each level $|k\rangle$ with two electrons, from $k = -k_F + Q$ to $k = +k_F + Q$, where $k_F = \pi/2a$ is the Fermi momentum, and Q is small, $Q \ll K_F$. For $Q = 0$, this is the ground state of the system. For $Q > 0$, this state $|\Psi(Q)\rangle$ carries a current.

4) Calculate the total energy E_0 for the ground state $|\Psi(Q = 0)\rangle$, for the current-carrying state $|\Psi(Q)\rangle$, and express their difference $\Delta E(Q)$ in powers of Q for small Q.

5) Calculate the total current $J(Q)$ of state $|\Psi(Q)\rangle$, again for small Q. Compare the result with the energy increase $\Delta E(Q)$, and discuss the possible physical connection between the two quantities.

Problem 2: A potential step in two dimensions

An electron beam (mass m, charge q, and spin 1/2) propagates in the xy plane. The potential energy is $V(x, y) = 0$ for $x < 0$ and $V(x, y) = U$ for $x > 0$ where U is a positive constant. Assume that the electrons move from the negative to the positive x direction and neglect electron-electron interactions.

1. Suppose first that the incident beam is parallel to the x axis, describe the electrons by traveling waves and find the probability that the electrons are reflected or transmitted by the potential step. Write the result as a function of the total energy E of the incident particles. Study only the case $E > U$.

2. Then suppose that the direction of the beam forms an angle α with the $-x$ axis. Calculate the reflection probability as a function of the angle α and of the total energy E of the incident particles. Find the angle β that the reflected beam forms with the $-x$ axis and the angle γ that the transmitted beam forms with the x axis. Determine the maximum angle α (let us call it $\alpha_M(E)$) for which there is a transmitted beam.

3. Finally suppose that in the region $x > 0$ there is a uniform magnetic field perpendicular to the xy plane (directed as $+z$) that couples only with the electron spin magnetic moment, so that the Hamiltonian in the region $x > 0$ is:

$$H = \frac{p_x^2 + p_y^2}{2m} + U + \mu_B \sigma_z B \tag{1}$$

   where (p_x, p_y) is the momentum of the particles, σ_z is the z Pauli matrix, $\mu_B = -\frac{q\hbar}{2m}$ is the Bohr magneton ($\hbar$ is the Planck constant divided by 2π) and B is the magnitude of the magnetic field. Using the same hypotheses as at point 2 and 3, find the maximum angle $\alpha_M^{\uparrow}(E)$ for which there is a transmitted beam for electrons with spin parallel to the magnetic field. Find also the maximum angle $\alpha_M^{\downarrow}(E)$ for electrons with spin anti-parallel to the magnetic field.

4. Find the range of angles $\Delta\alpha(E)$ for which the transmitted beam is completely spin polarized.

Problem 3: A free electron in an oscillating electric field

Consider a free electron, for simplicity in one dimension. Suppose at $t = 0$ a sinusoidal electric field is turned on, so that the Hamiltonian of the electron, for $t \geq 0$, denoting by $-e$ the charge of the electron and by E_{ac} the amplitude of the ac-field, is:

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + eE_{ac}\cos(\omega t + \phi)\hat{x} \ . \tag{2}$$

We assume that, at $t = 0$, the electron is in a given wave-packet $\psi_0(x) = \langle x|\psi_0\rangle$ which is centered in the origin and with zero average momentum, i.e., such that $\langle\hat{x}\rangle_{t=0} = \langle\psi_0|\hat{x}|\psi_0\rangle = 0$ and $\langle\hat{p}\rangle_{t=0} = \langle\psi_0|\hat{p}|\psi_0\rangle = 0$.

1) Using the Heisenberg representation of operators, $\hat{O}_H(t) = \hat{U}^\dagger(t,0)\hat{O}\hat{U}(t,0)$ (where $\hat{U}(t,0)$ is the (unitary) evolution operator), write down the Heisenberg's equations of motion for the operators $\hat{x}_H(t)$ and $\hat{p}_H(t)$, in terms of which you should be able to express in a simple way the expectation values of $\hat{x}$ and $\hat{p}$ at time t, i.e., find explicit analytic expressions for $\langle\hat{x}\rangle_t = \langle\psi(t)|\hat{x}|\psi(t)\rangle$, and $\langle\hat{p}\rangle_t = \langle\psi(t)|\hat{p}|\psi(t)\rangle$. Specialize your results, explicitly, to the two cases: a) $\phi = 0$ and b) $\phi = \pi/2$. What is the average momentum at time t in both cases? Discuss the possible presence of a "dc" component together with the obvious ac term. [N.B.: In deriving the Heisenberg's equations of motion, recall that while $\hat{U}(t,0)$ is not $e^{-i\hat{H}t/\hbar}$ because $\hat{H}$ is time-dependent, it still obeys the standard time-dependent Schrödinger equation.]

2) Apply the same procedure to calculate $\langle\hat{p}^2\rangle_t = \langle\psi(t)|\hat{p}^2|\psi(t)\rangle$.

3) Write down the average *kinetic* energy at time t, $E_{\rm kin}(t) = \langle\psi(t)|\hat{p}^2|\psi(t)\rangle/(2m)$, and the average *total* energy $E_{\rm tot}(t) = \langle\psi(t)|\hat{H}(t)|\psi(t)\rangle$. Plot and discuss these quantities in the two cases a) $\phi = 0$ and b) $\phi = \pi/2$ mentioned above.

You can indeed solve exactly the problem by writing down the full evolution operator $\hat{U}(t,0)$. Let us see how. Consider the unitary operator $\hat{T}(t) = e^{-i\Phi(t)}e^{ik(t)\hat{x}}e^{-ia(t)\hat{p}/\hbar}$, describing a translation by $a(t)$ followed by a boost of momentum $k(t)$, $\Phi(t)$ being a phase. The goal is to find $k(t)$, $a(t)$ and $\Phi(t)$ such that, if you make the *Ansatz*

$$\hat{U}(t,0) = \hat{T}(t)e^{-i\hat{H}_0 t/\hbar}\hat{T}(0)^{-1} \ ,$$

with $\hat{H}_0 = \hat{p}^2/2m$, this satisfies the time-dependent Schrödinger equation.

4) Write the equation that $\hat{T}(t)$ has to satisfy, and find corresponding expressions for $\dot{k}(t)$, $\dot{a}(t)$ and $\dot{\Phi}(t)$, which can be immediately integrated. Discuss the relationship between $k(t)$, $a(t)$, $\Phi(t)$ and the quantities studied in points 2) and 3) above. [Hint: It might be useful to remember that, for commutators, $[\hat{p}, f(\hat{x})] = -i\hbar f'(\hat{x})$ and $[\hat{x}, g(\hat{p})] = i\hbar g'(\hat{p})$, where f' and g' denotes derivatives of f and g, assuming you can Taylor-expand them.]