Optimal strokes for low Reynolds number swimmers: an example

François Alouges* Antonio DeSimone† Aline Lefebvre‡

Abstract

Swimming, *i.e.*, being able to advance in the absence of external forces by performing cyclic shape changes, is particularly demanding at low Reynolds numbers. This is the regime of interest for micro-organisms and micro- or nano-robots. We focus in this paper on a simple yet representative example: the three-sphere swimmer of Najafi and Golestanian [16]. For this system, we show how to cast the problem of swimming in the language of control theory, prove global controllability (which implies that the three-sphere swimmer can indeed swim), and propose a numerical algorithm to compute optimal strokes (which turn out to be suitably defined sub–Riemannian geodesics).

*Laboratoire de Mathématiques, Université Paris-Sud, 91405 Orsay cedex, France, Francois.Alouges@math.u-psud.fr

†SISSA-International School for Advanced Studies, via Beirut 2-4, 34014 Trieste, Italy, desimone@sissa.it

‡Laboratoire de Mathématiques, Université Paris-Sud, 91405 Orsay cedex, France, aline.lefebvre@math.u-psud.fr

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1 Introduction

The problem of swimming at low Reynolds numbers has attracted considerable attention in the recent literature, starting from the pioneering works of Taylor [18], Berg [5] and Purcell [17]. This problem is both puzzling and relevant *e.g.*, for biological systems and micro- or nano-robots. Indeed, due to the length and time scales involved, the motion of micro-swimmers is dominated by viscosity, while inertia is negligible. This implies that microorganisms, such as bacteria, must adopt swimming strategies completely different from those employed by larger organisms, such as fish. In particular, the observation that, in a flow regime obeying Stokes equations, a scallop cannot advance through the reciprocal motion of its valves is called the "scallop theorem" [17]. The mathematical explanation for this is the symmetry of the Stokes equations under time reversal: whatever forward motion will be produced by closing the valves, it will be exactly canceled by a backward motion upon reopening them.

This leads to the question of finding the simplest mechanisms capable of self propulsion at this scales. By this we mean the ability to advance by performing a cyclic shape change - *a stroke* - in the absence of external forces. Several proposals have been put forward and analyzed, see *e.g.*, [18, 17, 16, 2, 4]. A particularly simple example, due to Najafi and Golestanian [16], is the three-sphere swimmer. In its simplest form, it consists of three equal spheres of radius a moving along a straight line (see Fig. 1).

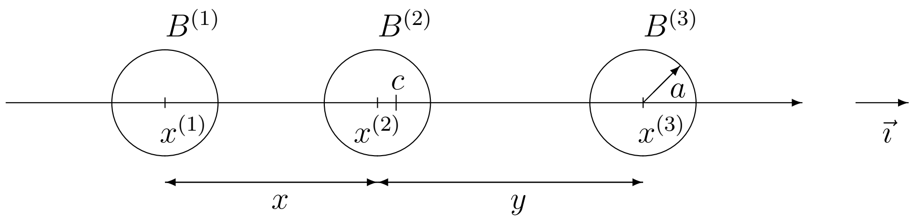

Figure 1: Swimmer's geometry and notation.

We call $\Omega := \cup_{i=1}^{3} B^{(i)}$ the union of the three open balls $B^{(i)}$, x and y the two distances between the centers of the balls and c the coordinate of the global center of mass. The state of our system is thus (x, y, c), its shape is (x, y) and its position will be c. A brief mathematical description of the system is the following: each ball is acted upon by a force $f^{(i)}$ (coming from the other balls) which is transmitted to the surrounding fluid, generating a flow solution to Stokes equations outside Ω. As a consequence, each ball moves at a velocity $u^{(i)}$. The relation between forces and velocities is linear (Stokes equations are linear) and given by

$$\begin{pmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{pmatrix} = S(x,y) \begin{pmatrix} f^{(1)} \\ f^{(2)} \\ f^{(3)} \end{pmatrix} \tag{1}$$

where S is known as the Oseen matrix [16]. In this context, self propulsion means that the total force acting on the system vanishes

$$f^{(1)} + f^{(2)} + f^{(3)} = 0. \tag{2}$$

It turns out that the center of mass c satisfies the ODE

$$\frac{dc}{dt} = V_x(x,y)\frac{dx}{dt} + V_y(x,y)\frac{dy}{dt}, \tag{3}$$

where (V_x, V_y) is a vector field that can be computed explicitly from S. The swimming problem is the following question: can one find force laws $f^{(i)} : [0,T] \to \mathbb{R}$ satisfying (2), which produce T-periodic paths in the space of shapes (x,y), and such that the average speed of the center of mass

$$\bar{c} = \frac{1}{T}\int_0^T \frac{dc}{dt}\,dt \tag{4}$$

does not vanish? A positive answer has been given in the physics literature [16] in the limiting regime of small spheres and small deformations.

We cast the swimming problem in the language of control theory. The controls are either the forces $f^{(i)}$ subject to the constraint (2), or the shape parameters x and y. The positive answer to the swimming problem is a consequence of the global controllability of the system, *i.e.*, the possibility of reaching an arbitrary point (x_1, y_1, c_1) of the space of states, starting from another arbitrary state (x_0, y_0, c_0), through self propulsion.

Theorem 1 *The three-sphere swimmer is a globally controllable system.*

Granted the possibility of swimming, the next interesting question is how to swim optimally, namely, to find the optimal stroke.

A classical notion of swimming efficiency is due to Lighthill [13]. It is defined as the inverse of the ratio between the average power expended by the swimmer during a stroke starting and ending at the shape (x_0, y_0) and the power that an external force would spend to translate the system rigidly at the same average speed $\bar{c}$:

$$\mathrm{Eff}^{-1} = \frac{\frac{1}{T}\int_0^T \sum_i f^{(i)} u^{(i)}}{6\pi\eta A \bar{c}^2}. \tag{5}$$

Here, η is the viscosity of the fluid, and $A = A(x_0, y_0)$ is the effective radius of the swimmer, which tends to $3a$ when the three spheres are infinitely far apart (see [16]). Our second main result is a numerical algorithm, namely, **Algorithm 1** in section 4, to compute the force laws that provide, for given $\bar{c}$, the stroke requiring minimal expended power, hence yielding maximal efficiency. We emphasize that, contrary to the standard approach in the Physics literature, we do not fix *a priori* the shape of the stroke, and then optimize over a few scalar parameters. Rather, we let the swimmer free to choose an optimal "gait".

The rest of the paper is organized as follows. In section 2 we describe the setting of the problem in detail. Section 3 contains the proof of Theorem 1. In section 4 we state the problem of optimal swimming and discuss the numerical strategy for its solution. Some numerical results are presented in section 5. Future directions are discussed in section 6.

2 Setting of the problem

2.1 Stokes equations

Let $\Omega = \cup_{i=1}^{3} B^{(i)}$ be the union of the three open balls, and assume that the flow in $\mathbb{R}^3 \setminus \Omega$ satisfies the (static) Stokes equation. This means that for given vectors $f^{(i)}$ there exists a unique pair (u, p), where the velocity of the fluid u is constant on each $\partial B^{(i)}$, and p is the pressure which satisfies

$$\left[\begin{array}{l} -\eta \Delta u + \nabla p = 0 \text{ in } \mathbb{R}^3 \setminus \Omega, \\ \operatorname{div} u = 0 \text{ in } \mathbb{R}^3 \setminus \Omega, \\ -\displaystyle\int_{\partial B^{(i)}} \sigma n = f^{(i)}, \\ u \to 0 \text{ at } \infty. \end{array}\right. \tag{6}$$

Here $\sigma = \eta\left(\nabla u + \nabla^t u\right) - p\mathrm{Id}$ is the Cauchy stress tensor, and n is the outer unit normal to the boundary of Ω. The constant trace of u on $\partial B^{(i)}$ represents the velocity $u^{(i)}$ of the ball $B^{(i)}$.

It is convenient to reformulate the problem as one in which the velocities $u^{(i)}$ are the given data and the forces $f^{(i)}$ are to be calculated, and show that there is a linear one-to-one correspondence between the $u^{(i)}$ and the $f^{(i)}$. Let $M = L^2(\mathbb{R}^3 \setminus \Omega)$, and H be the weighted Hilbert space

$$H = \left\{ u \in \mathcal{D}'(\mathbb{R}^3 \setminus \Omega) : \nabla u \ \text{ and } \frac{u}{\sqrt{1+|r|^2}} \in L^2(\mathbb{R}^3 \setminus \Omega)\,, u|_{\partial B^{(i)}} = 0 \right\} \tag{7}$$

endowed with the classical norm $||\cdot||_H$ defined by

$$||u||_H^2 = \int_{\mathbb{R}^3\setminus\Omega} |\nabla u|^2 \,, \tag{8}$$

which is equivalent (see [7, p. 117]) to the natural norm

$$|||u|||_H^2 = \int_{\mathbb{R}^3\setminus\Omega} \left(\frac{|u(r)|^2}{1+|r|^2} + |\nabla u|^2(r) \right) dr \,. \tag{9}$$

Here $r = (x_1, x_2, x_3)$ is the position vector of a point in $\mathbb{R}^3 \setminus \Omega$. Take now $\bar{u} \in C_0^\infty(\mathbb{R}^3 \setminus \Omega)$ satisfying the boundary conditions $\bar{u}|_{\partial B^{(i)}} = u^{(i)}$. It is well known (see [7, p. 154]) that there exists a unique solution (u, p) to the variational problem

$$\text{Find } (u,p) \in (\bar{u} + H) \times M \text{ such that } \forall (v,q) \in H \times M,$$
$$\left[\begin{array}{l} 2\displaystyle\int_{\mathbb{R}^3\setminus\Omega} \eta\, D(u)\cdot D(v) - \int_{\mathbb{R}^3\setminus\Omega} p \ \mathrm{div} v = 0, \\ \displaystyle\int_{\mathbb{R}^3\setminus\Omega} q \ \mathrm{div} u = 0, \end{array} \right. \tag{10}$$

where $D(u) = \frac{1}{2}(\nabla u + \nabla^t u)$. From the solution (u, p) one can compute the forces exerted on the balls as

$$f^{(i)} = -\int_{\partial B^{(i)}} \sigma n \,. \tag{11}$$

It is clear that the $f^{(i)}$ depend linearly on the $u^{(i)}$

$$f^{(i)} = \sum_j L_{ij} u^{(j)} \,, \tag{12}$$

and it easy to show that this relation is invertible because the L_{ij} define a symmetric and positive definite matrix. Indeed, let $(u^{(1)}, u^{(2)}, u^{(3)})$ and $(v^{(1)}, v^{(2)}, v^{(3)})$ be two sets of boundary conditions. We solve the corresponding Stokes problems and call the solutions (u, p) and (v, q) respectively. Moreover, we compute the corresponding forces $f^{(i)}$ and $g^{(i)}$ according to (11). Since u and v are divergence free, integrating by parts we get

$$\begin{aligned} \sum_{ij} (L_{ij} u^{(j)}) \cdot v^{(i)} &= \sum_i f^{(i)} \cdot v^{(i)} \\ &= 2\int_{\mathbb{R}^3\setminus\Omega} \eta\, D(u)\cdot D(v) \\ &= \sum_i g^{(i)} \cdot u^{(i)} \\ &= \sum_{ij} u^{(j)} \cdot (L_{ji} v^{(i)}) \,, \end{aligned}$$

and

$$\sum_i f^{(i)} \cdot u^{(i)} = \sum_{ij} (L_{ij} u^{(j)}) \cdot u^{(i)} = 2 \int_{\mathbb{R}^3 \setminus \Omega} \eta \, |D(u)|^2. \tag{13}$$

Finally, we define the solution of (6) for given data $f^{(i)}$ as the unique solution of (10) corresponding to the only set of Dirichlet data $u^{(i)}$ satisfying (12).

In the geometry of Fig. 1, forces and velocities are all directed along the axis of motion so that, from now onwards, we will denote by $f^{(i)}$ and $u^{(i)}$ their scalar components along that axis. Thus, from (12) and the properties of L_{ij}, we obtain that (1) holds and that the Oseen matrix S is symmetric and positive definite.

2.2 The ODEs describing swimming

Calling $(x_i)_i$ the positions of the centers of the three balls, we rewrite (1) as

$$\frac{d}{dt} \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{pmatrix} = S(x, y) \begin{pmatrix} f^{(1)} \\ f^{(2)} \\ f^{(3)} \end{pmatrix}, \tag{14}$$

and change variables to $x = x^{(2)} - x^{(1)}$, $y = x^{(3)} - x^{(2)}$, and $c = (x^{(1)} + x^{(2)} + x^{(3)})/3$. Setting $e_x = (-1, 1, 0)^t$, $e_y = (0, -1, 1)^t$, $e_c = (1/3, 1/3, 1/3)^t$ and $f = \left(f^{(1)}, f^{(2)}, f^{(3)}\right)^t$, we obtain

$$\left[\begin{array}{l} \dfrac{dx}{dt} = e_x \cdot Sf = Se_x \cdot f, \\ \dfrac{dy}{dt} = e_y \cdot Sf = Se_y \cdot f, \\ \dfrac{dc}{dt} = Se_c \cdot f, \end{array} \right. \tag{15}$$

because S is symmetric.

Now, since f satisfies (2), it is always possible to write it as

$$f = \alpha_x \frac{Se_y \times e_c}{Se_x \cdot (Se_y \times e_c)} - \alpha_y \frac{Se_x \times e_c}{Se_x \cdot (Se_y \times e_c)}. \tag{16}$$

This is because

1. the two vectors $Se_y \times e_c$ and $Se_x \times e_c$ are obviously orthogonal to e_c and never proportional, for if this were the case, there would exist $\lambda, \mu \in \mathbb{R}$ such that
$$S(\lambda e_x + \mu e_y) \times e_c = 0.$$

But, since e_c is orthogonal to both e_x and e_y, we would have

$$S(\lambda e_x + \mu e_y) \cdot (\lambda e_x + \mu e_y) = 0,$$

which is in contradiction with the fact that S is positive definite.

2. $Se_x \cdot (Se_y \times e_c)$ never vanishes. Indeed, if this were the case, then both $Se_x \times e_c$ and $Se_y \times e_c$ would be orthogonal to Se_x and Se_y. Since from the preceding remark, Se_x and Se_y are not collinear, then $Se_x \times e_c$ and $Se_y \times e_c$ should be collinear, which again from the preceding remark is not possible.

Using (16), system (15) becomes

$$\left[\begin{array}{l} \dfrac{dx}{dt} = \alpha_x, \\ \dfrac{dy}{dt} = \alpha_y, \\ \dfrac{dc}{dt} = \alpha_x \dfrac{Se_c \cdot (Se_y \times e_c)}{Se_x \cdot (Se_y \times e_c)} - \alpha_y \dfrac{Se_c \cdot (Se_x \times e_c)}{Se_x \cdot (Se_y \times e_c)} \end{array} \right. \tag{17}$$

which leads to

$$\frac{dc}{dt} = V_x(x,y)\frac{dx}{dt} + V_y(x,y)\frac{dy}{dt}, \tag{18}$$

with

$$V_x(x,y) = \frac{Se_c \cdot (Se_y \times e_c)}{Se_x \cdot (Se_y \times e_c)} = \frac{S^{-1}e_c \cdot (e_c \times e_y)}{S^{-1}e_c \cdot (e_x \times e_y)} \tag{19}$$

$$V_y(x,y) = -\frac{Se_c \cdot (Se_x \times e_c)}{Se_x \cdot (Se_y \times e_c)} = -\frac{S^{-1}e_c \cdot (e_c \times e_x)}{S^{-1}e_c \cdot (e_x \times e_y)}. \tag{20}$$

Therefore, if the system performs a stroke, *i.e.*, it follows a given closed curve $\gamma = (x, y)$ in the space of admissible shapes $\mathcal{S} = (2a, +\infty)^2$ defined by $\gamma : [0, T] \to \mathcal{S}$, then it experiences a global displacement of its center of mass which amounts to

$$\begin{aligned} \Delta c &= \int_0^T \left(V_x(x(t), y(t))\frac{dx}{dt}(t) + V_y(x(t), y(t))\frac{dy}{dt}(t) \right) dt, \\ &= \pm \oint_\gamma V \cdot dl \\ &= \pm \int_\omega \text{curl } V, \end{aligned} \tag{21}$$

where $V = (V_x, V_y)$ and $\omega \subset \mathcal{S}$ is the region enclosed by γ. Here, $\text{curl}V = \dfrac{\partial V_y}{\partial x} - \dfrac{\partial V_x}{\partial y}$ and the $-$ sign should be chosen if $t \mapsto \gamma(t)$ induces a clockwise orientation of γ. Notice also that the definition of $\mathcal{S}$ ensures that the balls do not overlap.

Thus, we can think of swimming as *the problem of controlling shape in order to produce a net displacement* Δc *different from 0 at the end of one stroke* γ. The force laws that are needed to produce an arbitrary path in the space of shapes $\gamma : [0,T] \to \mathcal{S}$, are easily recovered from (16) and (17):

$$f = \frac{dx}{dt}\frac{Se_y \times e_c}{Se_x \cdot (Se_y \times e_c)} - \frac{dy}{dt}\frac{Se_x \times e_c}{Se_x \cdot (Se_y \times e_c)}. \tag{22}$$

2.3 Swimming as a control problem

It is convenient to rewrite system (17) as

$$\frac{dX}{dt} = \alpha_x(t)F_x(X) + \alpha_y(t)F_y(X), \tag{23}$$

where $X = (x, y, c) \in \mathcal{S} \times \mathbb{R}$ and the vector-fields F_x and F_y are given by

$$F_x(X) = (1, 0, V_x(x, y))\,,\ F_y(X) = (0, 1, V_y(x, y))\,. \tag{24}$$

In what follows, we will denote by $\mathcal{X} = \mathcal{S} \times \mathbb{R}$ the set of admissible states. This system is globally controllable if, starting from any state $X_0 = (x_0, y_0, c_0) \in \mathcal{X}$, one can reach any other state $X_1 = (x_1, y_1, c_1) \in \mathcal{X}$ with a solution of (23) and suitable controls $t \mapsto (\alpha_x(t), \alpha_y(t))$. The system is locally controllable at X_0 if one can reach any point in a neighborhood of X_0. Since swimming means the ability to connect (x_0, y_0, c_0) and $(x_0, y_0, c_1 \neq c_0)$ with a solution of (23), we see that local controllability is a sufficient condition for swimming. In fact, it turns out that the three-sphere swimmer is globally controllable.

In order to proceed, we need to introduce basic notations and results of control theory applied to our situation. First, a sufficient condition for local controllability at X_0 is that the Lie algebra $Lie(F_x, F_y)(X_0) = \mathbb{R}^3$ (see, e.g., [1]). In particular this is true if

$$\det(F_x, F_y, [F_x, F_y])(X_0) \neq 0, \tag{25}$$

where $[F_x, F_y] = (F_x \cdot \nabla)F_y - (F_y \cdot \nabla)F_x$ is the Lie bracket of F_x and F_y. An easy computation shows that (since neither F_x nor F_y depend on c)

$$\det(F_x, F_y, [F_x, F_y])(X_0) = \text{curl}\ V(X_0) \tag{26}$$

and we recover (in view of (21)) that curl $V(X_0) \neq 0$ allows for swimming with any sufficiently small loop around (x_0, y_0).

Introducing the Martinet surface

$$\mathcal{M} = \{X \in \mathcal{X} : \det(F_x, F_y, [F_x, F_y])(X) = 0\}\,, \tag{27}$$

it is clear that the system is locally controllable outside $\mathcal{M}$ and therefore globally controllable on each connected component of $\mathcal{X} \setminus \mathcal{M}$.

3 Proof of Theorem 1

For the proof of theorem 1, which states that the three-sphere swimmer is globally controllable, we need the following two lemmas, the proofs of which are postponed to the next subsections.

Lemma 1 *The vector fields $F_x(X)$ and $F_y(X)$ are analytic functions of $X \in \mathcal{X}$.*

Lemma 2 *The set $\mathcal{X} \setminus \mathcal{M}$ is not empty.*

From the preceding lemmas, it is clear that the Martinet surface $\mathcal{M}$ is locally at most of dimension 2. Indeed, if $\det(F_x, F_y, [F_x, F_y])$ vanished in a neighborhood of a state $X_0 \in \mathcal{X}$, then by analyticity, it would vanish everywhere on $\mathcal{X}$, and $\mathcal{M}$ would be equal to $\mathcal{X}$ contradicting lemma 2. Now, since F_x and F_y do not depend on c, the Martinet surface is a cylinder with vertical axis and since $\mathrm{span}(F_x(X), F_y(X))$ is never a vertical plane, we deduce that (at least) one of the two vectors $F_x(X)$ or $F_y(X)$ is transverse to $\mathcal{M}$ at $X \in \mathcal{M}$. The global controllability on the connected components of $\mathcal{X} \setminus \mathcal{M}$ and the transverse vector field to pass from one connected component to another one prove the global controllability of the system and hence theorem 1.

Remark 1 *We notice that, in view of the c-invariance of our system, continuity of* $\mathrm{curl}\, V$ *could suffice to prove global controllability. However, analyticity of* $\mathrm{curl}\, V$ *gives extra information. For instance, we obtain that the set $\mathcal{M}$ is at most two-dimensional and hence its intersection with the space of shapes $\mathcal{S}$ is at most one-dimensional.*

3.1 Proof of Lemma 1

In view of formulas (19,20), the fact that $F_x(X)$ and $F_y(X)$ are analytic functions of X directly follows from the analyticity of $S^{-1}(x, y)$ with respect to (x, y) that we prove now. In fact, it is sufficient to show that for any triplet $U = (u^{(1)}, u^{(2)}, u^{(3)})^t$, the energy

$$\big(S^{-1}(x, y)U, U\big) = 2\int_{\mathbb{R}^3\setminus\Omega} \eta\, |D(u)|^2 \tag{28}$$

is analytic in (x, y) where (u, p) solves Stokes equations with boundary values $u^{(i)}$ on $\partial B^{(i)}$

$$\left[\begin{array}{l} -\eta\Delta u + \nabla p = 0 \text{ on } \mathbb{R}^3 \setminus \Omega\,, \\ \mathrm{div}(u) = 0 \text{ on } \mathbb{R}^3 \setminus \Omega\,, \\ u|_{\partial B^{(i)}} = u^{(i)} \text{ for } i = 1, 2, 3, \\ u \to 0 \text{ at infinity.} \end{array}\right. \tag{29}$$

In order to emphasize the dependence of the domain Ω on (x, y) we will write until the end of the section $\Omega_{(x,y)}$ instead of Ω. That u is analytic in $\mathbb{R}^3 \setminus \Omega_{(x,y)}$ follows from standard elliptic regularity (see, *e.g.*, [15]), but we stress that what concerns us here is that u is analytic with respect to (x, y), *i.e.*, with respect to deformations of the domain of the fluid flow. In order to proceed, we work near a point (x_0, y_0), set $\delta_x = x - x_0$, and $\delta_y = y - y_0$, and recall that a function X with values in a Banach space $\mathcal{B}$ is analytic at $(0, 0)$ if and only if one can write for any $\delta = (\delta_x, \delta_y)$ in a suitable neighborhood of $(0, 0)$

$$X(\delta) = \sum_{\alpha\in\mathbb{N}^2} \delta^\alpha X^{(\alpha)}\,, \tag{30}$$

where $X^{(\alpha)} \in \mathcal{B}$ satisfy the estimate

$$\exists C > 0,\ \rho > 0, \text{ s.t.} \, \forall\alpha \in \mathbb{N}^2,\ ||X^{(\alpha)}||_{\mathcal{B}} \leq \frac{C}{\rho^{|\alpha|}}\,. \tag{31}$$

Here, for a multiindex $\alpha = (\alpha_x, \alpha_y) \in \mathbb{N}^2$, and $\delta = (\delta_x, \delta_y) \in \mathbb{R}^2$, we have denoted by δ^α the quantity $\delta_x^{\alpha_x}\delta_y^{\alpha_y}$, and $|\alpha| = \alpha_x + \alpha_y$.

We then recast the Stokes problem on the fixed domain $\mathbb{R}^3 \setminus \Omega_{(x_0,y_0)}$, but with variable coefficients. Namely, we solve the following two problems on $\mathbb{R}^3 \setminus \Omega_{(x_0,y_0)}$

$$\left[\begin{array}{l} \Delta\phi = 0 \\ \phi|_{\partial B^{(1)}} = 1\,, \\ \phi|_{\partial B^{(2)}} = 0\,, \\ \phi|_{\partial B^{(3)}} = 0\,, \\ \phi \to 0 \text{ at infinity,} \end{array}\right. \qquad \left[\begin{array}{l} \Delta\psi = 0 \\ \psi|_{\partial B^{(1)}} = 0\,, \\ \psi|_{\partial B^{(2)}} = 0\,, \\ \psi|_{\partial B^{(3)}} = 1\,, \\ \psi \to 0 \text{ at infinity.} \end{array}\right. \tag{32}$$

Since ϕ and ψ are harmonic, as a consequence of classical elliptic regularity theory, they are analytic on $\mathbb{R}^3 \setminus \Omega_{(x_0,y_0)}$ (see [15]). Moreover, ϕ, ψ and all their derivatives are bounded functions on $\mathbb{R}^3 \setminus \Omega_{(x_0,y_0)}$, and satisfy the following decay estimate at infinity

$$\forall \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3,\ \exists C_\alpha > 0, \text{ s.t. } |\partial^\alpha \phi(r)| \leq \frac{C_\alpha}{|r|^{|\alpha|+1}}\,. \tag{33}$$

Now, we set for $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \Omega_{(x_0,y_0)}$

$$\theta(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (\bar{x}_1 - \delta_x \phi(\bar{x}_1, \bar{x}_2, \bar{x}_3) + \delta_y \psi(\bar{x}_1, \bar{x}_2, \bar{x}_3), \bar{x}_2, \bar{x}_3)\,,$$

and define

$$\bar{u}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = u(\theta(\bar{x}_1, \bar{x}_2, \bar{x}_3)),\ \forall (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \mathbb{R}^3 \setminus \Omega_{(x_0,y_0)}\,. \tag{34}$$

Since for $(\delta_x, \delta_y) = (0,0)$, $\theta = Id$, and θ is analytic in (δ_x, δ_y), θ admits an inverse ξ which is analytic in (δ_x, δ_y) near $(0,0)$, so that we may rewrite (34) as

$$u(x_1, x_2, x_3) = \bar{u}(\xi(x_1, x_2, x_3)),\ \forall (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \Omega_{(x,y)}. \tag{35}$$

Therefore, the analyticity of u in (δ_x, δ_y) is equivalent to the analyticity of $\bar{u}$. Calling $\bar{p} = p \circ \theta$, that (u,p) satisfy Stokes equations on $\mathbb{R}^3 \setminus \Omega_{(x,y)}$ is now equivalent to $(\bar{u}, \bar{p})$ satisfying

$$\left[\begin{array}{l} -\eta \sum_{ij} \left[\sum_k \bar{\partial}_{jk} \bar{u}_l \partial_i \xi_k \partial_i \xi_j + \bar{\partial}_j \bar{u}_l \partial_{ii} \xi_j \right] + \sum_j \bar{\partial}_j \bar{p} \partial_l \xi_j = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \mathbb{R}^3 \setminus \Omega_{(x_0,y_0)}\,,\ \text{for } l = 1,2,3, \\ \sum_{ij} \bar{\partial}_j \bar{u}_i \partial_i \xi_j = 0 \text{ on } \mathbb{R}^3 \setminus \Omega_{(x_0,y_0)}\,, \\ \bar{u}|_{\partial \bar{B}^{(i)}} = u^{(i)} \text{ for } i = 1,2,3, \end{array}\right. \tag{36}$$

in which we have used the notations $\partial_i = \dfrac{\partial}{\partial x_i}$, and $\bar{\partial}_i = \dfrac{\partial}{\partial \bar{x}_i}$. Notice that an explicit formula for the derivatives of ξ can be obtained by differentiation of $\xi \circ \theta = Id$. Indeed, one has

$$\nabla \xi(\theta(\bar{x}_1, \bar{x}_2, \bar{x}_3)) = \left(\bar{\nabla} \theta(\bar{x}_1, \bar{x}_2, \bar{x}_3)\right)^{-1}\,, \tag{37}$$

which can be computed explicitly from

$$\bar{\nabla}\theta = \begin{pmatrix} 1 - \delta_x \bar{\partial}_1 \phi + \delta_y \bar{\partial}_1 \psi & -\delta_x \bar{\partial}_2 \phi + \delta_y \bar{\partial}_2 \psi & -\delta_x \bar{\partial}_3 \phi + \delta_y \bar{\partial}_3 \psi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{38}$$

We insist on the fact that the Jacobian determinant $\det \bar{\nabla}\theta = 1 - \delta_x \bar{\partial}_1 \phi + \delta_y \bar{\partial}_1 \psi$ does not vanish for (δ_x, δ_y) small enough, uniformly in $\mathbb{R}^3 \setminus \Omega_{(x_0,y_0)}$ due to the uniform bound on the derivatives of ϕ and ψ. Therefore, the coefficients in the transformed Stokes equation (36) are indeed analytic in $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \delta_x, \delta_y)$. Setting

$$\left[\begin{array}{l} a_{jk} = \eta \sum_i \partial_i \xi_k \partial_i \xi_j \,, \\ b_j = \eta \sum_i \partial_{ii} \xi_j \,, \\ c_{jk} = \partial_j \xi_k \,, \end{array}\right. \tag{39}$$

the system (36) rewrites

$$\left[\begin{array}{l} R_{(\delta_x,\delta_y)}(\bar{u}, \bar{p}) = 0 \,, \\ \bar{u}|_{\partial \bar{B}^{(i)}} = u^{(i)} \text{ for } i = 1, 2, 3, \end{array}\right. \tag{40}$$

where the second order differential operator $R_{(\delta_x,\delta_y)}$ is given by

$$R_{(\delta_x,\delta_y)} = \begin{pmatrix} D_2 & 0 & 0 & \sum_j c_{1j}\bar{\partial}_j \\ 0 & D_2 & 0 & \sum_j c_{2j}\bar{\partial}_j \\ 0 & 0 & D_2 & \sum_j c_{3j}\bar{\partial}_j \\ \sum_j c_{1j}\bar{\partial}_j & \sum_j c_{2j}\bar{\partial}_j & \sum_j c_{3j}\bar{\partial}_j & 0 \end{pmatrix}, \tag{41}$$

with $D_2 = -\sum_j \left[\sum_k a_{jk}\bar{\partial}_{jk} + b_j \bar{\partial}_j\right]$. Since the coefficients are analytic in δ, we will write

$$a_{jk} = \sum_{\alpha \in \mathbb{N}^2} \delta^\alpha a_{jk}^{(\alpha)} \,, \tag{42}$$

$$b_j = \sum_{\alpha \in \mathbb{N}^2} \delta^\alpha b_j^{(\alpha)} \,, \tag{43}$$

$$c_{jk} = \sum_{\alpha \in \mathbb{N}^2} \delta^\alpha c_{jk}^{(\alpha)} \,. \tag{44}$$

Of particuliar importance for the sequel is the behavior of the functions $a_{jk}^{(\alpha)}$, $b_j^{(\alpha)}$, $c_{jk}^{(\alpha)}$, as $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ tends to infinity. We get from (33), (37), (38), and (39) the following estimates

$$\exists C > 0,\ \rho > 0,\ \text{s.t.} \quad \left[\begin{array}{l} ||a_{jk}^{(\alpha)}||_{L^\infty}, ||c_{jk}^{(\alpha)}||_{L^\infty}, ||\sqrt{1+|r|^2} b_j^{(\alpha)}||_{L^\infty} \le \dfrac{C}{\rho^{|\alpha|}} \,, \\ ||\sqrt{1+|r|^2}\bar{\partial}_l a_{jk}^{(\alpha)}||_{L^\infty}, ||\sqrt{1+|r|^2}\bar{\partial}_l c_{jk}^{(\alpha)}||_{L^\infty} \le \dfrac{C}{\rho^{|\alpha|}} \,. \end{array}\right. \tag{45}$$

Now, in order to deal with the non homogeneous boundary conditions, we consider a map $\bar{U}$ analytic in $\mathbb{R}^3 \setminus \Omega_{(x_0,y_0)}$ which obeys the boundary conditions for instance by solving

$$\left[\begin{array}{l} -\Delta \bar{U} = 0, \text{ on } \mathbb{R}^3 \setminus \Omega_{(x_0,y_0)} \,, \\ \bar{U}|_{\partial \bar{B}^{(i)}} = u^{(i)} \text{ for } i = 1, 2, 3, \end{array}\right. \tag{46}$$

and seek the solution as $\bar{u} = \bar{U} + \bar{v}$. Hence, $\bar{v}$ is solution of

$$\left[\begin{array}{l} R_{(\delta_x,\delta_y)}(\bar{v},\bar{p}) = -R_{(\delta_x,\delta_y)}(\bar{U},0) \\ \bar{v}|_{\partial\bar{B}^{(i)}} = 0 \text{ for } i = 1,2,3. \end{array}\right. \tag{47}$$

Unfortunately, the operator $R_{(\delta_x,\delta_y)}$ does not map $H \times M$ into itself and therefore, we rewrite our system as

$$\left[\begin{array}{l} R^{-1}_{(0,0)}R_{(\delta_x,\delta_y)}(\bar{v},\bar{p}) = -R^{-1}_{(0,0)}R_{(\delta_x,\delta_y)}(\bar{U},0) \text{ on } \mathbb{R}^3 \setminus \Omega_{(x_0,y_0)}\,, \\ \bar{v}|_{\partial\bar{B}^{(i)}} = 0 \text{ for } i = 1,2,3, \end{array}\right. \tag{48}$$

where by $R^{-1}_{(0,0)}(f,g)$ we mean the solution to the Dirichlet problem

$$\left[\begin{array}{l} R_{(0,0)}(\bar{w},\bar{q}) = (f,g), \text{ on } \mathbb{R}^3 \setminus \Omega_{(x_0,y_0)} \\ \bar{w}|_{\partial\bar{B}^{(i)}} = 0 \text{ for } i = 1,2,3. \end{array}\right. \tag{49}$$

That $R^{-1}_{(0,0)}R_{(\delta_x,\delta_y)}$ is invertible on $H \times M$ is clear. We now show that it depends on $\delta = (\delta_x,\delta_y)$ analytically (near $(0,0)$). Using the expansions (42,43,44) leads to

$$R^{-1}_{(0,0)}R_{(\delta_x,\delta_y)} = \sum_{\alpha\in\mathbb{N}^2} \delta^\alpha R^{-1}_{(0,0)}R^{(\alpha)}, \tag{50}$$

where the operators $R^{(\alpha)}$ are given by

$$R^{(\alpha)} = \begin{pmatrix} D_2^{(\alpha)} & 0 & 0 & \sum_j c^{(\alpha)}_{1j}\bar{\partial}_j \\ 0 & D_2^{(\alpha)} & 0 & \sum_j c^{(\alpha)}_{2j}\bar{\partial}_j \\ 0 & 0 & D_2^{(\alpha)} & \sum_j c^{(\alpha)}_{3j}\bar{\partial}_j \\ \sum_j c^{(\alpha)}_{1j}\bar{\partial}_j & \sum_j c^{(\alpha)}_{2j}\bar{\partial}_j & \sum_j c^{(\alpha)}_{3j}\bar{\partial}_j & 0 \end{pmatrix}, \tag{51}$$

with $D_2^{(\alpha)} = -\sum_j \left[\sum_k a^{(\alpha)}_{jk}\bar{\partial}_{jk} + b^{(\alpha)}_j\bar{\partial}_j\right]$, and it is sufficient to show that $R^{(\alpha)}$ satisfies (31). But, we have

$$||R^{-1}_{(0,0)}R^{(\alpha)}||_{\mathcal{L}(H\times M)} \le ||R^{-1}_{(0,0)}||_{\mathcal{L}(H'\times M,H\times M)}||R^{(\alpha)}||_{\mathcal{L}(H\times M,H'\times M)}$$

and we can estimate

$$||R^{(\alpha)}||_{\mathcal{L}(H\times M,H'\times M)} \le C\Big(\sum_{jk}||a^{(\alpha)}_{jk}\bar{\partial}_{jk}||_{\mathcal{L}(H,H')} + \sum_j ||b^{(\alpha)}_j\bar{\partial}_j||_{\mathcal{L}(H,H')} + \sum_{ij}||c^{(\alpha)}_{ij}\bar{\partial}_j||_{\mathcal{L}(H,M)} + \sum_{ij}||c^{(\alpha)}_{ij}\bar{\partial}_j||_{\mathcal{L}(M,H')}\Big)\,.$$

We consider each of the four terms separately.

$$
\begin{aligned}
||a_{jk}^{(\alpha)}\bar\partial_{jk}||_{\mathcal{L}(H,H')} &= \sup_{u,v\in H}\frac{\left|\int a_{jk}^{(\alpha)}\bar\partial_j u\bar\partial_k v+\bar\partial_k a_{jk}^{(\alpha)}\bar\partial_j u\, v\right|}{||u||_H||v||_H}\\
&\le \sup_{u,v\in H}\frac{||a_{jk}^{(\alpha)}||_{L^\infty}||\bar\partial_j u||_{L^2}||\bar\partial_k v||_{L^2}}{||u||_H||v||_H}\\
&\qquad +\frac{||\sqrt{1+|r|^2}\bar\partial_k a_{jk}^{(\alpha)}||_{L^\infty}||\bar\partial_j u||_{L^2}||\frac{v}{\sqrt{1+|r|^2}}||_{L^2}}{||u||_H||v||_H}\\
&\le C\left(||a_{jk}^{(\alpha)}||_\infty+||\sqrt{1+|r|^2}\bar\partial_k a_{jk}^{(\alpha)}||_\infty\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
||b_j^{(\alpha)}\bar\partial_j||_{\mathcal{L}(H,H')} &\le ||\sqrt{1+|r|^2}b_j^{(\alpha)}||_{L^\infty}\\
||c_{ij}^{(\alpha)}\bar\partial_j||_{\mathcal{L}(H,M)} &\le ||c_{ij}^{(\alpha)}||_{L^\infty}\\
||c_{ij}^{(\alpha)}\bar\partial_j||_{\mathcal{L}(M,H')} &\le ||c_{ij}^{(\alpha)}||_{L^\infty}+||\sqrt{1+|r|^2}\bar\partial_j c_{ij}^{(\alpha)}||_{L^\infty}
\end{aligned}
$$

Therefore, from the fact that the coefficients a_{jk}, b_j and c_{jk} satisfy (45), we deduce that $R_{(0,0)}^{-1}R_{(\delta_x,\delta_y)}$ is analytic in δ.

Moreover, since $R_{(0,0)}^{-1}R^{(0,0)}=Id$, (here, $R^{(0,0)}$ stands for $R^{(\alpha)}$ with $\alpha=(0,0)$) the theorem on analytic inverse implies that the inverse $\left(R_{(0,0)}^{-1}R_{(\delta_x,\delta_y)}\right)^{-1}$ also satisfies (30,31) for δ sufficiently small, which means that it is analytic in δ as well.

Since $(\bar U,0)$ is smooth, we infer that $R_{(0,0)}^{-1}R_{(\delta_x,\delta_y)}(\bar U,0)$ is analytic in δ (with values in $H\times M$), which leads to the analyticity of $(\bar v,\bar p)$ in $\delta=(\delta_x,\delta_y)$ and hence to the desired result.

3.2 Proof of Lemma 2

The proof of lemma 2 consists of two steps. In the first, we prove the estimate

$$
\exists C>0, x_0>0,\ \text{s.t.}\ \forall x,y>x_0,\ ||S(x,y)-S_\infty(x,y)||<\frac{C}{\min(x,y)^2} \tag{52}
$$

for the error term E in the asymptotic expansion for (1)

$$
\begin{pmatrix}u^{(1)}\\u^{(2)}\\u^{(3)}\end{pmatrix}=S(x,y)\begin{pmatrix}f^{(1)}\\f^{(2)}\\f^{(3)}\end{pmatrix}=(S_\infty(x,y)+E(x,y))\begin{pmatrix}f^{(1)}\\f^{(2)}\\f^{(3)}\end{pmatrix} \tag{53}
$$

where the matrix $S_\infty(x,y)$ is given by

$$S_\infty(x,y) = \frac{1}{\pi\eta}\begin{pmatrix} \frac{1}{6a} & \frac{1}{4x} & \frac{1}{4(x+y)} \\ \frac{1}{4x} & \frac{1}{6a} & \frac{1}{4y} \\ \frac{1}{4(x+y)} & \frac{1}{4y} & \frac{1}{6a} \end{pmatrix}. \tag{54}$$

We give an explicit short proof, in the interest of the reader, to keep the paper self-contained. Formulas like (52,54) are not new, and we first encountered them in [9]. More refined asymptotics show that, in fact, the error decay rate is cubic for the off-diagonal terms, and quartic for the diagonal ones, see [3]. In the second step, we construct a stroke for which we can prove, using (52), that the displacement of the center of mass is nonzero, making $\mathcal{M} = \mathcal{X}$ impossible.

Step 1. We note that for x, y sufficiently large, the matrix $S_\infty(x,y)$ is uniformly diagonal dominant and therefore there exist $\alpha > 0$, $x_0 > 0$ such that

$$\left\| S_\infty(x,y)\begin{pmatrix} f^{(1)} \\ f^{(2)} \\ f^{(3)} \end{pmatrix}\right\| \geq \alpha \left\| \begin{pmatrix} f^{(1)} \\ f^{(2)} \\ f^{(3)} \end{pmatrix}\right\|, \quad \forall x, y > x_0 \,. \tag{55}$$

Next, we recall that if u satisfies the homogeneous Stokes equations outside Ω, with $u|_{\partial B^{(i)}} = u^{(i)}$ a constant, then u can be written as

$$u(r) = \sum_{i=1}^{3} \int_{\partial B^{(i)}} G(r-r')t^{(i)}(r')\; dr' \,. \tag{56}$$

Here the Stokeslet

$$G(r) = \frac{1}{8\pi\eta}\left(\frac{1}{|r|} + \frac{r\otimes r}{|r|^3}\right) \tag{57}$$

is the fundamental solution of Stokes equation and $t^{(i)} = \sigma n|_{\partial B^{(i)}}$ is the force per unit area on $\partial B^{(i)}$.

We now construct an approximate solution to the three–sphere Stokes problem by using, as a building block, the solution to the outer Stokes problem with uniform Dirichlet data on one sphere (one–sphere Stokes solution). Let γ the boundary of a smooth bounded domain and consider the space $\mathcal{H}(\Gamma) = H^{-\frac{1}{2}}(\Gamma, \mathbb{R}^3)/\mathcal{R}$, where $\mathcal{R}$ is the equivalence relation $t\,\mathcal{R}\,t'$ iff $t - t' = \lambda n$ with $\lambda \in \mathbb{R}$ and n the unit normal to Γ (this is needed because the interior pressure p inside Γ is defined to within a constant, see [7,

p. 157]). We define the map Φ from $\mathcal{H}(\partial B^{(1)}) \times \mathcal{H}(\partial B^{(2)}) \times \mathcal{H}(\partial B^{(3)})$ to $H^{\frac{1}{2}}(\partial B^{(1)}, \mathbb{R}^3) \times H^{\frac{1}{2}}(\partial B^{(2)}, \mathbb{R}^3) \times H^{\frac{1}{2}}(\partial B^{(3)}, \mathbb{R}^3)$ given by

$$\begin{pmatrix} v^{(1)} \\ v^{(2)} \\ v^{(3)} \end{pmatrix} = \Phi \begin{pmatrix} g^{(1)} \\ g^{(2)} \\ g^{(3)} \end{pmatrix} = \begin{pmatrix} T_0^{(11)} & T_{-x}^{(12)} & T_{-(x+y)}^{(13)} \\ T_x^{(21)} & T_0^{(22)} & T_{-y}^{(23)} \\ T_{(x+y)}^{(31)} & T_y^{(32)} & T_0^{(33)} \end{pmatrix} \begin{pmatrix} g^{(1)} \\ g^{(2)} \\ g^{(3)} \end{pmatrix}, \tag{58}$$

where the operators $T_z^{(ij)} : \mathcal{H}(\partial B^{(j)}) \to H^{\frac{1}{2}}(\partial B^{(i)}, \mathbb{R}^3)$ are defined by

$$T_z^{(ij)} g^{(j)}(r) = \int_{\partial B^{(j)}} G(z\vec{\imath} + r - r') g^{(j)}(r')\, dr' \quad \text{for } r \in \partial B^{(i)}. \tag{59}$$

where $\vec{\imath}$ is the unit vector along the horizontal axis, see Fig. 1. It is clear from (56) that

$$\begin{aligned} \Phi(t^{(1)}, t^{(2)}, t^{(3)}) &= (u|_{\partial B^{(1)}}, u|_{\partial B^{(2)}}, u|_{\partial B^{(3)}}) \\ &= (u^{(1)}, u^{(2)}, u^{(3)}). \end{aligned}$$

Moreover, Φ defines an isomorphism because the Stokes problem is well posed and the integral representation (56) is unique. We shall see that Φ is also uniformly coercive for (x, y) sufficiently large.

It is well-known (see [7, eq. (5.31)] that $T_0^{(ii)}$ is coercive on $\mathcal{H}(\partial B^{(i)})$, meaning that there exists $\alpha_0 > 0$ such that

$$\forall g \in \mathcal{H}(\partial B^{(i)}),\ \left\langle g, T_0^{(ii)} g \right\rangle_{H^{-\frac{1}{2}}(\partial B^{(i)}), H^{\frac{1}{2}}(\partial B^{(i)})} \geq \alpha_0 ||g||^2_{\mathcal{H}(\partial B^{(i)})}, \tag{60}$$

while an easy estimate gives (due to the decay of G at infinity) for $z >> a$,

$$||T_z^{(ij)}||_{\mathcal{L}\left(\mathcal{H}(\partial B^{(j)}), H^{\frac{1}{2}}(\partial B^{(i)})\right)} \leq \frac{C}{|z|}. \tag{61}$$

The coerciveness of T_0 (60) together with the estimate (61) give the following uniform coerciveness estimate for $\Phi : \exists X > 0,\ \text{ s.t. } \forall x, y > X$,

$$\left\langle (g^{(1)}, g^{(2)}, g^{(3)}), \Phi(g^{(1)}, g^{(2)}, g^{(3)}) \right\rangle \geq \alpha \left\| (g^{(1)}, g^{(2)}, g^{(3)}) \right\|^2. \tag{62}$$

Now, for a given velocity V of the ball B, we call u_V the one–sphere Stokes solution outside B, namely, the solution of the homogeneous Stokes equations which vanishes at infinity and whose trace on ∂B is equal to V. Stokes formula gives the drag force F_V generated in this case as

$$F_V = -\int_{\partial B} \sigma(u_V) n = 6\pi a \eta V. \tag{63}$$

For the three values $U = (u^{(1)}, u^{(2)}, u^{(3)})^t$ of the velocity of the spheres, we compute the drags $F = (F^{(1)}, F^{(2)}, F^{(3)})^t$ solutions to the system $S_\infty(x,y)F = U$, and the corresponding velocities $V^{(i)} = \dfrac{F^{(i)}}{6\pi a\eta}$. We then compute

$$(w^{(1)}, w^{(2)}, w^{(3)}) = \Phi(t^{(1)} - \sigma(u_{V^{(1)}})n, t^{(2)} - \sigma(u_{V^{(2)}})n, t^{(3)} - \sigma(u_{V^{(3)}})n) \quad (64)$$

and get

$$\left\{\begin{array}{l} w^{(1)}(r) = u^{(1)} - V^{(1)} - u_{V^{(2)}}(-x\vec{\imath} + r) - u_{V^{(3)}}(-(x+y)\vec{\imath} + r), \\ w^{(2)}(r) = u^{(2)} - V^{(2)} - u_{V^{(1)}}(x\vec{\imath} + r) - u_{V^{(3)}}(-y\vec{\imath} + r), \\ w^{(3)}(r) = u^{(3)} - V^{(3)} - u_{V^{(1)}}((x+y)\vec{\imath} + r) - u_{V^{(2)}}(y\vec{\imath} + r). \end{array}\right. \quad (65)$$

Moreover, using (56), we infer for $r \in \partial B$ and $|z| >> a$ (since $u_V = V u_1$)

$$\begin{aligned} u_V(z\vec{\imath} + r) &= \int_{\partial B} G(z\vec{\imath} + r - r')\sigma(u_V)(r')n(r')\, dr' \\ &= G(z\vec{\imath}) \int_{\partial B} \sigma(u_V)(r')n(r')\, dr' + O\left(\frac{|V|}{z^2}\right) \\ &= \frac{F_V}{4\pi\eta|z|} + O\left(\frac{|V|}{z^2}\right) \end{aligned}$$

from the analytic expression of G and (63). Since $S_\infty F = U$, plugging these expressions into (65) leads to the uniform estimate

$$\exists C > 0, \text{ s.t. } \forall r \in \partial B^{(i)}, \ |w^{(i)}(r)| \leq \frac{C||(V^{(1)}, V^{(2)}, V^{(3)})||}{\min(x,y)^2}. \quad (66)$$

Since Φ satisfies the uniform coerciveness estimate (62), one deduces that

$$\begin{aligned} \alpha||t^{(i)} - \sigma(u_{V^{(i)}})n||^2_{\mathcal{H}(\partial B^{(i)})} &\leq \sum_{j=1}^{3} \left\langle t^{(j)} - \sigma(u_{V^{(j)}})n, w^{(j)} \right\rangle_{H^{-\frac{1}{2}}(\partial B^{(j)}), H^{\frac{1}{2}}(\partial B^{(j)})} \\ &\leq \sum_{j=1}^{3} ||t^{(j)} - \sigma(u_{V^{(j)}})n||_{\mathcal{H}(\partial B^{(j)})} ||w^{(j)}||_{H^{\frac{1}{2}}(\partial B^{(j)})}, \end{aligned}$$

from which we obtain

$$\begin{aligned} |f^{(i)} - F^{(i)}| &= \left| \int t^{(i)} - \sigma(u_{V^{(i)}})n \right| \\ &\leq ||t^{(i)} - \sigma(u_{V^{(i)}})n||_{\mathcal{H}(\partial B^{(i)})} ||1||_{H^{\frac{1}{2}}(\partial B^{(i)})} \\ &\leq C \sum_{j=1}^{3} ||w^{(j)}||_{H^{\frac{1}{2}}(\partial B^{(j)})} \\ &\leq \frac{C||(F^{(1)}, F^{(2)}, F^{(3)})||}{\min(x,y)^2}, \end{aligned}$$

from (66) and $V_i = \dfrac{F_i}{6\pi a\eta}$. Writing

$$\begin{aligned} ||S - S_\infty|| &= ||-S(S^{-1} - S_\infty^{-1})S_\infty|| \\ &\leq ||S||\,||S^{-1} - S_\infty^{-1}||\,||S_\infty||\,, \end{aligned}$$

for the matricial norm $||.||$ establishes (52).

Step 2. Let L and K be two numbers which are meant to tend to infinity with the constraint that

$$1 << K << L. \tag{67}$$

We consider the square loop in the space of shapes described by (x, y): $(L, L) \to (L + K, L) \to (L + K, L + K) \to (L, L + K) \to (L, L)$. The displacement of the center of mass generated by such a stroke is given by (21)

$$\Delta c = \int_L^{L+K} V_x(x, L) + \int_L^{L+K} V_y(L+K, y) + \int_{L+K}^{L} V_x(x, L+K) + \int_{L+K}^{L} V_y(L, y). \tag{68}$$

From symmetry, it is not difficult to see that $V_x(x, y) = -V_y(y, x)$ for all $x, y > r$, which implies

$$\int_L^{L+K} V_x(x, L) = \int_{L+K}^{L} V_y(L, y) \text{ and } \int_L^{L+K} V_y(L+K, y) = \int_{L+K}^{L} V_x(x, L+K), \tag{69}$$

and leads to

$$\Delta c = 2\int_L^{L+K} (V_x(x, L) - V_x(x, L + K))\, dx. \tag{70}$$

But, since $V_x(x, y) = \dfrac{\det(S(x, y)e_c, S(x, y)e_y, e_c)}{\det(S(x, y)e_x, S(x, y)e_y, e_c)}$, and using estimate (52) of $S(x, y)$ for large (x, y), we get

$$V_x(x, y) = \frac{a}{6}\left(-\frac{1}{x} - \frac{1}{x+y} + \frac{2}{y}\right) + O\left(\frac{1}{\min(x, y)^2}\right).$$

Therefore,

$$\begin{aligned} \frac{6}{a}\Delta c &= 2\int_L^{L+K} -\frac{1}{x+L} + \frac{1}{x+L+K} + \frac{2}{L} - \frac{2}{L+K} + O\left(\frac{1}{L^2}\right) \\ &= 2\ln\left(\frac{1+\frac{K}{L}}{(1+\frac{K}{2L})^2}\right) + \frac{4K^2}{L(L+K)} + O\left(\frac{K}{L^2}\right) \\ &= \frac{7K^2}{2L^2} + O\left(\frac{K}{L^2}\right). \end{aligned}$$

Taking $K = \sqrt{L}$ leads, for L sufficiently large, to a non zero displacement Δc proving lemma 2.

4 A numerical algorithm for computing optimal strokes

In this section, we propose an algorithm for the computation of a stroke of maximal efficiency. We recall that

$$\mathrm{Eff}^{-1} = \frac{\frac{1}{T}\int_0^T (f,u)}{6\pi\eta A\bar{c}^2}, \tag{71}$$

where we have set $(f,u) = \sum_i f^{(i)}u^{(i)}$, and remark that Eff is a non-dimensional quantity, invariant under an affine change of parametrization of time. Thus, we can set $T = 1s$ and $\bar{c} = \Delta c$. The problem we intend to solve numerically is to find optimal strokes. By this we mean to find, for each given initial shape $(x_0, y_0) \in \mathcal{S}$, the stroke $\gamma : [0,1] \to \mathcal{S}$ with

$$\gamma(0) = \gamma(1) = (x_0, y_0), \tag{72}$$

which performs a given displacement

$$\Delta c = \int_0^1 V(\gamma) \cdot d\gamma \tag{73}$$

with maximal efficiency. In view of (71), this means to solve the following constrained minimization problem:

$$\min_{\gamma \in \mathcal{A}(x_0,y_0,\Delta c)} \int_0^1 (f(\gamma(\tau)), u(\gamma(\tau)))\, d\tau, \tag{74}$$

where

$$\mathcal{A}(x_0,y_0,\Delta c) = \Big\{\gamma : [0,1] \to \mathcal{S} \text{ s.t. } \gamma(0) = \gamma(1) = (x_0,y_0) \\ \text{and } \int_0^1 V \cdot d\gamma = \Delta c\Big\}. \tag{75}$$

From (16), one has $f = \alpha_x U_x + \alpha_y U_y$ with

$$U_x = \frac{Se_y \times e_c}{Se_x \cdot (Se_y \times e_c)},\ U_y = -\frac{Se_x \times e_c}{Se_x \cdot (Se_y \times e_c)}. \tag{76}$$

Since $\dot\gamma = (\alpha_x, \alpha_y)$,

$$\begin{aligned}(f,u) &= (f, Sf) \\ &= g_{xx}\alpha_x^2 + 2g_{xy}\alpha_x\alpha_y + g_{yy}\alpha_y^2 \\ &= (G\dot\gamma, \dot\gamma)\end{aligned}$$

where the symmetric and positive definite matrix G is given by

$$G(x,y) = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \begin{pmatrix} (SU_x, U_x) & (SU_x, U_y) \\ (SU_y, U_x) & (SU_y, U_y) \end{pmatrix}. \tag{77}$$

Introducing the Lagrange multiplier λ associated with the constraint (73), the Euler-Lagrange equation for (74) is

$$-\dot{\overline{(G\dot\gamma)}} + \frac{1}{2}\begin{pmatrix} (\partial_x G\dot\gamma, \dot\gamma) \\ (\partial_y G\dot\gamma, \dot\gamma) \end{pmatrix} + \lambda \text{curl } V(\gamma)\dot\gamma^{\perp} = 0. \tag{78}$$

where $\partial_x G$ and $\partial_y G$ stand for the x and y derivatives of the matrix G.

Remark 2 *We wish to point out an interesting connection with sub–Riemannian geometry, which has guided our analysis, see [11]. Introducing for any path $\gamma \in \mathcal{S}$*

$$c(s) = c_0 + \int_0^s V(\gamma(\tau)) \cdot \dot\gamma(\tau)\, d\tau \,, \forall s \in [0,1]\,, \tag{79}$$

then $X = (\gamma, c)$ describes a curve in the three-dimensional space $\mathcal{X}$ whose tangent $\dot X$ is constrained to belong to a two-dimensional plane $T_X = span(F_x, F_y)$. The dissipation rate (f,u) gives, in the local basis (F_x, F_y) the square of the length of $\dot X$.

Therefore, with the metric given by (77) defined on T_X, our optimal strokes describe the shortest sub–Riemannian geodesics in $\mathcal{X}$ joining (x_0, y_0, c_0) to $(x_0, y_0, c_0+\Delta c)$. Equation (78) is the general equation for sub-Riemannian geodesics, along which the Hamiltonian $(G\dot\gamma, \dot\gamma)$ is constant. The existence of a minimizing geodesic joining any two points in state space follows from general theorems. Indeed, the system (23) being analytic and globally controllable, the Lie algebra $Lie(F_x, F_y)$ is of dimension 3 everywhere in state space (see [1]). This in turn implies the existence of minimizing geodesics [14, Theorem 1.19].

We now describe our algorithm for solving (74). It is based on the solution of the Cauchy problem for (78), with given trial value for λ, and a suitable shooting method. Namely, for given initial shape (x_0, y_0), and given trial initial velocity $(\dot x_0, \dot y_0)$, we compute the unique solution $(x(t), y(t))$ of the second order ODE (78) with a classical Runge-Kutta method. The functions $V(x, y)$ and $G(x, y)$ which appear as coefficients in (78) are computed at several points of $\mathcal{S}$ using a 3D axisymmetric finite element Stokes solver and then interpolated (see below for further details). In order to enforce the constraints (72) and (73), we introduce the function $\Psi : \mathbb{R}^3 \to \mathcal{X}$ defined by

$$\Psi(\dot{x}_0, \dot{y}_0, \lambda) = \left(x(1), y(1), \int_0^1 (V_x(x, y)\dot{x} + V_y(x, y)\dot{y})\, d\tau \right), \tag{80}$$

and rewrite the original problem (74) as

$$\text{Find a triplet } (\dot{x}_0, \dot{y}_0, \lambda) \text{ such that } \Psi(\dot{x}_0, \dot{y}_0, \lambda) = (x_0, y_0, \Delta c). \tag{81}$$

This is solved with the following algorithm.

Algorithm 1 *(shooting method)*

0. *Provide a target $\bar{Z} = (x_0, y_0, \Delta c)$ and a parameter N*
1. *Compute the values of V and G at several points of the space of shapes and interpolate the values*
2. *Start with an initial guess $\theta_0 = (\dot{x}_0, \dot{y}_0, \lambda)$*
3. *Loop for $n = 0 \cdots N - 1$*
   - *Compute $Z_n = \Psi(\theta_n)$ by solving (78) with a Runge-Kutta method*
   - *Compute $\theta_{n+1} = \theta_n + D\Psi(\theta_n)^{-1} \left(\frac{\bar{Z} - Z_n}{N - n} \right)$*
4. *(Newton's method) Loop for $n \geq N$ until θ_n converges to θ_∞*
   - *Compute $Z_n = \Psi(\theta_n)$ by solving (78) with a Runge-Kutta method*
   - *Compute $\theta_{n+1} = \theta_n + D\Psi(\theta_n)^{-1} \left(\bar{Z} - Z_n \right)$*
5. *Compute the (optimal) stroke γ_∞ from θ_∞*
6. *Compute the force laws from γ_∞ and (22)*

After the initializations, step 3. of the algorithm is a Newton type method for which the gap between initial guess and unknown target has been subdivided into N increments. A suitably large value has been chosen for N in order to obtain convergence of the algorithm. Moreover, steps 3. and 4. require the solution of a linear system at each iteration with coefficients given by the

3×3 matrix $D\Psi(\theta_n)$. The matrix $D\Psi(\theta)$ is obtained by finite differences from the approximation

$$D\Psi(\theta)\delta\theta \sim \Psi(\theta + \delta\theta) - \Psi(\theta), \tag{82}$$

by setting $\delta\theta = (\epsilon, 0, 0)$, $(0, \epsilon, 0)$ and $(0, 0, \epsilon)$ with ϵ small.

Depending on the initial guess θ_0 in step 2., the algorithm produces different geodesic strokes. Among the computed geodesic strokes, the one with maximal efficiency is selected as the optimal stroke.

5 Examples of optimal strokes

In this section we describe some numerical experiments demonstrating the significance of our approach and the reliability of the algorithm. For our tests, we have taken parameter N of Algorithm 1 equal to 30. For what concerns geometric and material parameters, our simulations describe, say, spheres of radius $a = 0.05\,mm$ swimming in a medium with the kinematic viscosity of water ($\nu = \frac{\eta}{\rho} = 1\,mm^2 s^{-1}$). Setting $T = 1\,s$, this results in a Reynolds number of the order $Re = a^2\nu = 0.0025$. The interpolation stage (step 1. of Algorithm 1) has been done by computing the quantities $V(x, y)$ and $G(x, y)$ at 50×50 equally spaced points in the region $[0.125\,mm, 0.7\,mm]^2$ of the space of shapes $\mathcal{S}$ with an axisymmetric finite element Stokes solver (based on FREEFEM [8]). The simulation domain has been restricted to a large bounding box of size $5\,mm \times 5\,mm$ around Ω on the boundary of which we have taken homogeneous Dirichlet boundary condition for the velocity. In fact, we only take forces that satisfy (2), hence the velocity is expected to decay like $1/|r|^2$ as r tends to infinity. We have then found the polynomial in $(1/x, 1/y)$ giving the best least square fit of these values (a polynomial of degree 4 in each variable proved to be sufficient). Eventually, curl$V$ has been computed by exact differentiation of this polynomial. The numerical results of this section have been validated by comparing with the results of direct finite element simulations, with an error in the predicted power consumption not exceeding one percent.

5.1 Optimal strokes versus square loops

For a given initial shape $(x_0, y_0) = (0.3\,mm, 0.3\,mm)$, and two different given displacements $\Delta c_1 = 0.001\,mm$ and $\Delta c_2 = 0.01\,mm$ we have compared the efficiency of our optimal strokes with the square one proposed by Najafi and Golestanian (NG stroke) [16]. In addition, we show the performance of a square loop in which the distances between the balls are increased, instead of being decreased, in the first two legs of the loop. The expended energy in all cases are given in Table 1 while the strokes are shown in Fig. 2. The optimal strokes are shown superimposed on the graph of curl$V$ in Fig. 3.

$\Delta c(mm)$	Optimal stroke	NG stroke	Naive stroke
0.001	0.0307	0.0405	0.0589
0.010	0.229	0.278	0.914

Table 1: Energy consumption (J).

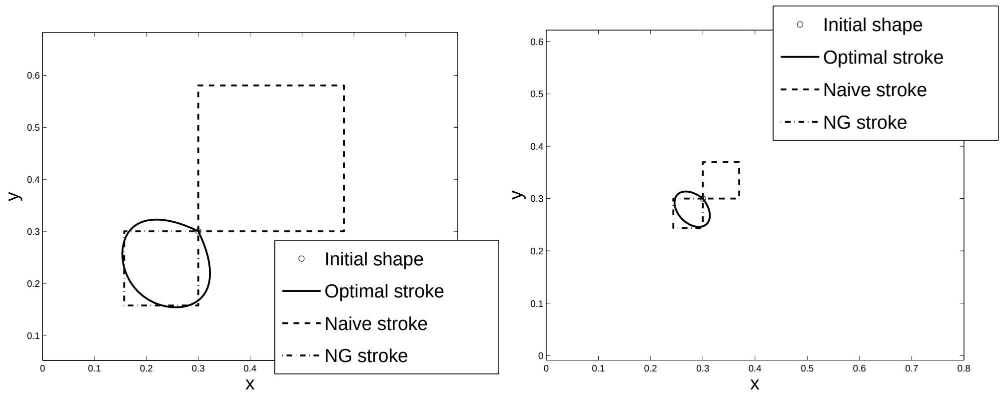

Figure 2: Optimal strokes and square strokes which induce the same displacement $\Delta c = 0.01\,mm$ (left) and $\Delta c = 0.001\,mm$ (right) in $T = 1\,s$.

The optimal stroke gives a noticeable efficiency improvement. It is remarkable that, in our regime, there is a drastic difference, for a given initial shape, between the two square loops. By contrast, in the limiting regimes of small spheres or small deformations, this difference disappears.

5.2 Multiplicity of geodesic strokes

As already mentioned, depending on the initial parameters θ_0 in our algorithm, one can converge to different geodesic strokes. This is due to the fact that the geodesics joining two points in $\mathcal{X}$ are not unique. These geodesics, when projected on $\mathcal{S}$ give more and more involved strokes, and the shortest one gives best efficiencies. In Fig. 4 we have shown three of them for the same parameters as before which have been named according to their shape. The corresponding expended energies are given in Table 2.

We remark that, for small Δc, curl $V(x, y)$ and $G(x, y)$ are essentially constant and equal to their values at (x_0, y_0). In this case, it is well known

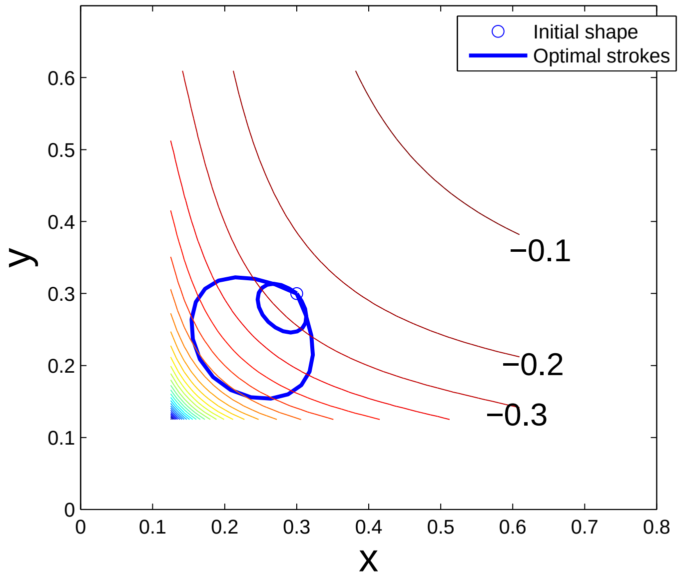

Figure 3: Optimal strokes (for $\Delta c = 0.01\,mm$ and $\Delta c = 0.001\,mm$ in $T = 1\,s$) and equally spaced level curves of curl$V$.

$\Delta c(mm)$	Drop	Bean	Pretzel
0.001	0.0307	0.0387	0.0637
0.010	0.229	0.451	0.529

Table 2: Energy consumption (J).

(from sub–Riemannian geometry) that geodesics project to ellipses. Therefore, in the small Δc regime, all geodesic strokes collapse to ellipses.

5.3 Swimming with many strokes

One can move of an amount Δc in time $T = 1\,s$ by executing n identical strokes, with each of which one moves by $\Delta c/n$. We have investigated the difference in terms of power consumption with respect to n and again $(x_0, y_0, \Delta c) = (0.3\,mm, 0.3\,mm, 0.01\,mm)$. To this aim, we have computed the energy consumption of the n-stroke movement relative to the one using a single stroke. The results are summarized in Fig. 5 where it is shown that this relative energy consumption reaches an asymptotic value for large n. As expected, the fewer the strokes, the bigger the efficiency, and reaching the target in just one stroke is the most preferable strategy, whenever possible.

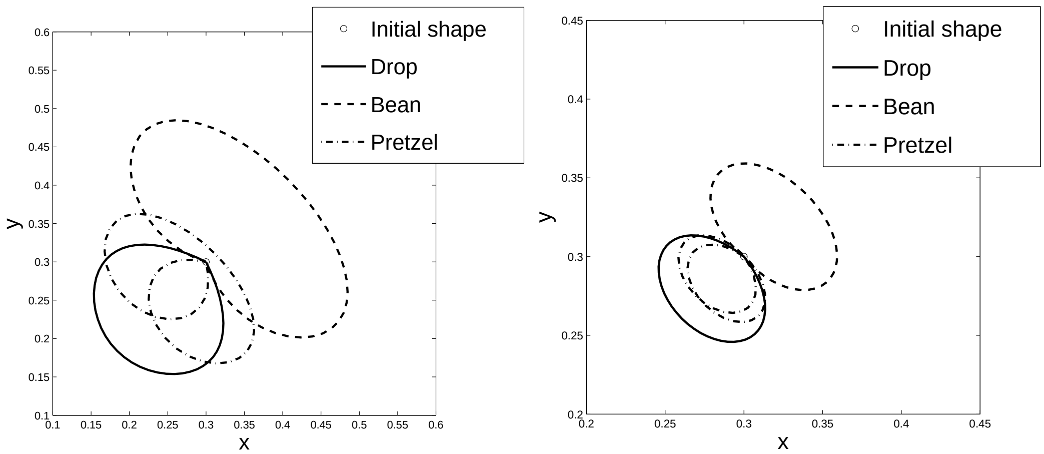

Figure 4: Three geodesic strokes (obtained by projecting geodesics onto the space of shapes $\mathcal{S}$) with $\Delta c = 0.01\,mm$ (left) and $\Delta c = 0.001\,mm$ (right) in $T = 1\,s$.

The opposite one, namely, performing many small shape changes around (x_0, y_0) is most inefficient.

If the shape of the system is constrained to lie in a subset $\bar{\mathcal{S}} \subset \mathcal{S}$, say, for biological or technological reasons, then the optimal swimming strategy requires exploring the largest allowed shape changes in $\bar{\mathcal{S}}$ leaving $(x_0, y_0, \Delta c)$ as adjustable parameters. Resolving the hydrodynamic interactions in this regime cannot be based on asymptotic formulas such as (54), and it calls for numerical tools such as the ones we have used in this section.

6 Discussion

We have shown how to formulate and solve numerically the problem of finding optimal strokes for low Reynolds number swimmers by focussing on the three-sphere swimmer of Najafi and Golestanian (a simple, yet representative example).

The theoretical part of our analysis shows how to address quantitatively swimming as the problem of controlling shape in order to produce a net displacement at the end of one stroke. By casting the problem in the language of control theory, we reduce the problem of swimming to the controllability of the system, and the search of optimal strokes to an optimal control problem leading to the computation of suitable sub–Riemannian geodesics. The numerical solution we find for the optimal stroke leads to an increase of efficiency exceeding 300% with respect to more naive proposals.

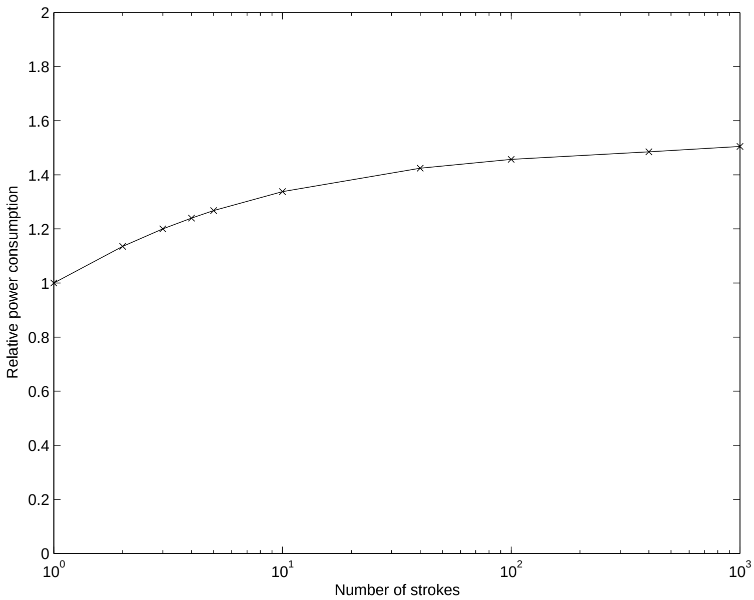

Figure 5: Relative expended energy versus number of strokes.

Much remains to be done. On the one hand, if one is interested in the optimal swimming strategy to reach a given target from a given initial position, the choice of a shape around which to fluctuate and the distance traveled with each stroke are parameters, to be optimized subject to suitable constraints. On the other hand, the three-sphere swimmer is just an example. Its simplicity enables us to carry out the analysis by using explicit formulas, while the study of biologically relevant swimmers will require more abstract mathematical tools.

We are working on extensions of our work in both of these two directions. However, we believe that our paper provides a significant head start. For both the questions of adjusting the stroke to a global optimality criterion, and of optimizing the stroke of complex swimmers, combining the numerical approach we advocate with the use of tools from sub–Riemannian geometry may prove extremely valuable. Useful inspiration can come from the sizable literature on the related field of control of swimmers in a perfect fluid, see *e.g.* [10] and the many references cited therein, and [6]. The literature on low Reynolds number swimmers is, by comparison, smaller, but growing at a fast pace, see *e.g.* [12, 19].

Acknowledgments: We thank A. Agrachev, R. Golestanian, B. Maury, G. Métivier, V. Šverák, and E. Trélat for very helpful discussions. This work was carried out during visits of FA to SISSA and of ADS to Paris XI. The financial support of these institutions is gratefully acknowledged. Additional support came from the EU through the MULTIMAT Marie Curie Research Training Network MRTN-CT-2004-505226 and the italian INdAM through the research project "Mathematical Challenges in Nanomechanics". Finally, we wish to thank the three anonymous referees for their many valuable comments, and A. Garroni and the staff of the library of the Mathematics Department 'G. Castelnuovo' of the University of Roma 'La Sapienza' for their kind assistance.

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