Periodic and conditionally periodic analogs of the many-soliton solutions of the Korteweg-de Vries equation

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A method of connecting the Korteweg-de Vries (KdV) equation, known from the theory of nonlinear waves, with the Schrödinger equation was discovered in 1967. This method is applied in the present paper to study a periodic problem. We find exact analytical formulae for a class of solutions \( u(x,t) \) such that at any moment in time \( t \) the potential \( u(x,t) \) of the Schrödinger operator has only a finite number of forbidden bands in the Bloch spectrum. We find in this connection all potentials with a finite number of bands. This class of solutions contains as a degenerate limiting case the well known \( N \)-soliton solutions of the \( KdV \) equation, which decrease rapidly as \( |x| \to \infty \).

INTRODUCTION

It is well known that the nonlinear Korteweg—de Vries (KdV) wave equation

\[
u_t = 6uu_{xx} - uu_{xxx}
\]

reduces to the inverse problem of scattering theory for the Schrödinger (Sturm-Liouville) operator

\[
L = -\frac{d^2}{dx^2} + u(x), \quad u(x) = u(x,t) \text{ for } t = \text{constant},
\]

if the solution \( u(x,t) \) decreases rapidly as \( |x| \to \infty \) (see [1,2]). The most effective study has in this case been made of the so-called "multisoliton" solutions which describe the interaction of a finite number of solitons—solutions of the kind \( u(x-ct) \). They have the form \( u(x,t) \) where at any time \( t \) the potential \( u \) is non-reflective (the reflection coefficient vanishes identically). Although the algebraic mechanism connecting the KdV equation with the Schrödinger operator continues to function also in the case of periodic boundary conditions, nobody had succeeded in applying it seriously to an effective study of the KdV equation until the recent work by the present authors [3,4] and by Its and Matveev [5].

The basis of this procedure is the fact, noted by one of us, [3] that a strictly periodic (and conditionally periodic) analog of the many-soliton solutions consists in those \( u(x,t) \) for which at any time \( t \) the potential \( u(x,t) \) has only a finite number of forbidden bands in the Bloch spectrum. Such a class of potentials, which we shall call in what follows finite-band potentials, contain as a degenerate limiting case all non-reflective potentials which decrease fast as \( |x| \to \infty \); all finite-band potentials and the corresponding solutions of the KdV equation can be found in terms of exact, albeit complicated, formulae. The solutions of the form \( u(x-ct) \) are in the periodic case potentials with a single forbidden band. This is a Weierstrass elliptic function \( 2\wp(x) + \text{constant} \). Even a consideration of their simplest perturbations leads to a two-band (i.e., two-forbidden-band) conditionally periodic potential \( u(x,t) \) with two, generally speaking, non-commensurate periods, where \( u(x + \Delta_1 + t + \Delta_2) = u(x,t) \) (i.e., after a period \( T = \Delta_2 \), the picture is re-established with a shift \( x \to x - \Delta_1 \)).

In this paper we describe a class of periodic and conditionally periodic finite-band potentials and the corresponding family of solutions of the KdV equation. Although many facts can easily be generalized also to the case of an infinite number of bands, to a large extent the results lose their effectiveness. We must note here that finite-band periodic potentials turn out to be relatively numerous among the periodic functions in contrast to the non-reflective Bargmann potentials: apparently one can approximate any smooth periodic potential by a finite-band one, although we have not proved this. We note that the procedure developed in the present paper is applicable also to other nonlinear equations which are "fully integrable" by the scattering theory method and which occur in a study of a periodic problem; it is now already known that their number is large [6-8] (Zakharov and Shabat [9] have developed a regular method to find them).

1. FINITE-BAND POTENTIALS AND INTEGRALS OF THE KdV EQUATION

Lax [10], using the procedure of [1], has noted that the basis for the connection between the KdV equation and the Schrödinger operator is the representation of the right-hand side \( 6uu' - uu'' \) as a commutator

\[
L = -\frac{d^2}{dx^2} + u, \quad A = \frac{d}{dx} + \frac{d}{dx}, \quad L = [A, L],
\]

whence it follows that the equations

\[
u_t = A \quad \text{and} \quad L = [A, L]
\]

are equivalent.

If \( \varphi \) is an eigenfunction, \( L \varphi = E \varphi \), we easily get from (1.1) the relation

\[(L-E)\varphi = (L-E)A\varphi.
\]

We fix two eigenfunction bases

\[
x = x_{\ast}, \quad \varphi = 1, \quad \varphi = i k, \quad k = E;
\]

\[
x = x_{\ast}, \quad \varphi = 1, \quad \varphi = i k, \quad k = E;
\]

\[
x = x_{\ast}, \quad c = 1, \quad s = 0.
\]

For a periodic potential \( u(x) \) with period \( T \) the translation operator produces when acting upon the eigenfunctions a shift over the period \( T \):

\[(\tilde{T} \varphi)(x) = \varphi(x + T).
\]

We obtain in both cases (1.4) and (1.4') a second rank matrix with a determinant equal to unity:

\[
\tilde{T}(x_{\ast}, k) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad |a| = |b| = 1
\]

in the base (1.4), or

\[
\tilde{T}(x_{\ast}, k) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad a_{11}a_{22} - a_{12}a_{21} = 1
\]

where the \( a_{ij} \) are real, \( k^2 = E \), in the base (1.4').

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For rapidly decreasing potentials one usually chooses $x_0 = \pm \infty$, $\mathbf{T} = \mathbf{T}(k)$ in the base from two exponents. In the case of a finite period the choice of the point $x_0$ is arbitrary and when we change $x_0$ (in the base (1.4)) we have the equation
\[
\frac{d^2}{dx^2} + (\mathfrak{Q}, \mathfrak{P}), \quad \mathfrak{P} = ik \begin{pmatrix} 0 & i \omega_k \\ -i \omega_k & 1 + i \xi_k \end{pmatrix}.
\]
(1.7)

The Bloch eigenfunctions $\phi_k(x, E, \mathbf{E})$ are determined by the conditions
\[
\mathcal{L}_k\phi_k = E\phi_k, \quad \mathcal{L}_k = \exp \left( \mp i\phi_k(E) \right) \mathfrak{P}_k, \quad \mathfrak{P}_k |_{x = \pm \infty} = 0,
\]
where the dispersion law $\mathfrak{P}(E)$ is determined in the allowed bands. The trace of the matrix $\mathfrak{I}$ is of the form
\[
\mathfrak{I} = \mathfrak{I} + 2a_k
\]
in the base (1.4) and is independent of $x_0$. The allowed bands are determined by the condition
\[
\left| \mathfrak{I} \right| = \left| \mathfrak{I}_{a_k} \right| < 1.
\]
(1.10)

We note that the eigenvalues of the matrix $\mathfrak{T}$ have the form $\mathfrak{a}_k = (\mathfrak{a}_R^2 - 1)^{1/2}$, or $\mathfrak{a}_R = \cos \mathfrak{P}(E)$. The points of the discrete spectrum $E_0$ of the periodic and the antiperiodic problems: $\psi(x + T) = \pm \psi(x)$, are determined by the conditions $\mathfrak{a}_R(E_0) = \pm 1$. They are the edges of the forbidden bands only when these levels are nondegenerate (for the matrix $\mathfrak{T}$ is a Jordan matrix for $E = E_0$). If $\mathfrak{a}_R = \pm 1$, but the matrix $\mathfrak{T}$ is diagonal (and equal to $\pm 1$) the forbidden band is collapsed to nothing. This is characterized by a condition similar to the non-reflectivity condition:
\[
b(x, t) = 0, \quad t = -E_0.
\]
(1.11)

The finite-band character of the potential means that all higher periodic and antiperiodic levels $E_n$ are twofold degenerate.

We find, clearly, from (1.6) that in the points of the spectrum $E_0$ of the periodic and antiperiodic problems: $\mathfrak{a}_R = \pm 1$, we have the equations
\[
\mathfrak{a}_R = \pm 1, \quad E = E_0,
\]
where $\mathfrak{a} = \mathfrak{a}_R + i\mathfrak{a}_I$, $|\mathfrak{a}| \neq 0$ in the non-degenerate points of the spectrum for all $x_0$.

If $\chi(x, E) = -i\mathfrak{I} \ln(\mathfrak{a})$, then $\chi$ will be independent of the point $x_0$ and will satisfy the Riccati equation which expresses, in particular, its imaginary part in terms of its real part:
\[
-i\chi' + \chi + u = E, \quad \chi = \frac{1}{2}(\ln \mathfrak{a})',
\]
(1.13)

and allows an asymptotic expansion as $E \to \infty$
\[
\chi(x, k') \sim k + \sum_{n=0}^{\infty} \frac{\chi_n(x)}{(2k)^n}.
\]
(1.14)

By virtue of (1.13) all functions $\chi_n(x)$ are polynomials in $u(x)$ and its derivatives with respect to $x$, while the $2m(k)$ total derivatives.

It is well known that all integrals
\[
\mathcal{I}(k) = \int \chi(x, k') \, dx, \quad \mathcal{I}_n = \int \chi_n(x) \, dx, \quad m \geq 0.
\]
(1.15)

are conserved by virtue of the KdV equation. Moreover (Lax and Gardner, 10, 11) all "higher KdV equations"
\[
\frac{\partial}{\partial x} \mathcal{I} = \frac{\partial}{\partial t} \mathcal{I} + \mathcal{N}(\mathcal{I}, \mathcal{I}),
\]
(1.16)

admit of a representation in the form (1.1):

\[
L = [A_n, L], \quad A_n = \frac{d^{n+1}}{dx^{n+1}} + \sum P_i \frac{d^n}{dx^n},
\]
(1.17)

where all $P_i$ are polynomials in $u$ and its derivatives with respect to $x$. The KdV equation itself is obtained for $m = 1$, and the operators $A_0, A_1, A_2$, and $A_3$, and the integrals $I_1, I_0, I_1,$ and $I_2$ take the following form:
\[
I_{1, 0} = \int u \, dx, \quad I_{1, 2} = \int u' \, dx, \quad A_2 = \frac{d}{dx},
\]
\[
I_1 = \int \left[ u^2 + \frac{1}{4} (u')^2 \right] \, dx, \quad A_1 = -4 \frac{d^2}{dx^2} + 3 \left( \frac{d}{dx} \frac{d}{dx} - \frac{1}{2} u \right),
\]
\[
I_0 = \int \left( \frac{u''}{2u'} \right)^2 - \frac{u'' u'''}{u'} \, dx, \quad A_0 = -16 \frac{d^4}{dx^4} - 20 (u^2 \frac{d^3}{dx^3} + \frac{d^2}{dx^2} u) + 30u \frac{d^2}{dx^2} + 5 \left( u'' \frac{d}{dx} + \frac{d}{dx} u'' \right).
\]
(1.18)

Any equation of the form $\tilde{u} = \mathcal{Q}(u, u', \ldots)$, where the right-hand side is a polynomial and can be written in the form of a commutator $[A, L] = \mathcal{Q}$, is of the form
\[
\tilde{u} = \sum \frac{c_n}{\delta u} \mathcal{Q}(\mathfrak{a}_n, \mathfrak{a}_n'), \quad \mathfrak{a}_n = \mathfrak{a}_n(\mathfrak{a}_n').
\]
(1.19)

Let some such equation be given. It turns out that all its periodic stationary solutions $u(x)$ are finite-band potentials, and we obtain thus all finite-band potentials. The conditionally periodic solutions of this equation are also finite-band potentials (see 13, 11). We shall indicate below the algorithm for integrating these equations:
\[
\sum \frac{c_n}{\delta u} \mathcal{Q}(\mathfrak{a}_n, \mathfrak{a}_n') = \text{const.}
\]
(1.20)

By virtue of (1.3) we have for the basis (1.4) of the eigenfunctions the equations
\[
\tilde{u} = A_0 \tilde{u} + \lambda \tilde{u}, \quad \lambda = A_0 \tilde{u} + \lambda \tilde{u},
\]
(1.21)

where the matrix
\[
\Lambda = \Lambda(\mathfrak{a}_n, \mathfrak{a}_n') = \frac{1}{\mu}(\mathfrak{a}_n, \mathfrak{a}_n')
\]
(1.22)

has a zero trace $\lambda + \bar{\lambda} = 0$ and has a polynomial dependence on $k$, on $u$, and on its derivatives with respect to $x$ in the point $x = x_0$. It is determined from the conditions $\tilde{u} = 0$ and $\tilde{u} = 0$ when $x = x_0$. It turns out that we have for the matrix $T(x_0, k)$ the equation
\[
\frac{\partial T}{\partial x} = [A_0, \tilde{T}].
\]
(1.23)

Comparing (1.23) with (1.7), we get from the condition
\[
\frac{\partial}{\partial x} \frac{\partial Q}{\partial x} = [A, \tilde{Q}],
\]
(1.24)

Equation (1.24) gives a new useful algebraic representation of the KdV equation and its higher analogs. For instance, for stationary solutions of Eqs. (1.16) we have Eq. (1.20), whence it follows that
\[
\frac{\partial Q}{\partial t} = [A, \tilde{Q}], \quad [A, \tilde{T}] = 0.
\]
(1.25)

Since $\text{Tr} \Lambda = 0$, the eigenvalues $\alpha_n$ are given by $\alpha_n(k) = \pm (\text{det} \Lambda)^{1/2}$, where $\text{det} \Lambda$ is a polynomial of $k^2 = E$, the zeroes of which (see below) are the boundaries of the bands, with coefficients depending on $u, u', \ldots, u(2N)$. These coefficients are also a complete set of commuting.
integrals of the Hamiltonian of Eq. (1.20) which also allows, by virtue of (1.25), a commutator representation with second rank matrices with coefficients which depend polynomially on k. Furthermore, for the matrix elements of the second Eq. (1.25) we get

\[ [A, \hat{T}]_i = (k - \lambda) b + (a - \delta) \mu = 0, \]

or

\[ 2 \delta b = 2 \alpha \mu. \]  

(1.26)

In the non-degenerate points of the periodic and antiperiodic problems we get from (1.26) by virtue of (1.25), a commutator representation

\[ \int \text{periodic and antiperiodic problems we get from (1.26) by virtue of (1.12) } \]

which branches at the band edges \( E_1 \). Inside the allowed bands the values of the same function \( \psi \) on different sheets then correspond to a pair of linearly independent functions \( \psi(x, x_0, E) \). One sees easily that the zeroes and poles of \( \psi \) can lie on the Riemann surface \( R \) only on the forbidden bands or their edges on the surface \( R \). It is clear that \( \psi_e \sim \exp(\pm i k(x - x_0)) \) as \( E \to \infty, k^2 = E \).

From (1.13) we get the following representation

\[ \psi(x, x_0, E) = \frac{\psi(x_0, E)}{\psi(x, E)} \exp \left\{ \int \frac{\psi(x, E)}{\psi(x_0, E)} \right\}. \]  

(2.1)

Moreover, there is for \( \psi \) a representation in the base \( (1.4') \):

\[ \psi = -i \psi(x_0, E) \psi(x, E). \]  

(2.2)

We get easily for \( \chi(x, E) \) an expression in terms of the matrix \( T \):

\[ \chi(x, E) = \frac{R(E)}{P(x, E)}, \quad \chi(x, E) = \frac{1}{P(x, E)} \]  

(2.4)

Here

\[ \chi(x, E) = \frac{R(E)}{P(x, E)}. \]

(2.5)

Moreover, it follows from (2.1) that \( \psi(x, E) \) has up to one pole \( \gamma_j(x_0) \) and one zero \( \gamma_j(x) \) in each of the forbidden bands or at their edges; more precisely, the function has on the Riemann surface \( R \) a pole on only one of the sheets: \( \gamma_j(x_0), \sigma_j \), where \( \sigma_j = \pm \).

From the condition that there be no pole on the other sheet \( \gamma_j(x_0), \sigma_j' \) and from Eqs. (2.2) and (2.4) we find that the quantity

\[ \chi(x_0, E) = \frac{R(E)}{P(x_0, E)}, \quad \chi(x, E) = \frac{1}{P(x, E)} \]  

(2.6)

has no pole when \( E = \gamma_j(x_0) \) and the sign in front of the radical \( {\text{R}}^{1/2} \) is equal to \( \sigma_j' \). Hence follows the equation

\[ \frac{dP(x_0, E)}{dx} = 2 \alpha \frac{dR(x_0, E)}{dx}. \]  

(2.6')

Solving (2.6) for \( \gamma_j \) we get

\[ \gamma_j = \pm 2 \alpha \sqrt{R(x_0)} \left( \gamma_j \right) \]  

(2.6')

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For the two-band case \((n = 2)\) these equations take the form
\[
\gamma'_1 = \frac{2iR(t_1)}{\gamma_1}, \quad \gamma'_2 = \frac{2iR(t_2)}{\gamma_2},
\]
and can be integrated by the substitution \((\alpha = 1, 2)\)
\[
(\gamma_1 - \gamma_1) \, dx = dz, \quad \tau = \frac{1}{2i} \int_{E}^{E} R(t) \, dt,
\]
\[
E < E_1 < E_2 < E_3 < E_4.
\]

The parameter \(x_0\) is chosen here such that \(Y_1(X_0) = E_2\); and can be integrated by the substitution \((\alpha! = 1, 2)\)
\[
(1.14) \quad \text{one can derive relations that express symmetric}
\]
\[
odds\text{properties the functions } Y_1 \text{ and } Y_2 \text{ are periodic in } \tau \text{ and possess the properties}
\]
\[
E_1 < Y_1 < E_2, \quad E_1 < Y_1 < E_3.
\]

From the asymptotic behavior of \(\chi_R\) as \(E \to \infty\) (Eq. (1.14)) one can derive relations that express symmetric polynomials in \(\gamma_1\) and \(\gamma_2\) in terms of \(u, u', u''\), ... . In particular, we have for \(n = 2\)

\[
a(x) = -2(1 + 12) + \sum E_j
\]
\[
\gamma_1 = \frac{1}{2} \sum \{w(E_1 + E_2) - A, \quad A = \frac{1}{2} \sum E_j + \frac{1}{2} \{E_1, E_2\}^2 \}
\]
\[
(2.6') \quad \gamma_2 = -\frac{1}{2} \sum m_j \Omega_j.
\]

Let us explain the geometrical meaning of Eqs. (2.6') and (2.6''), which are written on the Riemann surface \(R\). The forbidden band number \(j\) corresponds to the section \(l_j = [E_{2j}, E_{2j+1}]\) in the \(E\)-plane.

The Riemann surface \(R\) this section corresponds to the cycle \(a_j\) - a circle consisting of two sections \((l_j, +)\) and \((l_j, -)\), the ends of which are identical (see Fig. 1). The set of points \((\gamma_j, a_j)\) lies on the circles \(a_j\) and Eq. (2.6') holds for them. By varying \(x\) we get the motion of every point \((\gamma_j, a_j)\) along the circle \(a_j\) and the signs of \(a_j\) change from the upper sheet to the lower sheet. This cycle \(a_j\) lies on the circles \(a_j\) and Eq. (2.6') holds for all \(n \geq 2\) to give a different description of the same torus. We consider differentials on a Riemann surface which have no poles (of first order)
\[
\Omega'_m = \sum \frac{E_j \, dE}{\pi R(E)}; \quad m = 1, \ldots, n.
\]

normalized by the conditions
\[
\oint \Omega_m = 2 \pi \delta_m.
\]

We introduce cycles \(b_j\) on the Riemann surface which do not intersect the \(a_m\) with \(m \neq j\), while each \(b_j\) intersects \(a_j\) in one point, \(E_{2j}\) (see Fig. 2). We have the real matrix \(B_{mj}\):
\[
B_{mj} = \oint \Omega_m.
\]

It is known (Riemann) that \(B_{mj} = B_{jm}\), that the matrix \((B_{mj})\) is negative definite, and that it cannot be broken up into blocks (e.g., it cannot be diagonal). At \(n = 2\) this is the complete set of conditions for the matrix \(B_{mj}\).
the (second order) differential \( \Omega = (E_0 + q_1 E_0^{-1} + \ldots + q_n E_0^{-n}) \) is here normalized by the conditions

\[
\int \Omega = 0, \quad j = 1, \ldots, n. \tag{2.17}
\]

It is well known that one can easily get by using (1.14) the following representation for the potential \( u(x) \):

\[
u(x) = -2 \sum_{j=1}^{n} \gamma_j + \sum_{\alpha} E_{\alpha}, \tag{2.18}
\]

If \( \gamma_j = (\gamma_j', \alpha_j) \) are points on the Riemann surface, we can write \( \gamma_j \) as a numerical function of the parameters \( \eta_1, \ldots, \eta_n \) by virtue of (2.11) and (2.14):

\[
u(x) = \kappa_{n}(\eta_1, \ldots, \eta_n). \tag{2.19}
\]

Using (2.18) we get

\[
u(x) = -2 \sum_{j=1}^{n} \gamma_j + \sum_{\alpha} E_{\alpha}, \tag{2.20}
\]

by virtue of (2.16).

It is well known that the function \( \kappa(\eta_1, \ldots, \eta_n) \) is a standard algebraic function on the \( 2n \)-dimensional torus, given by the lattice (2.13). \[1\] We can follow Its and Matveev \[2\] and take for the function \( \kappa \) its logarithm of the \( \eta_j \) as a numerical function on the \( 2n \)-dimensional torus, with periods and, if we continue into the complex region, with periodicity of \( u(x) \).}

It follows from Eqs. (2.20) and (2.21) that, generally speaking, the potential \( u(x) \) is quasi-periodic with periods \( (T_1, \ldots, T_n) \), where

\[
T_i = 2\pi \frac{n_i}{P_i} \tag{2.22}
\]

where the matrix \( P^j \) is the inverse of the matrix \( B_{ij} \) of the periods and, if we continue into the complex region, with periods \( (T_1', \ldots, T_n') \) where

\[
T_i' = 2\pi \frac{n_i}{P_i} \tag{2.22'}
\]

The \( n - 1 \) relations

\[
\sum_{i} n_i T_i = 0, \tag{2.23}
\]

with \( n_i \) an integer, are necessary and sufficient for the periodicity of \( u(x) \). If, moreover, the \( n - 1 \) relations for the imaginary periods,

\[
\sum_{i} m_i T_i = 0, \tag{2.23'}
\]

are satisfied, we can express the potential in terms of elliptic functions. For the two-band case, \( n = 2 \), the compatibility condition for having both Eqs. (2.23) and (2.23') for five parameters \( (E_1, \ldots, E_5) \) gives us an enumerable set of three parameter families. One of them (Ince's case) has already been indicated at the end of Sec. 1 (see (1.30), (1.31), and (2.8')).

If we use Eqs. (2.2') and (1.31) for the potential \( u(x) = \delta R(x) \) we get the spectrum explicitly (surface \( R \)) and also the form of \( \gamma_1(x) \) and \( \gamma_0(x) \):

\[
R(E) = \frac{27}{4} \frac{x}{E_{0}} - \frac{21}{4} \frac{x}{E_{0}} - \frac{27}{4} \frac{x}{E_{0}} - \frac{27}{4} \frac{x}{E_{0}} - \frac{81}{4} \frac{x}{E_{0}}, \tag{2.24}
\]

or

\[
E_{1} = -3E_{0}, \quad E_{2} = -3E_{0}, \quad E_{3} = E_{0} = -3E_{0}, \quad E_{4} = -3E_{0}, \quad E_{5} = -3E_{0}, \quad \gamma_{1}(x) = \frac{3}{2} \left( \psi(x) \pm (g_{1} - 3g_{0}(x)) \right). \tag{2.25}
\]

It is, finally, relevant to note the general uniformity for the Bloch dispersion law \( p(E) \):

\[
p(E) = \pm \sqrt{E - E_{0} + \sum_{i=1}^{n} \frac{1}{2} \psi(x) \pm (g_{1} - 3g_{0}(x))}. \tag{2.25'}
\]

From the last equation, together with the form of the function \( \chi_{R} \) (see (2.3)) we easily get the statement which is the inverse of the result of Sec. 1: any finite-band potential satisfies one of the higher KdV equations (1.2). We note also that in the case of a potential which is periodic with period \( T \) it follows from the second Eq. (2.25) that the differential \( T' \) of the potential is the same as the differential \( \Omega \) occurring in Eqs. (2.16) and (2.17).

3. TIME-DEPENDENCE OF FINITE-BAND

POTENTIALS BY VIRTUE OF THE KdV EQUATION

We consider the "finite-band" solutions \( u(x, t) \) of the KdV equation which at any time \( t \) give a finite-band potential for the Schrödinger operator. If the finite-band potential \( u_{F}(x) \) is periodic with period \( T \) with \( T \to \infty \) and if \( u_{F}(x) \) decreases rapidly, the potential \( u_{F}(x) \) is non-reflective. The family of finite-band solutions of the KdV equation thus contains as a degenerate limiting case the many-soliton solutions. In that case the Riemann surface \( R \) of the root

\[
\left[ \prod_{i=1}^{n} (E - E_{i}) \right]^{1/2} \tag{3.1}
\]

is degenerate as for \( T \to \infty \) the band edges converge pairwise to one another, and in the limit the root can be taken. The parameters \( (\eta_1, \ldots, \eta_n) \), given by Eq. (2.11), have no meaning at all when \( T \to \infty \).

We now study the time-dependence of the potential \( u(x) \) by virtue of the KdV equation. Firstly, the band edges are integrals of the system. One can show that the derivatives \( \frac{d}{dt} \eta_{i}^{k} \) are constants, by virtue of any of Eqs. (1.20). One can easily evaluate these constants. We denote them by \( \dot{\eta}_{i}^{k} = W_{i}^{k} \) for the original KdV equation. We then get from Eqs. (2.20):

\[
u(x, t) = -2x(xU_{i} + tW_{i} + \eta_{i}^{k} + \ldots, xU_{i} + tW_{i} + \eta_{i}^{k}) + \text{const.} \tag{3.1'}
\]

It is, however, convenient to evaluate the time-dependence for the functions \( \gamma_{j} \) (or the points \( (\gamma_{j}, \sigma_{j}) \) on the cycles \( a_{j} \)). We get from Eq. (1.23) for the matrix \( T \) in the basis (1.4):
\[ \dot{x} = (\lambda_{1}\tau_{0})', \quad \Lambda = (\lambda_{1} + \mu_{1})/\mathbb{R}, \]
\[ \dot{\mathbb{R}} = \Lambda \mathbb{R} x^{2}. \]  
Moreover, for \( E = \gamma_{1}(x) \) after using (2.6) it follows from (3.4) by analogy with (2.6') that
\[ \dot{\gamma} = -4i\gamma_{1}(x)^{2} R^{\gamma}(\tau) \prod_{i} (\tau_{i} - \gamma_{i}). \]  
In the case of the KdV equation we have \( \Lambda = -2(u + 2E) \), i.e.,
\[ \dot{\gamma} = \pm 8i \left( \sum \gamma_{i} - \frac{1}{2} \sum E_{i} \right) R^{\gamma}(\tau) \prod_{i} (\tau_{i} - \gamma_{i}). \]  
Through the substitution (2.11) and (2.14) we can integrate Eqs. (3.5) and (3.5'), and the derivatives \( \eta_{k} = W_{k} \) can easily be expressed in terms of the periods of a few differentials on the Riemann surface \( \mathbb{R} \) with poles at infinity.

For the case of two forbidden bands we get, starting from Eq. (2.7), for the parameters \( (x_{0}, \tau_{0}) \):
\[ \dot{\gamma} = -4i \left( \gamma_{1}(x_{0}) - \frac{1}{2} \sum E_{i} \right), \]
\[ u(x, t) = -2(\gamma_{1}(x-x_{0}(t)) + \gamma_{1}(x-x_{0}(t) + \tau_{0}(t)) + \text{const.} \]  
Together with Eq. (2.8) this gives the final form of \( u(x, t) \) in the two-band case. In the particular case \( u(x, 0) = 6\mathbb{R}(x) \) we get from Eqs. (2.8') and (1.30), (1.31):
\[ u(x, t) = 2\theta(x - \beta_{1}(t)) + 2\theta(x - \beta_{2}(t)) + 2\theta(x - \beta_{3}(t)). \]
\[ \beta_{1} + \beta_{2} + \beta_{3} = 0, \]
\[ \int_{-\infty}^{x} \frac{dz}{12(\theta - 3\theta^{3}(z))} = t, \]
\[ \beta_{i} = \frac{1}{2} \theta^{-1}(-\theta(\beta_{i} - \beta_{i})), \quad (x - 3\theta^{3}(\beta_{i} - \beta_{i}))^{3/2}. \]  
In conclusion we note that the formulae given here can be improved upon in a number of cases but, in principle, they describe the whole dynamics of the finite-band solutions. The parameters \( \eta_{k} \) on the torus (determined apart from the lattice periods (2.13)) give "angle variables" which are canonically conjugate to the "action" variables formed from the eigenvalues of the Schrödinger operator by analogy of the work of Zakharov and Faddeev.\(^{[14]}\) It is relevant to draw attention to the complexity of the angle variables in the periodic case as compared to the fast decreasing case.