5. The functions $K_{\pm}(g, e)$, $g \in G_{\pm}$, can be rewritten in the form

$$K_{\pm}(g, e) = [(\mathrm{tr}\ (g) - 2)^3 (\mathrm{tr}\ (g) + 2)]^{-1/2}, \tag{22}$$

where a unique branch of the square root is chosen from the condition: $K_{\pm}(g, e) > 0$ for $\{\mathrm{Im}(\mathrm{tr}\, g) = 0,\ |\mathrm{tr}\, g| > 2\}$.

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COMPLETELY INTEGRABLE HAMILTONIAN SYSTEMS ASSOCIATED WITH MATRIX OPERATORS AND ABELIAN VARIETIES

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This paper deals with the integration of a periodic problem for nonlinear systems associated with matrix linear differential operators of first order by methods that generalize the methods of Novikov, the author, Matveev, and Its (see survey [1]). Examples of such systems that are of physical interest are the nonlinear Schrödinger equation [2], the equation of interaction of wave packets in nonlinear media [3], and the modified Korteweg–de Vries equation (complex). Manakov [4] has solved the n-dimensional generalization of the classical Euler problem of the motion of a rigid body by using the results obtained in this paper (a direct verification of the independence of the integrals obtained in [4] was given by Mishchenko and Fomenko [5]).

A general algorithm of construction of such nonlinear systems, together with a method of solution of the inverse problem (in the case of rapid decrease) for the corresponding linear operators, was found by Zakharov and Shabat [6]. We give another algorithm that is more convenient for the integration of the periodic problem. The basic objects of the investigation are matrix operators whose characteristic functions are meromorphic on a Riemann surface of finite genus. These are called finite-zone operators, and the surface itself is called the spectrum. The coefficients of these operators satisfy a Hamiltonian system of ordinary differential equations of the type of the stationary Korteweg–de Vries (KdV) equations (the Novikov equations). The problem of integrating this system is thus solved simultaneously with the inverse problem of spectral theory, i.e., the problem of finding all the finite-zone operators with given spectrum. The basic result is that the set of finite-zone operators with given spectrum is (to within a factorization with respect to the action of a commutative group) the Jacobian variety of the corresponding Riemann surface. The temporal dynamics for the nonlinear partial differential equations under consideration are completely calculated. Explicit formulas for the coefficients of the matrix operators found are given in terms of θ-functions.

The statements of the basic results in this paper were published in [7]; regarding the ideas of the proofs, see the survey [1, Chap. 3, §2]. The particular case of two-dimensional matrix operators was investigated independently by Its [8] by other methods. Krichever [9, 10] indicated an algebraic–geometric method of construction of nonlinear systems that generalize, in particular, the systems considered in this paper.

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Let F be the space of smooth complex-valued functions $f(\mathbf{x})$, $\mathbf{x} = (x^1, \ldots, x^n)$, of n variables $x^1, \ldots, x^n$ such that $\sum_i \partial f/\partial x^i = 0$. We use the notation $\partial f/\partial x^i \equiv f,_i$. Let A be the space of diagonal complex matrices of order n. In the space A we fix a basis of matrices A_k, $k = 1, 2, \ldots, n$, where $(A_k)^i_j = \delta^i_k\delta^k_j$. For each matrix $A \in \mathbf{A}$, where $A = (a_i\delta^j_i)$, we set $\partial f/\partial x_A = \sum_i a_i f,_i$. In particular, we have $f,_i = \partial f/\partial x_{A_i}$.

For each matrix $A \in \mathbf{A}$ suppose that we are given a matrix function $U_A(\mathbf{x}) = (u^j_{A_i})$, $i, j = 1, \ldots, n$, where all the matrix elements are functions in F. We consider the family of operators L_A on the space of vector-valued functions F^n depending on the parameter E,

$$L_A = \frac{\partial}{\partial x_A} + U_A - EA \equiv \frac{\partial}{\partial x_A} + Q_A, \quad A \in \mathbf{A}. \tag{1}$$

We require that the operators of the family constructed form a commutative algebra L for any E, i.e., if $A, B \in \mathbf{A}$, then

$$[L_A, L_B] = 0 \Leftrightarrow \frac{\partial Q_B}{\partial x_A} - \frac{\partial Q_A}{\partial x_B} = [Q_A, Q_B]. \tag{2}$$

Equating the coefficients of E in Eq. (2), we get the relation $[A, U_B] = [B, U_A]$, from which we have

$$U_A = [A, V], \quad A \in \mathbf{A}, \tag{3}$$

where $V = (v^j_i)$ is some matrix, and for definiteness we consider that the diagonal elements of V are zero. Now, equating the free terms in Eq. (2), we get a system of $n - 2$ nonlinear partial differential equations in the matrix V

$$\left[A, \frac{\partial V}{\partial x_B}\right] - \left[B, \frac{\partial V}{\partial x_A}\right] = [[A, V], [B, V]], \quad A, B \in \mathbf{A}. \tag{4}$$

System (4) can be rewritten in another form that is convenient for calculations,

$$v^j_{i,p} = v^j_p v^p_i, \ \text{if} \quad p \neq i, j, \quad v^j_{i,i} + v^j_{i,j} = -\sum_s v^j_s v^s_i. \tag{4'}$$

If V is a solution of system (4), then its matrix elements, and also any expressions depending only on V, can be regarded either as functions in F, or as functions of the single variable $x = x_A$, where the matrix $A \in \mathbf{A}$ is fixed. We pass from one notation to the other without special mention.

We now construct a collection of commutating dynamic systems on the variety of matrices V that are solutions of system (4). We consider the matrix differential equation

$$\frac{\partial \lambda}{\partial x_A} = [\lambda, Q_A], \quad Q_A = U_A - EA \equiv [A, V] - EA. \tag{5}$$

<u>LEMMA 1.</u> Equation (5) has a unique solution $\lambda = \lambda_B$, $B \in \mathbf{A}$, in the form of a formal series in powers of 1/E such that

$$\lambda_B = B + (\lambda_{1,B}/E) + (\lambda_{2,B}/E^2) + \ldots; \tag{6}$$

if $V = 0$, then

$$\lambda_B \equiv B. \tag{7}$$

The matrix elements of the matrix $\lambda_{i,B}$ can be expressed polynomially in terms of the matrix elements of the matrices $V; V,_j, \ldots, V,_{j_1 \ldots j_{i-1}}$ with constant coefficients depending on B.

<u>Definition 1.</u> The N-th KdV equation is defined to be the equation

$$[A, \dot{V} - \lambda_{N+1, B}] = 0, \quad A, B \in \mathbf{A}, \quad V = V(\mathbf{x}, t), \tag{8}$$

or an arbitrary sum of these equations

$$\sum_{k \leqslant N} [A, \dot{V} - \lambda_{k+1, B_k}] = 0, \quad B_k \in \mathbf{A}. \tag{8'}$$

Equation (8) [or (8')] defines a dynamic system on the variety of matrices V that are solutions of system (4). Equation (8) admits a commutative representation of the type of Lax or Novikov. We consider a matrix $\Lambda = \Lambda_{N,B}(x, E)$ of the form

$$\Lambda = BE^N + \lambda_{1,B}E^{N-1} + \ldots + \lambda_{N,B}. \tag{9}$$

LEMMA 2. Equation (8) is equivalent to the commutation relation

$$\left[L_A, \frac{\partial}{\partial t} + \Lambda\right] = 0 \Leftrightarrow \frac{\partial \Lambda}{\partial x_A} - \frac{\partial Q_A}{\partial t} = [\Lambda, Q_A]. \tag{10}$$

The proof follows at once from relations (7).

Representation (10) is a commutation representation of Eq. (8) on the matrices of order n that depend polynomially on the parameter E (the Novikov representation; regarding the equivalence of the representation of Lax and that of Novikov see [1, Chap. 2]). For Eq. (8') it is necessary to take a sum of matrices of the form (9) as the matrix Λ.

LEMMA 3. For different B, N the dynamic systems (8) commute with each other.

Proof. It suffices to show that the operators of the form $\partial/\partial t + \Lambda$ commute with each other, i.e., if $\Lambda_1 = \Lambda_{N_1,B_1}$, $\Lambda_2 = \Lambda_{N_2,B_2}$ and t_1, t_2 are the corresponding times, then we have the relation

$$\frac{\partial \Lambda_1}{\partial t_2} - \frac{\partial \Lambda_2}{\partial t_1} = [\Lambda_1, \Lambda_2]. \tag{11}$$

For this, it suffices to show that if Λ has form (9) and V is a solution to Eq. (8), then $\lambda = \lambda_A$, $A \in \mathbf{A}$, is a solution of the following equation:

$$\frac{\partial \lambda}{\partial t} = [\lambda, \Lambda]. \tag{12}$$

This is a consequence of relations (5) and the explicit form of Eq. (8). In particular, if $B = A_i$, where $(A_i)_l^k = \delta_i^k \cdot \delta_l^i$, and N = 2, then we get a system that is an analog of the nonlinear Schrödinger equation. In the following we use the explicit equation

$$\dot{v}_j^i \equiv \partial v_j^i / \partial t_i = v_{j,ii}^i - (v_j^i v_i^j)\, v_j^i. \tag{12'}$$

Definition 2. V is said to be a finite-zone potential for the operator L_A if L_A has for all E a characteristic vector function ψ, $L_A\psi = 0$, that is meromorphic on a Riemann surface Γ of finite genus which covers the E plane with n sheets. The surface Γ is called the spectrum of operator L_A.

Below, it is shown that the finite-zone operators obtained within the framework of our construction are such that their coefficients are periodic (or conditionally periodic) and the corresponding characteristic functions are Bloch functions, i.e., under a shift by a period they are multiplied by a scalar (see [1]).

Our aim is to find the stationary (i.e., independent of the time t) solutions of an equation of form (8) [or (8')]

$$[A, \lambda_{N+1,B}] = 0, \quad B \in \mathbf{A}, \tag{13}$$

$$[A, \sum_{k \leqslant N} \lambda_{k+1,B_k}] = 0, \quad B_k \in \mathbf{A}, \quad k = 1, \ldots, N. \tag{13'}$$

Equations (13), (13') represent systems of partial differential equations that, using Eqs. (4), can be rewritten as systems of n(n − 1) ordinary differential equations of order N each. For the integration of system (13) we use its representation of Lax type [which follows from (10)] on matrices that are polynomially dependent on E,

$$\frac{\partial \Lambda}{\partial x_A} = [\Lambda, Q_A], \tag{14}$$

where the matrix Λ is defined by Eq. (9), and relation (14) holds for any matrix $A \in \mathbf{A}$.

LEMMA 4. If V is a solution of system (13) [or (13')], then operator L_A is finite-zone for any $A \in \mathbf{A}$.

Proof. We consider the Riemann surface Γ of the algebraic function $w = w(E)$ given by the equation

$$R(w, E) = \det | w \cdot 1 - \Lambda | = 0. \tag{15}$$

From Eq. (14) it follows that for a change in x_A the matrix Λ remains similar to itself, consequently, its characteristic polynomial $R(w, E)$ does not depend on x_A. We construct a characteristic function ψ of operator L_A, requiring that it be a characteristic vector for the matrix Λ. Such a function exists, since the operator L_A and the operator of multiplication by matrix Λ commute, by relation (14). Then (under a suitable normalization) its coordinates can be rationally expressed in terms of the elements of the matrix $w \cdot 1 - \Lambda$, i.e., they are algebraic functions on the Riemann surface Γ given by Eq. (15). Hence, ψ can be extended to a meromorphic function on the Riemann surface $\Gamma \setminus \infty$. The function ψ is characteristic for all the operators in the commutative algebra L.

The surface Γ covers the E plane with n sheets. We calculate its genus, considering for simplicity that B is a matrix in general position, i.e., that its diagonal elements are pairwise distinct: $B = (b_i \delta_i^j)$, $b_i \neq b_j$. Then the i-th root of the characteristic polynomial $R(w, E)$ has for large n the asymptotic form

$$w_i(E) = b_i E^N \left(1 + O\left(\frac{1}{E}\right)\right). \tag{16}$$

From this it follows, in particular, that as $E \to \infty$ the surface Γ has n ordered points; we denote them by $\{1\}, \ldots, \{n\}$. The discriminant of the polynomial $R(w, E)$ has the form $\Delta = \prod_{i \neq j} (w_i - w_j) = \delta \cdot E^{Nn(n-1)} +$ smaller terms, where $\delta = \prod_{i \neq j} (b_i - b_j)$. Hence, the surface Γ has $Nn(n-1)$ branching points, from which we get its genus p,

$$p = N \frac{n(n-1)}{2} - (n-1). \tag{17}$$

Let

$$R(w, E) = \sum_{i,j} r_{ij} w^i E^j. \tag{18}$$

Then from the definition of matrix Λ it follows that the coefficients r_{ij} are polynomials in the elements of matrices $V, V', \ldots, V^{(N-1)}$. By Lemma 4, these polynomials are integrals of system (13). Among these integrals are "trivial" ones. Let π be the group of all diagonal nonsingular matrices of order n. The group π acts on the matrices V as follows:

$$V \mapsto \varepsilon V \varepsilon^{-1}, \quad \varepsilon \in \pi. \tag{19}$$

From the definition of system (4) it follows that the group π acts correctly on the solutions of this system. Further, from the uniqueness of coefficients $\lambda_{i,B}$ defined by Eqs. (6) and (6') it follows that these coefficients are transformed according to the law

$$\lambda_{i,B} \to \varepsilon \lambda_{i,B} \varepsilon^{-1}. \tag{20}$$

Therefore, π is the group of symmetries of system (13) [or (13')]. This gives the $n - 1$ integral of this system. We show below that system (13) is Hamiltonian. If we show that integrals $r_{ij}(V, V', \ldots)$ of this system are independent and are in involution, then we get from this the complete integrability of this system. Instead of this, we explicitly describe the structure of the invariant varieties of this system. For each Riemann surface Γ of form (15)

$$\sum r_{ij}^0 w^i E^j = 0$$

we consider the invariant variety M_Γ given by the intersection of the levels of the integrals $r_{ij}(V, V', \ldots)$

$$M_\Gamma = \{V \mid r_{ij}(V, V', \ldots) = r_{ij}^0\}. \tag{21}$$

We describe the variety M_Γ for any Γ. By Lemma 4, a point on the variety M_Γ is a finite-zone operator L_A (for an arbitrary matrix $A \in \mathrm{A}$) with spectrum Γ. Therefore, the problem of describing the varieties M_Γ is equivalent to the problem of finding all the finite-zone operators L_A with given spectrum Γ.

THEOREM. $M_\Gamma/\pi = J(\Gamma)$. Here M_Γ/π is the factor space with respect to the action of the group π, $J(\Gamma)$ is the Jacobi variety of the Riemann surface Γ.

Proof. We investigate the analytic properties of the characteristic vector function constructed. We construct a matrix-valued function $\Psi(x, y, P)$, where P is a point of the surface Γ, and x (and y) is the variable x_A, where the matrix $A \in \mathbf{A}$ is regarded as fixed [see the remark after the Eq. (4')]. Let E be different from a branching point, i.e., for given E the pair of operators $L_A = L$, Λ has exactly n linearly independent characteristic functions $\psi_1(x, E), \ldots, \psi_n(x, E)$. We place their coordinates in the matrix $\psi_i^j(x, E)$. Let $\varphi_i^j(x, E)$ be the inverse matrix; it exists, since the functions $\psi_1, \ldots, \psi_n$ are linearly independent. Then, if $P \in \Gamma$, $P = (E, k)$, k being the number of the sheet, then we set

$$\Psi_i^j(x, y, P) = \psi_k^j(x, E)\cdot\varphi_i^k(y, E). \tag{22}$$

This definition depends neither on the initial order of the characteristic functions $\psi_1, \ldots, \psi_n$, nor on their normalization. Thus, the function $\Psi_i^j(x, y, P)$ is a single-valued function on the Riemann surface Γ. We note that if

$$G(x, y, E) = \begin{cases} \mathrm{Tr}_P\Psi(x, y, P), & x \leqslant y \\ 0, & x > y, \end{cases}$$

then the function $G(x, y, E)$ is Green's matrix of operator L_A. We set $\Psi(x, x, P) = g(x, P)$.

LEMMA 5. Function $g(x, P)$ has the following properties:

a) $g(x, P)$ gives the spectral decomposition of matrix $\Lambda(x, E)$, i.e., $g^2 = g$, $g(x, (E, k))\cdot g(x, (E, l)) = 0$ for $k \neq l$ (k and l are the numbers of the sheets), $\mathrm{Tr}_P\, g(x, P) = 1$, $\mathrm{Tr}_P\, w(P) g(x, P) = \Lambda(x, E)$.

b) The matrix elements of matrix $g(x, P)$ are algebraic functions on surface Γ, and their poles are located precisely at the branching points of Γ.

c) The group of periods of function $g(x, P)$ coincides with the group of periods of potential $V(x)$.

d) $g(x, P)$ satisfies the equation

$$\frac{\partial g}{\partial x_B} = [g, Q_B], \qquad B \in \mathbf{A}. \tag{23}$$

e) $g(x, P)$ has as $P \to \{k\}$ an expansion of the form

$$g(x, P) = g_0^{(k)} + \frac{g_1^{(k)}}{E} + \frac{g_2^{(k)}}{E^2} + \ldots, \tag{24}$$

where

$$g_0^{(k)} = A_k, \quad g_1^{(k)} = -[A_k, V], \quad g_2^{(k)} = V_{,k} - \sum_{i \neq k} v_k^i v_i^k A_i + \left(\sum_s v_s^k v_k^s\right) A_k. \tag{25}$$

Proof. Since ψ is a characteristic vector of matrix Λ, Eq. (22) is for $x = y$ the expression for the projections of matrix Λ that is known from linear algebra. From this it is easy to get an explicit equation expressing the matrix g in terms of the matrix Λ: if $R(w, E) = w^n + I_1(E)w^{n-1} + \ldots + I_n$, then

$$g(x, P) = \frac{w^{n-1} + a_1 w^{n-2} + \ldots + a_{n-1}}{\partial R(w, E)/\partial w}, \tag{26}$$

where $a_k = I_k + I_{k-1}\Lambda + \ldots + \Lambda^k$, $k = 1, \ldots, n-1$. From this equation we get the assertion of part b), since the denominator of $\partial R(w, E)/\partial w$ has zeros at the branching points of the surface Γ. The part c) also follows from the Eq. (26). Equation (23) follows from Eq. (22) and from the fact that the matrix φ appearing in Eq. (22) satisfies the equation $\partial\varphi/\partial x_A = -\varphi Q_A$. Expansion (24) and Eq. (25) are obtained recurrently from Eq. (23) analogously to expansions (6). More precisely,

$$g_i^{(k)} = \lambda_{i, A_k}. \tag{27}$$

These expansions converge, by part b).

Another proof of the existence of expansions (24) can be obtained by the general methods indicated in the survey [11] of Gel'fand and Dikii for obtaining asymptotic expansions for resolvents.

COROLLARY. Functions $g_i^j(x, P)$ have in the part of the surface Γ at infinity only zeros, and they are located as follows: for $i \neq j$, $g_i^j(x, P)$ has a double zero at the points $P = \{k\}, k \neq i, j$, and a simple zero for k = i or k = j; $g_i^i(x, P)$ has a double zero for $P = \{k\}$, $k \neq \{i\}$, and $g_i^i(x, \{i\}) \equiv 1$.

A proof of the corollary is obtained immediately from Eq. (25).

LEMMA 6. Functions $\Psi_i^j(x, y, P)$ have the following properties on the Riemann surface Γ:

a) $\Psi_i^j(x, y, P)$ is meromorphic on $\Gamma \setminus \infty$, and its poles lie at the branching points of Γ.

b) The divisor of the zeros of $\Psi_i^j(x, y, P)$ can be decomposed into the sum of two divisors $d_i(y) + d^j(x)$.

c) As $P \to \{k\}$, $\Psi_i^j(x, y, P)$ has an asymptotic behavior of the form $\Psi_i^j(x, y, P) \sim \theta_{ik}^j(x, y, P) \cdot \exp\{(x-y)a_k E\}$, where θ_{ik}^j is meromorphic in a neighborhood of point P = {k}.

Proof. Let $c(x, y, E) = (c_i^j(x, y, E))$ be the matrix solution of the equation $L_A c = 0$ with initial condition $c(y, y, E) = 1$ (y is a fixed parameter). The matrix elements $c_i^j(x, y, E)$ are entire functions with respect to E. Considering the uniqueness of the solution, we have the equation

$$\Psi(x, y, P) = c(x, y, E(P))\, g(y, P). \tag{28}$$

The part a) of Lemma 6 follows from Eq. (28) and the part b) of Lemma 5. The matrix $\Psi(x, y, P)$ has rank 1; its columns are the characteristic functions of the operator L_A, acting with respect to the variable x, and they differ only by normalization, while the rows are the characteristic functions of the adjoint operator L_A^*, acting with respect to y, where the adjoint operator L_A^* is defined as follows (+ denotes transposition):

$$L_A^* = \frac{\partial}{\partial y} - Q_A^+. \tag{29}$$

Therefore,

$$\frac{\Psi_i^k(x, y, P)}{\Psi_j^k(x, y, P)} = \frac{g_i^k(y, P)}{g_j^k(y, P)}, \quad \frac{\Psi_i^k(x, y, P)}{\Psi_j^k(x, y, P)} = \frac{g_i^k(y, P)}{g_j^k(y, P)} \quad \text{does not depend on k.} \tag{30}$$

Part b) of the lemma follows from (30). We prove part c). We have:

$$\frac{\partial}{\partial x} \ln \Psi_i^j(x, y, P) = \sum_s (a_j E \delta_s^j - (a_j - a_s) v_s^j(x)) \frac{g_i^s(x, P)}{g_i^j(x, P)}. \tag{31}$$

We introduce the notation

$$\chi^j(x, P) = a_j E - \sum_s (a_j - a_s) v_s^j(x) \frac{g_i^s(x, P)}{g_i^j(x, P)} \tag{32}$$

(it does not depend on i). Integrating Eq. (31) and using the initial condition $\Psi_i^j(y, y, P) = g_i^j(y, P)$, we have:

$$\Psi_i^j(x, y, P) = g_i^j(y, P) \exp\left\{\int_y^x \chi^j(\xi, P)\, d\xi\right\}. \tag{33}$$

From expansions (25) it follows that the function $\chi^j(x, P)$ has as $P \to \{k\}$ an asymptotic behavior of the form

$$\chi^j(x, P) = a_k E + O(1). \tag{34}$$

Part c) of Lemma 6 follows from relations (34) and (33).

COROLLARY. If V is periodic with respect to x with period T, then characteristic function ψ constructed is a Bloch function, i.e.,

$$\psi(x+T, P) = \exp[p(P)]\cdot\psi(x, P), \tag{35}$$

where

$$p(P) = \int_{y}^{y+T} \chi^{j}(\xi, P)\, d\xi. \tag{36}$$

Proof. From Eq. (32) and the part c) of Lemma 5 it follows that the periods of χ^j and the periods of V are the same. We prove that the function p(P) does not depend on j. This is obvious from relation (30), since the right-hand side does not change under a shift by a period.

Remark. Similarly to Eq. (33), considering the dependence on y, it is easy to get the equation

$$\Psi_i^j(x, y, P) = g_i^j(x, P)\exp\left\{\int_x^y \chi_i(\xi, P)\, d\xi\right\}, \tag{37}$$

where

$$\chi_i(\xi, P) = -a_i E + \sum_s \frac{g_s^j(\xi, P)}{g_i^j(\xi, P)}(a_s - a_i)\, v_i^s(\xi). \tag{38}$$

Equating the right-hand sides of Eqs. (33) and (37) and taking the logarithmic derivative with respect to x of the identity obtained, we get the useful identity

$$\frac{\partial}{\partial x}\ln g_i^j(x, P) = \chi_i(x, P) + \chi^j(x, P). \tag{39}$$

LEMMA 7. The variational derivative of functional p{V} defined by Eq. (36) has the form

$$\frac{\delta p(P)}{\delta V(x)} = [A, g(x, P)]. \tag{40}$$

Proof. Suppose that matrix A has zero trace. We consider the two potentials: V and $\tilde{V} = V + \delta V$, where only the element $(\delta V)_j^i$ is different from zero in the matrix δV. We take an arbitrary value E that is different from a branching point; let (E, 1), . . ., (E, n) be the ordered preimage of the point E on Γ. Let $\tilde{\psi}_1 = \tilde{\psi}(x, (E, 1))$ be a Bloch function for the potential $\tilde{V}$, $\psi_2 = \psi(x, (E, 2)), \ldots, \psi_n = \psi(x, (E, n))$ — Bloch functions for the potential V. We consider the matrix whose first column is the vector $\tilde{\psi}_1 = (\tilde{\psi}_1^j)$, while the remaining ones are the vectors $\psi_2, \ldots, \psi_n$. We denote by D(x) the determinant of this matrix. We have the obvious identity

$$\frac{d}{dx}D(x) = (a_j - a_i)\,\delta v_j^i(x)\,\tilde{\psi}_1^j\,\eta_1^i(x), \tag{41}$$

where η_1^i is the cofactor of the element $\tilde{\psi}_1^i$. Let $\tilde{p}_1 = \tilde{p}(E, 1)$ for the potential $\tilde{V}$ and p_i = p(E, i) (i = 1, . . ., n) for the potential V. From the condition Tr A = 0 we get the condition $\Sigma p_i = 0$ (the unimodularity of the translation matrix). Integrating Eq. (41) over a period, we have:

$$(e^{\tilde{p}_1 + p_2 + \cdots + p_n} - 1)\, D(x) = \int_x^{x+T} (a_j - a_i)\,\psi_1^j(\xi)\,\eta_1^i(\xi)\,\delta v_j^i(\xi)\, d\xi. \tag{42}$$

To first order in δv_j^i, we get from this that

$$\delta p_1 = \int_x^{x+T} (a_j - a_i)\,\psi_1^j(\xi)\,\varphi_i^1(\xi)\,\delta v_j^i(\xi)\, d\xi. \tag{43}$$

Equation (43), by definition (22), implies Eq. (40) on the first sheet of surface Γ.

COROLLARY 1. Equation (8) is Hamiltonian.

Proof. In Eq. (8) we take $B = A_k$, where A_k is a basis matrix, $(A_k)^i_j = \delta^i_k \cdot \delta^k_j$. We construct the Hamiltonian for this equation. The function $\chi^j(x, P)$ has as $P \to \{k\}$ an expansion of the form

$$\chi^j(x, P) = a_k E + Q_0^{(k)} + \frac{Q_1^{(k)}}{E} + \ldots, \tag{44}$$

where coefficients $Q_0^{(k)}$ and $Q_1^{(k)}$ are polynomials in $V, V', \ldots$ and are determined from Eq. (32), considering (24) (for different j the polynomials $Q_i^{(k)}$ differ by a total derivative, therefore, the index j is omitted). We define functional $I_{k,N}\{V\}$, setting

$$I_{k,N}\{V\} = \int_T Q_{N+1}^{(k)}\, dx. \tag{45}$$

From Eqs. (40) and (36) it then follows that

$$\frac{\delta I_{k,N}\{V\}}{\delta V(x)} = [A, g_{N+1}^{(k)}], \tag{46}$$

where $g_{N+1}^{(k)}$ is defined by Eq. (24). By Eq. (27), Eq. (8) is equivalent to the equation

$$\dot{V} = (\mathrm{ad}\, A)^{-1} \frac{\delta I_{k,N}}{\delta V}, \tag{47}$$

which implies the Hamiltonian property, by the skew symmetry of operator ad A.

COROLLARY 2. Equation (13) [or (13')] is Hamiltonian.

Proof. By (47), these equations are the equations for the extremals for some functional $I\{V\}$ that is a linear combination of the functionals $I_{k,N}\{V\}$, therefore, they are Hamiltonian.

We now return to the analytic properties of the characteristic functions. Let $\mathfrak{A}$ be the Abel mapping of the k-th symmetric power $S^k\Gamma$ of the surface Γ into its Jacobian $J(\Gamma)$:

$$\mathfrak{A}: S^k\Gamma \to J(\Gamma), \quad k = 1, 2, \ldots \tag{48}$$

The mapping $\mathfrak{A}$ is constructed as follows. Let α_i, β_j (i, j = 1, . . ., p) be the canonical basis of cycles on the Riemann surface Γ (of genus p) that have intersection indices of the form $\alpha_i \circ \alpha_j = \beta_i \circ \beta_j = 0$, $\alpha_i \circ \beta_j = \delta_{ij}$; $\omega_1, \ldots, \omega_p$ is a basis of differentials of the first kind on Γ, normalized by the conditions $\oint_{\alpha_k} \omega_l = 2\pi i \delta_{kl}$. Let $B_{kl} = \oint_{\beta_k} \omega_l$ be the matrix of periods, $P_0 \in \Gamma$ a fixed point. The Abel mapping is constructed as follows:

$$[\mathfrak{A}(P_1, \ldots, P_k)]_l = \sum_{s=1}^{k} \int_{P}^{P_s} \omega_l. \tag{48'}$$

$\mathfrak{A}$ is a birational isomorphism for k equal to the genus p of the surface Γ. Let D_w be the branching divisor on Γ, Σ the "divisor of infinity," $\Sigma = \sum_{i=1}^{n} \{i\}$.

LEMMA 8. a) The degrees of the divisors d_i and d^j (i, j = 1, . . ., n) are equal to the genus p of the surface Γ.

b) On the Jacobian variety $J(\Gamma)$ we have the relation

$$\mathfrak{A}(d_i(x)) + \mathfrak{A}(d^j(x)) = \mathfrak{A}(D_w) - \mathfrak{A}(2\Sigma - \{i\} - \{j\}). \tag{49}$$

Proof. The equality of the degrees of divisors $d_i(x)$ and $d^j(y)$ is obvious, by the equal standing of operators L_A and L_A^*. By Lemma 6 and Lemma 5, and also by the corollary of the latter, the divisor of the function $g_i^j(x, P)$ has the form

$$d_i(x) + d^j(x) + 2\Sigma - \{i\} - \{j\} - D_w. \tag{50}$$

The degree of the divisor of the poles D_W is equal to $Nn(n-1)$ (see above), and the degree of the divisor $2\Sigma - \{i\} - \{j\}$ is equal to $2(n-1)$. Since function $g_i^j(x, P)$ is algebraic on the surface Γ, the degree of divisor $d_i(x) + d^j(x)$ is equal to $Nn(n-1) - 2(n-1)$, which proves, by Eq. (17), the part a) of the lemma. The part b) follows from Eq. (50) and the classical theorem of Abel.

On the Jacobian variety $J(\Gamma)$ with coordinates $\eta_1, \ldots, \eta_p$

$$\eta(x) = \mathfrak{A}(d^j(x)). \tag{51}$$

LEMMA 9. $\eta(x)$ is a rectilinear winding of torus $J(\Gamma)$, i.e.,

$$\eta(x) = \eta(y) + (x-y)\mathbf{Z}, \tag{52}$$

where $\mathbf{Z}$ is a constant vector.

Proof. We consider the function $\widetilde{\Psi}^j(x, y, P)$, defined by the equation

$$\widetilde{\Psi}^j(x, y, P) = \frac{\Psi_i^j(x, y, P)}{g_i^j(y, P)} \tag{53}$$

(which does not depend on i). According to (33), we have:

$$\widetilde{\Psi}^j(x, y, P) = \exp\int_y^x \chi^j(\xi, P)\, d\xi. \tag{54}$$

From Eqs. (25) and (32) it follows that if $P \to \{j\}$, then $\chi^j(\xi, P) = a_j E + O(1/E)$. Therefore, as $P \to \{j\}$ we have

$$\widetilde{\Psi}^j(x, y, P) \sim \exp[a_j E(x-y)]\left(1 + O\left(\frac{1}{E}\right)\right), \tag{55}$$

and as $P \to \{k\},\ k \neq j$,

$$\widetilde{\Psi}^j(x, y, P) \sim \frac{v_k^j(x)}{v_k^j(y)} \exp[a_k E(x-y)]\left(1 + O\left(\frac{1}{E}\right)\right). \tag{56}$$

Moreover, from Eq. (53) and Lemma 6, the divisor of the function $\widetilde{\Psi}^j(x, y, P)$ has the form $d^j(x) - d^j(y)$. From arguments of the type used in a lemma of Akhiezer (see [12] and [1, Chap. 2, §3]) it follows that the zeros of $d^j(x)$ of the function $\widetilde{\Psi}^j(x, y, P)$ are determined according to the poles of $d^j(y)$ from Eq. (52), where the vector $\mathbf{Z}$ is defined as follows. Let Ω_k be the Abelian differential (of the second kind) on the surface Γ having a unique double pole for $P = \{k\}$ and such that as $P \to \{k\}$

$$\Omega_k = -(dz/z^2) + O(z), \tag{57}$$

normalized by the conditions

$$\oint_{\alpha_j} \Omega_k, \quad j = 1, \ldots, p. \tag{58}$$

Let the vector Z_k have the coordinates Z_k^i, where

$$Z_k^j = \oint_{\beta_j} \Omega_k. \tag{59}$$

Then the vector $\mathbf{Z}$ has the coordinates $Z^j = \sum_{k=1}^n a_k Z_k^j$.

COROLLARY. The configuration of the divisors $d_i(x)$, $d^j(x)$ for any x is determined to within a linear equivalence by the specification of one of them (e.g., d^1) for $x = x_0$.

We return to the proof of the theorem. We assign to a solution $V(x)$ of Eq. (13) [or (13')] a point η on the Jacobi variety $J(\Gamma)$, taking

$$\eta = \mathfrak{A}(d^1(x_0)). \tag{60}$$

According to the corollary of Lemma 9, we can construct the whole configuration $\mathfrak{A}(d_i(x))$, $\mathfrak{A}(d^j(x))$ for any x, i.e., we can find the location of the zeros of the function $\Psi_i^j(x, y, P)$ for any x, y, i, j. From Lemma 9 it follows (see Eqs. (55), (56) and below) that the functions $\widetilde{\Psi}^j$ are uniquely determined by specifying η. The functions $\Psi_i^j(x, y, P)$, hence also $g_i^j(x, P)$, can be constructed for this data to within a multiple that does not depend on P, i.e., for given η it is possible to construct a function $\hat{g}_i^j(x, P)$ such that

$$\hat{g}_i^j(x, P) = \varepsilon_i^j(x)\, g_i^j(x, P). \tag{61}$$

Matrix $\hat{g}_i^j(x, P)$ must satisfy the conditions of Lemma 5. From relation $\hat{g}^2 = \hat{g}$ we get that $\varepsilon_i^j = \varepsilon_j/\varepsilon_i$. Since $\hat{g}$ must satisfy an equation of form (23), ε_i does not depend on x, i.e., $\hat{g} = \varepsilon g \varepsilon^{-1}$, where $\varepsilon = (\varepsilon_i \delta_i^j) \in \pi$. We show that the corresponding potentials V also differ. Indeed, this follows from Eqs. (25), since the matrix elements of V are principal terms of the expansion of g_i^j at infinity. Thus, by giving a point on J(Γ) the potential V is regenerated to within the action of the group π, i.e., the mapping $M_\Gamma/\pi \to J(\Gamma)$ is injective. For the proof of surjectivity we can use the explicit formulas for the potential obtained below. It is simpler to use dimensional arguments: Eq. (13) is a Hamiltonian system with $Nn(n-1)/2$ degrees of freedom. The integrals r_{ij} [Eq. (18)] are independent, since dim $M_\Gamma \leqslant N\frac{n(n-1)}{2}$ from the injectivity proved above. Hence, $\dim M_\Gamma = N\frac{n(n-1)}{2}$ and dim $M_\Gamma/\pi = N\frac{n(n-1)}{2} - (n-1) =$ the genus of the surface Γ = dim J(Γ).

By commutativity (Lemma 3), the variety M_Γ is invariant for dynamic systems of form (8). We calculate the trajectories of these systems on $M_\Gamma/\pi = J(\Gamma)$ (we recall that the dynamic systems of form (8) commute with the action of the group π).

LEMMA 10. The trajectories of dynamic system (8) on the torus J(Γ) are rectilinear windings, i.e.,

$$\eta(\tau) = \eta(\sigma) + (\tau - \sigma)W, \tag{62}$$

W being a constant vector.

Proof. For a dynamic system of form (8) we consider the corresponding operator

$$M = \frac{\partial}{\partial \tau} + \hat{\Lambda}, \quad \hat{\Lambda} = \Lambda_{\hat{N}, \hat{B}}. \tag{63}$$

From the commutation condition (11) we have:

$$\frac{\partial \Lambda}{\partial \tau} = [\Lambda, \hat{\Lambda}], \quad \text{i.e.,} \quad [M, \Lambda] = 0. \tag{64}$$

Analogously, we have

$$[M, L_A] = 0. \tag{65}$$

By relations (64), (65), M is a finite-zone operator with the same spectrum Γ; its characteristic function has the form $\Psi_i^j(\tau, \sigma, x, y, P)$. Let x = y (in the following we omit the dependence on x = y). For $\tau = \sigma$ we get the function $g = g(\tau, P)$. The assertions of Lemma 5, Lemma 6 (except for part c)), and Lemma 8 for the functions $\Psi(\tau, \sigma, P)$ and $g(\tau, P)$ are true automatically. We investigate the asymptotic behavior of functions $\Psi_i^j(\tau, \sigma, P)$. Let $\hat{\Lambda} = (\hat{\lambda}_i^j)$. Then we have an equation analogous to (32) and (33),

$$\Psi_i^j(\tau, \sigma, P) = g_i^j(\sigma, P) \exp \int_\sigma^\tau \zeta^j(\xi, P)\, d\xi, \tag{66}$$

where

$$\zeta^j(\xi, P) = -\sum_s \hat{\lambda}_s^j(\xi, E) \frac{g_i^s(\xi, P)}{g_i^j(\xi, P)}.$$

As $P \to \{k\}$ the function $\zeta^j(\tau, P)$ has asymptotic behavior of the form

$$\zeta^j(\tau, P) = -\hat{b}_k E^{\hat{N}} + O(1). \tag{67}$$

The function $\hat{\Psi}^j(\tau,\sigma,P)=\frac{\Psi_i^j(\tau,\sigma,P)}{g_i^j(\sigma,P)}$ has asymptotic behavior as P → {j} of the form

$$\hat{\Psi}^j(\tau,\sigma,P)=\exp[\hat{b}_j E^{\hat{N}}(\sigma-\tau)]\left(1+O\left(1/E\right)\right), \tag{68}$$

and, as $P\to\{k\}$, $j\neq k$,

$$\hat{\Psi}^j(\tau,\sigma,P)=\frac{v_k^j(\tau)}{v_k^j(\sigma)}\exp[\hat{b}_k E^{\hat{N}}(\sigma-\tau)]\left(1+O\left(1/E\right)\right). \tag{69}$$

From this, as in Lemma 9, we get Eq. (62). The vector W is computed as follows: let $\Omega_{k,\hat{N}}$ be the Abel differential (of the second kind) with unique pole of order $\hat{N}+1$ at the point P = {k}, and in a neighborhood of it

$$\Omega_{k,\hat{N}}=-\hat{N}\frac{dz}{z^{\hat{N}+1}}+O(z) \tag{70}$$

(z is a local parameter), normalized by the conditions

$$\oint_{\alpha_j}\Omega_{k,\hat{N}}=0,\quad j=1,\ldots,p. \tag{71}$$

Suppose that the vector $\mathbf{W}_k$ has coordinates W_k^l, where

$$W_k^l=\oint_{\beta_l}\Omega_{k,\hat{N}},\quad l=1,\ldots,p. \tag{72}$$

Then $\mathbf{W}=\sum_{k=1}^{n}\hat{b}_k\mathbf{W}_k$. Lemma 10 makes it possible to construct convenient coordinates on the Jacobian variety J(Γ).

We derive formulas giving an explicit expression for the potential V in terms of θ-functions. Let η be a given point on J(Γ). We introduce the notation:

$$\mathfrak{A}(D_w)=\hat{w},\quad \mathfrak{A}\{j\}=[j],\quad \mathfrak{A}(\Sigma)=\sigma, \tag{73}$$

$$\eta_j=-\eta+\hat{w}-2\sigma+[1]+[j],\quad \eta^i=\eta+[i]-[1], \tag{74}$$

$$\bar{\theta}(\eta)=\theta(\eta-\mathbf{K}),\quad \frac{\partial\bar{\theta}(\eta)}{\partial x_p}=\sum_l Z_p^l\bar{\theta}_{,l}(\eta), \tag{75}$$

$$\frac{\partial\bar{\theta}(\eta)}{\partial t_p}=\sum_l W_p^l\bar{\theta}_{,l}(\eta), \tag{76}$$

where the vector $\mathbf{W}_p$ is determined by Eq. (72) for $\hat{N}=2$; θ is a Riemann function (see [13]), and K is the vector of Riemann constants

$$K_j=\frac{1}{2}-\frac{1}{2}B_{jj}+\sum_{\substack{k=1\\k\neq j}}^{p}\oint_{\alpha_k}\left(\int_{P_0}^{P}\omega_j\right)\omega_k(P). \tag{77}$$

Let

$$\xi_k^{(i)}=\begin{cases}\int_{P_0}^{\{i\}}\Omega_k, & k\neq i,\\ \lim_{P\to\{i\}}\left(\int_{P_0}^{P}\Omega_k-E\right), & k=i,\end{cases} \tag{78}$$

where Ω_k is determined by conditions (57), (58), and $P_0\in\Gamma$ is the reference origin appearing in the definition of the Abel transform (48'). Further, let

$$\mu_k^{(i)} = \begin{cases} \int\limits_{P_0}^{\{i\}} \Omega_{k,2}, & k \neq i, \\ \lim\limits_{P\to\{k\}} \left(\int\limits_{P_0}^{P} \Omega_{k,2} - E^2\right), & k = i. \end{cases} \tag{79}$$

Then for the dependence of the potential V on the variables $x_1, \ldots, x_n$ we have the following expression (where $x_{1_0} = \ldots = x_{n_0} = 0$):

$$v_j^i(x_1, \ldots, x_n) = \omega_j^i \prod_k \exp\left[(\xi_k^{(j)} - \xi_k^{(i)}) x_k\right] \frac{\bar\theta(x_k \mathbf{Z}_k + \eta_j + [j])}{\bar\theta(x_k \mathbf{Z}_k + \eta^i + [i])}, \tag{80}$$

where ω_j^i are constants that are independent of x and are connected by the relations

$$\frac{\omega_p^i \omega_j^p}{\omega_j^i} = \frac{\partial}{\partial x_p}\left\{\ln \frac{\bar\theta(\eta_j + [j])}{\bar\theta(\eta^i + [i])}\right\} \cdot \left[\frac{\bar\theta(\eta^p + [p])}{\bar\theta(\eta_p + [p])}\right]^n, \quad p \neq i, j, \tag{81}$$

$$\omega_j^i \omega_i^j = \left[\frac{\bar\theta([i] + \eta^i)\,\bar\theta([j] + \eta^j)}{\bar\theta([i] + \eta_i)\,\bar\theta([j] + \eta_j)}\right]^n \times$$

$$\times \left\{-\frac{\partial}{\partial t_p} \ln \frac{\bar\theta(\eta_j + [j])}{\bar\theta(\eta^i + [i])} + \mu_i^{(i)} - \mu_i^{(j)} + \left[2(\xi_i^{(j)} - \xi_i^{(i)}) + \frac{\partial}{\partial x_i} \ln \frac{\bar\theta([j] + \eta_j)}{\bar\theta([i] + \eta^i)}\right]^2 + \frac{\partial^2}{\partial x_i^2} \ln \frac{\bar\theta([j] + \eta_j)}{\bar\theta([i] + \eta^i)}\right\}. \tag{82}$$

<u>Proof.</u> From identity (39) and expansions (25) it follows that as $P \to \{i\}$

$$\chi_j(x, P) + \chi^i(x, P) = \left(\frac{v_i^{j'}}{v_i^j}\right) + O\left(\frac{1}{E}\right). \tag{83}$$

On the other hand, from Eq. (54) it follows that

$$\chi^i(x, P) = \frac{\partial}{\partial x} \ln \tilde\Psi^i(x, y, P). \tag{84}$$

Similarly,

$$\chi_j(y, P) = \frac{\partial}{\partial y} \ln \underset{\sim}{\Psi}_j(x, y, P), \tag{84'}$$

where the function $\underset{\sim}{\Psi}_j(x, y, P) = \Psi_j^i(x, y, P)/g_j^i(x, P)$ has divisor $d_j(y) - d_j(x)$ and exponential asymptotic behavior at infinity. The functions $\tilde\Psi^i$ and $\underset{\sim}{\Psi}_j$ are determined uniquely with regard to normalization by giving a point η on $J(\Gamma)$ (see the proof of the theorem). Their expression in terms of θ-functions is as follows:

$$\tilde\Psi^i(x_k, 0, P) = \exp\left[x_k\left(\int\limits_{P_0}^{P} \Omega_k - \xi_k^{(i)}\right)\right] \frac{\bar\theta(\mathfrak{A}(P) + x_k \mathbf{Z}_k + \eta^i)\,\bar\theta([i] + \eta^i)}{\bar\theta(\mathfrak{A}(p) + \eta^i)\,\bar\theta([i] + x_k \mathbf{Z}_k + \eta^i)}, \tag{85}$$

$$\underset{\sim}{\Psi}_j(0, x_k, P) = \exp\left[-x_k\left(\int\limits_{P_0}^{P} \Omega_k - \xi_k^{(j)}\right)\right] \frac{\bar\theta(\mathfrak{A}(P) + \eta_j)\,\bar\theta([j] + \eta_j + x_k \mathbf{Z}_k)}{\bar\theta(\mathfrak{A}(P) + \eta_j + x_k \mathbf{Z}_k)\,\bar\theta([j] + \eta_j)}, \tag{85'}$$

where $k = 1, \ldots, n$. Considering (83), (84), and (84'), we get Eq. (80) from the Eqs. (85) and (85'). Equation (81) follows from Eqs. (4') and Eq. (80). Equation (82) follows from Eq. (12').

The inclusion of the temporal dynamics is carried out from Lemma 10 with the aid of the substitution $\eta \mapsto \eta + t \cdot \mathbf{W}$, where the vector W is defined by Eq. (72).

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STABILITY OF EQUILIBRIUM IN A POTENTIAL FIELD

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We consider the motion described by the equation

$$\ddot{x}(t) = -u'(x(t)), \quad x \in \mathbf{R}^n, \tag{1}$$

where $u' = \operatorname{grad} u$, in a neighborhood of the critical point $x = 0$ of the potential function. Position $x = 0$ is stable if this point is a strict local minimum of the function u. We investigate the opposite case: point $x = 0$ is not even a nonstrict minimum, i.e., set $U^- = \{x\colon u(x) < u(0)\}$ is not empty and its closure contains this point.* If we assume only infinite differentiability of the potential, then stability is possible, as is shown by the following example (Painlevé): $u(x) = \exp(-|x|^{-1}) \sin(|x|^{-1})$. Here U^- is a sequence of spherical shells with radii converging to zero.

We show that under quite weak restrictions on the potential function it is possible to guarantee instability of the equilibrium and to estimate the rate of deviation of the trajectory in dependence on the magnitude of an initial perturbation (Theorem 2).† In the case of planar motion ($n = 2$) these restrictions can be made minimal.

THEOREM 1. Let $n = 2$, and suppose that point 0 is not a nonstrict minimum of function u. Position $x = 0$ is unstable if one of the following conditions holds:

A) function $u(x_1, x_2)$ can be expanded in a convergent power series in x_1 and x_2 in a neighborhood of zero;

*The author thanks V. I. Arnol'd and N. N. Kolesnikov, who called his attention to this problem.
†Apparently, this problem was first considered by Appel (Rational Mechanics, Vol. 2). Chetaev proved instability for the case of a homogeneous potential function [7].

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