Ground states of a two-dimensional electron in a periodic magnetic field

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The two-dimensional Pauli Hamiltonian for an electron with spin 1/2 in a transverse magnetic field has the following property: addition of any doubly periodic (but not small) increment to the homogeneous field leaves the ground state, i.e., the lower Landau level, fully degenerate, despite the loss of symmetry, and is separated from the next level by a finite gap (all the remaining levels spread out to form a continuous spectrum). For an integral or rational flux

\[ \phi = \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{5\pi}{2}, \ldots \]

the aggregate of the magnetic Bloch functions of the ground state can be explicitly obtained in terms of elliptic functions, and forms an \( N \)-dimensional linear space if the quasimomenta are fixed. The dependence on the quasimomenta forms a topologically nontrivial object (vector bundle over a torus), from whose structure it is possible to determine the possible decays into magnetic bands under small potential (periodic) perturbations.

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1. INTRODUCTION

The great difficulties encountered when attempts are made to describe the states of an electron in a periodic external field, when another magnetic field (also periodic or uniform) is present in addition to the periodic potential, are well known. In the presence of a magnetic field, the discrete symmetry group of the external field, consisting of shifts by the vectors of the main translations of the lattice, generates in general a noncommutative symmetry group of the Hamiltonian, since the translations are accompanied by a "gauge" winding of the \( \phi \) functions by a phase factor. The basis translations commute only when the magnetic-field flux through all the two-dimensional elementary cells has an integer number of quanta. In this case (or in a rational one that reduces to an whole-number enlargement of the lattice) it is possible to define a magnetic analog of the Bloch functions that are eigenfunctions for the basis magnetic translations. Even in this case, however (see below), the topology of the family of the Bloch functions turns out to be much more complicated than in a purely potential field. In the case of an irrational number of flux quanta there are many other difficulties, which will be dealt with in the text. In particular, it is difficult to define spectral characteristics that would vary continuously with the field.

This paper is based on the use of an analytic observation that arises in instanton theory, and some fragments of which date back to the paper of Atiyah and Singer. We started from an interpretation of this observation, given by Aharonov and Casher\(^3\) in the language of the ground state of an electron in a localized field on a plane. It was found possible to apply these premises to the important case of a periodic field, i.e., to find all the ground states for a two-dimensional Pauli operator in a periodic magnetic field directed along the \( z \) axis; the number of ground states turned out to be "the same" as in a uniform field with the same flux. This means that the degeneracy of the ground state is not lifted by the periodic magnetic field, despite the loss of translational symmetry. The curious topological structure of the family of Bloch functions (in the case of an integer or rational number of flux quanta) makes it possible to investigate the decay of the ground state into magnetic band when a periodic potential is turned on.

We consider the nonrelativistic Pauli Hamiltonian with spin 1/2 for a two-dimensional case, when the magnetic field, which depends only on the variables \( x \) and \( y \), is doubly periodic and directed along the \( z \) axis:

\[ H = \frac{1}{2} \left[ \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2 + \frac{1}{c^2} \left( \frac{\partial}{\partial T} \right)^2 + \frac{1}{c^2} \left( \frac{\partial}{\partial \phi} \right)^2 \right] + \frac{1}{c^2} \left( \frac{\partial}{\partial T} \right)^2 + \frac{1}{c^2} \left( \frac{\partial}{\partial \phi} \right)^2 \]

\[ + \frac{1}{2} \left( \frac{\partial}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial}{\partial y} \right)^2 + \left( \frac{\partial}{\partial T} \right)^2 + \left( \frac{\partial}{\partial \phi} \right)^2 \]

The Pauli operator is

\[ \psi = (\psi_1, \psi_2) \]

and commutes with \( \phi \). This operator reduces to a pair of scalar operators \( \hat{H}_x \) on functions of the type \( \eta \) by

\[ \eta = (\psi_1, \psi_2) \]

where \( \eta = 1 \). The magnetic flux

\[ \phi = \oint \frac{\partial}{\partial y} \psi \; dx dy \]

is the principal "topological" characteristic. To be sure, a topological interpretation of the flux (in a characteristic class) is meaningful only for whole-number fluxes. There are no known reasonable characteristics of the spectrum of the operator \( H \) (i.e., characteristics that vary regularly with the value of the field \( \phi \)) if \( \phi/2\pi \) is an irrational number.

To study the ground states of an electron in a periodic field \( B \), we use the observations of Aharonov and Casher, who considered the ground states for the Pauli operator in a finite field concentrated in a finite region of the \( x, y \) plane. The operator \( H_0 \) is of the
The condition they exist) we obtain in fact the ground states. From the condition $(H_s, \phi) = 0$ it follows that $(A_s^* \phi, A_s^* \psi) = 0$, or $A_s^* \psi = 0$. The order of the equation is lowered in analogy with the situation in instanton theory.

Next, a state with $A_s^* \psi = 0$ is possible only under the condition $\epsilon \phi > 0$. Let $\epsilon \phi > 0$ and $n = 1$. We make the substitution:

$$\psi(x, y) = \exp(-\epsilon \phi(x, y)) \phi(x, y).$$

From the condition of quadratic integrability of $\epsilon$, together with the easily calculated asymptotic form of $\phi$ for a finite field $B$, it follows that $\phi$ is an analytic (and entire) function and has the properties of degree

$$A=(a_1, a_2, \ldots, a_k).$$

At integer $N = 2 \pi \phi$ the complete basis of the ground states is obtained in the form of "magnetic-Bloch" functions—the eigenvectors of the magnetic translations (10) with eigenvectors of unity modulus:

$$T_\tau \Phi = \exp(2i\pi \tau \phi) \Phi.$$

Starting from the postulate (8), we obtain by simple calculation

$$\Gamma_s \Phi = \Phi \exp \left( -i \omega_0 n \phi^2 \right) \exp \left( 2i \pi \phi (\delta \tau_2 / 2) \right) \exp \left( \delta \pi \omega_0 \phi \right) \exp \left( i \pi \delta \phi \right).$$

We choose the solution of Eq. (2) in the form

$$\phi = (2\pi)^{-1} \int \delta(x-\xi) \delta(y-\xi') \phi' \; d\xi' \; d\xi,$$

where $K$ is the unit cell and its area is $|K| = T_1 T_2$. The ground states $R_s = 0$, as will be explained later, must be sought for any integer value of the magnetic flux $(2\pi)^{-1} \phi = N$ in the form

$$\phi = \sum_{a=0}^{N-1} \phi_a \exp(ia\theta),$$

where the conditions on the constants $A = (a_1, a_2, \ldots, a_k)$ will be indicated later (see (14) below). The need for choosing the analytic function $\phi$ in the form of a product of $a$ functions and an exponential will also be made clear later.

When shifted by the basis periods of the lattice $T_1$ and $T_2$

$$(x, y) \rightarrow (x+T_1, y), \quad (x, y) \rightarrow (x+T_2),$$

the function $\phi$ acquires an increment

$$\phi(x, y) \rightarrow \phi(x+T_1, y) \exp[-i \pi \phi(x, y)].$$

In the case $T_1 T_2 = \pi$ and even that in a quite indirect manner, by going over to a sphere instead of a plane. This transition alters the entire spectrum, and preserves only the exact solutions for the ground states—the zero modes.

It is of interest to ascertain whether the spectrum of the operator $H$ on a plane (for a finite field $B$) has a gap that separates the ground states from the remaining ones. Of course, this question can have, even for the whole-number flux, different answers for a plane $R^2$ and for a sphere $S^2$, where the spectrum is always discrete.30

2. GROUND STATES IN A PERIODIC FIELD

We recall first the elliptic functions we need. Let for simplicity the lattice be rectangular and let the lattice points be given by $x_m = m T_1 + i n T_2$. The so-called $\phi$ function is given in the form of the infinite product

$$\phi(z) = \prod_{m,n} \left(1 - \frac{z}{x_m y_n} \right) \exp \left( \frac{z}{x_m y_n} \right) \right)$$

and has the translational properties

$$\phi(z+T_1) = \phi(z),$$

$$(z+T_1, z+T_2),$$

$$\phi(-z) = \phi(z),$$

$$\phi(z) = \phi(-z).$$

From the condition $(2\pi)^{-1} \phi = N$ it follows by virtue of (13) that all the functions (8) are eigenfunctions for both magnetic translations $T_1$ and $T_2$. The requirement that the eigenvalues of $T_1$ and $T_2$ have a unity modulus imposes the following conditions on the constants:

$$\Delta = (a_1, a_2, \ldots, a_k).$$

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At fixed \( L \) of the type \( \alpha \), form an \( N \)-dimensional linear complex space. We obtain by this procedure all the Bloch functions from one fixed \( \alpha \). Then the product \( \phi = \phi_1 \phi_2 \) of them.

At fixed \( \alpha_1 \) and \( \alpha_2 \) we obtain an \( N \)-dimensional family of magnetic-Bloch eigenfunctions \( \phi_{\alpha_1 \alpha_2} = \text{const} \), where \( N = (2\pi)^N \). This family is obtained from one function \( \phi \) with parameters \( a = (a_1, a_2, \ldots, a_N) \) in the following manner: we take any elliptic (doubly periodic) magnetic-Bloch eigenfunction \( \phi(x) \) with poles in part of the points \( a_1, \ldots, a_N \) such that the product \( \phi \alpha \) has no poles. Then the product \( \phi \alpha \) is again a magnetic-Bloch function of the type \( \phi \), accurate to a constant factor, with the same quasi-momenta \( \alpha_1 \) and \( \alpha_2 \). These functions \( \chi \) form an \( N \)-dimensional linear complex space. We obtain by this procedure all the Bloch functions from one of them.

The family of Bloch functions \( \phi_{\alpha_1 \alpha_2} \) at arbitrary \( \lambda \) and fixed \( \alpha_1 \) and \( \alpha_2 \) is thus a linear \( N \)-dimensional complex space designated \( \mathcal{C}^N(\alpha_1, \alpha_2) \). Simple mathematical considerations show that under the conditions (14) the family \( \phi \) is the complete basis of the solution \( H \psi = 0 \) at the flux \( N = (2\pi)^N \).

3. SOME CONCLUSIONS

1. If the magnetic flux is rational, \( (2\pi)^N \psi_0 = N \psi_0 \), where \( N \) and \( M \) are integer mutually prime numbers, then enlargement of the lattice permits reduction of the problem to an integer flux, by increasing the dimension of the unit cell by \( M \) times, \( K = M \mathbf{K} \) (nothing is further dependent on further enlargement). For the cell \( M \mathbf{K} \), the flux is equal to \( 2\pi N/M \). By the same token we obtain the complete basis of ground states \( \psi_0 \) in the space of functions on the \( x, y \) plane. We take large integers \( N \) and \( M \) (such that the product \( N \mathbf{K} \) is divisible by \( M \) and impose on the state periodic boundary conditions with periods \( T_1 \) in \( x \) and \( T_2 \) in \( y \) [the periodicity is understood to be relative to the magnetic translations (10)]. We obtain a certain number \( D \) of states, proportional to \( e^{2\pi i \mathbf{K} \cdot \mathbf{L}} \), in analogy with the case of constant \( B \) (see Ref. 7). Dividing this number by the area of the large cell \( T_1 T_2 |K| \), we conclude that the number of ground states per unit area is proportional to the magnetic flux and is equal to \( (2\pi |K|)^{-1} \), where \( |K| \) is the area of the unit cell of the initial lattice with periods \( T_1 \) and \( T_2 \). We see that this number (at least defined only for a rational flux ratio) varies continuously with the field and runs through all real values of the flux.

2. Comparison with the case of a uniform field \( B = B_0 \) shows that in an alternating field there are at the zero level "just as many" (see (17) and (18) below) eigenfunctions \( \phi_{\alpha_1 \alpha_2} \) as in the uniform field. This circumstance is quite unexpected. The point is that the group of magnetic translations (10) in a uniform magnetic field \( B_0 \) is a continuous group (isomorphic to the known nilpotent Heisenberg-Weyl group); each Landau level realizes an irreducible infinite representation of this continuous group.

The spectrum is therefore in fact "discrete": the Heisenberg-Weyl group has only one infinite-dimensional irreducible representation that does not reduce to a commutative group if the representation of the center is specified as multiplication by a nonzero number fixed by the value of the magnetic flux through a unit area.

On going to a periodic field \( B = B_0 + M \mathbf{B} \), where \( M = \psi_0 \), we lose the continuous Heisenberg-Weyl symmetry. We are left with only the discrete subgroup \( \Gamma \) of the magnetic translations (10), which conserves the Hamiltonian \( H \). The restriction of the irreducible representation with a continuous group to the subgroup \( \Gamma ") already ceases to be irreducible. Therefore even a small periodic perturbation would broaden the Landau level into a magnetic band in accord with some expansion of the Landau level in the irreducible representations of the group \( \Gamma ") next, in the case of an irrational flux through the unit cell, certain difficulties would arise, since the expansion of the Landau level in irreducible representations of the discrete group \( \Gamma ") is not unique, and these expansions are difficult to classify; in the rational case it is impossible to find good spectral characteristics that vary regularly together with the field \( B \).

In our case, in the absence of an electric field, despite the loss of continuous symmetry, no broadening of the level occurs. We are left with energy degeneracy that is not connected with any irreducible representations of the group \( \Gamma ") . It is clear that there is no such degeneracy at the higher energy levels.

3. The Hilbert space \( \mathcal{H} \) spans the states \( \psi_0 \), in analogy with the "discrete" spectrum; in the complete Hilbert space \( \mathcal{H} \) all the quadratically integrable functions \( (x, y) \), the subspace \( \mathcal{H}_0 \) is singled out by the direct term \( \mathcal{H} = \mathcal{H}_0 \), in analogy with the expansion in the direct sum of the subspaces of the Landau levels for the case of a uniform field \( B = B_0 \). Accurately speaking this means, in particular, that in the space \( \mathcal{H}_0 \) of the ground states it is possible to choose not a Bloch basis but a quadratically integrable one (the "Wannier basis"); the integral is taken over the unit cell of the reciprocal lattice, and all the pair differences \( a - a' \) are (fixed):

\[
W(x, y) = \int e^{2\pi i \mathbf{k} \cdot \mathbf{r}_j} \psi_0 \delta \Phi
\]

where \( \delta \Phi \) is given by (8), and

\[
\mathbf{p} + \mathbf{p}_0 = 2\pi \mathbf{T} + \mathbf{r}_j + \sum_{r} \phi_i + \text{const}
\]
Here \( W(x,y) \) is an \( N \)-dimensional space. All the possible magnetic translations of the space of the functions \( W(x,y) \) yield a complete basis at the zero level, where the shifted spaces \( W_{m,n} \) are pairwise orthogonal:

\[
W_{m,n}, (x',y') = (T_f^{-n})(T_x^{-m}) W(x,y),
\]

\[
(W_{m,n}, W_{m',n'}) = 0, \text{ if } (m,n) \neq (m',n').
\]

For finite periodic perturbations \( \Delta B \) of the uniform magnetic field \( B_0 \), we can rigorously prove that the zero level remains separated from the next level by a finite barrier. This result has been established for any rational field. It is therefore valid also for an irrational one. The degeneracy follows from the results of Sec. 1. We have found by direct calculations the finite perturbation of the basis set of functions of the Landau ground state when a periodic magnetic field is added to the uniform one (without change of the flux); the perturbed functions have the same energy. We have so far not proved rigorously that for sufficiently large perturbations \( \Delta B \) of the uniform field \( B_0 \) the next level cannot drop to the zero level and touch it from above; it appears, however, that in a certain sense this is "almost" always the case.

4. As follows obviously from the conclusion, the Pauli operator can be replaced by a zero-spin Schrödinger operator, where the potential \( v(x,y) \) is proportional to the magnetic field: \( v = evB \). All the results are then preserved. If we add a purely potential (albeit small) periodic increment \( v(x,y) \), disturbing the connection between \( x \) and \( B \), the degeneracy is lifted: for an integer flux \( 2\pi N^2 \), \( N \) is a rational number (for a rational one \( N^2 M \) reduces to a whole-number enlargement of the lattice) the space of the ground states spreads to form a single magnetic band or several magnetic bands (for details see Sec. 4 below).

The result is a dispersion law \( \varepsilon = \varepsilon(p_1, p_2) \), where \( p_1 \) and \( p_2 \) are the quasimomenta (for the enlarged lattice in the rational case) and \( i=1,2, \ldots, N \).

5. The set of Bloch functions \( \psi \) forms a complex "vector bundle" (i.e., a linear space that depends on the parameters: see Ref. 4) with layer \( C^\infty(p_1, p_2) \), where the base \( \{i, \ldots, N\} \) is a two-dimensional torus defined as the factor of the \( p \)-plane over the reciprocal lattice. This bundle is topologically nontrivial.

6. The results given in the preceding subsections for the spectrum are valid also for an irrational magnetic field \( (2\pi\tau)^2 eB \), since they depend continuously on \( \phi \) in our case. It is possible to identify directly the eigenfunctions of the ground state in this case, too. To be sure, the Bloch functions \( \psi \) no longer have any meaning, since no enlargement of the lattice will reduce the problem to an integer flux. It is possible, however, to propose the following prescription: we consider a uniform field \( B_0 \) and obtain for its some eigenfunction basis (for any, possibly irrational, value of the flux) in the form \( \psi = \exp(-iQg) \tilde{\psi}(x) \), where \( \tilde{\psi} \) takes the form (7) and \( g(x) \) is analytic, with \( \tilde{\psi} \) a continuous index. Having obtained this basis, we perform the following operations: 1) we replace the function \( \tilde{\psi} \) by \( \varphi \) in accord with (7), replacing \( B_0 \) by \( B \) where the fluxes \( \phi \) and \( \tilde{\psi} \) coincide, and 2) we replace in the quantity \( \psi \) defined by (14) the magnetic field \( B_0 \) by \( B \).

This pair of operations is performed, strictly speaking, on the functions \( \psi \) that are meaningless in the irrational case. The answer, however, is written in the following form: the entire space of the solutions of the equation \( H, \phi = 0 \) in the field \( B \) is obtained from the analogous space of solutions in the uniform field \( B_0 \), with the same flux \( \phi_0 = \phi \), by multiplying by a single general function

\[
\psi = \exp[\gamma(x,y)]. \quad B \to B
\]

\[
\rho(x,y) = \exp\left\{ \frac{\tau}{2\pi} \sum_{l=-\infty}^{\infty} \left[ \text{sgn}(z-2l)(B_0 - B) \right] \delta(x-l) \delta(y)
\right\}
\]

\[
\quad - \frac{\tau}{2} \left[ \text{sgn}(z-2l) \left( B_0 - B \right) \right] \delta(x-l) \delta(y)
\]

\[
\quad + (i\pi) \left[ \text{sgn}(z-2l) \left( B_0 - B \right) \right] \delta(x-l) \delta(y)
\]

\[
\gamma = x + iy
\]

It follows from the results of this research that the case of a periodic field \( B \) is more natural than the class of finite localized fields, considered by Aharanov and Casher, among which there is no uniform field. The complexity of the phenomena in the periodic case and, in particular, the appearance of a nontrivial bundle in the description of the magnetic-Bloch functions (see Sec. 3, Item 5) takes place already in a uniform field if it is correctly considered.

4. PERTURBATION BY A WEAK POTENTIAL FIELD

We discuss now the question of the spreading of the ground state to form a magnetic band upon addition of a weak potential perturbation \( u(x,y) \) (electric field) periodic with the same lattice. As already indicated, the operator \( H \) at a fixed value of the spin is scalar. For a rational flux \( N^2 M = N^2 M^2 + N^2 M N \), with \( N \) and \( M \) mutually prime, we change to a lattice \( MT_1 \) and \( MT_2 \). Then the quasimomenta \( p_1 \) and \( p_2 \) are defined accurate to the addition of integer multiples \( 2\pi M T_1 \) and \( 2\pi M T_2 \), respectively. The Bloch functions \( \psi \) of the lattice \( MT_1 \) and \( MT_2 \) for an unperturbed operator \( H \) form a linear space of dimensionality \( MN \) at arbitrary fixed \( p_1 \) and \( p_2 \) from a unit cell \( K^* \) with area

\[
|K| = 4\pi N^2 M^2 T_1 T_2.
\]

The perturbation of the Hamiltonian \( H = H + u(x,y) \), where

\[
u(x + T_1, y) = u(x, y + T_2) = u(x,y)
\]

defines a Hermitian form \( \tilde{i} \) on the functions (8):

\[
\tilde{i}(\psi, \psi') = \int dx dy \psi^*(x,y) \psi(x,y')
\]

\[
A = (x_1, \ldots, x_N), \quad \tilde{A} = (y_1, \ldots, y_N), \quad |K| = T_1 T_2.
\]

From the periodicity of the perturbation \( u(x,y) \) with periods \( T_1 \) and \( T_2 \) it follows that:

1. the form \( \tilde{i} \) differs from zero only when the quasi-
We consider the vectors
\[ \sum_{k=1}^{M} C_k \]
by virtue of (15);

2) it follows from the structure of the magnetic-translational group that the matrix of the form \( L \) on the linear space of the functions (8) of dimensionality \( MN \) (at constant \( p_1 \) and \( p_2 \)) has in a natural basis a block-diagonal form consisting of \( M \times N \) blocks, on all of which these forms are equivalent (in particular, they have the same eigenvalues \( \epsilon \)).

To prove the last fact it is convenient to start with the minimum necessary enlargement of the lattice \( L_1, T_1, L_2, T_2 \), where \( L_2 = M \). Then the quasimomenta \( \beta_1 \) and \( \beta_2 \) are defined in modulo of the lattice \([2\pi(L_1T_1)^{1/2}, 2\pi(L_2T_2)^{1/2}] \). For the magnetic translations (10) we have
\[ T_1T_2 = (T_1' T_2'), \quad \epsilon_{T_1T_2} = \exp \left[ 2\pi i \beta_1 \beta_2 \right], \quad \gamma = \gamma_0. \]

We fix arbitrary values of the quasimomenta \( \beta_1 \) and \( \beta_2 \) and the vector \( \mathbf{c}_1 \), which is the eigenvector for \((T_1')^T \) and \((T_2')^T \) with eigenvalues
\[ \exp \left[ i(p_1, T_1) \right], \quad \exp \left[ i(p_2, T_2) \right]. \]
We consider the vectors
\[ c_i = (p_1, T_1) c_1, \] where \( q = 0, 1, \ldots, L_1 - 1 \), \( s = 0, 1, \ldots, L_2 - 1 \). From (20) we get
\[ (T_1')^T c_i = \exp \left[ i(p_1 + 2\pi M s T_1') T_1 L_1 \right] c_i. \]

Since everything is periodic with periods \( T_1 \) and \( T_2 \), the operators \( T_1 ' \) and \( T_2 ' \) commute with the perturbed Hamiltonians. Consequently the quadratic form \( \epsilon \) is the same (equivalent) for all the quasimomenta obtained by the transformation
\[ (p_1, p_2) \rightarrow (p_1 + 2\pi M s T_1', p_2 + 2\pi M t T_2'). \]

By virtue of (23) we can change to the reciprocal lattice
\[ (2\pi M^{-1} T_1', 2\pi M^{-1} T_2'), \] i.e., each of the quasimomenta \( \beta_1 \) and \( \beta_2 \) from the unit cell \( K^* \) with area \( |K^*| = 4\pi M^{-1} T_1' T_2' \) must be set in correspondence with a direct sum of all \( N \)-dimensional spaces with quadratic form \((p_1, p_2)\), of which there are exactly \( M \) by virtue of (22), where \( M = L_1 L_2 \). The block character of the form on the \( MN \) dimensional space is now obvious, and the preservation of the \( M \)-fold degeneracy for a potential perturbation periodic in \( T_1 \) and \( T_2 \) follows from obvious arguments.

The metric on the family of functions (6) is given by
\[ \epsilon(p_1, p_2) = \int \frac{1}{2} \left[ \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2} \right] \]
The dispersion law for a perturbed Hamiltonian is defined as the extremas of the quadratic function \( \mathcal{C}(\mathbf{c}_1, \mathbf{c}_2) \) under the condition \( \mathcal{C}(\mathbf{c}_1, \mathbf{c}_2) = \text{const} \):
\[ \mathcal{C}(\mathbf{c}_1, \mathbf{c}_2) = q^2(q_0, k_0) = \text{const}, \]

\[ \text{[the gradient is with respect to the variables } A \text{ and } A^{-1}, a, \text{ and at a fixed quasimomentum]}. \]

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The dispersion law \( \epsilon(p_1, p_2) \), as indicated above, is \( M \)-fold degenerate. We have in fact, generally speaking, \( N \) different eigenvalues \( \epsilon(p_1, p_2), \ldots, \epsilon(p_1, p_2) \), each of which is \( M \)-fold. This is the consequence of the rational character of the flux with denominator \( M \). We recall that the quasimomenta \( \beta_1 \) and \( \beta_2 \) are defined here in modulo \( 2\pi M^{-1} T_1' \) and \( 2\pi M^{-1} T_2' \).

The simplest cases are:
1) \( N = 1, M = 1 \). The dispersion law is defined as a single-valued continuous function on a torus, \( \epsilon(p_1, p_2) \); there is no degeneracy at fixed \( p_1 \) and \( p_2 \). Perturbations lead, obviously, to only one magnetic band.

2) \( N > 1, M = 1 \). Here too, there is only one magnetic band; at fixed \( p_1 \) and \( p_2 \) we have \( M \)-fold degeneracy ("vector bundling" on a torus). Only one magnetic band is produced (this result was first obtained by Zak for a uniform field \( B_0 \)).

For \( N > 1 \) (and arbitrary \( M \)) there are produced \( M \)-fold degenerate eigenvalues \( \epsilon(p_1, p_2), \ldots, \epsilon(p_1, p_2) \); it can be assumed that \( \epsilon = \epsilon \), at \( \pi \). The equality of two eigenvalues of complex Hermitian matrix \( \epsilon \) is specified by three real conditions. Therefore for a two-parameter family of matrices \( \epsilon(p_1, p_2) \) we can find a "typical" case assume all the eigenvalues to coincide at arbitrary \( p_1 \) and \( p_2 \). If even two branches coincide accidentally at the point \( p_1, p_2 \) they cannot be shifted on going around this point—the branches are separated by virtue of the general properties of Hermitian matrices.

We shall move by varying \( p_1 \) (under the condition \( p_2 = \text{const} \)), where \( -\pi / M T_1 < p_2 < \pi / M T_1 \) and track the eigendirection initially numbered \( j \). On going through the entire period we arrive, generally speaking to only one eigendirection \( \gamma_j \). We obtain a permutation of the eigendirections (monodromy) \( \gamma_1 \rightarrow \gamma_1, \ldots, N \rightarrow \gamma_N \).

Analogously, by varying \( p_2 \) (with \( p_1 \) constant) we obtain after the circling the permutation \( \gamma_j \rightarrow -\gamma_j \). The permutations \( \gamma_j \) and \( \gamma_j \) commute: \( \gamma_j \gamma_j = \gamma_j \gamma_j \). We can therefore obtain for both permutations \( \gamma_j \) and \( \gamma_j \) a common minimum breakdown of the set of eigendirections (1, \ldots, \( N \)) into cycles of length \( n_1, \ldots, n_N \), where

\[ \sum n_i = N. \]

Within the limit of each cycle \( \gamma_i \), the branches \( \epsilon(p_1, p_2) \) cannot be subdivided, since they are shifted after circling over the periods. In this case there are produced exactly \( \hbar \) magnetic bands, corresponding to the different cycles \( \gamma_i \), although the energy values may overlap.

The possible types of decay of the zeroth level into degenerate eigenvalues \( \epsilon(p_1, p_2), \ldots, \epsilon(p_1, p_2) \) is described by breakdown of the number \( N \) into terms
\[ N = \sum n_i. \]

A more detailed classification of these types is determined by the commutation group of the permutations \( \gamma_i \)

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and \( \gamma \).

The principal spectral characteristic in each magnetic band with number \( j \) is the density of the number of states per unit area in the interval \( dE \), which we designate by \( \rho_j(E) dE \), where

\[
\int (\sum \rho_j) dE = \frac{e^2}{2\hbar}\frac{1}{d^2}\frac{1}{4\pi^2}. \tag{1}
\]

Here \( M \) is the multiplicity of the degeneracy at fixed \( p_1 \) and \( p_2 \). Irrational numbers \((2\pi)^{-}\frac{1}{e}\) can be approximated by rational ones \( N_i/M_i \), where the numerator \( N_i \) and the denominator \( M_i \) increase if \( N_i/M_i \rightarrow (2\pi)^{-}\frac{1}{e} \).

It is easily seen that with increasing \( N \), and \( M \), the dispersion law breaks up into more and more magnetic bands, so that all the characteristics, with the exception of the total density of the number of states per unit area \( \sum \rho_j dE \), become meaningless.

At large \( N \) it is possible to pose the natural problem of calculating the statistical weights of various numbers of magnetic bands produced in the decay, bearing in mind the percentage of the configurations \( \gamma_1 \) and \( \gamma_2 \) with the number of bands in the given interval \( k \) (i.e., the common cycles of a pair of commuting permutations) relative to the total number of configurations, i.e., classes of conjugacy of the commuting permutations. This is a purely combinatorial problem.

1) The reader must be warned that the paper by Aharonov and Casher contains wrong references to the known work of Atiyah and Singer. Fortunately, these references have no bearing on the matter.

2) The quadrature non-integrability of one of the eigenfunctions in Ref. 3 points to the absence of a gap in this case.

Nonlinearity of current-voltage characteristic and magnetoresistance of the interface between the superconducting and normal phases

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We consider the dc resistance of the interface between the normal and superconducting phases at current densities of the order of the Ginzburg-Landau critical value. Account is taken of the effect of an external constant magnetic field on the interface resistance.

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1. Measurements of the resistance of superconductors in the intermediate state have shown that it exceeds the resistance of the purely normal phase.\(^1\) It was subsequently established that the additional resistance is due to the finite depth of penetration of the electric field, \( \xi \), into the superconductor. This depth greatly exceeds the coherence length \( \xi(0) \) and the depth \( X(T) \) of penetration of the electric field in the superconductor, and is due to processes that are peculiar to superconductors and are connected with the relaxation of the invariance of the populations of electron-like and hole-like branches of the excitation energy spectrum.\(^2\)

\[^1\] NORDITA, Denmark.

The population difference between the electron and hole branches of the spectrum leads to a difference between the chemical potential of the pairs, \( \mu_p = h/2|\psi_p| \) (per particle) from the chemical potential \( \mu_p = -e\phi \) of the quasiparticles (\( \phi \) is the phase shift of the order parameter and \( \psi \) is the scalar potential of the electric field). As a result, the gauge-invariant scalar potential

\[ \phi = \frac{\psi}{\frac{e}{\hbar c}} \]

differs from zero. The characteristic length over which the difference between the populations of the spectrum branches in the interior of the superconductor relaxes is in fact the depth of penetration of the electric field. Near the critical temperature, the depth...